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A family of entire functions with Baker domains

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Abstract. In his paper [The iteration of polynomials and transcendental entire functions. *J. Aust. Math. Soc. (Series A)* **30** (1981), 483–495], Baker proved that the function f defined by

$$f(z) = z + \frac{\sin\sqrt{z}}{\sqrt{z}} + c$$

has a Baker domain for c sufficiently large. In this paper we use a novel method to prove that f has a Baker domain for all $c > 0$. We also prove that there exists an open unbounded set contained in the Baker domain on which the orbits of points under f are asymptotically horizontal.

1. *Introduction*

Let f be a meromorphic function which is not rational of degree one and denote by f^n, $n \in \mathbb{N}$, the nth iterate of f. The *Fatou set*, $F(f)$, is defined to be the set of points, $z \in \mathbb{C}$, such that the sequence $(f^n)_{n\in\mathbb{N}}$ is well defined, meromorphic and forms a normal family in some neighbourhood of z. The complement, $J(f)$, of $F(f)$ is called the *Julia set* of f. An introduction to the properties of these sets can be found in, for example, [**2**] for rational functions and [**3**] and [**6**] for transcendental meromorphic functions.

The set $F(f)$ is completely invariant so for any component U of $F(f)$ there exists, for each $n \in \mathbb{N}$, a component U_n of $F(f)$ such that $f^n(U) \subset U_n$. If $U_p = U$ for some minimal $p \in \mathbb{N}$, then we say that U is a periodic component of period p. There are five possible types of periodic components (see [**3**, Theorem 6]). In particular, U is called a *Baker domain* if there exists $z_0 \in \partial U$ such that $f^{np}(z) \to z_0$ as $n \to \infty$, for $z \in U$, where $f^p(z_0)$ is not defined (see [**7**] for a survey article on Baker domains). In this case, there is at least one component U_k, $1 \le k \le p$, with the property that $f^{np}(z) \to \infty$ as $n \to \infty$ for $z \in U_k$. If U is a Baker domain of a transcendental *entire* function f, then $f^n(z) \to \infty$ as $n \to \infty$ for $z \in U$ and, moreover, U is simply connected (see [**1**, Theorem 5]).

In [**1**] Baker considered the transcendental entire function f defined by

$$f(z) = z + \frac{\sin\sqrt{z}}{\sqrt{z}} + c. \tag{1.1}$$

He showed that, for sufficiently large positive constant c, f has a Baker domain. In [**4**] we show that this function has sparsely distributed singular values. Here we show that the function f defined in (1.1) has a Baker domain for all $c > 0$, thus extending the original result of Baker. We prove the following.

THEOREM 1.1. *For all $c > 0$, the function f defined in (1.1) has an invariant Baker domain U, symmetrical about the real axis and containing (x_c, ∞) for some $x_c > 0$.*

In [**1**] Baker constructed a domain D symmetric about the real axis with a (truncated) parabolic boundary. He proved that $f(D) \subset D$ by showing that the distance between $z + c = x + c + iy \in D$ and any point $z_0 \in \partial D$ is greater than $|\sin(\sqrt{z})/\sqrt{z}|$ whenever

$$\tfrac{1}{2}c(x + 1 + \tfrac{1}{2}c)^{-1/2} > e|z|^{-1/2} \quad \text{for } z = x + iy \in D.$$

This inequality is true whenever c is greater than approximately 6 and so f has a Baker domain with the properties in Theorem 1.1 for such c.

A proof of this type (finding a set D that is invariant under f) can be given for certain smaller values of c (for example, $c > 1$). However, when $0 < c < 1$, a serious problem arises in the application of Baker's method since no invariant parabolic domain exists. This fact will be explained in more detail in the proof of Lemma 3.4 and the remarks that follow it.

Therefore, in order to establish the result for all $c > 0$ we use a more general approach; namely, we find a domain G which is not invariant under f but which instead has the property that $\mathbb{C} \setminus \bigcup_{n\in\mathbb{N}} f^n(G)$ contains an open set. Furthermore, G contains an invariant curve Γ such that $f^n(z) \to \infty$ as $n \to \infty$ for all z in Γ. We are grateful to the referee for pointing out that this approach was also used by Morosawa in [**5**].

In this paper we also prove the following.

THEOREM 1.2. *Let the function f be as defined in (1.1) and let U be the invariant Baker domain of f shown to exist in Theorem 1.1. There exists an open unbounded set V contained in U such that, for any given $z \in V$, there exists a constant $\eta_z \in \mathbb{R}$ such that:*

(i) $\Re(f^n(z)) \to \infty$ *as* $n \to \infty$; *and*

(ii) $\Im(f^n(z)) \to \eta_z$ *as* $n \to \infty$.

Moreover, if $z \in V$ then
$$\begin{cases} \eta_z > 0, & \text{if } \Im(z) > 0,\\ \eta_z = 0, & \text{if } \Im(z) = 0,\\ \eta_z < 0, & \text{if } \Im(z) < 0.\end{cases}$$

Remark. Theorems 1.1 and 1.2 concern the function f as defined in (1.1). It should be noted that, by making only minor changes to the proofs, these theorems can easily be generalized to the broader family of functions

$$f(z) = z + a\frac{\sin(b\sqrt{z})}{\sqrt{z}} + c,$$

where a, b, $c > 0$.

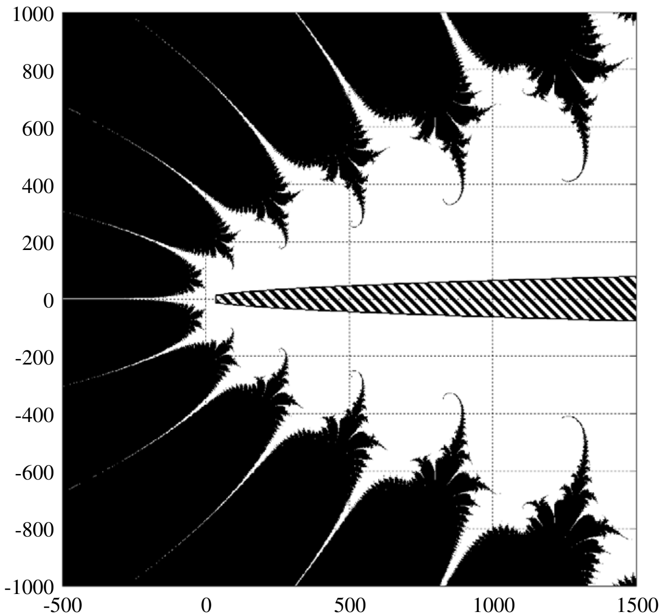

FIGURE 1. Julia set of f for $c = 6$.

Remark. The geometry of the Baker domain of f is dependent on the value of c. The effect of varying c can be illustrated by considering the fixed point equation on the positive real axis, $\mathbb{R}^+$, given by

$$\frac{\sin\sqrt{x}}{\sqrt{x}} = -c.$$

When $c > 1$, we have $(\sin\sqrt{x})/\sqrt{x} > -c$ for all $x \in \mathbb{R}^+$ and so there are no fixed points on the positive real axis. In this case the Baker domain contains the whole positive real axis.

Now we consider the point $x_0 = (3\pi/2)^2$ as c decreases towards zero. When $c < c_0 \approx 0.2$ we have $(\sin\sqrt{x_0})/\sqrt{x_0} < -c$ so there exists a bounded interval $(0, x'_c)$ of the real line that contains a finite number of fixed points of f.

As the value of c decreases continuously from c_0 towards 0, x'_c increases. There exists a sequence $\{c_n\}_{n\in\mathbb{N}}$ of values of c tending to zero from above as n tends to infinity with the following properties: for each $n \in \mathbb{N}$, as c decreases continuously through c_n, a parabolic fixed point of f appears on the real line to the right of the other real fixed points, and this new real fixed point instantaneously bifurcates into a pair of fixed points that remain on the real axis. The left-hand fixed point in the pair is attracting and the right-hand fixed point in the pair is repelling. Each attracting fixed point is associated with an attracting domain and each repelling fixed point is in $J(f)$. Hence the Baker domain (symmetric about the real axis) appears to move to the right as c decreases, with more and more attracting Fatou components appearing to the left of the Baker domain.

This behaviour is supported by computer experiments. Figures 1 and 2 present the results of computer experiments to estimate the shape of the Julia set (black) for a large value of c and a small value of c respectively.

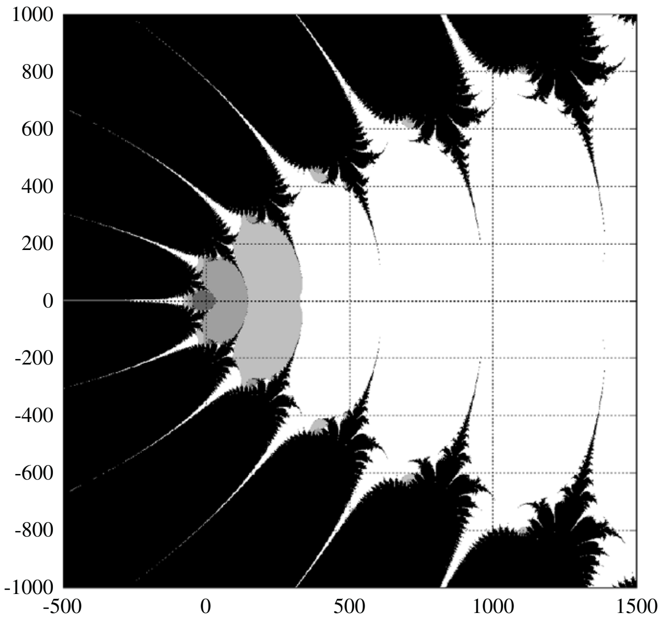

FIGURE 2. Julia set of f for $c = 0.05$.

In Figure 1, we have $c = 6$ and the invariant parabolic domain used by Baker has been shaded on the diagram. In Figure 2, we have $c = 0.05$ and there are three attracting domains corresponding to attracting fixed points on the real line. These attracting domains, together with their pre-images, are shaded in grey.

2. *Strategy for proving Theorem 1.1*

To prove Theorem 1.1, we consider the function g defined by

$$g(w) = \sqrt{f(w^2)} = \sqrt{w^2 + \frac{\sin w}{w} + c}. \tag{2.1}$$

Throughout this paper $c > 0$ is fixed and $\sqrt{}$ denotes the principal square root.

Now, for $K \geq 0$ and $L > 0$ define the open half-strip $R(K, L)$ by

$$R(K, L) = \{w \mid \Re(w) > K, |\Im(w)| < L\}. \tag{2.2}$$

We show that for any given $L > 0$ there exist $K > 0$ and $L' > 0$ such that

$$g^n(R(K, L')) \subset R(K, L) \quad \text{for all } n \in \mathbb{N}. \tag{2.3}$$

It should be noted that we do not insist that $R(K, L)$ be contained in $R(K, L')$. With $h(w) = w^2$, we deduce that

$$f^n(h(R(K, L'))) = h(g^n(R(K, L'))) \subset h(R(K, L)) \quad \text{for all } n \in \mathbb{N}. \tag{2.4}$$

Thus, by Montel's theorem, $V = h(R(K, L')) \subset F(f)$. Since V is unbounded and connected, we deduce that there exists a single unbounded component U of $F(f)$ such that $V \subset U$.

Finally, there exists $K_0 > 0$ such that

$$f(x) > x + \frac{c}{2} \quad \text{for all } x \geq K_0, \tag{2.5}$$

and $\Gamma = (K_0, \infty) \subset V \subset U$. Then Γ is invariant under f and $f^n(x) \to \infty$ as $n \to \infty$. It follows that $f^n(z) \to \infty$ as $n \to \infty$ for all $z \in U$, so U is a Baker domain of f. The symmetry of U in the real axis follows from the fact that $f(\overline{z}) = \overline{f(z)}$ for $z \in \mathbb{C}$.

3. *Preliminary results*

We can write the function g defined in (2.1) as

$$\begin{aligned} g(w) &= w\sqrt{1 + \frac{\sin w}{w^3} + \frac{c}{w^2}} \\ &= w\left(1 + \frac{1}{2}\left(\frac{\sin w}{w^3} + \frac{c}{w^2}\right) - \frac{1}{8}\left(\frac{\sin w}{w^3} + \frac{c}{w^2}\right)^2 + \cdots\right) \\ &= w + \frac{\sin w}{2w^2} + \frac{c}{2w} + B(w), \end{aligned} \tag{3.1}$$

say, provided

$$\left|\frac{\sin w}{w^3} + \frac{c}{w^2}\right| < 1.$$

Writing $w = u + iv$, we can express g in terms of its real and imaginary parts as

$$g(w) = u + iv + \delta u + i\delta v, \tag{3.2}$$

where

$$\delta u = \Re\left(\frac{\sin w}{2w^2}\right) + \Re\left(\frac{c}{2w}\right) + \Re(B(w)) \tag{3.3}$$

and

$$\delta v = \Im\left(\frac{\sin w}{2w^2}\right) + \Im\left(\frac{c}{2w}\right) + \Im(B(w)). \tag{3.4}$$

Let $L > 0$ be fixed for the rest of this paper. Since g is symmetric in the sense that $g(\overline{w}) = \overline{g(w)}$, it is sufficient to consider the properties of iterates of points in the set $R^+(K, L)$ defined by

$$R^+(K, L) = \{w \mid \Re(w) > K, 0 < \Im(w) < L\}. \tag{3.5}$$

We begin by establishing two properties of B that will be required later.

LEMMA 3.1. *There exist positive constants A_1 and $K_1 > \max\{1, \sqrt{K_0}\}$ such that*

$$|B(w)| < \frac{A_1}{u^3} \quad \textit{for all } w \in R^+(K_1, L). \tag{3.6}$$

Proof. From (3.1) we have

$$B(w) = w \sum_{m=2}^{\infty} \frac{1}{2^m m!}\left(\prod_{k=1}^{m}(-2k+3)\right)\left(\frac{\sin w}{w^3} + \frac{c}{w^2}\right)^m. \tag{3.7}$$

Defining b_m by

$$b_m = \frac{1}{2^m m!}\prod_{k=1}^{m}(-2k+3) \quad \text{for all } m \in \mathbb{N}, \tag{3.8}$$

we note that

$$|b_m| = \left|\frac{1}{2^m m!}\prod_{k=1}^{m}(-2k+3)\right| = \left|\prod_{k=1}^{m}\frac{(-2k+3)}{2k}\right| < 1 \quad \text{for all } m \in \mathbb{N}. \tag{3.9}$$

Thus

$$\begin{aligned} |B(w)| &\le |w| \sum_{m=2}^{\infty} |b_m| \left|\frac{\sin w}{w^3} + \frac{c}{w^2}\right|^m \\ &< |w| \sum_{m=2}^{\infty} \left|\frac{\sin w}{w^3} + \frac{c}{w^2}\right|^m \\ &= |w| \left|\frac{\sin w}{w^3} + \frac{c}{w^2}\right|^2 \sum_{m=0}^{\infty} \left|\frac{\sin w}{w^3} + \frac{c}{w^2}\right|^m. \end{aligned} \tag{3.10}$$

Clearly, there exists $K_1 > \max\{1, \sqrt{K_0}\}$ such that

$$\left|\frac{\sin w}{w^3} + \frac{c}{w^2}\right| < \frac{1}{2} \quad \text{for all } w \in R^+(K_1, L), \tag{3.11}$$

and so, by (3.10),

$$|B(w)| < 2|w| \left|\frac{\sin w}{w^3} + \frac{c}{w^2}\right|^2 = \frac{2}{|w|^3}\left|\frac{\sin w}{w} + c\right|^2 \quad \text{for all } w \in R^+(K_1, L). \tag{3.12}$$

It easily follows that there exists $A_1 > 0$ such that

$$|B(w)| < \frac{A_1}{u^3} \quad \text{for all } w \in R^+(K_1, L), \tag{3.13}$$

and this completes the proof. □

Next, we derive an estimate for the derivative of $B(w)$.

LEMMA 3.2. *There exists a positive constant A_2 such that*

$$|B'(w)| \le \frac{A_2}{|w|^4} \quad \textit{for all } w \in R^+(K_1, L). \tag{3.14}$$

Proof. From equations (3.7) and (3.8) we have

$$B(w) = w \sum_{m=2}^{\infty} b_m \left(\frac{\sin w}{w^3} + \frac{c}{w^2}\right)^m. \tag{3.15}$$

Differentiating and re-arranging we have

$$B'(w) = \left(\frac{\sin w}{w^3} + \frac{c}{w^2}\right)^2 \sum_{m=0}^{\infty} b_{m+2}\left(\frac{\sin w}{w^3} + \frac{c}{w^2}\right)^m$$
$$+ w\left(\frac{\cos w}{w^3} - \frac{3\sin w}{w^4} - \frac{2c}{w^3}\right)\left(\frac{\sin w}{w^3} + \frac{c}{w^2}\right)$$
$$\times \sum_{m=0}^{\infty}(m+2)b_{m+2}\left(\frac{\sin w}{w^3} + \frac{c}{w^2}\right)^m,$$

so

$$|B'(w)| \leq \left|\frac{\sin w}{w^3} + \frac{c}{w^2}\right|^2 \sum_{m=0}^{\infty} |b_{m+2}|\left|\frac{\sin w}{w^3} + \frac{c}{w^2}\right|^m$$
$$+ |w|\left(\left|\frac{\cos w}{w^3}\right| + 3\left|\frac{\sin w}{w^4}\right| + 2\left|\frac{c}{w^3}\right|\right)\left|\frac{\sin w}{w^3} + \frac{c}{w^2}\right|$$
$$\times \sum_{m=0}^{\infty} |(m+2)b_{m+2}|\left|\frac{\sin w}{w^3} + \frac{c}{w^2}\right|^m. \tag{3.16}$$

By (3.9) and (3.11),

$$\sum_{m=0}^{\infty} |b_{m+2}|\left|\frac{\sin w}{w^3} + \frac{c}{w^2}\right|^m < 2 \quad \text{for all } w \in R^+(K_1, L). \tag{3.17}$$

Now note that, for $m \geq 2$,

$$|mb_m| = \frac{m}{2^m m!}\prod_{k=2}^{m}(2k-3) = \frac{1}{2}\prod_{k=2}^{m}\frac{2k-3}{2k-2} < \frac{1}{2} \tag{3.18}$$

and so, by (3.11),

$$\sum_{m=0}^{\infty} |(m+2)b_{m+2}|\left|\frac{\sin w}{w^3} + \frac{c}{w^2}\right|^m < 1 \quad \text{for all } w \in R^+(K_1, L). \tag{3.19}$$

It follows from (3.16)–(3.19) that

$$|B'(w)| < \frac{2}{|w|^4}\left|\frac{\sin w}{w} + c\right|^2 + \frac{1}{|w|^4}\left(|\cos w| + 3\left|\frac{\sin w}{w}\right| + 2c\right)\left|\frac{\sin w}{w} + c\right|$$
$$\text{for all } w \in R^+(K_1, L). \tag{3.20}$$

Thus, there exists $A_2 > 0$ such that

$$|B'(w)| \leq \frac{A_2}{|w|^4} \quad \text{for all } w \in R^+(K_1, L),$$

and this completes the proof. □

With these properties of $B(w)$ established, we proceed to obtain estimates for δu and δv valid in $R^+(K_1, L)$. First we consider δu.

LEMMA 3.3. *Let δu be as defined in (3.3) and (3.1). Then there exists a positive constant A_3 such that*

$$\left|\delta u - \frac{c}{2u}\right| \le \frac{A_3}{u^2} \quad \textit{for all } w \in R^+(K_1, L). \tag{3.21}$$

Proof. We first note that

$$\begin{aligned}\Re\left(\frac{\sin w}{2w^2}\right) &\le \left|\frac{\sin w}{2w^2}\right| \\ &< \frac{\sqrt{\sin^2 u + \sinh^2 L}}{2u^2 + 2v^2} \\ &< \frac{\sqrt{1 + \sinh^2 L}}{2}\frac{1}{u^2} \\ &= \frac{\cosh L}{2}\frac{1}{u^2} \quad \text{for all } w \in R^+(K_1, L).\end{aligned} \tag{3.22}$$

Secondly, we note that

$$\left|\Re\left(\frac{c}{2w}\right) - \frac{c}{2u}\right| = \left|\frac{cu}{2|w|^2} - \frac{c}{2u}\right| = \frac{cv^2}{2u^3 + 2uv^2} < \frac{cL^2}{2}\frac{1}{u^3} \quad \text{for all } w \in R^+(K_1, L). \tag{3.23}$$

Lastly, from Lemma 3.1, we have

$$|\Re(B(w))| < \frac{A_1}{u^3} \quad \text{for all } w \in R^+(K_1, L). \tag{3.24}$$

The result follows from (3.3) and (3.22)–(3.24). □

Remark. Lemma 3.3 implies that when the real part of the orbit of any point is sufficiently large, and the imaginary part remains bounded above by L and below by 0, then the orbit will continue to move to the right. In Lemmas 3.4, 4.1 and 4.2 below, we will show that for points contained in the narrower strip $R^+(K, L')$, the imaginary part of their orbits does indeed remain bounded in this way.

Now we consider δv.

LEMMA 3.4. *Let δv be as defined in (3.4) and (3.1). Then there exists a positive constant A_4 such that*

$$\left|\delta v - \frac{\cos u \sinh v}{2u^2} + \frac{cv}{2u^2}\right| \le A_4\frac{v}{u^3} \quad \textit{for all } w \in R^+(K_1, L). \tag{3.25}$$

Proof. We consider the terms on the right-hand side of equation (3.4). Firstly, we observe that there exists $\alpha_1 > 0$ such that

$$\begin{aligned}&\left|\frac{\cos u \sinh v}{2u^2} - \Im\left(\frac{\sin w}{2w^2}\right)\right| \\ &= \left|\frac{\cos u \sinh v}{2u^2} - \frac{(u^2 - v^2)\cos u \sinh v - 2uv \sin u \cosh v}{2|w|^4}\right| \\ &= \left|\frac{(3u^2v^2 + v^4)\cos u \sinh v + 2u^3 v \sin u \cosh v}{2u^2(u^2 + v^2)^2}\right| \\ &\le \alpha_1\frac{v}{u^3} \quad \text{for all } w \in R^+(K_1, L).\end{aligned} \tag{3.26}$$

Next, we observe that

$$0 < \frac{cv}{2u^2} + \Im\left(\frac{c}{2w}\right) = \frac{cv^3}{2u^4 + 2u^2v^2} \le \frac{cv^3}{2u^4} \le \frac{cL^2}{2}\frac{v}{u^3} \quad \text{for all } w \in R^+(K_1, L). \tag{3.27}$$

Finally, we consider the term $\Im(B)$. Clearly $B(\mathbb{R}) \subset \mathbb{R}$, and so the image under B of any curve segment starting on the real axis also starts on the real axis. Let $w = u + iv \in R^+(K_1, L)$ and consider the line segment Γ with the parametrization

$$\gamma(t) = u + it, \quad 0 \le t \le v.$$

Since $B(\Gamma)$ is some curve joining the point $B(u) \in \mathbb{R}$ to the point $B(u + iv)$, we have

$$|\Im(B(u + iv))| \le \text{Length}(B(\Gamma)) = \int_0^v |B'(\gamma(t))|\, dt \le v \times \max_\Gamma\{|B'|\}. \tag{3.28}$$

Now, by Lemma 3.2, we have

$$|B'(w)| \le \frac{A_2}{|w|^4} \le \frac{A_2}{u^4} \quad \text{for all } w \in R^+(K_1, L).$$

Substituting this estimate into (3.28), we have

$$|\Im(B(w))| \le v \times \frac{A_2}{u^4} < A_2\frac{v}{u^3} \quad \text{for all } w \in R^+(K_1, L). \tag{3.29}$$

The result follows from (3.26), (3.27) and (3.29). □

Remark. Note that, in the case $c > 1$, when v is sufficiently small the magnitude of the third term on the left-hand side of (3.25) is greater than that of the second and so, when v is positive and sufficiently small and u is sufficiently large, δv is necessarily negative. This observation, together with Lemma 3.3, means that orbits in $R^+(K_1, L)$ move down towards the real axis and to the right so that $R^+(K_1, L)$ is invariant under g. In this case it is possible to use a simpler proof that f has a Baker domain by finding an invariant half-strip for g, which corresponds to an invariant parabolic domain for f in the z-plane.

When $0 < c < 1$, however, the magnitude of the third term can be less than that of the second for any $v > 0$ and δv can be positive for suitable values of u. This means that there is no invariant half-strip in the w-plane. This explains why the more general approach presented in this paper is necessary.

4. *Proof of Theorem 1.1*

To prove Theorem 1.1, we show that we can choose K so that (2.3) holds.

We consider a point $w_0 = u_0 + iv_0$ in a half-strip of the form $R^+(K, L)$. We denote $g^n(w_0)$ by $w_n = u_n + iv_n$.

We begin by showing in Lemma 4.1 that if K is sufficiently large, then the growth of the imaginary part of the orbit of w_0 over a particular number of iterates is quite small. In Lemma 4.2 we will use this result to show the much stronger result that the imaginary part actually decreases in a certain sense that will be made precise. Finally, we show that the results of Lemmas 4.1 and 4.2 can be applied repeatedly along the whole forward orbit of w_0.

LEMMA 4.1. *There exist positive constants $K_2 \geq K_1$ and A_5 such that for any L' satisfying $0 < L' \leq L$, and any $w_0 \in R^+(K_2, L'/2)$, if M_0 is the smallest integer such that $u_0 \leq 2\pi M_0$, then there exists a positive integer $n(M_0)$ depending on w_0 such that $u_{n(M_0)-1} \leq 2\pi M_0 < u_{n(M_0)}$, and for n in $\{1, \ldots, n(M_0)\}$ we have:*
(i) $c/(8u_0) < u_n - u_{n-1} < 3c/(4u_0) < \pi$;
(ii) $|v_n - v_0| < A_5 v_0/u_0$; and
(iii) $0 < v_n < L'$.

Proof. We begin by setting

$$A_5 = \max\left\{\frac{1}{2}, \frac{17\pi}{c}\right\} \times \left(\frac{\sinh L}{L} + c + A_4\right), \tag{4.1}$$

and

$$K_2 = \max\left\{K_1, 6\pi, A_5, \frac{4A_3}{c}, \frac{c}{\pi}\right\}. \tag{4.2}$$

We proceed by constructing an induction argument to show that properties (i)–(iii) hold whenever $u_{n-1} \leq 2\pi M_0$. We begin by considering the case $n = 1$.

Since $w_0 \in R^+(K_2, L'/2)$ and $K_2 \geq K_1$, we can use Lemmas 3.3 and 3.4 to estimate δu_0 and δv_0, respectively. By Lemma 3.3,

$$\left|\delta u_0 - \frac{c}{2u_0}\right| \leq \frac{A_3}{u_0^2} < \frac{A_3}{K_2 u_0} \leq \frac{c}{4u_0},$$

since $u_0 > K_2 \geq 4A_3/c$. Thus,

$$\frac{c}{4u_0} < \delta u_0 = u_1 - u_0 < \frac{3c}{4u_0} < \frac{3c}{4K_2} < \pi,$$

so (i) holds for $n = 1$.

Lemma 3.4 gives

$$\begin{aligned} |v_1 - v_0| = |\delta v_0| &\leq \left|\frac{\cos u_0 \sinh v_0}{2u_0^2} - \frac{c v_0}{2u_0^2}\right| + A_4\frac{v_0}{u_0^3} \\ &< \frac{1}{2u_0^2}(\sinh v_0 + c v_0 + A_4 v_0), \end{aligned}$$

as $u_0 > K_2 > 2$. Since

$$\sinh v_0 < v_0\frac{\sinh L}{L},$$

we can write

$$\begin{aligned} |v_1 - v_0| &< \frac{v_0}{2u_0^2}\left(\frac{\sinh L}{L} + c + A_4\right) \\ &< \frac{v_0}{u_0}\frac{1}{2}\left(\frac{\sinh L}{L} + c + A_4\right) \leq A_5\frac{v_0}{u_0} \end{aligned} \tag{4.3}$$

and so (ii) is true for $n = 1$.

Since $A_5 v_0/u_0 < A_5 v_0/K_2 \le v_0$, it follows that $0 < v_1 < 2v_0 < L'$ and so (iii) is true for $n = 1$.

Now we suppose that

$$\frac{c}{8u_0} < u_n - u_{n-1} < \frac{3c}{4u_0} < \pi, \quad |v_n - v_0| < A_5\frac{v_0}{u_0} \quad \text{and} \quad 0 < v_n < L' \tag{4.4}$$

for all $n \in \{1, \ldots, k\}$ where k is any positive integer with the property that $u_k \le 2\pi M_0$.

We then claim that

$$\frac{c}{8u_0} < u_{k+1} - u_k < \frac{3c}{4u_0} < \pi, \quad |v_{k+1} - v_0| < A_5\frac{v_0}{u_0} \quad \text{and} \quad 0 < v_{k+1} < L'. \tag{4.5}$$

First note that, since $w_k \in R^+(K_1, L)$, it follows from Lemma 3.3 that

$$\left|\delta u_k - \frac{c}{2u_k}\right| \le \frac{A_3}{u_k^2} < \frac{A_3}{K_2 u_k} \le \frac{c}{4u_k}$$

since $u_k > K_2 \ge 4A_3/c$. Note that $u_k < 2u_0$ since $u_0 > K_2 \ge 6\pi$ and $u_k < u_0 + 2\pi$. Thus, by (4.4),

$$\frac{c}{8u_0} < \frac{c}{4u_k} < \delta u_k = u_{k+1} - u_k < \frac{3c}{4u_k} < \frac{3c}{4u_0} < \frac{3c}{4K_2} \le \frac{3c}{4}\frac{\pi}{c} < \pi.$$

This proves the first part of (4.5).

By (4.4) again, $0 < v_n < L'$ and $u_n > u_0 > K_2$ for all $n \in \{1, \ldots, k\}$, so $w_n \in R^+(K_2, L)$ for all $n \in \{0, \ldots, k\}$. Thus, by Lemma 3.4,

$$\begin{aligned} |v_{k+1} - v_0| = \left|\sum_{n=0}^{k} \delta v_n\right| &\le \sum_{n=0}^{k}\left(\frac{\sinh v_n}{2u_n^2} + \frac{c v_n}{2u_n^2} + A_4\frac{v_n}{u_n^3}\right) \\ &\le \frac{1}{2u_0^2}\sum_{n=0}^{k}(\sinh v_n + c v_n + A_4 v_n) \end{aligned}$$

where, again, we have used the fact that $u_0 > 2$.

Since $0 < v_n < L' \le L$ for all $n \in \{0, \ldots, k\}$, we can write

$$0 < \sinh v_n < v_n \frac{\sinh L}{L},$$

and hence

$$|v_{k+1} - v_0| < \frac{1}{2u_0^2}\sum_{n=0}^{k} v_n\left(\frac{\sinh L}{L} + c + A_4\right). \tag{4.6}$$

By (4.4) we have

$$v_n \le v_0\left(1 + \frac{A_5}{u_0}\right) < v_0\left(1 + \frac{A_5}{K_2}\right) \le 2v_0 \tag{4.7}$$

for all $n \in \{0, \ldots, k\}$, so

$$|v_{k+1} - v_0| < \frac{v_0}{u_0^2}\left(\frac{\sinh L}{L} + c + A_4\right)(k+1). \tag{4.8}$$

Now we derive an upper bound for $k+1$. Since $u_k - u_0 < 2\pi$ (because $u_k \leq 2\pi M_0$ is a condition on the choice of k) and $u_n - u_{n-1} > c/(8u_0)$ for n in $\{1, \ldots, k\}$, by (4.4), we have

$$k < \frac{2\pi}{c/(8u_0)} = \frac{16\pi u_0}{c}.$$

Since, by (4.2), $\pi u_0/c \geq \pi K_2/c \geq 1$, we have

$$k+1 < \frac{16\pi u_0}{c} + 1 \leq \frac{17\pi u_0}{c}. \tag{4.9}$$

Substituting this estimate into (4.8) gives

$$|v_{k+1} - v_0| < \frac{v_0}{u_0}\left(\frac{\sinh L}{L} + c + A_4\right)\frac{17\pi}{c} \leq A_5\frac{v_0}{u_0}.$$

This proves the second part of (4.5).

Since $A_5 v_0/u_0 < A_5 v_0/K_2 \leq v_0$, it follows that $0 < v_{k+1} < 2v_0 < L'$. Thus the proof of (4.5) is complete.

Lastly we show that the required positive integer $n(M_0)$ exists. We have shown that (i)–(iii) are true for all n such that $u_{n-1} \leq 2\pi M_0$. It follows from (i) that there exists n such that $u_n > 2\pi M_0$. We take $n(M_0)$ to be the smallest such n. We have shown that (i)–(iii) hold for all $n \leq n(M_0)$ as required. □

In Lemma 4.1 the choice of L' was arbitrary (so long as $0 < L' \leq L$). Now we set $L' = L/2$ and consider any w_0 in $R^+(K_2, L'/2) = R^+(K_2, L/4)$. Since the hypotheses of Lemma 4.1 are met, we see that w_n is in $R^+(K_2, L/2)$ for all n in $\{0, \ldots, n(M_0)\}$. Next we consider the forward orbit of $w_{n(M_0)}$. Note that it follows from property (i) of Lemma 4.1 that $u_{n(M_0)} < 2\pi M_0 + \pi$. Since $w_{n(M_0)}$ is in $R^+(K_2, L/2)$ and since K_2 is independent of L', we can apply the arguments of Lemma 4.1 again (this time with $L' = L$) to show that there exists a positive integer $n(M_0+1) > n(M_0)$ depending on w_0 such that $u_{n(M_0+1)-1} \leq 2\pi(M_0+1) < u_{n(M_0+1)}$ and for n in $\{n(M_0)+1, \ldots, n(M_0+1)\}$ we have

$$c/(8u_{n(M_0)}) < u_n - u_{n-1} < 3c/(4u_{n(M_0)}) < \pi, \tag{4.10}$$

$$|v_n - v_{n(M_0)}| < A_5 v_{n(M_0)}/u_{n(M_0)} \tag{4.11}$$

and

$$0 < v_n < L' = L. \tag{4.12}$$

Thus we see that if w_0 is in $R^+(K_2, L/4)$, then w_n lies in $R^+(K_2, L)$ for all n in $\{n(M_0), \ldots, n(M_0+1)\}$. In the following lemma we use (4.10)–(4.12) to prove the stronger property that there exists $K_3 > 0$ such that for any $w_0 \in R^+(K_3, L/4)$ we in fact have $v_{n(M_0+1)} < v_{n(M_0)}$, which implies that both $w_{n(M_0)}$ and $w_{n(M_0+1)}$ lie in the narrower strip $R^+(K_3, L/2)$.

LEMMA 4.2. *There exists a positive constant* $K_3 \geq K_2$ *such that for any* $w_0 \in R^+(K_3, L/4)$ *we have*

$$v_{n(M_0+1)} < v_{n(M_0)}. \tag{4.13}$$

Proof. We begin by defining

$$\kappa = \max\{K_2, A_3, 2A_5, 9\pi\}. \tag{4.14}$$

Let w_0 be a point in $R^+(\kappa, L/4)$. By Lemma 4.1 and (4.12) we see that $0 < v_n < L$ for all n in $\{n(M_0), \ldots, n(M_0+1)\}$. Hence by Lemma 3.4

$$\begin{aligned} v_{n(M_0+1)} - v_{n(M_0)} = \sum_{n=n(M_0)}^{n(M_0+1)-1} \delta v_n \le \frac{1}{2} \sum_{n=n(M_0)}^{n(M_0+1)-1} \frac{\sinh v_n \cos u_n}{u_n^2} \\ - \frac{c}{2} \sum_{n=n(M_0)}^{n(M_0+1)-1} \frac{v_n}{u_n^2} \\ + A_4 \sum_{n=n(M_0)}^{n(M_0+1)-1} \frac{v_n}{u_n^3}. \end{aligned} \tag{4.15}$$

We derive bounds for each of the three sums in turn, to show that the right-hand side of (4.15) is negative for all w_0 in $R^+(\kappa, L/4)$ with sufficiently large modulus, and this is sufficient to prove Lemma 4.2.

First we derive an upper bound for the third sum. By Lemma 4.1, (4.2), (4.11) and (4.12) we have, for all n in $\{n(M_0), \ldots, n(M_0+1)\}$,

$$0 < v_n \le v_{n(M_0)} + A_5 \frac{v_{n(M_0)}}{u_{n(M_0)}} = v_{n(M_0)} \left(1 + \frac{A_5}{u_{n(M_0)}}\right) < 2v_{n(M_0)},$$

so that we can write

$$A_4 \sum_{n=n(M_0)}^{n(M_0+1)-1} \frac{v_n}{u_n^3} < 2A_4 v_{n(M_0)} \sum_{n=n(M_0)}^{n(M_0+1)-1} \frac{1}{u_n^3}.$$

Next, by (4.10), we see that $u_n \ge u_{n(M_0)}$ for all n in $\{n(M_0), \ldots, n(M_0+1)\}$ and so

$$A_4 \sum_{n=n(M_0)}^{n(M_0+1)-1} \frac{v_n}{u_n^3} < 2NA_4 \frac{v_{n(M_0)}}{u_{n(M_0)}^3},$$

where N is the number of terms in the sum; that is,

$$N = n(M_0+1) - n(M_0). \tag{4.16}$$

Now, by (4.10), $Nc/(8u_{n(M_0)}) < u_{n(M_0+1)} - u_{n(M_0)} < 3\pi$ and so

$$N < \frac{24\pi u_{n(M_0)}}{c}. \tag{4.17}$$

Thus

$$A_4 \sum_{n=n(M_0)}^{n(M_0+1)-1} \frac{v_n}{u_n^3} < \frac{48\pi A_4}{c} \frac{v_{n(M_0)}}{u_{n(M_0)}^2}. \tag{4.18}$$

Next we obtain a lower bound for the second sum on the right-hand side of (4.15). Note that $v_n > 0$ for all n in $\{n(M_0), \ldots, n(M_0+1) - 1\}$ by (4.12).

Using (4.11) again we have $v_n > v_{n(M_0)} - A_5 v_{n(M_0)}/u_{n(M_0)}$ for all n in

$$\{n(M_0), \ldots, n(M_0+1)-1\}$$

so, by (4.14),

$$\frac{c}{2}\sum_{n=n(M_0)}^{n(M_0+1)-1} \frac{v_n}{u_n^2} > \frac{c}{2} v_{n(M_0)}\left(1 - \frac{A_5}{u_{n(M_0)}}\right)\sum_{n=n(M_0)}^{n(M_0+1)-1} \frac{1}{u_n^2} > \frac{c}{4} v_{n(M_0)} \sum_{n=n(M_0)}^{n(M_0+1)-1} \frac{1}{u_n^2}.$$

Since $u_{n(M_0)} > K_2 \geq 6\pi$, it follows from (4.10) that for all n in $\{n(M_0), \ldots, n(M_0+1)-1\}$ we have $u_n < u_{n(M_0)} + 3\pi < 2u_{n(M_0)}$. Hence

$$\frac{c}{2}\sum_{n=n(M_0)}^{n(M_0+1)-1} \frac{v_n}{u_n^2} > \frac{c}{16}\frac{v_{n(M_0)}}{u_{n(M_0)}^2} N.$$

By Lemma 4.1, $u_{n(M_0)} < 2\pi M_0 + \pi$, and so $u_{n(M_0+1)} - u_{n(M_0)} > \pi$. Thus, by (4.10),

$$N > \frac{4\pi u_{n(M_0)}}{3c} \tag{4.19}$$

and so

$$\frac{c}{2}\sum_{n=n(M_0)}^{n(M_0+1)-1} \frac{v_n}{u_n^2} > \frac{\pi}{12}\frac{v_{n(M_0)}}{u_{n(M_0)}}. \tag{4.20}$$

Lastly we consider the first sum on the right-hand side of (4.15). This is the most delicate of the three estimates, as it involves showing that there is significant ‘cancellation’ amongst the terms of the sum. Here we estimate the size of the sum by using orders of magnitude, the asymptotic order terms being valid as w_0 tends to infinity in $R^+(\kappa, L/4)$ for all n in $\{n(M_0), \ldots, n(M_0+1)-1\}$.

We start by estimating $\sinh v_n$. From (4.11) we deduce that

$$v_n = v_{n(M_0)} + O\left(\frac{v_{n(M_0)}}{u_{n(M_0)}}\right).$$

Since $0 < v_n < L$ and $0 < v_{n(M_0)} < L$, we deduce by the mean value theorem that

$$\sinh v_n = \sinh v_{n(M_0)} + O\left(\frac{v_{n(M_0)}}{u_{n(M_0)}}\right).$$

We use this expression to write the term inside the first sum on the right-hand side of (4.15) as

$$\frac{\sinh v_n \cos u_n}{2u_n^2} = \frac{(\sinh v_{n(M_0)} + O(v_{n(M_0)}/u_{n(M_0)}))\cos u_n}{2u_n^2}.$$

Now, for all n in $\{n(M_0), \ldots, n(M_0+1)-1\}$ we have $2\pi M_0 < u_{n(M_0)} \leq u_n \leq 2\pi(M_0+1)$ so $u_n = u_{n(M_0)} + O(1)$, and so

$$u_n^2 = u_{n(M_0)}^2 + O(u_{n(M_0)})$$

giving

$$\frac{1}{u_n^2} = \frac{1}{u_{n(M_0)}^2}\left(1 + O\left(\frac{1}{u_{n(M_0)}}\right)\right).$$

Substituting for $1/u_n^2$ we have

$$\begin{aligned}\frac{\sinh v_n \cos u_n}{2u_n^2} &= \frac{(\sinh v_{n(M_0)} + O(v_{n(M_0)}/u_{n(M_0)}))\cos u_n}{2u_{n(M_0)}^2}\left(1+O\left(\frac{1}{u_{n(M_0)}}\right)\right)\\ &= \frac{\sinh v_{n(M_0)}\cos u_n + O(v_{n(M_0)}/u_{n(M_0)})}{2u_{n(M_0)}^2} + O\left(\frac{v_{n(M_0)}}{u_{n(M_0)}^3}\right)\\ &= \frac{\sinh v_{n(M_0)}\cos u_n}{2u_{n(M_0)}^2} + O\left(\frac{v_{n(M_0)}}{u_{n(M_0)}^3}\right).\end{aligned}$$

Thus

$$\sum_{n=n(M_0)}^{n(M_0+1)-1}\frac{\sinh v_n \cos u_n}{2u_n^2} = \frac{\sinh v_{n(M_0)}}{2u_{n(M_0)}^2}\sum_{n=n(M_0)}^{n(M_0+1)-1}\cos u_n + \sum_{n=n(M_0)}^{n(M_0+1)-1} O\left(\frac{v_{n(M_0)}}{u_{n(M_0)}^3}\right). \tag{4.21}$$

The second sum on the right-hand side of (4.21) consists of N terms. By Lemma 3.3 we have $|\delta u_n - c/(2u_n)| \le A_3/u_n^2$ and so $\delta u_n = c/2u_n + O(1/u_n^2)$ as u_0 tends to infinity. Hence, we can write

$$N = (4\pi u_{n(M_0)})/c + O(1). \tag{4.22}$$

Thus,

$$\sum_{n=n(M_0)}^{n(M_0+1)-1}\frac{\sinh v_n \cos u_n}{2u_n^2} = \frac{\sinh v_{n(M_0)}}{2u_{n(M_0)}^2}\sum_{n=n(M_0)}^{n(M_0+1)-1}\cos u_n + O\left(\frac{v_{n(M_0)}}{u_{n(M_0)}^2}\right).$$

Finally, since $v_{n(M_0)}$ is bounded, we have

$$\sum_{n=n(M_0)}^{n(M_0+1)-1}\frac{\sinh v_n \cos u_n}{2u_n^2} = O\left(\frac{v_{n(M_0)}}{u_{n(M_0)}^2}\right)\sum_{n=n(M_0)}^{n(M_0+1)-1}\cos u_n + O\left(\frac{v_{n(M_0)}}{u_{n(M_0)}^2}\right). \tag{4.23}$$

We now show that

$$\sum_{n=n(M_0)}^{n(M_0+1)-1}\cos u_n = O(1). \tag{4.24}$$

It is easy to deduce from Lemma 3.3 and the fact that $u_n = u_{n(M_0)} + O(1)$ that

$$\delta u_n = \frac{c}{2u_n} + O\left(\frac{1}{u_n^2}\right) = \frac{c}{2u_{n(M_0)}} + O\left(\frac{1}{u_{n(M_0)}^2}\right) = O\left(\frac{1}{u_{n(M_0)}}\right), \tag{4.25}$$

so

$$(\delta u_n)^2 = O\left(\frac{1}{u_{n(M_0)}^2}\right)$$

and

$$\frac{1}{\delta u_n} = \left(\frac{c}{2u_{n(M_0)}} + O\left(\frac{1}{u_{n(M_0)}^2}\right)\right)^{-1} = \frac{2u_{n(M_0)}}{c} + O(1). \tag{4.26}$$

Thus

$$\begin{aligned}\sum_{n=n(M_0)}^{n(M_0+1)-1} \cos u_n &= \sum_{n=n(M_0)}^{n(M_0+1)-1} \frac{\delta u_n}{\delta u_n} \cos u_n \\ &= \frac{2u_{n(M_0)}}{c} \sum_{n=n(M_0)}^{n(M_0+1)-1} \delta u_n \cos u_n + \sum_{n=n(M_0)}^{n(M_0+1)-1} O(1)\delta u_n \cos u_n \\ &= \frac{2u_{n(M_0)}}{c} \sum_{n=n(M_0)}^{n(M_0+1)-1} \delta u_n \cos u_n + \sum_{n=n(M_0)}^{n(M_0+1)-1} O\left(\frac{1}{u_{n(M_0)}}\right). \quad (4.27)\end{aligned}$$

Using (4.22) we can simplify the second sum on the right-hand side of (4.27) and write

$$\sum_{n=n(M_0)}^{n(M_0+1)-1} \cos u_n = O(u_{n(M_0)}) \sum_{n=n(M_0)}^{n(M_0+1)-1} \delta u_n \cos u_n + O(1). \quad (4.28)$$

We now show that

$$\sum_{n=n(M_0)}^{n(M_0+1)-1} \delta u_n \cos u_n = O\left(\frac{1}{u_{n(M_0)}}\right).$$

Since $|\cos' x| = |\sin x| \leq 1$, we have

$$\int_{u_n}^{u_{n+1}} \cos x \, dx = \delta u_n \cos u_n + O((\delta u_n)^2) = \delta u_n \cos u_n + O\left(\frac{1}{u^2_{n(M_0)}}\right).$$

Now (see Figure 3), it is easy to see that

$$\left| \int_{2\pi M_0}^{u_{n(M_0)}} \cos x \, dx \right| < \delta u_{n(M_0)-1} = O\left(\frac{1}{u_{n(M_0)}}\right)$$

and

$$\left| \int_{2\pi(M_0+1)}^{u_{n(M_0+1)}} \cos x \, dx \right| < \delta u_{n(M_0+1)-1} = O\left(\frac{1}{u_{n(M_0)}}\right).$$

So, by (4.22),

$$\begin{aligned}0 = \int_{2\pi M_0}^{2\pi(M_0+1)} \cos x \, dx &= \int_{2\pi M_0}^{u_{n(M_0)}} \cos x \, dx + \int_{u_{n(M_0)}}^{u_{n(M_0+1)}} \cos x \, dx - \int_{2\pi(M_0+1)}^{u_{n(M_0+1)}} \cos x \, dx \\ &= \sum_{n=n(M_0)}^{n(M_0+1)-1} \delta u_n \cos u_n + N \times O\left(\frac{1}{u^2_{n(M_0)}}\right) + O\left(\frac{1}{u_{n(M_0)}}\right) \\ &= \sum_{n=n(M_0)}^{n(M_0+1)-1} \delta u_n \cos u_n + O\left(\frac{1}{u_{n(M_0)}}\right). \quad (4.29)\end{aligned}$$

Thus

$$\sum_{n=n(M_0)}^{n(M_0+1)-1} \delta u_n \cos u_n = O\left(\frac{1}{u_{n(M_0)}}\right),$$

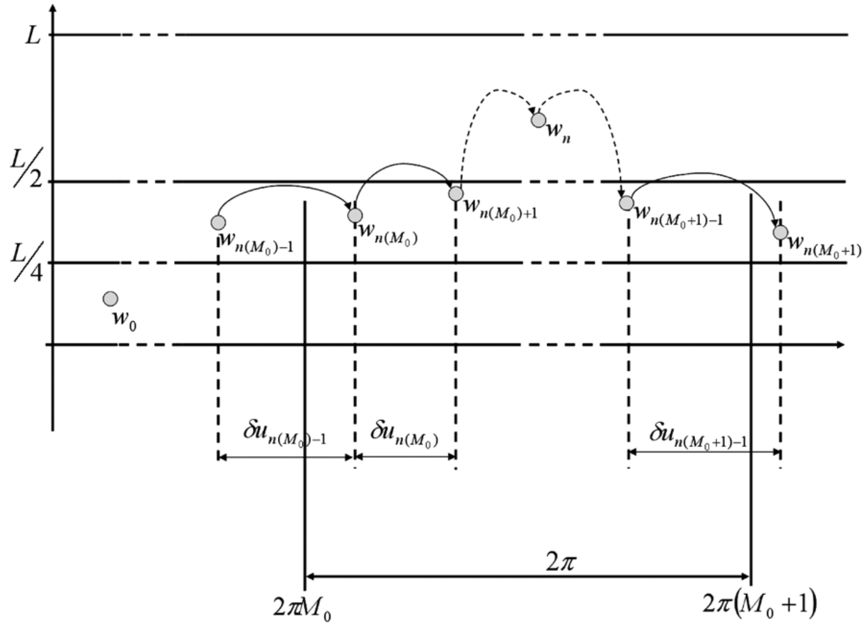

FIGURE 3. Labelling of points in the orbit of w_0.

and so, by (4.28),

$$\sum_{n=n(M_0)}^{n(M_0+1)-1} \cos u_n = O(1).$$

It now follows from (4.23) that

$$\sum_{n=n(M_0)}^{n(M_0+1)-1} \frac{\sinh v_n \cos u_n}{2u_n^2} = O\left(\frac{v_{n(M_0)}}{u_{n(M_0)}^2}\right). \tag{4.30}$$

From (4.15), (4.18), (4.20) and (4.30) we have

$$v_{n(M_0+1)} - v_{n(M_0)} \leq -\frac{\pi}{12}\frac{v_{n(M_0)}}{u_{n(M_0)}} + O\left(\frac{v_{n(M_0)}}{u_{n(M_0)}^2}\right) \quad \text{as } w_0 \to \infty \text{ in } R^+(\kappa, L/4) \tag{4.31}$$

and this completes the proof of Lemma 4.2. □

In the discussion following Lemma 4.1 we showed that, for w_0 in $R^+(K_2, L/4)$, the points w_n lie in $R^+(K_2, L)$ for all n in $\{n(M_0), \ldots, n(M_0+1)\}$.

It follows from Lemma 4.2 that for any w_0 in $R^+(K_3, L/4)$ the points $w_{n(M_0)}$ and $w_{n(M_0+1)}$ are both in fact in $R^+(K_3, L/2)$. Thus, $w_{n(M_0+1)}$ meets the hypotheses of Lemma 4.1 (with $L' = L$) and so there exists $n(M_0+2)$ such that $u_{n(M_0+2)-1} \leq 2\pi(M_0+2) < u_{n(M_0+2)}$ with conditions analogous to (4.10)–(4.12) being satisfied for all n in $\{n(M_0+1), \ldots, n(M_0+2)\}$.

This enables us to apply the method of Lemma 4.2 to $w_{n(M_0+1)}$ to show that $v_{n(M_0+2)} < v_{n(M_0+1)}$ and hence $w_{n(M_0+2)} \in R^+(K_3, L/2)$. By repeatedly applying the methods of Lemmas 4.1 and 4.2 in this way we deduce that for each $M \geq M_0$ there exists $n(M)$ such that $u_{n(M)-1} \leq 2\pi M < u_{n(M)}$ and, for each n in $\{n(M)+1, \ldots, n(M+1)\}$,

$$c/(8u_{n(M)}) < u_n - u_{n-1} < 3c/(4u_{n(M)}) < \pi, \tag{4.32}$$

$$|v_n - v_{n(M)}| < A_5 v_{n(M)}/u_{n(M)} \tag{4.33}$$

and

$$0 < v_n < L' = L. \tag{4.34}$$

Thus the whole forward orbit of w_0 lies in $R^+(K_3, L)$.

We recall that g is symmetric in the sense that $g(\overline{w}) = \overline{g(w)}$ so that equivalent properties to (4.32)–(4.34) hold for any w_0 in

$$R^-(K_3, L/4) = \{w \mid \Re(w) > K_3, -L/4 < \Im(w) < 0\}.$$

Setting $K = K_3$ and $L' = L/4$, and using (2.5) together with the fact that $K \geq \sqrt{K_0}$, we see that (2.3) holds. Thus the proof of Theorem 1.1 is complete.

5. *Proof of Theorem 1.2*

To prove Theorem 1.2 we let the unbounded open set V from the statement of Theorem 1.2 be $h(R(K, L'))$ with $K = K_3$ and $L' = L/4$ from the proof of Theorem 1.1 where, as in Section 1, $h(w) = w^2$.

We start by considering the real line. It follows from (2.5), since $K \geq \sqrt{K_0}$, that the conclusions (i) and (ii) of Theorem 1.2 hold on $(K^2, \infty) \subset h(R(K, L'))$ with $\eta_x = 0$.

Now we consider points in $h(R(K, L')) \setminus \mathbb{R}$. Since g is symmetric in the sense that $g(\overline{w}) = \overline{g(w)}$ for all $w \in \mathbb{C}$, to prove Theorem 1.2 for a general point in the set $h(R(K, L')) \setminus \mathbb{R}$, it is sufficient to consider only the iterates of points w in $R^+(K, L')$ under g.

For any w_0 in $R^+(K, L')$, by (4.32), $u_n = \Re(w_n)$ tends to infinity as n tends to infinity and $v_n = \Im(w_n)$ remains positive and bounded, by (4.34). Hence $\Re(z_n) = u_n^2 - v_n^2$ tends to infinity as n tends to infinity and this proves conclusion (i) of Theorem 1.2 for any z_0 in $h(R(K, L'))$.

Next we prove Theorem 1.2(ii). Since $h \circ g(w) = f \circ h(w) = f(w^2)$, it is sufficient to show that

$$\Im((h \circ g^n(w))) = \Im((g^n(w))^2) \to \eta_{w^2} \quad \text{as } n \to \infty, \text{ for all } w \in R^+(K, L'), \tag{5.1}$$

where η_{w^2} is some positive constant depending on w^2.

As in the previous section, for any fixed $w_0 = u_0 + iv_0 \in R^+(K, L')$ we denote the nth iterate under g by $w_n = u_n + iv_n$. Thus we can write condition (5.1) as

$$u_n v_n \to \tfrac{1}{2}\eta_{w_0^2} \quad \text{as } n \to \infty, \text{ for all } w_0 \in R^+(K, L'). \tag{5.2}$$

To prove Theorem 1.2(ii), we note that for each $n \geq n(M_0)$, there exists some M satisfying $M \geq M_0$ such that $n = n(M) + d$ where $0 \leq d < n(M+1) - n(M)$. By (4.32) and (4.33), we have

$$u_n = 2\pi M + O(1) = 2\pi M\left(1 + O\left(\frac{1}{M}\right)\right) \quad \text{as } M \to \infty, \tag{5.3}$$

and

$$v_n = v_{n(M)} + O\left(\frac{v_{n(M)}}{M}\right) = v_{n(M)}\left(1 + O\left(\frac{1}{M}\right)\right) \quad \text{as } M \to \infty, \tag{5.4}$$

so

$$u_n v_n = 2\pi M v_{n(M)}\left(1 + O\left(\frac{1}{M}\right)\right) \quad \text{as } M \to \infty. \tag{5.5}$$

Thus, to prove Theorem 1.2(ii), it is sufficient to show that $M v_{n(M)}$ tends to some positive limit as M tends to infinity. We proceed by deriving a recursive expression for $v_{n(M)}$.

It follows from (4.15), (4.18), (4.20) and (4.30) and the observations immediately following the proof of Lemma 4.2 that

$$v_{n(M+1)} - v_{n(M)} = \sum_{k=n(M)}^{n(M+1)-1} \delta v_k = O\left(\frac{v_{n(M)}}{u_{n(M)}^2}\right) - \sum_{k=n(M)}^{n(M+1)-1} \frac{c v_k}{2u_k^2} \quad \text{as } M \to \infty. \tag{5.6}$$

We consider the sum on the right-hand side of (5.6) and we derive a more accurate estimate than that found in Lemma 4.2.

It follows from (5.3) and (5.4) that

$$-\sum_{k=n(M)}^{n(M+1)-1} \frac{c v_k}{2u_k^2} = -\frac{c}{2} N\left(\frac{v_{n(M)}}{4\pi^2 M^2} + O\left(\frac{v_{n(M)}}{M^3}\right)\right) \quad \text{as } M \to \infty,$$

where $N = n(M+1) - n(M)$. By Lemma 3.3 and (4.32) we can estimate N by

$$N = \frac{8\pi^2 M}{c} + O(1) \quad \text{as } M \to \infty.$$

Putting all these observations together, we have

$$-\sum_{k=n(M)}^{n(M+1)-1} \frac{c v_k}{2u_k^2} = -v_{n(M)}\left(\frac{1}{M} + O\left(\frac{1}{M^2}\right)\right) \quad \text{as } M \to \infty.$$

Substituting for the sum in equation (5.6) and re-arranging, gives

$$\begin{aligned} v_{n(M+1)} &= v_{n(M)}\left(1 - \frac{1}{M} + O\left(\frac{1}{M^2}\right)\right) \\ &= v_{n(M)} \frac{M-1}{M}\left(1 + O\left(\frac{1}{M^2}\right)\right) \quad \text{as } M \to \infty, \end{aligned}$$

so there exists a sequence A_M, $M \geq M_0$, and a constant $A > 0$ such that

$$v_{n(M+1)} = v_{n(M)} \frac{M-1}{M}\left(1 + \frac{A_M}{M^2}\right) \quad \text{for } M \geq M_0, \tag{5.7}$$

where $|A_M| < A$ for all $M \geq M_0$. Let M' be such that $M' \geq M_0$ and $|A_M|/M^2 < 1$ for all $M \geq M'$. It follows from (5.7) that

$$v_{n(M)} = v_{n(M')} \frac{M'-1}{M-1} \prod_{j=M'}^{M-1} \left(1 + \frac{A_j}{j^2}\right), \tag{5.8}$$

and so, by (5.5),

$$u_n v_n = 2\pi M v_{n(M')} \frac{M'-1}{M-1} \left(1 + O\left(\frac{1}{M}\right)\right) \prod_{j=M'}^{M-1} \left(1 + \frac{A_j}{j^2}\right) \quad \text{as } M \to \infty.$$

Now the infinite product

$$\prod_{j=M'}^{\infty} \left(1 + \frac{A_j}{j^2}\right)$$

is convergent with a strictly positive limit, Q say. Thus

$$u_n v_n \to 2\pi v_{n(M')}(M'-1)Q \quad \text{as } n \to \infty,$$

since M tends to infinity as n tends to infinity. Setting $\eta_{z_0} = 2\pi v_{n(M')}(M'-1)Q > 0$, we have shown that the imaginary parts of the sequence of iterates $f^n(z_0)$, where $z_0 = w_0^2$ in $h(R^+(K, L'))$, tend to a positive limit which depends on the initial point z_0. This concludes the proof of Theorem 1.2.

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