On a problem of Gabriel and Ulmer

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Abstract

We present a locally finitely presentable category with a finitely presentable regular generator $\mathcal{G}$ and a finitely presentable object A, such that A is not a coequalizer of morphisms whose domains and codomains are finite coproducts of objects in $\mathcal{G}$, thereby settling a problem by Gabriel and Ulmer. We also show that in λ-orthogonality classes in $\mathbf{Alg}_{\mathcal{S}}\,\tau$ (category of $\mathcal{S}$-sorted τ-algebras) for a λ-ary signature τ, λ-presentable objects have a presentation by less than λ generators and relations and use this to exhibit an example of a reflective subcategory of a locally finitely presentable category which is closed under directed colimits, but not a $\aleph_0$-orthogonality class, disproving a characterization of λ-orthogonality classes in the book by Adámek and Rosický.

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Introduction

Notation and preliminary results are from (Adámek, Herrlich, Strecker [2]) and (Adámek, Rosický [3]). Throughout, λ will be a regular cardinal, and all subcategories are considered to be full. For a concrete category $\mathbf{C}$ over $\mathbf{Set}$ resp. $\mathbf{Set}^{\mathcal{S}}$ (the category of $\mathcal{S}$-sorted sets) with free objects, $|\_|$ will denote the usual forgetful functor (which we tend to leave out notationally), and $F_{\mathbf{C}}$ the usual free functor. For better readability, terms and their term functions will be notationally identified.

[1] Home-page: http://www.dcs.ed.ac.uk/home/jan . This work was supported by the Studienstiftung des deutschen Volkes and is dedicated to my father on the occasion of his 60^{th} birthday.

Recall that an object A is called *λ-presentable* if $\hom(A, \_)$ preserves λ-directed colimits, and a category $\mathbf{A}$ is called *locally λ-presentable* if it is cocomplete and has a set $\mathcal{A}$ of λ-presentable objects such that every object in $\mathbf{A}$ is a λ-directed colimit of objects from $\mathcal{A}$ (or equivalently, if it is isomorphic to a category of models of a limit theory in the logic L_λ, [3, ch 1.B,5.B]).

It is well-known that in a variety of λ-ary algebras, the λ-presentable objects are precisely the algebras presentable by less than λ generators and less than λ equations [3, 3.13]. We generalize the latter notion to more general categories: First recall that a set $\mathcal{G} \subseteq \mathrm{Ob}\,\mathbf{C}$ of objects in a cocomplete category $\mathbf{C}$ is a regular generator if for every object $X \in \mathrm{Ob}\,\mathbf{C}$ the canonical morphism

$$\coprod_{G\in\mathcal{G}} \coprod_{f\in\hom(G,X)} G \to X$$

factoring through the cocone of all morphisms $f \in \hom(G, X)$ $(G \in \mathcal{G})$ is a regular epimorphism [1]. Let $\mathbf{C}$ be a cocomplete category with a λ-presentable regular generator $\mathcal{G}$. Call an object C of $\mathbf{C}$ *λ-$\mathcal{G}$-presented* if there exists a coequalizer

$$\coprod_{j\in J} G_j \rightrightarrows \coprod_{i\in I} G_i \to C$$

with $G_i, G_j \in \mathcal{G}$ and $\mathrm{card}\, I, \mathrm{card}\, J < \lambda$.
Let $\mathbf{Alg}_{\mathcal{S}}\,\tau$ be the category of $\mathcal{S}$-sorted total algebras for a λ-ary signature τ, and $\mathbf{C}$ a reflective subcategory closed under λ-directed colimits. Note that a set $\mathcal{G}$ of representatives (w.r.t. isomorphism) of the class $\{F_{\mathbf{C}}X : X \in \mathbf{Set}^{\mathcal{S}} \wedge \sharp X < \lambda\}$ is a λ-presentable regular generator (where $F_{\mathbf{C}}$ is the usual functor sending a set to the free algebra generated by it, and for $X = (X_s)_{s\in\mathcal{S}} \in \mathbf{Set}^{\mathcal{S}}$ we define $\sharp X := \sum_{s\in\mathcal{S}} \mathrm{card}\, X_s)$. In this situation call the λ-$\mathcal{G}$-presented objects λ-presented.
Thus in varieties the λ-presented objects are exactly the algebras presentable by less than λ generators and less than λ equations in the usual way.

Gabriel and Ulmer prove in [5] the following characterization of λ-presentable objects in locally presentable categories with a λ-presentable regular generator:

Proposition 1 ([5, 7.6]) *Let $\mathbf{C}$ be a locally presentable category and $\mathcal{G} \subseteq \mathrm{Ob}\,\mathbf{C}$ a λ-presentable regular generator. Then the λ-presentable objects are exactly the retracts of λ-$\mathcal{G}$-presented objects.*
If, additionally, regular epimorphisms are closed under composition in $\mathbf{C}$, then the λ-presentable objects are exactly the λ-$\mathcal{G}$-presented objects.

In [5, 7.7e] Gabriel and Ulmer state that they do not know an example of a locally presentable category $\mathbf{C}$ with a λ-presentable regular generator $\mathcal{G}$ and a λ-presentable object A, which is not λ-$\mathcal{G}$-presented. An example of this kind is provided in this note. In a category $\mathbf{A}_* = \mathbf{Alg}\,\mathbf{\Sigma}$ of single-sorted total algebras for some specific signature Σ consisting of two nullary and countably many unary operations we construct an $\aleph_1$-orthogonal, hence reflective, subcategory $\mathbf{C}_*$ such that

- $\mathbf{C}_*$ is closed under directed colimits in $\mathbf{A}_*$, hence a locally finitely presentable category with a finitely presentable regular generator $\mathcal{G}_* := \{F_{\mathbf{C}_*}X : X \in \mathbf{Set}^{\mathcal{S}} \wedge \sharp X < \aleph_0\}$, and
- $\mathbf{C}_*$ contains a finitely presentable object C which is not finitely $\mathcal{G}_*$-presented.

We also show how to obtain an analoguous example of a category $\bar{\mathbf{A}} = \mathbf{Alg}\,\mathbf{\Omega}$ with a subcategory $\bar{\mathbf{C}}$ and an object $\bar{C}$ where Ω consists of finitely many finitary operations.

Since in any λ-orthogonality class of a category $\mathbf{Alg}_{\mathcal{S}}\,\tau$ with a λ-ary signature τ an object is λ-presentable iff it is λ-presented (Proposition 3, a generalization of the situation in (quasi-)varieties), this subcategory $\mathbf{C}_*$ cannot be an $\aleph_0$-orthogonality class in $\mathbf{A}_*$. This disproves the first part of theorem [3, 1.39] stating that a subcategory of a locally λ-presentable category is a λ-orthogonality class iff it is reflective and closed under λ-directed colimits.

1 Results

One easily obtains the following "single-step" modification of the orthogonal-reflection construction in [3, 1.37]:

Proposition 2 *Let $\mathbf{A}$ be a cocomplete category and $\mathcal{M} \subseteq \operatorname{Mor}\mathbf{A}$ a set of morphisms with λ-presentable domains and codomains. Then for every $A \in \operatorname{Ob}\mathbf{A}$ there exists a limit ordinal i_* and a diagram $(b_{i,j} : B_i \to B_j)_{i \leq j < i_*}$ such that*

- *$B_0 = A$*
- *$b_{i,i+1} : B_i \to B_{i+1}$ for $i < i_*$ is defined either by*
  - *a pushout*

$$\begin{tikzcd} M \arrow[r, "m"] \arrow[d, "h_0"'] & M' \arrow[d, "h_1"] \\ B_i \arrow[r, "b_{i,i+1}"'] & B_{i+1} \end{tikzcd}$$

of a span $B_i \xleftarrow{f} M \xrightarrow{m} M'$ with $m \in \mathcal{M}$, or by

- a coequalizer $M' \overset{p}{\underset{q}{\rightrightarrows}} B_i \overset{b_{i,i+1}}{\longrightarrow} B_{i+1}$ *of a pair* (h_0, h_1) *for which there exists* $(m : M \to M') \in \mathcal{M}$ *with* $h_0 \circ m = h_1 \circ m$.

- *For every limit ordinal* $j \leq i_*$, $(b_{i,j} : B_i \to B_j)_{i<j}$ *is a colimit cocone for the diagram* $(b_{i,i'} : B_i \to B_{i'})_{i \leq i' < j}$.
- *For each* $i < i_*$, b_{i,i_*} *is a* $\mathcal{M}^\perp$*-reflection arrow. In particular we have a reflection arrow* $b_{0,i_*} : A \to B_{i_*}$.

Thus the λ*-orthogonality class* $\mathcal{M}^\perp$ *is reflective in* $\mathbf{A}$.

The next result generalizes the corresponding fact in (quasi-)varieties:

Theorem 3 *Let* $\mathbf{C}$ *be a* λ*-orthogonality class in a category* $\mathbf{Alg}_\mathcal{S}\,\tau$ *for a* λ*-ary signature* τ*. Then an object is* λ*-presentable in* $\mathbf{C}$ *iff it is* λ*-presented in* $\mathbf{C}$.

PROOF.

$\Leftarrow$**:** This follows directly from Proposition 1.

$\Rightarrow$**:** By Proposition 1 every λ-presentable object in $\mathbf{C}$ is retract of a λ-presented object in $\mathbf{C}$, it is hence sufficient to show that in $\mathbf{C}$ the class of λ-presented objects in $\mathbf{C}$ is closed under coequalizers:

Let $F_\mathbf{C}Y \overset{f}{\underset{g}{\rightrightarrows}} F_\mathbf{C}X \overset{e}{\longrightarrow} B$, $F_\mathbf{C}\bar{Y} \overset{\bar{f}}{\underset{\bar{g}}{\rightrightarrows}} F_\mathbf{C}\bar{X} \overset{\bar{e}}{\longrightarrow} \bar{B}$ and $\bar{B} \overset{h}{\underset{k}{\rightrightarrows}} B \overset{c}{\longrightarrow} A$ be $\mathbf{C}$-coequalizers with $\sharp X, \sharp Y, \sharp\bar{X} < \lambda$ ($\sharp\bar{Y} < \lambda$ is not needed). To show that A is λ-presented we apply Lemma 5 with $(q : F_\mathbf{C}(\bar{X} + \bar{X}) \to B) := [h \circ \bar{e}, k \circ \bar{e}]$ (the brackets denote the factorising morphism from the coproduct - up to an isomorphism) and obtain a coequalizer $F_\mathbf{C}Y' \overset{f'}{\underset{g'}{\rightrightarrows}} F_\mathbf{C}X' \overset{e'}{\longrightarrow} B$ with $\sharp X' < \lambda$ and $\sharp Y' < \lambda$, thus $\sharp(Y' + \bar{X}) < \lambda$, and morphisms $\bar{h}, \bar{k} : F_\mathbf{C}\bar{X} \to F_\mathbf{C}X'$ such that $e' \circ \bar{h} = h \circ \bar{e}$ and $e' \circ \bar{k} = k \circ \bar{e}$ (see (1)).

Then $F_\mathbf{C}(Y' + \bar{X}) \overset{[f',\bar{h}]}{\underset{[g',\bar{k}]}{\rightrightarrows}} F_\mathbf{C}X' \overset{c \circ e'}{\longrightarrow} A$ is a coequalizer in $\mathbf{C}$: $c \circ e'$ is obviously epimorphic. Let $a : F_{\mathbf{C}}X' \to A'$ be given with $a \circ [f', \bar{h}] = a \circ [g', \bar{k}]$.

$$
\begin{tikzcd}
 & & & F_{\mathbf{C}}\bar{X} \arrow[d, "\bar{e}"] \arrow[ddll, dashed, shift right, "\bar{h}"'] \arrow[ddll, dashed, shift left, "\bar{k}"] \\
F_{\mathbf{C}}Y \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] \arrow[d, hook] & F_{\mathbf{C}}X \arrow[d, hook] \arrow[drr, "e"] & & \bar{B} \arrow[d, shift right, "h"'] \arrow[d, shift left, "k"] \\
F_{\mathbf{C}}Y' \arrow[r, shift left, "f'"] \arrow[r, shift right, "g'"'] & F_{\mathbf{C}}X' \arrow[rr, "e'"] \arrow[dr, "a"] & & B \arrow[d, "c"] \arrow[dl, dashed, "\bar{a}"] \\
 & & A' & A \arrow[l, dashed, "\tilde{a}"]
\end{tikzcd}
\tag{1}
$$

Since we have $a \circ f' = a \circ g'$, there exists $\bar{a} : B \to A'$ with $\bar{a} \circ e' = a$. This implies $\bar{a} \circ h \circ \bar{e} = \bar{a} \circ k \circ \bar{e}$, and so $\bar{a} \circ h = \bar{a} \circ k$, since $\bar{e}$ is an epimorphism. Thus there exists $\tilde{a} : A \to A'$ such that $\bar{a} = \tilde{a} \circ c$, i. e. $a = \tilde{a} \circ c \circ e'$.

$\square$

Remark 4 *One can also show that in a λ-orthogonality class in a category $\mathbf{Alg}_{\mathcal{S}}\,\tau$ for a λ-ary signature τ, the λ-small objects are exactly the λ-presented objects (see [7]), where an object A is λ-small if* $\hom(A, \_)$ *sends λ-directed colimits to episinks (as defined in [4]). For further characterizations of smallness conditions on objects in categories of algebras, see [7].*

Lemma 5 *Let $\mathbf{C}$ be a λ-orthogonality class in a category $\mathbf{Alg}_{\mathcal{S}}\tau$ for a λ-ary signature τ. Let $X, Y, \tilde{X} \in \mathbf{Set}^{\mathcal{S}}$ with $\sharp X < \lambda$, $\sharp Y < \lambda$ and $\sharp\tilde{X} < \lambda$. Let $F_{\mathbf{C}}Y \underset{g}{\overset{f}{\rightrightarrows}} F_{\mathbf{C}}X \overset{e}{\to} B$ be a coequalizer in $\mathbf{C}$ and $(q : F_{\mathbf{C}}\tilde{X} \to B) \in \mathbf{C}$.*
Then there exist $X', Y' \in \mathbf{Set}^{\mathcal{S}}$ with $\sharp X' < \lambda$, $\sharp Y' < \lambda$, $Y \subseteq Y'$ and $X \subseteq X'$, a coequalizer $F_{\mathbf{C}}Y' \underset{g'}{\overset{f'}{\rightrightarrows}} F_{\mathbf{C}}X' \overset{e'}{\to} B$ in $\mathbf{C}$ and $(q' : F_{\mathbf{C}}\tilde{X} \to F_{\mathbf{C}}X') \in \mathbf{C}$, such that

the following diagram commutes (let $u_{\_}$ be the universal morphisms):

$$\begin{tikzcd}
 & F_{\mathbf{C}}Y \arrow[rr, shift left, "f"] \arrow[rr, shift right, "g"'] \arrow[ddd, hook] & & F_{\mathbf{C}}X \arrow[ddd, hook] \arrow[dddrr, "e"] & & F_{\mathbf{C}}\tilde{X} \arrow[dddll, dashed, "q'"'] \arrow[ddd, "q"] \\
Y \arrow[ur, "u_Y"] \arrow[d, hook] & & X \arrow[ur, "u_X"] \arrow[d, hook] & & & \\
Y' \arrow[dr, "u_{Y'}"'] & & X' \arrow[dr, "u_{X'}"] & & & \\
 & F_{\mathbf{C}}Y' \arrow[rr, shift left, "f'"] \arrow[rr, shift right, "g'"'] & & F_{\mathbf{C}}X' \arrow[rr, "e'"'] & & B
\end{tikzcd}$$

PROOF. Let the conditions in the premiss of the above statement be fulfilled and write $\mathbf{A} := \mathbf{Alg}_{\mathcal{S}}\tau$.

There exists $\mathcal{M} \subseteq \operatorname{Mor}\mathbf{A}$, such that the domain and codomain of every morphism in $\mathcal{M}$ are λ-presentable and such that $\mathbf{C} = \mathcal{M}^{\perp}$. Let $F_{\mathbf{C}}Y \overset{f}{\underset{g}{\rightrightarrows}} F_{\mathbf{C}}X \xrightarrow{\bar{e}} K$ be the coequalizer in $\mathbf{A}$. Set $B_0 := K$ and let the reflection $B_{i_*} := B$ of K in $\mathbf{C}$ be constructed as in Proposition 2. We have a colimit $(b_{i,i_*} : B_i \to B_{i_*})_{i<i_*}$ in $\mathbf{A}$ of the diagram $(b_{i,j} : B_i \to B_j)_{i \leq j < i_*}$, and for the objects B_i constructed from spans there exists a pushout in $\mathbf{A}$

$$\begin{tikzcd}
P_i \arrow[r, "m_i"] \arrow[d, "f_i"'] & P_i' \arrow[d, "f_i'"] \\
B_{i-1} \arrow[r, "b_{i-1,i}"'] & B_i
\end{tikzcd}$$

By supposition on $\mathcal{M}$, we have $R_i, R_i', S_i, S_i' \in \mathbf{Set}^{\mathcal{S}}$, each of cardinality less than λ, and coequalizers $F_{\mathbf{A}}R_i \overset{\nu_i}{\underset{\sigma_i}{\rightrightarrows}} F_{\mathbf{A}}S_i \xrightarrow{\mu_i} P_i$ and $F_{\mathbf{A}}R_i' \overset{\nu_i'}{\underset{\sigma_i'}{\rightrightarrows}} F_{\mathbf{A}}S_i' \xrightarrow{\mu_i} P_i'$ in $\mathbf{A}$. It is easy to see that R_i', S_i' can be chosen such that $R_i \subseteq R_i'$ and $S_i \subseteq S_i'$. Then the following diagram commutes, where the universal arrows $\bar{u}$ are w.l.o.g. inclusions, as well as the arrows without labels. Let r_i, r_i' be reflection arrows and let $e_i := R(b_{i-1,i_*} \circ f_i \circ \mu_i)$ and $e_i' := R(b_{i,i_*} \circ f_i' \circ \mu_i')$ for the reflector R.

$$
\begin{tikzcd}
 & & R_i \arrow[r, hook] \arrow[dl, hook, "\bar{u}_{R_i}"'] & R_i' \arrow[dr, hook, "\bar{u}_{R_i'}"] & & \\
 & F_{\mathbf{A}} R_i \arrow[rrr, hook] \arrow[dd, shift right, "\nu_i"'] \arrow[dd, shift left, "\sigma_i"] & & & F_{\mathbf{A}} R_i' \arrow[dd, shift right, "\nu_i'"'] \arrow[dd, shift left, "\sigma_i'"] & \\
 & & S_i \arrow[r, hook] \arrow[dll, "u_{S_i}"'] \arrow[dl, hook, "\bar{u}_{S_i}"] & S_i' \arrow[dr, hook, "\bar{u}_{S_i'}"'] \arrow[drr, "u_{S_i'}"] & & \\
F_{\mathbf{C}} S_i \arrow[dddrrr, bend right, "e_i"'] & F_{\mathbf{A}} S_i \arrow[l, "r_i"] \arrow[rrr, hook] \arrow[d, "\mu_i"] & & & F_{\mathbf{A}} S_i' \arrow[r, "r_i'"'] \arrow[d, "\mu_i'"] & F_{\mathbf{C}} S_i' \arrow[dddll, bend left, "e_i'"] \\
 & P_i \arrow[rrr, "m_i"] \arrow[d, "f_i"] & & & P_i' \arrow[d, "f_i'"] & \\
 & B_{i-1} \arrow[rrr, "b_{i-1,i}"] \arrow[drr, "b_{i-1,i_*}"'] & & & B_i \arrow[dl, "b_{i,i_*}"] & \\
 & & & B & &
\end{tikzcd}
\tag{2}
$$

For $j < i_*$ and $b \in B_j$ we define $i_b := \min\{i \leq j : b \in b_{i,j}[B_i]\}$ (note that $b_{j,j} = id_{B_j}$). Then we either have $i_b = 0$, or in the reflection construction a span belongs to i_b (because in $\mathbf{A}$ λ-directed colimit-sinks and regular epimorphisms are (jointly) surjective). In the latter case it is easy to see that, by construction of pushouts in $\mathbf{A}$, there exist $U_b \subseteq B_{i_b-1}$ and $V_b \subseteq F_{\mathbf{A}} S_{i_b}'$ with $\operatorname{card} U_b, \operatorname{card} V_b < \lambda$ (because τ is λ-ary), such that $b = b_{i_b,j}(x_b)$ for some $x_b \in \langle b_{i_b-1,i_b}[U_b] \cup f_{i_b}' \circ \mu_{i_b}'[V_b]\rangle_{B_{i_b}}$.
Now define recursively W_α for ordinals α:

$\alpha = 0$: $W_0 := q \circ u_{\tilde{X}}[\tilde{X}]$
Successor ordinal: For any ordinal α set $W_{\alpha+1} := \bigcup\limits_{b \in W_\alpha : i_b > 0} U_b \cup f_{i_b} \circ \mu_{i_b}[S_{i_b}]$.
Limit ordinal: For a limit ordinal β set $W_\beta := \bigcup\limits_{\alpha<\beta} W_\alpha$.

It is easy to see that the sequence of the W_β is stationary for $\beta \geq \lambda$ and that $\operatorname{card} W_\lambda < \lambda$. Hence for $J := \{i_b : b \in W_\lambda\}$ (note $0 \in J$) and $X' := \coprod\limits_{j \in J} S_j'$ (with $S_0' := X$) we have $\operatorname{card} J < \lambda$ and $\sharp X' < \lambda$. Set $e' := ([e_j']_{j \in J} : F_{\mathbf{C}} \coprod\limits_{j \in J} S_j' \to B)$ (with $e_0' := e$).

We now define $q' : F_{\mathbf{C}}\tilde{X} \to F_{\mathbf{C}}X'$ as follows: Let $x \in u_{\tilde{X}}[\tilde{X}] \subseteq F_{\mathbf{C}}\tilde{X}$. By construction of J, we have $q(x) \in \langle \bigcup_{j\in J} b_{j,i_*} \circ f'_j \circ \mu'_j[F_{\mathbf{A}}S'_j]\rangle^{\mathbf{A}}_B$. Choose $y_x \in F_{\mathbf{A}} \coprod_{j\in J} S'_j$ with $[b_{j,i_*} \circ f'_j \circ \mu'_j]_{j\in J}(y_x) = q(x)$ and set $q'(x) := r_{F_{\mathbf{A}}(\coprod_{j\in J} S'_j)}(y_x)$ (for the reflection arrow of $F_{\mathbf{A}}(\coprod_{j\in J} S'_j)$).
So far we have shown that the following parts of the diagram in the statement of the lemma commute:

$$\begin{tikzcd}
X \arrow[r, "u_X"] \arrow[d, hook] & F_{\mathbf{C}}X \arrow[d, hook] \arrow[dr, "e"] & \\
X' \arrow[r, "u_{X'}"'] & F_{\mathbf{C}}X' \arrow[r, "e'"'] & B
\end{tikzcd}
\qquad
\begin{tikzcd}
 & F_{\mathbf{C}}\tilde{X} \arrow[dl, "q'"'] \arrow[d, "q"] \\
F_{\mathbf{C}}X' \arrow[r, "e'"'] & B
\end{tikzcd}$$

If we can show that e' is a strict epimorphism in $\mathbf{C}$, it is even a regular epimorphism by [5, 1.4], i. e. we have $Y' \in \mathbf{Set}^{\mathcal{S}}$ and $f', g' \in \hom_{\mathbf{C}}(F_{\mathbf{C}}Y', F_{\mathbf{C}}X')$ such that $F_{\mathbf{C}}Y' \overset{f'}{\underset{g'}{\rightrightarrows}} F_{\mathbf{C}}X' \xrightarrow{e'} B$ is a coequalizer in $\mathbf{C}$. Since B is λ-presentable by [3, 1.16], Y' then can be chosen to satisfy $\operatorname{card} Y' < \lambda$ by [5, 6.6e], and it is easy to see that furthermore Y' can be chosen such that the following diagram commutes.

$$\begin{tikzcd}
Y \arrow[r, "u_Y"] \arrow[d, hook] & F_{\mathbf{C}}Y \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] \arrow[d, hook] & F_{\mathbf{C}}X \arrow[d, hook] \\
Y' \arrow[r, "u_{Y'}"] & F_{\mathbf{C}}Y' \arrow[r, shift left, "f'"] \arrow[r, shift right, "g'"'] & F_{\mathbf{C}}X'
\end{tikzcd}$$

So it remains to show:

Observation 5.1 *e' is a strict epimorphism in $\mathbf{C}$.*

PROOF of Observation. $e' = [e'_i]_{i\in J}$ is an epimorphism, since $e'_0 = e$ is epimorphic. Let $(h' : F_{\mathbf{C}}X' \to A') \in \mathbf{C}$ be given with

$$\forall t, t' \in \operatorname{Mor}\mathbf{C} : (e' \circ t = e' \circ t' \Rightarrow h' \circ t = h' \circ t'). \tag{3}$$

For $i \in J$ set $h'_i := h' \circ \iota_i$ for the canonical $\iota_i : F_{\mathbf{C}}S'_i \to F_{\mathbf{C}} \coprod_{j\in J} S'_j$. We need $\bar{h} \in \hom_{\mathbf{C}}(B, A)$ with $\bar{h} \circ e' = h'$. We have $\bar{h}$ with

$$\bar{h} \circ e = h' \circ \iota_0, \tag{4}$$

because $e = e' \circ \iota_0$ is the coequalizer of (f, g) and $e' \circ \iota_0 \circ f = e' \circ \iota_0 \circ g$ implies $h' \circ \iota_0 \circ f = h' \circ \iota_0 \circ g$ by (3).

$$\begin{tikzcd}
F_{\mathbf{C}}X \arrow[d, hook, "\iota_0"'] \arrow[dr, "e"] & \\
F_{\mathbf{C}} \coprod_{j\in J} S'_j \arrow[r, "e'"] \arrow[dr, "h'"'] & B \arrow[d, dashed, "\bar{h}"] \\
& A'
\end{tikzcd}$$

It remains to show $\bar{h} \circ e' = h'$. We show by transfinite induction on k that for every $k \in J$ we have $\bar{h} \circ e'_k \circ r'_k = h'_k \circ r'_k$ (this is obviously sufficient).

$k = 0$: The statement holds by (4), because $e = e'_0$.

Induction step: Let $k \in J$; suppose we have $h' \circ r'_j = \bar{h} \circ e' \circ r'_j$ for each $j \in J$ with $j < k$. We need to show $\bar{h} \circ e'_k \circ r'_k = h'_k \circ r'_k$. For every $z \in S_k \subseteq F_{\mathbf{A}} S'_k$ we have, by construction of J, $z' \in F_{\mathbf{A}} \coprod_{k > j \in J} S'_j$ with $e'(r_{F_{\mathbf{A}}(\coprod_{j\in J} S'_j)}(z')) = e'(r'_k(z))$ (where the canonical $F_{\mathbf{A}} \coprod_{k>j\in J} S'_j \to F_{\mathbf{A}} \coprod_{j \in J} S'_j$ is w.l.o.g. considered to be an inclusion.) By (3) this implies $h'(r_{F_{\mathbf{A}}(\coprod_{j\in J} S'_j)}(z')) = h'_k(r'_k(z))$, and thus

$$\begin{aligned}
\bar{h}(e'_k(r'_k(z))) &= \bar{h}(e'(r_{F_{\mathbf{A}}(\coprod_{j\in J} S'_j)}(z'))) \\
&= h'(r_{F_{\mathbf{A}}(\coprod_{j\in J} S'_j)}(z')) \\
&= h'_k(r'_k(z))
\end{aligned}$$

using the induction hypothesis. Thus we have

$$\bar{h} \circ e'_k \circ r'_k \circ \varphi = h'_k \circ r'_k \circ \varphi \tag{5}$$

(with the inclusion $\varphi : F_{\mathbf{A}} S_k \hookrightarrow F_{\mathbf{A}} S'_k$). Since $e'_k \circ r'_k$ factorizes through μ'_k (see (2)), we have $e'_k \circ r'_k \circ \nu'_k = e'_k \circ r'_k \circ \sigma'_k$ which by (3) implies $h'_k \circ r'_k \circ \nu'_k = h'_k \circ r'_k \circ \sigma'_k$. Since μ'_k is the coequalizer in $\mathbf{A}$ of (ν'_k, σ'_k), there exists $\tilde{h}$, such that $\tilde{h} \circ \mu'_k = h'_k \circ r'_k$. This implies

$$\begin{aligned}
\tilde{h} \circ m_k \circ \mu_k &\overset{(2)}{=} \tilde{h} \circ \mu'_k \circ \varphi \\
&= h'_k \circ r'_k \circ \varphi \\
&\overset{(5)}{=} \bar{h} \circ e'_k \circ r'_k \circ \varphi \\
&\overset{(2)}{=} \bar{h} \circ b_{k,i_*} \circ f'_k \circ \mu'_k \circ \varphi \\
&\overset{(2)}{=} \bar{h} \circ b_{k,i_*} \circ f'_k \circ m_k \circ \mu_k
\end{aligned}$$

and thus $\tilde{h} \circ m_k = \bar{h} \circ b_{k,i_*} \circ f'_k \circ m_k$, since μ_k is an epimorphism. But $A' \perp m_k$ then implies $\tilde{h} = \bar{h} \circ b_{k,i_*} \circ f'_k$, which again leads to

$$\begin{aligned} h'_k \circ r'_k &= \tilde{h} \circ \mu'_k \\ &= \bar{h} \circ b_{k,i_*} \circ f'_k \circ \mu'_k \\ &= \bar{h} \circ e'_k \circ r'_k. \end{aligned}$$

□

To provide a solution to the problem of Gabriel and Ulmer we define categories $\mathbf{A}_*$ and $\mathbf{C}_*$ and a $\mathbf{C}_*$-object C as follows:

- $\mathbf{A}_*$:=$\mathbf{Alg}\,\boldsymbol{\Sigma}$ with $\Sigma := \{\varrho, \sigma, \kappa\} \cup \{\varphi_n : n \in \mathbb{N}_{>0}\}$ (ϱ, σ nullary and κ, φ_n unary operations),
- $\mathbf{C}_*$ is the full subcategory of $\mathbf{A}_*$ consisting of those Σ-algebras that satisfy the following formulas:

(1) $\varrho = \sigma \Rightarrow \big(\exists!(x_1, x_2, x_3, ...)\big)\big(\bigwedge_{n \geq 1}\big)\big(\varphi_n(x_n) = \varrho \wedge \kappa(x_{n+1}) = x_n\big)$

(2) $(\forall x, y)\big(\varphi_n(x) = \varphi_n(y) \Rightarrow \kappa(x) = \kappa(y)\big)$ for each $n \geq 1$

Thus $\mathbf{C}_*$ is the orthogonality class $\mathcal{M}^\perp$ for $\mathcal{M} := \{q\} \cup \{q_n : n \in \mathbb{N}_{>0}\}$ where

- $q : E \to E'$ is the (unique) morphism having domain E and codomain E' where
  - · E is the quotient of the initial Σ-algebra 0 under the relation $\varrho = \sigma$ and
  - · E' is the Σ-algebra given by generators $e_1, e_2, e_3, ...$ and relations $\varrho = \sigma$, and, for all $n \geq 1$, $\varphi_n(e_n) = \varrho$ and $\kappa(e_{n+1}) = e_n$.
- $q_n : A_n \to A'_n$ (for each $n \geq 1$) is the obvious quotient morphism with
  - · A_n the Σ-algebra given by generators a, b and the relation $\varphi_n(a) = \varphi_n(b)$ and
  - · A'_n the Σ-algebra given by generators a, b and the relations $\varphi_n(a) = \varphi_n(b)$ and $\kappa(a) = \kappa(b)$.

Also we define C to be the Σ-algebra given by generators $c_1, c_2, c_3, ...$ and relations $\varrho = \sigma$, $\varphi_2(c_1) = c_1$ and, for all $n \geq 1$, $\varphi_n(c_n) = \varrho$ and $\kappa(c_{n+1}) = c_n$. We note $C \in \mathrm{Ob}\,\mathbf{C}_*$.

Theorem 6 ***a)*** $\mathbf{C}_*$ *is reflective and closed under directed colimits in* $\mathbf{A}_*$, *hence a locally finitely presentable category with a finitely presentable regular generator* $\mathcal{G}_* := \{F_{\mathbf{C}_*}X : X \in \mathbf{Set}^{\mathcal{S}} \wedge \sharp X < \aleph_0\}$, *and*
b) C *is finitely presentable, but not finitely* $\mathcal{G}_*$*-presented in* $\mathbf{C}_*$.

PROOF.
a) The second part follows from the first by the second part of [3, 1.39].

$\mathbf{C}_*$ is reflective by [3, 1.37], because it is an $\aleph_1$-orthogonality class. We note that q is a reflection arrow.
$\mathbf{C}_*$ is closed under directed colimits:
Let $(d_i : D_i \to A)_{i\in I}$ be the $\mathbf{A}_*$-colimit of a directed diagram (with morphisms $d_{ij} : D_i \to D_j$) in $\mathbf{C}_*$. We need to show $A \in \mathrm{Ob}\,\mathbf{C}_*$. We have $A \in \{q_n : n \in \mathbb{N}_{>0}\}^{\perp}$, because $\aleph_0$-orthogonality classes are closed under directed colimits [3, 1.35]. So we are left to show that $\hom_{\mathbf{A}_*}(q, A)$ is bijective. One can show as in the proof of [3, 1.35] that $\hom_{\mathbf{A}_*}(q, A)$ is surjective, because $\hom_{\mathbf{A}_*}(q, D_i)$ is surjective for each i, and for each $m \in \mathcal{M}$, $\mathrm{dom}\, m$ is finitely presentable.
So we have to show that $\hom_{\mathbf{A}_*}(q, A)$ is injective:
Let $f, g : E' \to A$ be given with $f \circ q = g \circ q$. To show $f = g$ it is sufficient to show that $f(e_n) = g(e_n)$ for each $n \in \mathbb{N}$. For given $n \in \mathbb{N}$ define $l : K \to E'$ by $l(k) = e_{n+1}$, where K is the Σ-algebra given by the generator k and relations $\varrho = \sigma$ and $\varphi_{n+1}(k) = \varrho$. K is finitely presentable in the variety $\mathbf{A}_*$, because it is finitely presented. Therefore we have $i \in I$ and f', g' such that $d_i \circ f' = f \circ l$ and $d_i \circ g' = g \circ l$.

$$\begin{tikzcd} & K \arrow[r, shift left, "f'"] \arrow[r, shift right, "g'"'] \arrow[d, "l"] & D_i \arrow[d, "d_i"] \\ E \arrow[r, "q"] & E' \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & A \end{tikzcd} \tag{6}$$

This yields

$$\varphi_{n+1}(f'(k)) = f'(\varphi_{n+1}(k)) = f'(\varrho) = \varrho = g'(\varrho) = \varphi_{n+1}(g'(k)).$$

Now $D_i \perp q_{n+1}$ implies $\kappa(f'(k)) = \kappa(g'(k))$, thus

$$f(e_n) = f(\kappa(e_{n+1})) = f(\kappa(l(k))) = d_i(\kappa(f'(k))) = d_i(\kappa(g'(k))) = g(e_n).$$

b) The proof of this part in many places uses the fact that all operations in Σ are at most unary.
E is finitely presentable in $\mathbf{A}_*$ as a finite colimit of finitely presentable objects by [3, 1.16], so its reflection E' is finitely presentable in $\mathbf{C}_*$ (since $\mathbf{C}_*$ is closed under directed colimits). Since C is the regular quotient of E' in $\mathbf{C}_*$ by the relation $\varphi_2(e_1) = e_1$, it is also finitely presentable in $\mathbf{C}_*$.
It remains to show that there exists no $\mathbf{C}_*$-coequalizer $F_{\mathbf{C}_*}Y \rightrightarrows F_{\mathbf{C}_*}X \to C$ with finite sets X, Y. Let $(h : F_{\mathbf{C}_*}X \to C) \in \mathrm{Mor}\,\mathbf{C}_*$ with finite X; we will show that h is not a regular epimorphism in $\mathbf{C}_*$.

Case 1: $c_1 \notin h[F_{\mathbf{C}_*}X]$. Let $c : E' \to C$ be the quotient morphism of the $\mathbf{A}_*$-coequalizer which exists because C is the regular quotient of E' in $\mathbf{A}_*$ by the relation $\varphi_2(e_1) = e_1$.

Claim 6.1 *For every $x \in X$ there exists $x' \in E'$ with $c(x') = h(x)$, such that there exists no term $\vec{t}$ with $\vec{t}(\varphi_2(e_1)) = x'$ (i. e. $x' \notin \langle \varphi_2(e_1) \rangle_{E'}^{\mathbf{A}_*}$ for the $\mathbf{A}_*$-subalgebra of E' generated by $\varphi_2(e_1)$).*

PROOF of Claim. Since c is surjective we have y such that $c(y) = h(x)$. Suppose there exists a term $\vec{s}$ with $\vec{s}(e_1) = y$. Let n be maximal such that there exists a term $\vec{u}$ with $\vec{s} = \vec{u} \circ \varphi_2^n$ (n exists, because terms of finitary operations have only finite length). By maximality of n there exists no term $\vec{t}$ with $\vec{u} = \vec{t} \circ \varphi_2$. Let $x' := \vec{u}(e_1)$. Then there does not exist a term $\vec{t}$ with $x' = \vec{t}(\varphi_2(1))$, either, by construction of E': $\langle e_1 \rangle_{E'}^{\mathbf{A}_*}$ is given by the generator e_1 and the relations $\varrho = \sigma$ and $\varphi_1(e_1) = \varrho$. Thus, for every term $\vec{v}$, $\vec{t} \circ \varphi_2(e_1) = \vec{v}(e_1)$ in E' implies $\vec{t} \circ \varphi_2 = \vec{v}$. $\square$

The chosen x' define a morphism $h' : F_{\mathbf{C}_*}X \to E'$. By the way of choosing the x', and because we have $e_1 \notin h[F_{\mathbf{C}_*}X]$ by assumption, we know that for every $y \in h'[F_{\mathbf{C}_*}X]$ there is no term $\vec{t}$ with $\vec{t}(\varphi_2(e_1)) = y$. The congruence of c in $\mathbf{A}_*$ is contained in the reflexive hull of

$$\langle \{ (\varphi_2^n(e_1), \varphi_2^m(e_1)) : n, m \in \mathbb{N} \} \rangle_{(E')^2}^{\mathbf{A}_*}.$$

Thus $c|_{h'[F_{\mathbf{C}_*}X]}$ is injective. Since by construction we have $h = c \circ h'$, this implies

$$\forall f, g \in \operatorname{Mor} \mathbf{C}_* : (h \circ f = h \circ g \Rightarrow h' \circ f = h' \circ g).$$

But h' does not factor through h (we even have $\hom_{\mathbf{C}_*}(C, E') = \emptyset$, because E' has no φ_2-fixpoint), i. e. h is not strict and thus not a regular epimorphism in $\mathbf{C}_*$.

Case 2: $c_1 \in h[F_{\mathbf{C}_*}X]$. Suppose that h is a regular epimorphism in $\mathbf{C}_*$. Let $N := \{ n \in \mathbb{N}_{>0} : c_n \in h[F_{\mathbf{C}_*}X] \}$. N is non-empty by supposition and finite because otherwise C would be generated as a Σ-algebra by the finite $h[X]$, but C is obviously not finitely generated. So $\bar{n} := \max N$ exists. Let $h = (F_{\mathbf{C}_*}X \xrightarrow{\bar{h}} h[F_{\mathbf{C}_*}X] \overset{i}{\hookrightarrow} C)$ be the (Surjective, Injective)-factorization of h in $\mathbf{A}_*$. It follows easily that i is a reflection arrow. Now consider the following

$\mathbf{A}_*$-pushout P:

$$\begin{tikzcd}
E \arrow[r, "q"] \arrow[d, "d"'] & E' \arrow[d, "g"] & \\
h[F_{\mathbf{C}_*}X] \arrow[r, "f"] \arrow[rrd, hook, "i"'] & P \arrow[rd, "r_P"] & \\
 & & C
\end{tikzcd}$$

(where d is the unique morphism $E \to h[F_{\mathbf{C}_*}X]$). Since we have $\mathbf{C}_* = (\{q\} \cup \{q_n : n \in \mathbb{N}_{>0}\})^{\perp}$ one can consider P as the first step in the orthogonal-reflection construction of $R(h[F_{\mathbf{C}_*}X]) \cong C$ (for the reflector R) (see Proposition 2). So by Proposition 2 we have a reflection arrow $r_P : P \to C$ with $r_P \circ f = i$. Now we have $f(c_{\bar{n}}) \notin g[E']$ (because it is easy to see that otherwise we would have $c_{\bar{n}} \in d[E]$, which is obviously not the case). In particular we have $g(c_{\bar{n}}) \neq f(c_{\bar{n}})$.
We also have $f(c_{\bar{n}}) \notin \langle \kappa[P] \rangle_P^{\mathbf{A}_*}$: Since (f, g) is jointly surjective as a colimit-sink in $\mathbf{A}_*$, it is sufficient to show that we have $f(c_{\bar{n}}) \notin \langle \kappa \circ f \circ h[F_{\mathbf{C}_*}X] \rangle$ and $f(c_{\bar{n}}) \notin \langle \kappa \circ g[E'] \rangle$.

- We have $f(c_{\bar{n}}) \notin \langle \kappa \circ f \circ h[F_{\mathbf{C}_*}X] \rangle$, because otherwise we would have

  $$\begin{aligned} c_{\bar{n}} = r_P(f(c_{\bar{n}})) \in r_P[\langle \kappa \circ f \circ h[F_{\mathbf{C}_*}X] \rangle] &\subseteq \langle \kappa \circ r_P \circ f \circ h[F_{\mathbf{C}_*}X] \rangle \\ &= \langle \kappa \circ h[F_{\mathbf{C}_*}X] \rangle, \end{aligned}$$

  i. e. $c_{\bar{n}} = \vec{t} \circ \kappa(y)$ for some $y \in h[F_{\mathbf{C}_*}X]$ and some term $\vec{t}$. This is easily seen to imply $y = c_{\bar{n}+1}$, contradicting the maximality of $\bar{n}$.
- We have $f(c_{\bar{n}}) \notin \langle \kappa \circ g[E'] \rangle$: Otherwise we would in particular have $f(c_{\bar{n}}) \in g[E']$.

By the lemma below, $g(c_{\bar{n}}) \neq f(c_{\bar{n}})$ then implies $r_P(g(c_{\bar{n}})) \neq r_P(f(c_{\bar{n}}))$, i. e. we have two different elements $a \in C$ with $\varphi_{\bar{n}}(a) = 0$. But this contradicts the injectivity of $\varphi_{\bar{n}}$ on C.

Lemma 6.1 *Let $x \in A \in \mathrm{Ob}\,\mathbf{A}_*$. If we have $x \notin \langle \kappa[A] \rangle_A^{\mathbf{A}_*}$, then for every $y \in A$ we have:*

$$r_A(x) = r_A(y) \Rightarrow x = y.$$

PROOF of lemma. The reflection $B_{i_*} := RA$ of $B_0 := A$ is iteratively constructed in $\mathbf{A}_*$ from pushouts of spans, coequalizers of pairs and directed colimits (see Proposition 2, also for the notation used in the following). We prove the above implication inductively for each of these construction steps by making use of the fact that every strictly decreasing sequence of ordinal numbers has only finitely many members: Let $x, y \in A$ with $x \notin \langle\kappa[A]\rangle_A^{\mathbf{A}_*}$ and $r_A(x) = r_A(y)$. To obtain $x = y$ we show that for every $j \leq i_*$ we have the implication

$$b_{0,j}(x) = b_{0,j}(y) \Rightarrow \exists i < j : b_{0,i}(x) = b_{0,i}(y).$$

Let us first note that from $x \notin \langle\kappa[A]\rangle_A^{\mathbf{A}_*}$ it is easy to obtain inductively (via the three construction steps and by definition of the q, q_n) that for every $i \leq i_*$ we have $b_{0,i}(x) \in B_i \setminus \langle\kappa[B_i]\rangle_{B_i}^{\mathbf{A}_*}$. Now we show the above implication inductively:

- Pushouts of spans: Let

$$\begin{tikzcd} M \arrow[r, "m"] \arrow[d, "h_0"'] & M' \arrow[d, "h_1"] \\ B_i \arrow[r, "b_{i,i+1}"'] & B_{i+1} \end{tikzcd}$$

  be the pushout of a span $B_i \overset{h_0}{\leftarrow} M \overset{m}{\rightarrow} M'$ with $m \in \mathcal{M}$. Let $b_{i,i+1}(b_{0,i}(x)) = b_{i,i+1}(b_{0,i}(y))$. We need to show $b_{0,i}(x) = b_{0,i}(y)$.
  · Suppose $m = q$. Since q is injective and monomorphisms in $\mathbf{A}_*$ are easily seen to be pushout-stable, $b_{i,i+1}$ is injective.
  · Suppose $m = q_n$ for some $n \in \mathbb{N}$. By construction of the pushout we have $b_{i,i+1}(b_{0,i}(x)) = b_{i,i+1}(b_{0,i}(y))$ iff $b_{0,i}(x) = b_{0,i}(y)$ or there exist $j \in \mathbb{N}_{\geq 2}, x_1, ..., x_j \in \operatorname{dom} q_n$ such that

$$b_{0,i}(x) = h_0(x_1) \wedge q_n(x_1) = q_n(x_2) \wedge h_0(x_2) = h_0(x_3) \wedge ... \wedge h_0(x_j) = b_{0,i}(y).$$

  Suppose $b_{0,i}(x) \neq b_{0,i}(y)$. W.l.o.g. the x_ν are mutually distinct: If we have $x_\nu = x_\mu$ for $\mu > \nu$, we can remove $x_\nu, ..., x_\mu$ from the list. But $q_n(x_1) = q_n(x_2)$ for $x_1 \neq x_2$ implies $x_1 = \vec{t}(\kappa(\iota))$ for some term $\vec{t}$ und $\iota \in \{a, b\} \subseteq A_n$ and thus $b_{0,i}(x) = h_0(x_1) \in \langle\kappa[B_i]\rangle$, contradicting the above observation.
- Coequalizers of pairs: Let $M' \overset{h_0}{\underset{h_1}{\rightrightarrows}} B_i \overset{b_{i,i+1}}{\rightarrow} B_{i+1}$ be the $\mathbf{A}_*$-coequalizer of a pair (h_0, h_1) with $h_0 \circ m = h_1 \circ m$ for some $(m : M \to M') \in \mathcal{M}$. Let $b_{i,i+1}(b_{0,i}(x)) = b_{i,i+1}(b_{0,i}(y))$. We need to show $b_{0,i}(x) = b_{0,i}(y)$.
  · Suppose $m = q$: By construction of a coequalizer in $\mathbf{A}_*$ we have $b_{i,i+1}(b_{0,i}(x)) = b_{i,i+1}(b_{0,i}(y))$ iff $b_{0,i}(x) = b_{0,i}(y)$ or there exist $j \in \mathbb{N}_{\geq 1}$, $x_1, ..., x_j \in E'$ and $\nu_1, ..., \nu_j \in \{0, 1\}$ such that

$$b_{0,i}(x) = h_{\nu_1}(x_1) \wedge h_{\neg\nu_1}(x_1) = h_{\nu_2}(x_2) \wedge ... \wedge h_{\neg\nu_j}(x_j) = b_{0,i}(y)$$

  (with the notation $\neg 0 := 1, \neg 1 := 0$).
  Suppose we have the latter case. Since $\{e_n : n \in \mathbb{N}\} = \kappa[\{e_n : n \in \mathbb{N}_{\geq 1}\}]$ generates E' (as a Σ-algebra), we have $x_1' \in E'$

and a term $\vec{t}$ with $\vec{t}(\kappa(x'_1)) = x_1$, and thus $b_{0,i}(x) = h_{\nu_1}(\vec{t}(\kappa(x'_1))) = \vec{t}(\kappa(h_{\nu_1}(x'_1)))$.

· Suppose $m = q_n$ for some $n \in \mathbb{N}$: This case is clear, since q_n is an epimorphism in $\mathbf{A}_*$, and so $b_{i,i+1}$ is an isomorphism.

- Directed Colimits: Let $(b_{j,\alpha} : B_j \to B_\alpha)_{j<\alpha}$ be the colimit in $\mathbf{A}_*$ of the directed diagram $(b_{j,j'} : B_j \to B_{j'})_{j<\alpha}$ with $i < \alpha$. Let $b_{i,\alpha}(b_{0,i}(x)) = b_{i,\alpha}(b_{0,i}(y))$ Then there exists a $j < \alpha$ with $i < j$, such that $b_{i,j}(b_{0,i}(x)) = b_{i,j}(b_{0,i}(y))$, because in $\mathbf{A}_*$ directed colimits are concrete.

□

Remark 7 *We now sketch how to modify the above example so that one needs only finitely many finitary operations:*
We define categories $\bar{\mathbf{A}}$ and $\bar{\mathbf{C}}$ as follows:

- $\bar{\mathbf{A}}$:=$\mathbf{Alg}\,\mathbf{\Omega}$ *with $\Omega := \{\alpha, \beta, \gamma, \delta\}$ where α is binary, β is unary and γ and δ are nullary operations,*
- $\bar{\mathbf{C}}$ *is the full subcategory of $\bar{\mathbf{A}}$ consisting of those Ω-algebras that satisfy the following formulas:*

(1) $\alpha(\beta^n\gamma, x) = \alpha(\beta^n, \gamma, y) \Rightarrow \alpha(\gamma, x) = \alpha(\gamma, y)$ *for all* $n \geq 1$
(2) $\Big(\exists!(x_1, x_2, x_3, ...)\Big)\Big(\forall n \geq 1\Big)\Big(\alpha(\beta^n\gamma, x_n) = \gamma \wedge \alpha(\gamma, x_{n+1}) = x_n\Big)$

Then we have a functor $G_a : \bar{\mathbf{A}} \to \mathbf{A}_*$ assigning to each Ω-algebra its Σ-reduct, where Σ is viewed as a subset of the set of derived operations of Ω in the following way:

- $\varrho := \gamma$, $\sigma := \delta$
- $\kappa := \alpha(\gamma, \_)$
- $\varphi_n := \alpha(\beta^n\gamma, \_)$

Since we can view any Σ-algebra as a partial Ω-algebra via the above identifications, we also have a functor $F_a : \mathbf{A}_* \to \bar{\mathbf{A}}$ assigning to each Σ-algebra A the free Ω-algebra over the partial Ω-algebra corresponding to A (Grätzer [6, §28]).
Let $G_c : \bar{\mathbf{C}} \to \mathbf{C}_*$ resp. $F_c : \mathbf{C}_* \to \bar{\mathbf{C}}$ be the restrictions of G_a resp. F_a. Then G_c is right adjoint to F_c. G_c preserves directed colimits, thus F_c preserves finitely presentable objects. So $\bar{\mathbf{C}}$ is locally finitely presentable: it is cocomplete as a small-orthogonality class in $\bar{\mathbf{A}}$, and $\{F_{\bar{\mathbf{C}}}1\}$ is a finitely presentable regular generator.
Now $\bar{C} := F_cC$ is finitely presentable. Since it is easy to see that F_c reflects finitely presented objects, $\bar{C}$ is not finitely presented.

Corollary 8 *There exists a subcategory of a locally λ-presentable category which is reflective and closed under λ-directed colimits, but not a λ-orthogonality class.*

Remark 9 • *For a direct proof that the subcategory* $\mathbf{C}$ *of* $\mathbf{A}$ *in Theorem 6 is not an* $\aleph_0$*-orthogonality class, see [7].*

- *As I was told by Prof. Adámek and Prof. Rosický after completion of this work, they recently have been informed by M. Hébert that it is implicit in [8] that a subcategory of a locally* λ*-presentable category which is reflective and closed under* λ*-directed colimits, need not be a* λ*-orthogonality class.*

Problem 10 *Characterize* λ*-orthogonality classes in locally* λ*-presentable categories by closure properties.*

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