Bell, Group and Tangle

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The "Bell" of the title refers to bipartite Bell states, and their extensions to, for example, tripartite systems. The "Group" of the title is the Braid Group in its various representations; while "Tangle" refers to the property of entanglement which is present in both of these scenarios. The objective of this note is to explore the relation between Quantum Entanglement and Topological Links, and to show that the use of the language of entanglement in both cases is more than one of linguistic analogy.

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Entanglement is to Quantum Theory what Number Theory is to Mathematics: – The subject is fundamental and the problems are easily understood - but the solutions are elusive.

I. INTRODUCTION

The objective of this talk is to introduce the concept of Quantum Entanglement as well as some elements of Topological Entanglement such as braids and links, and explore the relation between these ideas.

Although bipartite entanglement has been well analyzed, with useful and easily-computed measures available, extending the analysis beyond the two-subspace regime is currently an open problem.

In this note we first explore the problem of extending the Von Neumann entropy-type of entanglement measure to a tripartite system, showing how a naïve extension fails.

We relate this case to the theory of links, and on the basis of this analogy introduce the concept of Braid Groups.

We then discuss the relation between braid representations and unitary entanglement-producing operators in quantum mechanics, illustrating by the examples of the Hopf Link as a quantum entangled bipartite system, and the Borromean Rings as an entangled tripartite system.

II. QUANTUM ENTANGLEMENT

A. Vector Spaces and Entanglement

A basic operation for vectors is addition. For mathematicians therefore, vector addition presents no surprises. For physicists, vector addition is such a remarkable property that in quantum mechanics the phenomena it gives rise to it go by many names, superposition rule, interference, entanglement, ...

Entanglement is a property of the vectors in *direct product spaces*, the simplest case being that of a *bipartite* space, $V_1 \otimes V_2$. If the space V_1 has basis $\{v^1_i : i = 1 \dots m\}$ and V_2 basis $\{v^2_j : j = 1 \dots n\}$ then $V_1 \otimes V_2$ has basis $\{v^1_i \otimes v^2_j : i = 1 \dots n, j = 1 \dots n\}$.

Since a vector of $V_1 \otimes V_2$ is a sum of products of basis vectors of V_1 and V_2, it need not necessarily be itself a *product* of a vector of V_1 and a vector of V_2. If it is not, we say that it is an *entangled* vector.

In this note we shall take our examples from 2-spaces i.e. whose elements are qubits.

Example II.1 *Take each of the vector spaces V_1, V_2 as 2-dimensional with basis*

$$e_1 = |0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad e_2 = |1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{1}$$

then $e_1 \otimes e_2 + e_2 \otimes e_2 = (e_1 + e_2) \otimes e_2 = |0, 1\rangle + |1, 1\rangle$ *is* factorizable , *therefore* not *entangled,or sometimes called* separable*; while* $e_1 \otimes e_2 + e_2 \otimes e_1$ *is entangled.*

B. Local transformations

Although sometimes referred to as a "resource", entanglement is rather peculiar in that it is *not* invariant under unitary transformations.

Example II.2 *Consider the unitary transformation* $U \equiv \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$.

One easily evaluates $U|0, 0\rangle = \frac{1}{\sqrt{2}}(|0, 0\rangle + |1, 1\rangle)$ *which is a (intuitively, maximally) entangled state (Bell state).*

The situation is different for *local* transformations:

Definition II.3 *A local unitary transformation U on the bipartite space $V_A \otimes V_B$ is one of the form $U_A \otimes U_B$, where U_A (resp. U_B) is a unitary transformation on V_A (resp. V_B).*

Clearly, a local unitary transformation leaves a factorizable state factorized. Conversely, entangled states remain entangled - since (local) unitary transformations are invertible (with local inverses). More generally, it can be shown that local transformations leave measures of entanglement, such as that introduced in section(II D), invariant.

C. States, pure and mixed

1. Pure States

Vectors correspond to *pure* states. For example $|\psi\rangle = \sum_i |i\rangle$ or $\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. We may equally represent a pure state $|\psi\rangle$ by the Operator (Projector) $|\psi\rangle\langle\psi|$ which projects onto that state. In the vector form above, the state ρ is represented by a matrix:

$$\rho = \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \begin{bmatrix} \alpha^* & \beta^* \end{bmatrix} = \begin{bmatrix} \alpha\alpha^* & \alpha\beta^* \\ \beta\alpha^* & \beta\beta^* \end{bmatrix} \qquad \rho^2 = \rho. \tag{2}$$

NOTE: trace $\rho = 1$ (Normalization) and ρ is Positive i.e. ρ is a Hermitian matrix with (semi-) positive eigenvalues (for a pure state the only non-zero eigenvalue is 1).

2. Mixed States

We take the preceding properties as our general *definition* of a (mixed) state; that is,

Definition II.4 *ρ is a state if it is a positive matrix of trace 1.*

Note: One may readily show that ρ is a (convex) sum of pure states (not a unique sum). A general state is also referred to as a *density matrix*.

D. Measures of Entanglement

1. Entropy of a state (Von Neumann entropy)

Definition II.5 *The Entropy of the state ρ is given by $E(\rho) = -\mathrm{tr}(\rho \log \rho) = -\sum_i \lambda_i \log(\lambda_i)$*

For qubits we conventionally take logs to base 2.

Example II.6 Pure state

Every pure state has entropy zero. Since every *pure (qubit) state is unitarily equivalent to* $\rho = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ *the entropy is* $E(\rho) = 1\log 1 + 0\log 0 = 0$.

Example II.7 Mixed state

For a general qubit (unitarily equivalent to)

$$\rho = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \quad \lambda_1 \geq 0 \;\; \lambda_2 \geq 0 \quad \lambda_1 + \lambda_2 = 1 \tag{3}$$

we have $E(\rho) = -\lambda_1 \log(\lambda_1) - \lambda_2 \log(\lambda_2)$ *where we may express the entropy in terms of a single parameter* λ $(0 \leq \lambda \leq 1)$ $E(\rho) = -\lambda \log \lambda - (1-\lambda)\log(1-\lambda)$. *It is easy to show that* $0 \leq E(\rho) \leq 1$ *with maximum entropy value* 1 *for* $\lambda = 1/2$ *- defining a* maximally random state *(taking logs to base 2).*

2. Intuitive Measures of Entanglement

Intuitively a satisfactory measure of the amount of entanglement $\mathcal{E}$ for a two-qubit bipartite system should satisfy the criteria of the following table (for pure states):

State	Entangled?	Entanglement measure $\mathcal{E}$
(1)State $\frac{1}{\sqrt{2}}(\|0,0\rangle + \|0,1\rangle)$	No	0
(2)Bell State $\frac{1}{\sqrt{2}}(\|0,0\rangle + \|1,1\rangle)$	Yes	1
(3)State $\sqrt{\lambda}(\|0,0\rangle + \sqrt{1-\lambda}\|1,1\rangle)$	Yes	$0 \leq \mathcal{E} \leq 1$

It turns out that the (Von Neumann) Entropy gives a measure of entanglement for *pure* states; but **not** directly, as all pure states have entropy zero.

We must first take the *Partial Trace* over one subsystem of the bipartite system.

3. Partial Trace

Definition II.8 *If* $V = V_A \otimes V_B$ *then* $tr_B(Q_A \otimes Q_B) = Q_A tr(Q_B)$. *Extend to sums by linearity.*

For example, if $Q_A = |u_1\rangle\langle u_2|$ $\quad Q_B = |v_1\rangle\langle v_2|$ $\quad$ then $\quad tr_B(Q_A \otimes Q_B) = |u1\rangle\langle u2|\langle v2|v1\rangle$.

Example II.9 Non-entangled state

The density matrix corresponding to $|\alpha\rangle = \frac{1}{\sqrt{2}}(|0,0\rangle + |0,1\rangle)$ *is*

$$\rho_\alpha = \frac{1}{2}(|0,0\rangle + |0,1\rangle)(\langle 0,0| + \langle 0,1|) = \begin{bmatrix} 1/2 & 1/2 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$tr_B(\rho_\alpha) = \frac{1}{2}(|0\rangle\langle 0| + |0\rangle\langle 0|) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

The resulting partially-traced state is a pure state, which has entropy zero.

Example II.10 Maximally entangled state

Consider the (intuitively, maximally) entangled state (Bell State):

$$\begin{aligned} |\beta\rangle &= (1/\sqrt{2})(|0,0\rangle + |1,1\rangle) && (4)\\ \rho_\beta &= (1/2)(|0,0\rangle + |1,1\rangle)(\langle 0,0| + \langle 1,1|) && (5)\\ tr_B(\rho_\beta) &= (1/2)(|0\rangle\langle 0| + |1\rangle\langle 1|) && (6)\\ &= \begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix} && (7) \end{aligned}$$

The resulting (reduced) density matrix has maximum entropy 1.

Example II.11 **Entangled state**

Consider the entangled state :

$$\begin{aligned} |\gamma\rangle &= (1/\sqrt{3})(|0,0\rangle + |0,1\rangle) + |1,0\rangle) && (8)\\ \rho_\gamma &= (1/3)(|0,0\rangle + |0,1\rangle) + |1,0\rangle)(\langle 0,0| + \langle 0,1| + \langle 1,0|) && (9)\\ tr_B(\rho_\gamma) &= (1/2)(|0\rangle\langle 0| + |1\rangle\langle 1|) && (10)\\ &= \begin{bmatrix} 2/3 & 1/3 \\ 1/3 & 1/3 \end{bmatrix} && (11) \end{aligned}$$

The resulting (reduced) density matrix has entropy 0.55.

Example II.12 Interpolating entangled state

Consider the pure state interpolating between entangled and non-entangled states :

$$|\Theta\rangle = \cos\theta(|0,0\rangle + \sin\theta|1,1\rangle). \qquad (12)$$

The reduced density matrix is $\begin{bmatrix} \cos^2\theta & 0 \\ 0 & \sin^2\theta \end{bmatrix}$

The corresponding entanglement measure *varies from* 0 *for* $\theta = 0, \pi/2$ *via a maximum of* 1 *for* $\theta = \pi/4$ *(Bell state).*

4. Entanglement of Formation

The previous examples indicate that using the entropy of the reduced density matrix gives a useful measure of entanglement for pure states. It can be shown that partially tracing over either subspace for a bipartite system gives the same result. Therefore we may define:

Definition II.13 *The measure of entanglement (sometimes referred to as* entanglement of formation*) for a bipartite pure state is the average of the entropies of the two reduced density matrices (that is, partially traced over each of the two subsystems).*

Although a general state is not a unique sum of pure states, we may define:

Definition II.14 *The entanglement* $\mathcal{E}(\rho)$ *of a mixed bipartite state* $\rho \in V_A \otimes V_B$ *is given by* $\mathcal{E}(\rho) = \min\{\sum_i \lambda_i \mathcal{E}(\psi_i) | \rho = \sum_i \lambda_i \psi_i\}$ *where the* ψ_i *are pure states in* $V_A \otimes V_B$*.*

The foregoing calculation involves taking the minimum of an infinite set; however, it has been shown that in the case of bipartite states the entanglement (as defined herein) may be obtained from an equally appropriate measure of entanglement, called the *concurrence*[1]; and this latter is obtainable as a simple function of the eigenvalues of the 4×4 matrix ρ.

The above definition (II.14) extends readily to multipartite mixed states; we thus concentrate in this note on describing entanglement measures for multipartite *pure* states.

Thus encouraged, the problem now remains to define a measure of entanglement for tripartite (and higher) states.

E. Tripartite states

Since in the bipartite case one obtains the entanglement by tracing out each space and then averaging, it would seem appropriate in the tripartite case to define the entanglement measure as the average of the three bipartite entanglements obtained by tracing out each of the three subspaces in turn. To see how this naïve approach would work, consider the following example:

Example II.15

$$|\psi\rangle = \frac{1}{\sqrt{3}}(|1,0,0\rangle + |0,1,0\rangle + |0,0,1\rangle) \tag{13}$$

$$\rho_\psi = \frac{1}{3}(|1,0,0\rangle + |0,1,0\rangle + |0,0,1\rangle)(\langle 1,0,0| + \langle 0,1,0| + \langle 0,0,1|) \tag{14}$$

Due to the symmetry of this state, the three partial traces are equal, each giving the matrix

$$\begin{bmatrix} 1/3 & 0 & 0 & 0 \\ 0 & 1/3 & 1/3 & 0 \\ 0 & 1/3 & 1/3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

At this point it would therefore seem reasonable to define an entanglement measure as the average of the three (equal) bipartite measures. In this case the concurrence of each reduced density matrix is 2/3, giving an average concurrence of 2/3, corresponding to an entanglement of .55, which seems reasonable enough.

However, the "success" of this approach is short-lived, as the next example shows.

We consider the following tripartite analogue of a Bell state (often referred to as a GHZ state[2]), which we intuitively expect to be maximally entangled.

Example II.16

$$|\Psi\rangle = \frac{1}{\sqrt{2}}(|0,0,0\rangle + |1,1,1\rangle) \tag{15}$$

$$\rho_\Psi = \frac{1}{2}(|0,0,0\rangle + |1,1,1\rangle)(\langle 0,0,0| + \langle 1,1,1|) \tag{16}$$

Again this state has three equal partial traces, each giving the matrix

$$\begin{bmatrix} 1/2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/2 \end{bmatrix}$$

which corresponds to a separable (non-entangled) state (concurrence=0). This is certainly not what we intuitively expect for the state $|\Psi\rangle$.

This situation, where we have three subspaces clearly linked (entangled) but each projected subspace is not linked, mirrors the well-known topological feature of the *Borromean Rings* (see Figure 1). Here we see that all the links are certainly what one reasonably term entangled; however, if we remove any link the resulting two are no longer entangled. From the analogy with the previous quantum example it would therefore seem profitable to explore the *topological* properties of links, and see if we can relate these to the unitary transformations which produce entanglement.

III. BRAIDS, KNOTS AND LINKS

In this section we explore braid groups[3], as introduced by Artin[4], considering them as a generalisation of the better known symmetric groups, and their relation to links.

A. Symmetry Group

The symmetry group S_n (sometimes called the permutation group) is defined as the the set of $n!$ permutations on n distinct objects, combining according to the rule illustrated by

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 2 & 4 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 2 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 3 & 4 \end{pmatrix} \tag{17}$$

for the case of S_4. A diagrammatic representation of the resultant permutation is found in Figure 2. The symmetric group S_n has a presentation in terms of $n-1$ adjacent transpositions[1], $\{s_i \; i = 1 \ldots n-1\}$ where s_1 sends the i to $i+1$ and $i+1$ to i. This rather mysterious presentation is:

$$s_i s_j = s_j s_i \quad |i-j| > 1 \tag{18}$$

$$s_i s_i = I \tag{19}$$

$$s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} \tag{20}$$

where Eq.20 plays an important role in the generalization to the *Braid group*, in which context it is known as the *braiding relation* or the *Yang-Baxter condition*.

The foregoing presentation can be implemented by the $n \times n$ matrix representation:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ & & \ddots & & & \\ & & & s & & \\ & & & & \ddots & \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

[1] The right-hand side of Eq.(17) is an adjacent transposition.

where s is the 2×2 matrix $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ whose $(1,1)$ element is in the (i,i) position of the matrix representing s_i, $(i = 1 \dots n-1)$.

B. Braid group

The braid group is like the symmetric group, but in three dimensions, so you must imagine the arrow joining the elements of the permuted set to go "over" or "under" each other. A diagrammatic representation of the elements σ_1 and σ_1^{-1} of B_4 is given by Figure 3. Since now clearly $\sigma_1^2 \neq 1$, all the (non-trivial) Braid groups are infinite dimensional. Just as for the symmetric group, the braid group B_n has a presentation in terms of $n-1$ generators σ_i (and their inverses). This presentation is:

$$\sigma_i \sigma_j = \sigma_j \sigma_i \quad |i-j| > 1 \tag{21}$$

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \tag{22}$$

where notably the analogue of Eq.(19) is absent. Eq.(22) is known as the *braiding relation* or the *Yang-Baxter* condition, as was noted above.

C. Knots and Links

Of particular interest to us is the fact that, as shown by Alexander[5], *all* knots and links may be obtained from elements of a braid group by the simple expedient of joining the the "dots";that is, join 1 to 1, 2 to 2, and so on.

Example III.1 *For the braid group B_2 with one generator σ_1, in Figure 4 we can see that performing this action with σ_1 gives the unknot.*

Example III.2 *Similarly, σ_1^2 in the braid group B_2 gives the Hopf Link, as in Figure 5.*

More complicated examples are the Olympic Symbol, Figure 6 and the Borromean Rings, as previously noted, Figure 8.

IV. UNITARY REPRESENTATIONS OF BRAID GROUPS AND ENTANGLEMENT

In order to relate the action of the braid group to unitary transformations on quantum systems, we shall associate each initial point of the braid group description as, for example, in Figure 3, with a qubit. A generic unitary representation of the braid group which satisfies the relation Eq.(21) can in principle be obtained from the following:

$$\hat{\sigma}_i = I \times \cdots \times U \times I \cdots \times I \tag{23}$$

where $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and U is a 4×4 unitary matrix occupying the $(i, i+1)$ position in the product. Of course it is more difficult to satisfy Eq.(22), the braiding, or Yang-Baxter, relation. We describe one form for B_2 in the following.

A. Unitary representation for B_2

In a sense finding a unitary representation for B_2 is a trivial exercise, as in this case there are effectively no relations on the single generator σ_1. Thus any unitary matrix will do; but for our purpose we require a 4×4 unitary matrix - since it is acting on the two-qubit space - and we should like it to mimic the Hopf link; that is, the unitary representative $\hat{\sigma}_1^2$ should produce a maximally entangled state from a (generic) non-entangled state. We define a (continuous parameter) unitary transformation matrix as follows:

$$u(\theta) = \begin{bmatrix} \cos\theta & 0 & 0 & -\sin\theta \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ \sin\theta & 0 & 0 & \cos\theta \end{bmatrix}. \tag{24}$$

Defining $\hat{\sigma}_1 = u(\pi/8)$ so that

$$\hat{\sigma}_1^2 = U \equiv 1/\sqrt{2} \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

whence one evaluates $U|0,0\rangle = \frac{1}{\sqrt{2}}(|0,0\rangle + |1,1\rangle)$ as in Example(II.2).

The analogous unlinked diagram corresponds (trivially) to $\hat{\sigma}_1 \hat{\sigma}_1^{-1} = I$.

Although this unitary representation produces a maximally entangled (Bell) state from the generic separable state, it is not a true representation of B_2 since $U^8 = I$; it is rather a unitary representation of the quasi-symmetric group of the Appendix (Section VII) on one generator, with R-exponent $= 16$. Further, the unitary operator $u(\theta)$ produces, as expected, a continuous range of entanglements as in Fig.(7); however, the transformation corresponding to $\hat{\sigma}_1$, $u(\pi/8)$, produces an intermediate value of entanglement, not obviously corresponding to the "unknot".

B. Unitary representation for B_3

Finding unitary representations for B_3 and beyond is less trivial than for B_2, since now the braiding relation Eq.(22) must be satisfied. Using $u(\theta)$ as in Eq.(24, we write the representation for B_3 as

$$\hat{\sigma}_1 = u(\theta) \times I, \quad \hat{\sigma}_2 = I \times u(\theta).$$

One calculates that the braiding relation Eq.(22) is satisfied only for $\theta = 0, \pi/4, 3\pi/4, \pi$ in the range $[0, \pi]$. Choosing the value $\theta = \pi/4$ gives for $u(\pi/4)$ the matrix U of Example(II.2). Taking as our paradigm for tripartite entanglement the Borromean Rings of Figure(1) whose braid representation is $(\sigma_1 {\sigma_2}^{-1})^3$ from Figure(8), we evaluate

$$(\hat{\sigma}_1 \hat{\sigma}_2^{-1})^3 |0,0,0\rangle = -\frac{1}{2}(|0,0,0\rangle + |0,1,1\rangle + |1,0,1\rangle + |1,1,0\rangle).$$

This state is entanglement-equivalent, as discussed in Definition(II.3) to $\frac{1}{\sqrt{2}}(|0,0,0\rangle + |1,1,1\rangle)$, the tripartite analogue of a Bell state (GHZ state) as can be seen by use of the *local transformation*[2] $V = v \otimes v \otimes v$ where $v = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$,

$$V \frac{1}{\sqrt{2}}(|0,0,0\rangle + |1,1,1\rangle) = -\frac{1}{2}(|0,0,0\rangle + |0,1,1\rangle + |1,0,1\rangle + |1,1,0\rangle).$$

The geometric action of "removing" one link corresponds algebraically to $\sigma_1 \sigma_2 {\sigma_2}^{-1} {\sigma_1}^{-1} \sigma_2 {\sigma_2}^{-1} = I$ which has no effect on the initial non-entangled generic state - see Figure(9).

V. CONCLUSIONS

We have attempted to display a relation between quantum entanglement and the topological properties of links. With this in mind we gave an elementary introduction to the quantum entanglement of bipartite systems, illustrating by a simple example how a naïve extension to the tripartite case fails. Nevertheless, the attempt to mimic the partial trace approach of the bipartite definition suggests an analogy with properties of links, especially of the Borromean Rings in the tripartite case.

With that in mind, we introduced the topic of braid groups, which provide a description of links in general. Starting with the bipartite case, we showed that a unitary representation of the two-string braid group B_2 gives an analogy between the Hopf Link and a maximally entangled (Bell)

[2] The same transformations have been used in a similar context by the authors of Reference[7]

state. Use of a continuous unitary representation of B_2 gives intermediate values of entanglement. Proceeding to the B_3 case, we showed that the braid word description of the Borromean Rings does indeed produce (an equivalent of) the maximally-entangled tripartite GHZ state.

The braid group approach provides an illustrative bridge between a geometric picture of entanglement and the algebraic description of the quantum state, which may prove of value in elucidating some of the properties of the tripartite, and higher, cases.

VI. REFERENCES

This note has been designed to be read without references. I have however included in the following some seminal papers, and also some general online sites which contain references for further reading. The analogy with Borromean rings was explored independently by Kauffman and Lomonaco [6]; and equivalent unitary representations of the braid groups producing entanglement were given by Chen, Xue and Ge in reference [7]

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VII. APPENDIX: SYMMETRY GROUPS, BRAID GROUPS AND HECKE ALGEBRA

We summarize here the presentations of some of the groups discussed in the text.

1. **Symmetric Group** S_n

   Generators: $\{s_i : \ i = 1 \ldots n-1\}$

   Relations: $s_i^2 = I \quad s_i s_j = s_j s_i \ \ (|i-j| \geq 2) \quad s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$

   Matrix Representation: $s_i = 1 \times \cdots \times s \times 1 \times 1$ where $s = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ replaces each element 1 along the diagonal.

2. **Braid Group** B_n

   Generators: $\{\sigma_i : \ i = 1 \ldots n-1\}$

   Relations: $\sigma_i\sigma_j = \sigma_j\sigma_i \ \ (|i-j| \geq 2) \qquad \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$

3. **Hecke algebra**

   This is a q-deformation of S_n and has the relations of B_n plus an additional one: Generators: $\{\sigma_i : \ i = 1 \ldots n-1\}$

   Relations: $\sigma_i\sigma_j = \sigma_j\sigma_i \ \ (|i-j| \geq 2) \quad \sigma_i^2 = (1-q)\sigma_i + q \quad \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$

   Matrix Representation (Burau): $\sigma_i = 1 \times \cdots \times \sigma \times 1 \times 1$ where $\sigma = \begin{bmatrix} 1-q & q \\ 1 & 0 \end{bmatrix}$ replaces each element 1 along the diagonal.

4. **Quasi-symmetric group**

   Generators: $\{\sigma_i : \ i = 1 \ldots n-1\}$

   Relations: $\sigma_i\sigma_j = \sigma_j\sigma_i \ \ (|i-j| \geq 2) \quad \sigma_i^R = I \quad \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$

   Matrix Representation : $\sigma_i = 1 \times \cdots \times \sigma \times 1 \times 1$ where $\sigma = \begin{bmatrix} 1-q & q \\ 1 & 0 \end{bmatrix}$ and $(-q)^R = I$.

VIII. FIGURES

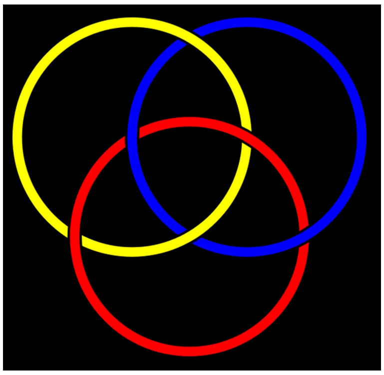

FIG. 1: Borromean Rings

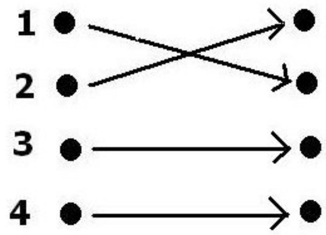

FIG. 2: An element of S_4 (s_1)

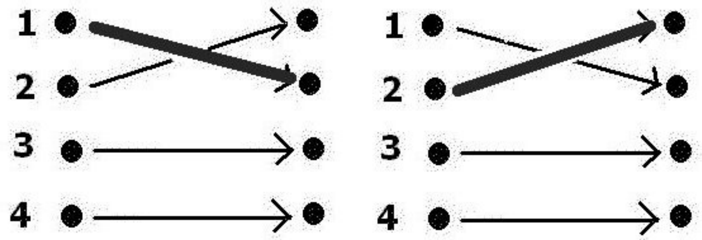

FIG. 3: σ_1 and σ_1^{-1} of B_4

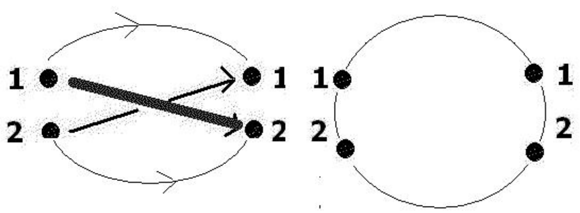

FIG. 4: In B_2, σ_1 produces the unknot

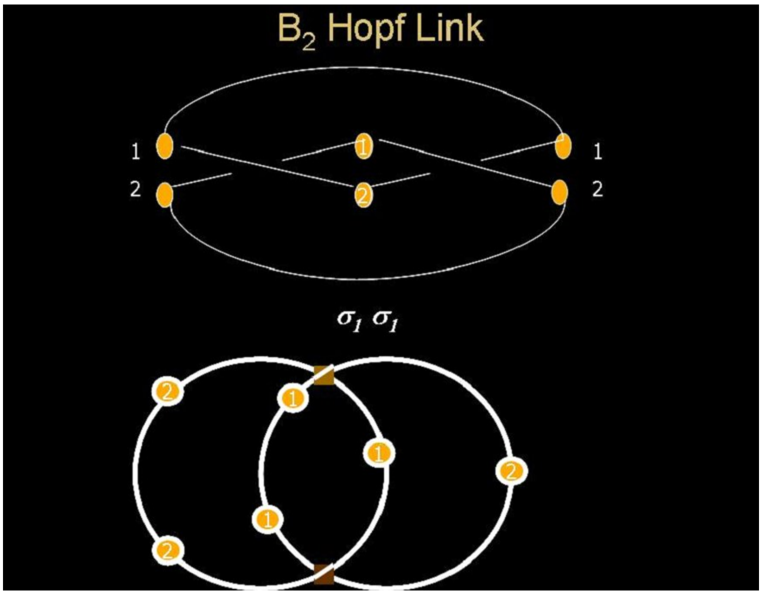

FIG. 5: In B_2, σ_1^2 produces the Hopf link

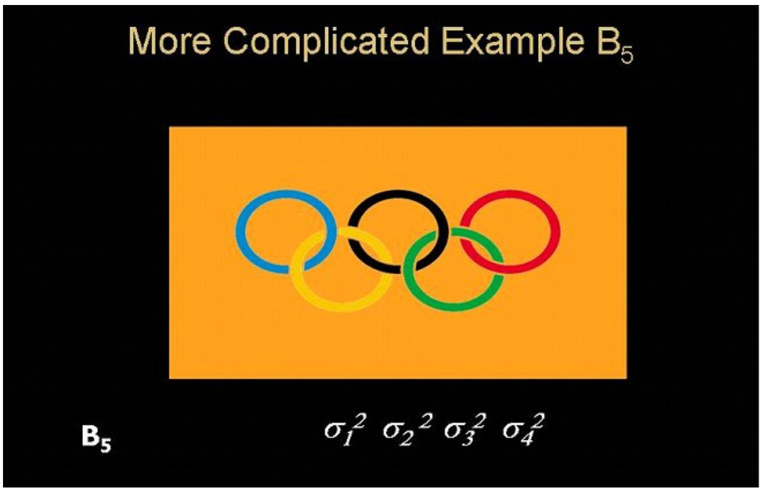

FIG. 6: The Olympic Symbol in B_5

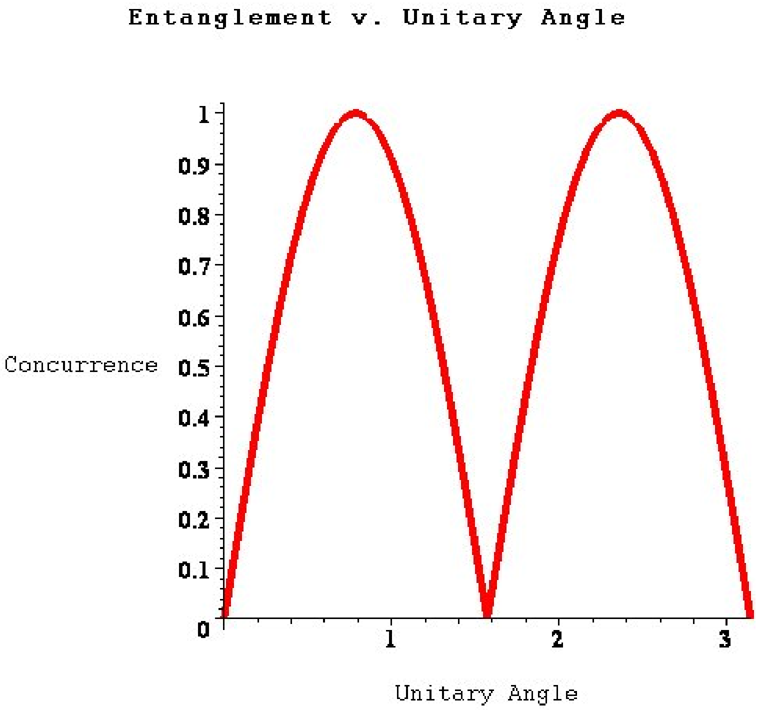

FIG. 7: Entanglement(Concurrence) v. Unitary Angle 0 to π

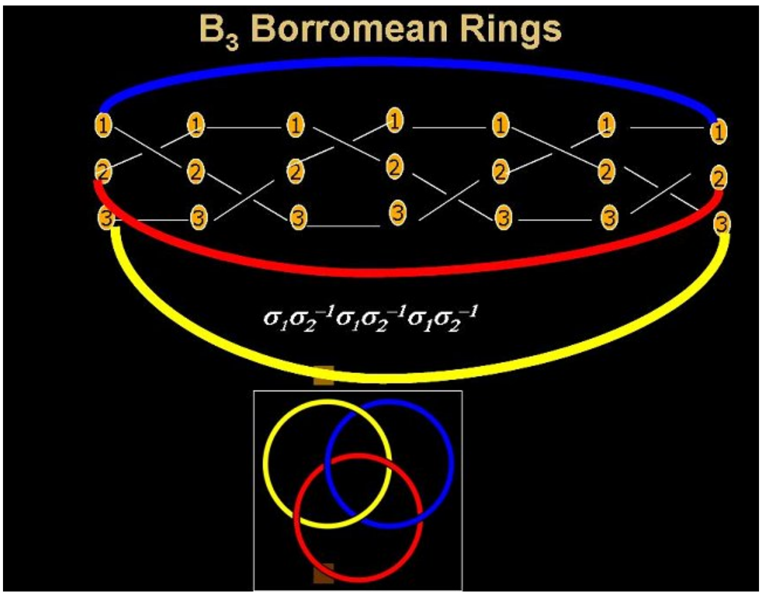

FIG. 8: Borromean Rings from B_3

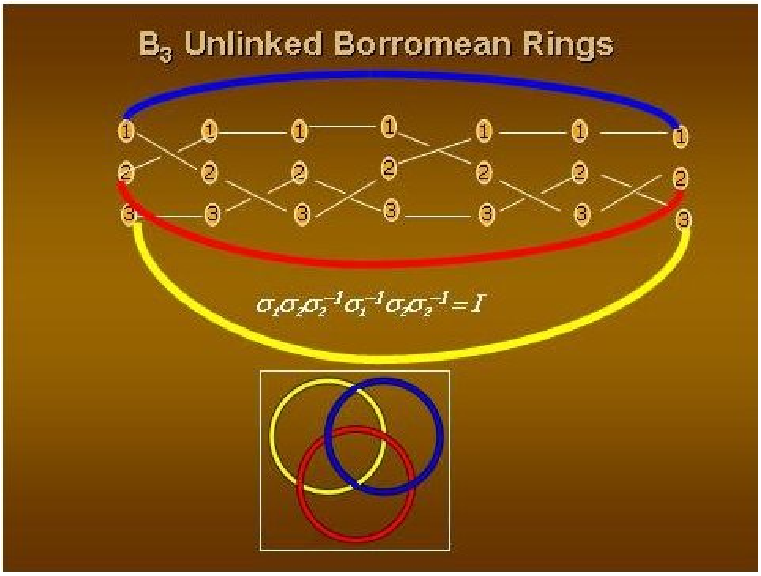

FIG. 9: Unlinked Borromean Rings from B_3