A counterexample to a continued fraction conjecture

A COUNTEREXAMPLE TO A CONTINUED FRACTION CONJECTURE

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Abstract. It is known that if $a \in \mathbb{C} \setminus (-\infty, -1/4]$ and $a_n \to a$ as $n \to \infty$, then the infinite continued fraction with coefficients $a_1, a_2, \dots$ converges. A conjecture has been recorded by L. Jacobsen et al, taken from the unorganized portions of Ramanujan's notebooks, that if $a \in (-\infty, -1/4)$ and $a_n \to a$ as $n \to \infty$, then the continued fraction diverges. Counterexamples to this conjecture for each value of a in $(-\infty, -1/4)$ are provided. Such counterexamples have already been constructed by Glutsyuk, but the examples given here are significantly shorter and simpler.

1. Introduction

For each $n \in \mathbb{N}$, let a_n be a non-zero complex number and let t_n be the Möbius tranformation $t_n(z) = a_n/(1+z)$; then the continued fraction

$$\mathbf{K}(a_n|\,1) = \cfrac{a_1}{1 + \cfrac{a_2}{1 + \cfrac{a_3}{1 + \cdots}}}, \tag{1.1}$$

is considered to converge if the sequence with nth term equal to the n-fold composition $t_1 \cdots t_n(0)$ converges within the extended complex plane $\mathbb{C}_\infty$. We identify the continued fraction (1.1) with the sequence $t_1, t_2, \dots$ of Möbius transformations. A problem derived from the private notebooks of Ramanujan is posed in [3, page 38], which asks whether, for a given complex number $a \neq -1/4$ and a sequence $a_1, a_2, \dots$ that converges to a, the continued fraction $\mathbf{K}(a_n|\,1)$ diverges if and only if $a \in (-\infty, -1/4)$. In this paper it is demonstrated that $\mathbf{K}(a_n|\,1)$ may or may not converge if $a \in (-\infty, -1/4)$, thereby proving the conjecture to be false. Glutsyuk has already provided such examples in [4], but our methods are significantly shorter and simpler. Our conclusions are summarized in a theorem, whose proof is postponed until §3.

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Theorem 1.1. *If $a \in (-\infty, -1/4)$ then there are sequences a_n of real numbers that converge to a for which $\mathbf{K}(a_n|\,1)$ converges and there are sequences a_n of real numbers that converge to a for which $\mathbf{K}(a_n|\,1)$ diverges.*

2. Iteration of a single Möbius Transformation

To understand the dynamics of the sequence $t_1 \cdots t_n$, where $t_n(z) = a_n/(1+z)$ and $a_n \to a$ as $n \to \infty$, one must first understand the dynamics of the sequence formed through iterating the Möbius map $t(z) = a/(1+z)$. The theory of iteration of a single Möbius transformation is well known (see, for example, [1] or [5]) and it is independent of continued fractions. We elaborate briefly on this theory.

The conjugacy type of a given Möbius transformation $f(z) = (Az + B)/(Cz + D)$ may be determined from the conjugation invariant quantity $T(f) = (A+D)^2/(AD - BC)$: if $T(f) \in [0, 4)$ then f is *elliptic*; if $T(f) = 4$ then f is *parabolic*; otherwise f is *loxodromic*. Therefore t is elliptic if $a \in (-\infty, -1/4)$; parabolic if $a = -1/4$; and otherwise loxodromic.

If t is loxodromic and $a_n \to a$ as $n \to \infty$, it follows from the general theory (see [2] or [6]) that $\mathbf{K}(a_n|\,1)$ converges. If t is parabolic and $a_n \to a$ as $n \to \infty$, $\mathbf{K}(a_n|\,1)$ may converge or it may diverge, and it is easy to construct examples of both circumstances. This leaves the situation of Theorem 1.1, when t is elliptic. Elliptic maps are by definition conjugate to Möbius maps of the form $z \mapsto e^{i\theta} z$, where $\theta \in (0, 2\pi)$, hence $t^n(0)$ diverges (since 0 is not a fixed point of t), that is, $\mathbf{K}(a|\,1)$ diverges. Thus, for one part of Theorem 1.1 we may choose a_n to be the constant sequence $a, a, \ldots$. The other part of Theorem 1.1 is proved in §3.

3. Proof of Theorem 1.1

We need a preliminary lemma.

Lemma 3.1. *The subset of $(-\infty, -1/4)$ consisting of those numbers $a \in (-\infty, -1/4)$ for which $t(z) = a/(1+z)$ is a map of finite order is a dense subset of $(-\infty, -1/4)$.*

Proof. Let t be conjugate to $g(z) = e^{i\theta} z$, $\theta \in (0, 2\pi)$; then

$$-1/a = T(t) = T(g) = 4\cos^2 \tfrac{1}{2}\theta. \tag{3.1}$$

The maps t and g are of finite order if and only if θ is a rational multiple of π, and rational multiples of π are dense in $(0, 2\pi)$. The result is assured by continuity of the correspondence (3.1). □

Proof of Theorem 1.1. We construct a sequence a_n that converges to $a \in (-\infty, -1/4)$ for which $\mathbf{K}(a_n|\,1)$ converges. By Lemma 3.1, we may choose a sequence $\alpha_1, \alpha_2, \dots$ in $(-\infty, -1/4)$ that converges to a for which each map $s_n(z) = \alpha_n/(1+z)$ is of finite order. Let $\epsilon_1, \epsilon_2, \dots$ be a sequence in $(0,1)$ that converges to 0 for which $\sum \epsilon_n$ diverges. Define $t_n(z) = (1-\epsilon_n)\alpha_n/(1+z)$, for $n = 1, 2, \dots$. One may easily verify that

$$t_n s_n^{-2} t_n(z) = z + \epsilon_n. \tag{3.2}$$

Since s_n is of finite order, the two equal quantities in equation (3.2) are also equal to the m-fold composition $t_n s_n \cdots s_n t_n(z)$, where $m = \text{order}(s_n)$.

For each n, choose an integer N_n such that $N_n\epsilon_n$ is greater than the maximum element from the finite set

$$\{\, |t_{n+1}s_{n+1}^q(0)| \,:\, 0 \leqslant q \leqslant \text{order}(s_{n+1}) - 2,\ t_{n+1}s_{n+1}^q(0) \neq \infty \,\}. \tag{3.3}$$

Let ϕ_n represent the string of maps $t_n, s_n, \dots, s_n, t_n$, in which s_n occurs $\text{order}(s_n) - 2$ times. The continued fraction corresponding to the sequence of Möbius maps

$$\phi_1, \dots, \phi_1, \phi_2, \dots, \phi_2, \dots \tag{3.4}$$

is the example we require, where the string ϕ_n occurs in the continued fraction N_n times. To see that (3.4) provides an example of the required form, notice that the coefficients a_n arise from the maps s_n or t_n, thus certainly $a_n \to a$ as $n \to \infty$. It remains to demonstrate that the continued fraction converges (to ∞). This is true as

$$(t_1 s_1^{-2} t_1)^{N_1} \cdots (t_n s_n^{-2} t_n)^{N_n}(z) = z + \sum_{i=1}^{n} N_i\epsilon_i,$$

by equation (3.2), hence

$$\begin{aligned} &(t_1 s_1^{-2} t_1)^{N_1} \cdots (t_n s_n^{-2} t_n)^{N_n} (t_{n+1} s_{n+1}^{-2} t_{n+1})^p t_{n+1} s_{n+1}^q(0) \\ &= t_{n+1}s_{n+1}^q(0) + p\epsilon_{n+1} + \sum_{i=1}^{n} N_i\epsilon_i \\ &> \sum_{i=1}^{n-1} N_i\epsilon_i, \end{aligned}$$

by (3.3), where $0 \leqslant p < N_{n+1}$ and $0 \leqslant q \leqslant \text{order}(s_{n+1}) - 2$. Therefore, the continued fraction converges to ∞. □

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