A product formula and combinatorial field theory

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We treat the problem of normally ordering expressions involving the standard boson operators a, $a^\dagger$ where $[a, a^\dagger] = 1$. We show that a simple product formula for formal power series — essentially an extension of the Taylor expansion — leads to a double exponential formula which enables a powerful graphical description of the generating functions of the combinatorial sequences associated with such functions — in essence, a combinatorial field theory. We apply these techniques to some examples related to specific physical Hamiltonians.

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1 Introduction

The normally ordered form of an expression involving ladder or, more generally, field operators is defined as one in which all annihilation operators are moved to the right using the appropriate commutation rules. Such expressions play an important role wherever the Fock space representation is used. In field theory and in many body quantum mechanics, normal order enters the formalism through the Wick theorem [1]. This enables us to represent field–theoretical m–point Green functions or statistical m–point correlation functions (both defined as the vacuum mean values of products of the field operators) by expressions within which all initially taken operators are contracted. Another advantage of the normally ordered operators is seen if their matrix elements are calculated in the coherent states representation [2]. If the coherent states are introduced as eigenstates of the annihilation operators such matrix elements automatically become functions of the complex variables and provide us the Fock–Bargmann representation of quantum mechanical quantities.

Quantum optics is one branch of quantum physics where this approach is a basic tool widely used in numerous applications.

The intersection between the boson normal ordering problem and combinatorics was discovered more than thirty years ago. The first seminal result [4] was that, for both bosons and fermions, ordering a general string (*i.e.*, a product of nonnegative integer powers) of the canonical creation and annihilation operators results in an expansion in which the coefficients are combinatorial numbers called *rook numbers*. The second seminal result [5] expressed the boson string $\left(a^\dagger a\right)^n$, where $[a, a^\dagger] = 1$, as

$$\left(a^\dagger a\right)^n = \sum_{k=1}^{n} S(n,k) a^{\dagger k} a^k \,, \tag{1}$$

that is in terms of classical combinatorial numbers known as *Stirling numbers of the second kind*. Both results, especially the second one, being the "physicists version" of the relation

$$\left(x \frac{\mathrm{d}}{\mathrm{d}x}\right)^n = \sum_{k=1}^{n} S(n,k) x^k \frac{\mathrm{d}^k}{\mathrm{d}x^k} \tag{2}$$

known to mathematicians much earlier, have inspired investigations which lead one to conclude that the relation between the normal ordering problem and combinatorics is not accidental. Generalizations of (1) to the normally ordered expressions being more complicated than $\left(a^\dagger a\right)^n$, [6] – [9], have led to the introduction of new classes of combinatorial numbers, whose properties were elucidated using standard methods of combinatorial analysis — recurrences, generating functions and graph representations. Normal ordering for noncanonical operators has also been investigated, in particular q-bosons, leading to q–generalizations of Stirling and Bell numbers [10], [11], [12]. This work has also revealed the relation between matrix elements taken in the coherent states representation and the Bell numbers, [13], [14], [6], [7].

As noted above, all these results indicate a fundamental connection between the general normal ordering problem and combinatorics. Most of field theoretical calculations employ operator expressions reduced to the normally ordered form. This leads one to believe that the methods of combinatorial analysis will be useful in understanding and solving problems of quantum physics. Our main goal in this work is to justify this statement. We shall provide the reader with a general method for constructing normally ordered expressions and shall explain how to link them to well-known combinatorial problems. We shall also present analogies between our methods and those of standard field theory, in particular Feynman diagrams. Finally, we will illustrate this approach using examples arising in one mode boson normal ordering.

2 The product formula

Let $f(x) = \sum\limits_{n=0}^{\infty} f_n x^n/n!$ and $g(x) = \sum\limits_{n=0}^{\infty} g_n x^n/n!$ be two formal power series, also called the exponential generating functions (egf) of sequences $\{f_n\}_{n=0}^{\infty}$ and

$\{g_n\}_{n=0}^{\infty}$, respectively. Then

$$f\left(\lambda\frac{\mathrm{d}}{\mathrm{d}x}\right)g(x)\bigg|_{x=0} = g\left(\lambda\frac{\mathrm{d}}{\mathrm{d}x}\right)f(x)\bigg|_{x=0} = \sum_{n=0}^{\infty} f_n \cdot g_n\,\frac{\lambda^n}{n!}\,, \tag{3}$$

which is a straightforward consequence of the elementary relation

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left(\frac{x^m}{m!}\right)\bigg|_{x=0} = \delta_{nm}\,, \tag{4}$$

reducing the usual Cauchy product of series to the point–wise Hadamard product.

If we apply this result to a function $F(\hat{w})$ of an operator $\hat{w}$ then for any indeterminate λ

$$F(\lambda\hat{w}) = F\left(\lambda\,\frac{\mathrm{d}}{\mathrm{d}x}\right)\mathrm{e}^{x\hat{w}}\bigg|_{x=0}\,. \tag{5}$$

On taking the normal form $\mathcal{N}[\,\cdot\,]$ of the both sides

$$\mathcal{N}[F(\lambda\hat{w})] = F\left(\lambda\frac{\mathrm{d}}{\mathrm{d}x}\right)\mathcal{N}(\mathrm{e}^{x\hat{w}})\bigg|_{x=0}\,. \tag{6}$$

We emphasize that on the left–hand side above the functional and operator aspects are mixed while on the right–hand side they are distinct. The functional aspects are given by a (formal) series in usual derivatives while the operator aspects are described by an universal expression — namely the normally ordered exponential of $\hat{w}$, in general a *word* $\hat{w}(a,a^\dagger)$ in terms of the operators a, $a^\dagger$. This means that (6), which we shall call the *product formula*, enables us to reformulate the general normal ordering problem into a normal ordering of $\mathrm{e}^{x\hat{w}}$. To calculate it explicitly still remains a non-trivial mathematical task but the problem is tractable for a large class of physically interesting examples. In this note we shall consider the cases where $\hat{w}(a,a^\dagger)$ is either a product of positive powers of a and $a^\dagger$, or a power of $a+a^\dagger$.

3 The double−exponential formula

Using the notation

$$\mathcal{N}(\mathrm{e}^{x\hat{w}}) \equiv\, :G_{\hat{w}}(x,a,a^\dagger):\,, \tag{7}$$

where the symbol : : denotes that the function $G_{\hat{w}}(x,a,a^\dagger)$ is normally ordered assuming that $a^\dagger$ and a *commute*, [2], [3], we can rewrite (6) as

$$\mathcal{N}[F(\lambda\hat{w})] = F\left(\lambda\frac{\mathrm{d}}{\mathrm{d}x}\right):G_{\hat{w}}(x,a,a^\dagger):\bigg|_{x=0}\,. \tag{8}$$

For many physical applications the general form of F is given by

$$F(x) = \exp\left(\sum_{m=1}^{\infty} L_m\frac{x^m}{m!}\right) \tag{9}$$

and G may also be written in exponential form

$$:G_{\hat{w}}(x,a,a^\dagger) := :\exp\left(\sum_{n=1}^{\infty} V_n^{(\hat{w})}(a,a^\dagger)\frac{x^n}{n!}\right): \,. \tag{10}$$

Substituting (9) and (10) in (8) we arrive at the so-called *double–exponential formula*

$$\mathcal{N}[F(\lambda\hat{w})] = \exp\left(\sum_{m=1}^{\infty} \frac{L_m}{m!}\,\lambda^m\,\frac{\mathrm{d}^m}{\mathrm{d}x^m}\right)\cdot :\exp\left(\sum_{n=1}^{\infty} V_n^{(\hat{w})}(a,a^\dagger)\,\frac{x^n}{n!}\right):\Bigg|_{x=0}\,, \tag{11}$$

which is a convenient starting point for our analysis. Eqn. (11) means that using only simple derivations we can find solution to the normal ordering problem. Thus the function F defines the sequence $\{L_m\}_{m=0}^{\infty}$ while the operator $\hat{w}$ defines the sequence $\{V_n\}_{n=0}^{\infty}$. Examples of the solutions to the latter problem are [9]

$$\begin{aligned}
\hat{w} = a^\dagger a\,, \qquad & \mathcal{N}\big[\exp(xa^\dagger a)\big] = :\exp\big[a^\dagger a(\mathrm{e}^x-1)\big]:\,, \\
& V_n^{(\hat{w})}(a,a^\dagger) = a^\dagger a\,, \\
\hat{w} = (a^\dagger)^r a\,, \qquad & \mathcal{N}\left[\exp\big(x(a^\dagger)^r a\big)\right] = \\
& =:\exp\left[a^\dagger a\sum_{n=1}^{\infty}(a^\dagger)^{(r-1)n}(r-1)^n\frac{\Gamma\left(n+1/(r-1)\right)}{\Gamma\left(1/(r-1)\right)}\frac{x^n}{n!}\right]:\,, \\
& V_n^{(\hat{w})}(a,a^\dagger) = a^\dagger a(a^\dagger)^{(r-1)n}(r-1)^n\frac{\Gamma\left(n+1/(r-1)\right)}{\Gamma\left(1/(r-1)\right)}\,, \\
\hat{w} = a + a^\dagger\,, \qquad & \mathcal{N}\big[\mathrm{e}^{x(a+a^\dagger)}\big] =: \mathrm{e}^{x^2/2}\mathrm{e}^{x(a+a^\dagger)}:\,, \\
& V_1^{(\hat{w})}(a,a^\dagger) = a + a^\dagger\,, \quad V_2^{(\hat{w})}(a,a^\dagger) = 1\,, \\
& V_n^{(\hat{w})}(a,a^\dagger) = 0 \quad \text{for } n > 2\,.
\end{aligned} \tag{12}$$

4 Multivariate Bell polynomials

Applying the *double exponential formula* requires some effort. It is easy to see that it works effectively if we deal with monomials in both exponentials but leads to rather tedious calculations in more complicated cases. It is more practical to expand both exponents as formal power series; a general method for this is the theory of *multivariate Bell polynomials* [15], [16].

Multivariate Bell polynomials arose from the question of constructing the Taylor–Maclaurin expansion of the composite function $f(g(x))$. For any $f(x) = \sum_{n=1}^{\infty} f_n\frac{x^n}{n!}$ and $g(x) = \sum_{n=1}^{\infty} g_n\frac{x^n}{n!}$ given as formal power series one gets

$$f(g(x)) = [f\circ g](x) = \sum_{n=1}^{\infty} F_n[f;g]\,\frac{x^n}{n!}\,, \tag{13}$$

where

$$F_n[f;g] = \sum_{k=1}^{n} B_{nk}(g_1, g_2, \ldots, g_{n-k+1}) f_k \,. \tag{14}$$

The coefficients B_{nk} are certain polynomials in the Taylor coefficients g_i called multivariate Bell polynomials, or sometimes more simply but unprecisely[1]), Bell polynomials.

The multivariate Bell polynomials are closely related to combinatorial numbers. They satisfy

$$B_{n,k}(g_1, \ldots, g_{n-k+1}) = \sum_{\{\nu_i\}}{}'' \frac{n!}{\prod\limits_{j=1}^{n} [\nu_j!\,(j!)^{\nu_j}]} \, g_1^{\nu_1} g_2^{\nu_2} \ldots g_{n-k+1}^{\nu_{n-k+1}} \,, \tag{15}$$

where the summation $\sum\limits_{\{\nu_i\}}{}''$ is over all possible non-negative $\{\nu_i\}$ which are partitions of an integer n into sum of k integers, *e.i.*,

$$\sum_{j=1}^{n} j\nu_j = n \,, \quad \sum_{j=1}^{n} \nu_j = k \,. \tag{16}$$

From (15) and (16) one may show that the multivariate Bell polynomials satisfy, for a and b arbitrary constants, the homogeneity relation

$$B_{n,k}(ab^1 g_1, ab^2 g_2, \ldots, ab^{n-k+1} g_{n-k+1}) = a^k b^n \, B_{n,k}(g_1, g_2 ..., g_{n-k+1}) \,. \tag{17}$$

Recalling (12) one expects that the multivariate Bell polynomials of especial use to us are those from exponential generating functions

$$\begin{aligned} \exp\bigl(g(x)\bigr) &= \sum_{n=1}^{\infty} Y_n[g] \frac{x^n}{n!} \,, \\ Y_n[g] &= \sum_{k=1}^{n} B_{nk}(g_1, g_2, \ldots, g_{n-k+1}) \,, \end{aligned} \tag{18}$$

obtained from (13) and (14) for $f_k = 1$, $k = 1, 2, \ldots$.

It may also be seen from (15) and (16) that $B_{n,k}(1, \ldots, 1)$ are the Stirling numbers of the second kind. In such a case (14) gives

$$\exp\bigl(u(\exp x - 1)\bigr) = \sum_{n=1}^{\infty} \left(\sum_{k=1}^{n} S(n,k) u^k \right) \frac{x^n}{n!} \,, \tag{19}$$

[1]) The Bell, or exponential, polynomials are defined as $B_n(u) = \sum\limits_{k=0}^{n} S(n,k)u^k$, *i.e.*, as polynomials which coefficients are the Stirling numbers of the second kind and $B_n(1) = B_n$ are the Bell numbers.

and

$$\exp\big(\exp x - 1)\big) = \sum_{n=1}^{\infty} B_n \frac{x^n}{n!}\,, \tag{20}$$

with B_n denoting the Bell numbers.

The second useful case is $\exp\left(g_1 x + g_M \dfrac{x^M}{M!}\right)$, *i.e.*, the case when only two g's do not vanish. We have

$$\begin{aligned} \exp\left(g_1 x + g_M \frac{x^M}{M!}\right) &= \sum_{n=0}^{\infty} H_n^{(M)}(g_1, g_M)\frac{x^n}{n!}\,, \\ H_n^{(M)}(g_1, g_M) &= n! \sum_{r=0}^{[n/M]} \frac{g_1^{n-Mr} g_M^r}{(n-Mr)!r!(M!)^r}\,, \end{aligned} \tag{21}$$

where $H_n^{(M)}(g_1, g_M)$ are called the two variable Hermite–Kampé de Fériet polynomials. They are generalizations of the standard Hermite polynomials, [17].

Another important property is the *inversion formula*. This states that the following two expressions

$$Y_n\,[g] = \sum_{k=1}^{n} B_{n,k}(g_1, \ldots, g_{n-k+1}) \tag{22}$$

and

$$g_n = \sum_{j=1}^{n} (-1)^j (j-1)! B_{n,j}\big(Y_1[g], \ldots, Y_{n-j+1}[g]\big)\,, \tag{23}$$

are inverse to each other if this notion is understood in the following way: an arbitrary series $\{Y_n[g]\}_{n=1}^{\infty}$ can be obtained in the form (22) if we choose the series $\{g_n\}_{n=1}^{\infty}$ only as in (23). It means that we are really able to change the double exponential formula into power series and *vice versa*. Many other properties of the multivariate Bell polynomials are also known, together with their explicit forms for some basic (elementary) functions.

5 The coherent states representation and the analogy to field theory

As mentioned in the Introduction using the coherent states representation of the normally ordered strings of a and $a^\dagger$ allows us to dispense with operators and deal with functions of a complex variable z and its conjugate z^*. Taking the coherent states mean value of the *double exponential formula* we get

$$\big\langle z\big|\mathcal{N}[F(\lambda\hat{w})]\big|z\big\rangle = \exp\!\left(\sum_{m=1}^{\infty} \frac{L_m}{m!}\,\lambda^m \frac{\mathrm{d}^m}{\mathrm{d}x^m}\right) \cdot \exp\!\left(\sum_{n=1}^{\infty} V_n^{(\hat{w})}(z, z^*)\frac{x^n}{n!}\right)\Bigg|_{x=0}\,, \tag{24}$$

which, for the case $z = 1$ and for $V_n^{(\hat{w})} := V_n^{(\hat{w})}(1,1)$, becomes

$$\big\langle 1\big|\mathcal{N}[F(\lambda\hat{w})]\big|1\big\rangle = \exp\!\left(\sum_{m=1}^{\infty} \frac{L_m}{m!}\,\lambda^m \frac{\mathrm{d}^m}{\mathrm{d}x^m}\right) \cdot \exp\!\left(\sum_{n=1}^{\infty} n^{(\hat{w})} \frac{x^n}{n!}\right)\Bigg|_{x=0}\,, \tag{25}$$

identical with the *counting formula* of [18], used there in order to enumerate the Feynman–type diagrams in zero–dimensional analogues of the field theoretical models.

The field theoretical analogy may be pushed further on recalling the functional formalism of the field theory, [19]. A basic quantity which defines any field theory model is the generating functional of the Green functions. Physically it is interpreted as the vacuum–vacuum transition amplitude of the time–ordered exponential of the quantum field operator in the Heisenberg picture $\hat{\phi}_{\mathrm{H}}(x)$ coupled to an external current $J(x)$

$$G(J) = \left(0 \left| T \exp\left(\mathrm{i} \int \mathrm{d}^D x \hat{\phi}_{\mathrm{H}}(x) J(x)\right)\right| 0\right) \tag{26}$$

from which the m–point Green functions are got as the m–th functional derivatives with respect to J. Passing to the interaction picture one obtains

$$G(J) = \langle 0 | T \exp\left(\mathrm{i} \int \mathrm{d}^D x \left(\mathcal{S}_{\mathrm{int}}(\hat{\phi}_{\mathrm{I}}(x)) + \hat{\phi}_{\mathrm{I}}(x) J(x)\right)\right) | 0 \rangle, \tag{27}$$

where $\mathcal{S}_{\mathrm{int}}$ is an interaction Lagrangian density. Rewriting $G(J)$ in terms of a functional integral

$$G(J) = \int [\mathrm{d}\phi] \exp\left(\mathrm{i}\left(S_0 + S_{\mathrm{int}} + \phi J\right)\right), \tag{28}$$

with $S_0 = \int \mathrm{d}^D x \mathcal{L}_0$ and $S_{\mathrm{int}} = \int \mathrm{d}^D x \mathcal{L}_{\mathrm{int}}$ denoting free (bilinear) and interaction action functionals for the field $\phi(x)$, and $\phi J = \int \mathrm{d}^D x \phi(x) J(x)$, respectively, one notes that $G(J)$ is an analogue of the partition function of statistical mechanics. Another expressions for the generating functional of the Green functions are those given using functional differential operators

$$\begin{aligned} G(J) &= \exp\left(\frac{1}{2}\,\frac{\delta}{\delta\phi}\,\Delta\frac{\delta}{\delta\phi}\right) \exp\left(\mathrm{i}\left[S_{\mathrm{int}}(\phi) + \phi J\right]\right)\Big|_{\phi=0} = \\ &= \exp\left[\mathrm{i} S_{\mathrm{int}}\left(-\mathrm{i}\frac{\delta}{\delta J}\right)\right] \cdot \exp\left[-\frac{1}{2}\, J \Delta J\right], \end{aligned} \tag{29}$$

where $\Delta = \Delta(x, y)$ is the causal Green function of the free field equation generated by a free action, S_{int} is, as previously, an interaction action and we use abbreviations

$$\begin{aligned} \frac{\delta}{\delta\phi}\,\Delta\,\frac{\delta}{\delta\phi} &= \int \mathrm{d}^D x\, \mathrm{d}^D y\, \frac{\delta}{\delta\phi(x)}\, \Delta(x, y) \frac{\delta}{\delta\phi(y)}\,, \\ J \Delta J &= \int \mathrm{d}^D x\, \mathrm{d}^D y J(x) \Delta(x, y) J(y)\,, \\ J\phi &= \int \mathrm{d}^D x J(x) \phi(x)\,. \end{aligned} \tag{30}$$

Equivalence of both formulas of (29) comes from the identity

$$D\left(-\mathrm{i}\frac{\delta}{\delta J}\right)T(J) = T\left(-\mathrm{i}\frac{\delta}{\delta \phi}\right)\big[D(\phi)\exp\big(\mathrm{i}\phi J\big)\big]\Big|_{\phi=0} \tag{31}$$

satisfied by arbitrary functionals D and T having formal Taylor expansions around zero [20]. Here we remark that while in the derivation of the first equality of (29) the normal ordering is extensively used, the second equality, as well as (31), may be obtained by manipulating the functional integral representation of the Green functions generating functional. Although much simpler that the standard advanced field theoretical methods, our approach give essentially the same formulae, which are directly applicable to the time evolution operator or to the *partition function integrand* [21]. This we consider as a strong argument in favour of the present method.

6 The counting formula, graphs and combinatorial field theory

In the effort to understand the meaning of perturbation expansions in quantum physics, both in quantum mechanics and in quantum field theory, it is important to know their large order behaviour. Solving the problem for the coupling constant perturbation series one finds that the number of the Feynman diagrams, contributing order by order to the perturbation series coefficients, grows factorially, [22], [23], [24]. The factorial growth occurs because of the combinatorial reasons which (at least qualitatively and for the large orders asymptotics) explain why such a behaviour does not essentially depend on details of the model if one disregards the problem of *renormalons*[2]). Enumeration of diagrams for more complicated models is echoed by that of their zero–dimensional analogues, [25], [26]. Taking all diagrams of a given order to have equal values shows that this is the number of graphs which determines the divergence of the (renormalized) perturbation series. Being able to estimate these numbers order by order we may investigate a perturbation series, estimate its asymptotic character and investigate resummation methods. The idea is to search for generating functions of given sequences of (combinatorial) numbers, *i.e.,* to look for functions which enumerate the same weighted objects as the initial formal series do. If found in analytic form such functions may be treated as generalized sums of formal, *i.e.,* divergent and so analytically meaningless, series. We shall call such an approach a "combinatorial field theory" emphasizing that the notion should be understood in a two senses. The first sense is to obtain a generating function for the perturbation series, using any of the standard methods. The second, or inverse sense, is to construct interaction which leads to a postulated set of combinatorial numbers — following the statement of reference [18] that *given a sequence of numbers* $\{a_n\}$*, it is always possible to find a set of Feynman rules that reproduce that sequence.*

[2]) Renormalons are a peculiar subclass of the Feynman diagrams which renormalized values grow factorially with the order of the graph. Renormalons appear in renormalizable models and are known to influence perturbation expansions of the nonabelian gauge theories.

The zero–dimensional analogue of (29) taken for $J = 0$ (*i.e.*, the generating functional for the *bubble* Feynman graphs in the field–theoretical jargon) is a particular case of the right hand side of (25). So the latter formula may be considered as a recipe which counts the Feynman–like graphs in a model for which the potential is $\sum_{n=1}^{\infty} V_n \frac{x^n}{n!}$ and for which we deal with "multilegged propagators" of strenghts L_m, [26], [27]. Using (19) and (20) we have for g's related both to $\{L_m\}$ and $\{V_n\}$

$$e^{ug(x)} = \sum_{n=0}^{\infty} \frac{x^n}{n!} \sum_{k=1}^{n} u^k Y_{nk}([g]) = \sum_{n=0}^{\infty} \frac{x^n}{n!} Y_n([g], u)\,, \tag{32}$$

and for (25) we get

$$\left\langle 1 \middle| \mathcal{N}[F(\lambda \hat{w})] \middle| 1 \right\rangle = Z(L, V, \lambda) = \sum_{n=0}^{\infty} A_n(L, V) \frac{\lambda^n}{n!}\,, \tag{33}$$

using a notation

$$A_n(L, V) = Y_n([L]) \cdot Y_n([V])\,. \tag{34}$$

The standard example which illustrates how the counting formula works is the the zero–dimensional analogue of the anharmonic oscillator or ϕ^4 model. It gives

$$\exp\left(\frac{a}{2!}\frac{\mathrm{d}^2}{\mathrm{d}x^2}\right) \cdot \exp\left(\frac{-gx^4}{4!}\right)\Big|_{x=0} = \frac{1}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{\Gamma(2n + \frac{1}{2})}{n!} \left(\frac{-ag}{4!}\right)^n . \tag{35}$$

This is a consequence of

$$\exp\left(y \frac{\mathrm{d}^2}{\mathrm{d}x^2}\right) x^n = H_n(x, y)\,, \tag{36}$$

where $H_n(x, y) = H_n^{(2)}(x, y)$ are the Hermite–Kampé de Fériet polynomials, quoted in the Sect.4, (21). For $x = 0$ they take nonvanishing values only for even n

$$H_{2k}^{(2)}(0, y) = \frac{(2k)!}{k!} \frac{y^k}{2^k}\,. \tag{37}$$

The expansion in (35) may be equally considered either as the expansion in the number of vertices, *i.e.*, in g^n or as the expansion in the number of lines, *i.e.*, in a^n. In (33) the expansion parameter is λ which counts derivatives and, in a graph representation, produces lines or "propagators". We shall adopt this convention in what follows; however, each approach may be translated into the other using (31).

The series at the right hand side of (35) is divergent which poses the question as to its meaning. The answer is given if one notices that the same series is got when one calculates a formal power series expansion in g for the integral

$$I(a, g) = \frac{1}{\sqrt{\pi a^{1/2}}} \int_{-\infty}^{\infty} \mathrm{d}x \exp\left(-\frac{x^2}{\sqrt{a}} - \frac{gx^4}{4!}\right). \tag{38}$$

The integral above does not define a function analytical for $g = 0$ because it does not exist for $g < 0$. Being well defined for $g > 0$ it admits an analytic continuation to $g \in \{C^2 \setminus [0, -\infty)\}$ which is given by

$$I(a,g) = \frac{2}{\sqrt{2\pi}} \left(\frac{3}{ag}\right)^{1/2} \exp\left(\frac{3}{ag}\right) K_{1/4}\left(\frac{3}{ag}\right), \tag{39}$$

where K is the Macdonald function, *i.e.*, a Bessel function of the imaginary argument. This allows to give a meaning to the series in (35) — it is an asymptotic expansion of (39) for $g \to 0$, $g \in \{C^2 \setminus [0, -\infty)\}$. So the series under consideration may be treated as summable in a generalized sense (nonuniquely, *modulo* functions obeying zeroth asymptotic expansions) to the properly defined function.

As explained in the Sect.4 we are able to find $\{A_n\}_{n=0}^{\infty}$ either constructing the multivariate Bell polynomials for sequences $\{L\}_{n=0}^{\infty}$ and $\{V\}_{n=0}^{\infty}$ or by identifying factors in the exponential formula with known exponential generating functions. Formula (33) also gives A_n a combinatorial interpretation. Namely, the number A_n is obtained as the number of all, connected and disconnected, graphs with the same number n of labelled lines. The Feynman–like graphs representation is however different from the conventional one of field theory. For the first thing, we classify graphs using the number of lines, not vertices. This does not matter for the simplest examples having monomials in both exponentials, but for more complicated normally ordered expressions our method leads to an alternative description. We refer the reader to [9] and [28] for details; here we list only the rules of the graph construction and define those multiplicity factors which are necessary in order to understand pictorial illustrations of examples in the next section. They are:
— a line starts from a white dot, the *origin*, and ends at a black dot, the *vertex*,
— we associate strengths V_k with each vertex receiving k lines and multipliers L_m with a white dot which is the origin of m lines,
— to count such graphs we calculate their multiplicity due to the labelling of lines and the factors L_m and V_k.

7 Examples

We illustrate our approach using as examples the $\hat{w} = a^\dagger a$ and $\hat{w} = a^\dagger + a$. We first consider the normal ordering problem

$$\mathcal{N}\left[\exp\left(\lambda a^\dagger a + \frac{\lambda^M}{M!}(a^\dagger a)^M\right)\right], \tag{40}$$

related to Hamiltonians $\lambda a^\dagger a + \dfrac{\lambda^M}{M!}(a^\dagger a)^M$, in particular for $M = 2$ to the Kerr–type Hamiltonian

$$\mathcal{H} = \lambda a^\dagger a \left(1 + \frac{\lambda}{2} a^\dagger a\right). \tag{41}$$

In our previously introduced notation

$$L_1 = 1\,, \quad L_M = 1 \quad \text{for some } M > 1\,, \quad \text{and} \quad L_m = 0 \quad \text{otherwise}\,, \tag{42}$$

and

$$F(x) = \exp\left(x + \frac{x^M}{M!}\right); \qquad V_n^{(\hat{w})} = 1 \quad \text{for} \quad n = 1, 2, \ldots . \tag{43}$$

In order to get the Taylor–Maclaurin expansion of $F(x)$ we recall (21) and expand (43) in terms of the two variable Hermite–Kampé de Fériet polynomials $H_n^{(M)}(x, y)$

$$F(x) = \exp\left(x + \frac{x^M}{M!}\right) = \sum_{n=0}^{\infty} H_n^{(M)}(1, \tfrac{1}{M!})\, \frac{x^n}{n!} . \tag{44}$$

From (12), (33) and (34) this yields

$$A_n = H_n^{(M)}\left(1, \tfrac{1}{M!}\right) \cdot B_n , \tag{45}$$

where B_n are the Bell numbers. Combinatorially B_n count all the partitions of an n–set and $H_n^{(M)}(1, \frac{1}{M!})$ count partitions of an n–set into singletons and M–tons. For $M = 2$

$$H_n^{(2)}(1, \tfrac{1}{2}) = \left(\frac{\mathrm{i}}{\sqrt{2}}\right)^n H_n\left(-\frac{\mathrm{i}}{\sqrt{2}}\right) = 1,\ 2,\ 4,\ 10,\ 26,\ 76,\ 232,\ \ldots . \tag{46}$$

are the *involution* numbers expressible using Hermite polynomials $H_n(x)$ and the initial terms of A_n are: 1, 4, 20, 150, 1352, 15428, ... For $M = 3$ the coefficients A_n are: 1, 2, 10, 75, 527, 6293, ..., *etc.* In both cases the series diverge, as may seen from the $n \to \infty$ asymptotics, [29]

$$B_n \sim n! \frac{\exp\bigl(\exp r(n) - 1\bigr)}{[r(n)]^{n+1}\sqrt{2\pi \exp r(n)}} , \tag{47}$$

where $r(n) \sim \log n - \log(\log n)$, applying d'Alembert criterion to $A_n/n!$.

We now obtain the generating function of the sequence$\{A_n/n!\}_{n=0}^{\infty}$. First consider the *partition function integrand* corresponding to the Hamiltonian related to (40) for the particular value $z = 1$:

$$\begin{aligned}
&\Big\langle 1 \Big| \mathcal{N}\Big[\exp\left(-\beta\left(\lambda a^\dagger a + \tfrac{\lambda^M}{M!}(a^\dagger a)^M\right)\right)\Big]\Big| 1 \Big\rangle = \\
&\qquad = \sum_{n=0}^{\infty} H_n^{(M)}\left(-\beta, \tfrac{-\beta}{M!}\right) \cdot B_n \frac{\lambda^n}{n!} = \\
&\qquad = \frac{1}{\mathrm{e}} \int_0^\infty \mathrm{d}x \sum_{n=0}^{\infty} H_n^{(M)}\left(-\beta, \tfrac{-\beta}{M!}\right) \frac{(x\lambda)^n}{n!} \left(\sum_{k=0}^{\infty} \frac{\delta(x-k)}{k!}\right) = \\
&\qquad = \frac{1}{\mathrm{e}} \int_0^\infty \mathrm{d}x \exp\left[-\beta\left(x\lambda + \frac{(x\lambda)^M}{M!}\right)\right] \left(\sum_{k=0}^{\infty} \frac{\delta(x-k)}{k!}\right) = \\
&\qquad = \frac{1}{\mathrm{e}} \sum_{k=0}^{\infty} \frac{1}{k!} \exp\left[-\beta\left(k\lambda + \frac{(k\lambda)^M}{M!}\right)\right] ,
\end{aligned} \tag{48}$$

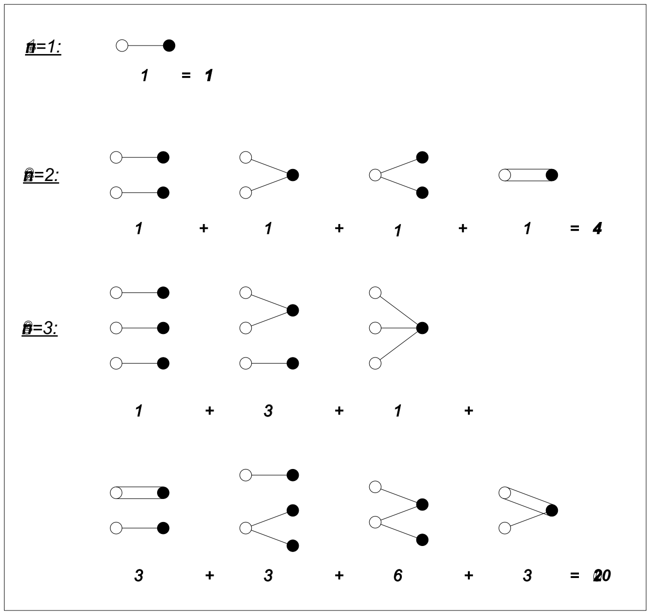

Fig. 1. Lowest order Feynman–type graphs for the Kerr–type Hamiltonian with $n = 1, 2, 3$ lines. The number below each graph is its multiplicity.

where we have expressed the Bell numbers as the Stieltjes moments of the so-called *Dirac comb*, equivalent to the representation of the Bell numbers by the Dobiński formula. Applying formal manipulations we obtain a series which is convergent for positive β and which is identical with the expression which we would get if calculated $\left\langle 1\middle|\mathcal{N}\left[\exp\left(-\beta(\lambda a^\dagger a + \frac{\lambda^M}{M!}(a^\dagger a)^M)\right)\right]\middle|1\right\rangle$ from the definition of the Glauber–Klauder–Sudarshan coherent state $|z = 1\rangle$.

Calculation can be repeated for an arbitrary coherent state using generalized Dobiński formulae, [31], for the Bell polynomials

$$B_k(|z|^2) = \sum_{l=0}^{k} S(k,l)|z|^{2l} = \int_0^\infty \mathrm{d}x\, x^k \mathrm{e}^{-|z|^2} \sum_{l=0}^{\infty} \frac{|z|^{2l}\delta(x-l)}{l!}\,. \tag{49}$$

It leads to

$$\left\langle z\middle|\mathcal{N}\left[\exp\left(-\beta(\lambda a^\dagger a + \frac{\lambda^M}{M!}(a^\dagger a)^M)\right)\right]\middle|z\right\rangle = \sum_{n=0}^{\infty} H_n^{(M)}\left(-\beta, \frac{-\beta}{M!}\right) B_n(|z|^2)\frac{\lambda^n}{n!} = \\ = \mathrm{e}^{-|z|^2}\sum_{k=0}^{\infty}\frac{|z|^{2k}}{k!}\exp\left[\beta\left(k\lambda + \frac{(k\lambda)^M}{M!}\right)\right], \tag{50}$$

As mentioned the series (50) has a physical interpretation — for $\beta = 1/kT$ it is the *partition function integrand* for the Hamiltonians related to (40). Moreover, the series (50) may be integrated with respect to $|z^2|$ term by term which leads to the partition function expressed in terms of the Jacobi theta functions and their generalizations [30] as expected. This we consider as a further argument which justifies the procedure undertaken and suggests that the method will also be effective in more complicated applications.

The second example is the word $\hat{w} = a + a^\dagger$ and the normal ordering problem of $\mathcal{N}\left(\exp \frac{1}{M!}\,(\lambda\hat{w})^M\right)$. Thus we consider

$$F(x) = \exp\left(\frac{x^M}{M!}\right), \quad M = 1,\, 2,\, 3,\, \ldots . \tag{51}$$

Because

$$\mathcal{N}\left(\mathrm{e}^{x\hat{w}}\right) = :G_{\hat{w}}(x, a, a^\dagger): = :\mathrm{e}^{x^2/2}\mathrm{e}^{x(a+a^\dagger)}:, \tag{52}$$

we have

$$V_1^{(\hat{w})}(a, a^\dagger) = a + a^\dagger\,, \quad V_2^{(\hat{w})}(a, a^\dagger) = 1\,, \quad V_n^{(\hat{w})}(a, a^\dagger) = 0 \quad \text{for}\ \ n > 2\,. \tag{53}$$

In order to get the expansion of (52) we use the modified Hermite polynomials

$$h_n(x) = \left(-\frac{\mathrm{i}}{\sqrt{2}}\right)^n H_n\left(\frac{\mathrm{i}x}{\sqrt{2}}\right), \quad \exp\left(2x + \frac{x^2}{2}\right) = \sum_{n=0}^{\infty} \frac{h_n(2)}{n!}\, x^n\,. \tag{54}$$

In general we get

$$\begin{aligned} Z_M(L, V, \lambda) &= \exp\left(\frac{\lambda^M}{M!}\frac{\mathrm{d}^M}{\mathrm{d}x^M}\right) \cdot \exp\left(2x + \frac{x^2}{2}\right)\Big|_{x=0} = \\ &= \sum_{n=0}^{\infty} \frac{h_{Mn}(2)}{n!}\left(\frac{\lambda^M}{M!}\right)^n, \end{aligned} \tag{55}$$

from which it is seen that the values of A_n are $A_{Mn} = \dfrac{(Mn)!}{(M!)^n n!}\, h_{Mn}(2)$ and zero otherwise. The particular cases for which the generating functions are known in closed forms are:

$$\begin{aligned} &M = 1\,, \quad A_n = h_n(2) = 1,\, 2,\, 5,\, 14,\, 43,\, 142,\, 499,\, 1850,\, \ldots\,, \quad n = 0,\, 1,\, 2,\, \ldots\,, \\ &\qquad Z_1(L, V, \lambda) = \exp\left(2\lambda + \frac{\lambda^2}{2}\right), \end{aligned} \tag{56}$$

and

$$\begin{aligned} &M = 2\,, \quad A_{2n} = 1,\, 5,\, 129,\, 7485,\, 755265,\, 116338005,\, \ldots\,, \\ &\qquad Z_2(L, V, \lambda) = \sum_{n=0}^{\infty} \frac{h_{2n}(2)}{n!}\left(\frac{\lambda^2}{2!}\right)^n = \frac{1}{(1-\lambda^2)^{1/2}} \exp\left(\frac{2\lambda^2}{1-\lambda^2}\right), \end{aligned} \tag{57}$$

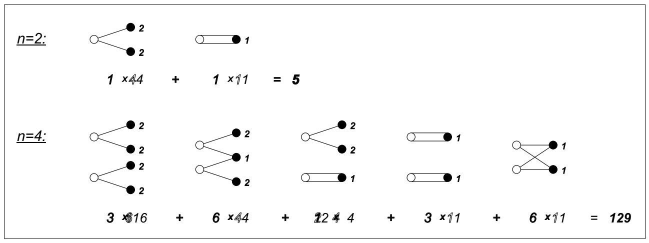

Fig. 2. Lowest order Feynman–type graphs for the superfluidity–type example with $n = 2, 4$ lines. The number below each graph is (multiplicity) $\times \prod_k$(vertex factor)$V_k = 2\delta_{k,1} + \delta_{k,2}$).

known as the Doetsch equality. Physically the latter example corresponds to a (special case of) single mode superfluidity–type Hamiltonian $\mathcal{H} \sim (a + a^\dagger)^2$), [32], while mathematically it is the solution to the normal ordering of the exponential of the general $su(1,1)$-Lie algebra element, [33].

A closed expression has also been found recently, [34], for the case $M = 3$:

$$
\begin{aligned}
Z_3(L,V,\lambda) &= \sum_{n=0}^{\infty} \frac{h_{3n}(2)}{n!} \left(\frac{\lambda^3}{3!}\right)^n = \\
&= \frac{1}{(1-\phi\lambda^3)^{1/2}} \exp\left(\phi^3 \frac{\lambda^3}{6} - \phi^4 \frac{\lambda^6}{8}\right) {}_2F_0\left(\frac{1}{6}, \frac{5}{6}; -; \frac{3\lambda^6}{2(1-\phi\lambda^3)^3}\right),
\end{aligned}
\tag{58}
$$

where $\phi(\lambda) = \dfrac{1 - \sqrt{1 - 4\lambda^3}}{\lambda^3}$ and ${}_2F_0$ is a formal series for (2,0)–generalized hypergeometric (MacRobert) function. In contrast to the $M = 1$ and $M = 2$ cases the series in (58) is divergent; however it is asymptotic and Borel summable to the function of well-defined analytic structure — the Airy function [35].

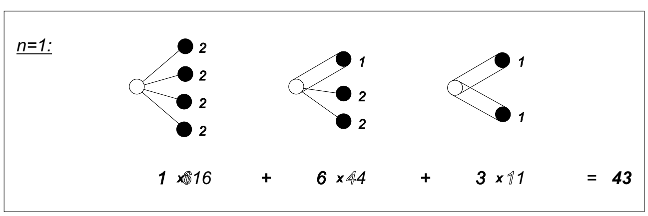

Fig. 3. Feynman–type graphs for the case $\mathcal{N}\left(\exp\left(\frac{1}{3!}\lambda(a + a^\dagger)^3\right)\right)$ with $n = 4$ lines.

The case $\mathcal{N}\left(\exp\left(\frac{y}{4!}(\lambda\hat{w})^4\right)\right)$ is interesting because it corresponds to the $g\hat{x}^4/4!$ interaction. We demonstrated in the Sect.6 that for its zero–dimensional analogue the counting formula leads to a divergent series. Solving the problem according to our scheme we do not follow the rule (55) but exploit the knowledge of

$\mathcal{N}\left(\exp\frac{(\lambda\hat{w})^2}{2!}\right)$ given by Eqn. (57). Using (6), (36), (37) and (57) we have

$$\mathcal{N}\left(\exp\left[\frac{y}{4!}\,(\hat{w})^4\right]\right) = \exp\left(\frac{y}{4!}\,\frac{\mathrm{d}^2}{\mathrm{d}x^2}\right)\mathcal{N}\big(\exp(x\hat{w}^2)\big)\Big|_{x=0} \tag{59}$$

and subsequently

$$\begin{aligned}\left\langle 1\middle|\mathcal{N}\left[\exp\left(\frac{y}{4!}\,(\lambda\hat{w})^4\right)\right]\middle|1\right\rangle &= \exp\left(\frac{y}{4!}\,\frac{\mathrm{d}^2}{\mathrm{d}x^2}\right)\left[\frac{1}{(1-2x^2)^{1/2}}\,\exp\left(\frac{4x^2}{1-2x^2}\right)\right]\Big|_{x=0} = \\ &= \sum_{n=0}^{\infty}\frac{h_{4n}(2)}{n!}\left(\frac{y\lambda^4}{2(4!)}\right)^n .\end{aligned} \tag{60}$$

The series is divergent for any $y \neq 0$ and according to our best knowledge up to now no closed form of its generating function has been found. Investigating its relation to the example of the Sect.6 we find

$$\begin{aligned}\left\langle 0\middle|\mathcal{N}\left[\exp\left(\frac{y}{4!}\,(\lambda\hat{w})^4\right)\right]\middle|0\right\rangle &= \lim_{z\to 0}\left\langle z\middle|\mathcal{N}\left[\exp\left(\frac{y}{4!}\,(\lambda\hat{w})^4\right)\right]\middle|z\right\rangle = \\ &= \lim_{z\to 0}\exp\left(\frac{y\lambda^4}{4!}\,\frac{\mathrm{d}^4}{\mathrm{d}x^4}\right)\exp\left(2x\mathrm{Re}\,z+\frac{x^2}{2}\right)\Big|_{x=0} = \\ &= \lim_{z\to 0}\exp\left(\frac{y\lambda^4}{4!}\,\frac{\mathrm{d}^4}{\mathrm{d}x^4}\right)\sum_{n=0}^{\infty}\frac{x^n}{n!}\,H_n^{(2)}\big((2\mathrm{Re}\,z),1\big)\Big|_{x=0} = \\ &= \frac{1}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{\Gamma(2n+\frac{1}{2})}{n!}\left(\frac{2^2y\lambda^4}{4!}\right)^n ,\end{aligned} \tag{61}$$

which generalized sum equals to (39) in view of (35). A similar technique may be used to find expansions of analogues of the Green functions and to investigate their summability by Borel or Padé methods but this important and interesting topic is out of the scope of this note.

8 Conclusions

The main thrust of this article has been to show the intimate relationship between the normal ordering problem and combinatorics, both of series and of graphs. We have related the graphical approach to that of Feynman diagram in field theory. Basing our analysis on a simple product formula, we have applied these combinatorial methods to investigate some simple problems of field theory and many body quantum physics. The close connection between the normal–ordering problem of the operators appearing in the theory and the associated combinatorial algebra is evident. These combinatorial methods open the possibility of investigating solutions to a variety of field theoretical problems. Differing from, and more simple than, the standard methods, this approach encourages us to believe that it may fruitfully be applied to more complex problems than those which we have treated for illustration in this article, and that it may well become a supplement to the standard methods of field theory, which are at present dominated by complex and functional analysis.

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