



Quantifying the Lucky Droplet Model for Rainfall

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Abstract

It is difficult to explain rainfall from ice-free clouds, because the timescale for the onset of rain showers is shorter than the mean time for collisions between microscopic water droplets. It has been suggested that raindrops are produced from very rare ‘lucky’ droplets, which undergo a large number of collisions on a timescale which is short compared to the mean time for the first collision. This work uses large deviation theory to develop estimates for the timescale for the onset of a rain shower, as a function of the collision rate coefficients. The growth history of the fast-growing droplets which do become raindrops is discussed. It is shown that their first few collisions are always approximately equally spaced in time, regardless of how the mean time for collisions varies as a function of the number of collisions.

Keywords Large-deviation theory · Rainfall · Smoluchowski equation · Instantaneous gelation

1 Introduction

There are some fundamental challenges in understanding rainfall from clouds [1, 2]. When ice is present, the Bergeron mechanism [1] helps to explain how precipitation can be initiated, by condensation of water vapour onto ice crystals. When no ice is present, however, rainfall appears to rely on collision and coalescence of microscopic water droplets. It has proven to be difficult to understand how the rate of these collisions can be sufficient to explain rainfall [3].

The collisions occur mainly as a result of microscopic water droplets with different radius falling at different terminal velocities. An important aspect of the growth of raindrops is that, because larger droplets fall more rapidly and have a larger cross-section, the collisions happen more frequently as the droplet grows. If τ_n is the mean time between the first $n - 1$ and the first n collisions, it is expected that τ_n decreases rapidly as a function of n . According

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to a model which is discussed in detail below, the sum

$$\langle T \rangle = \sum_{n=1}^{\mathcal{N}} \tau_n \quad (1)$$

which is the mean time for \mathcal{N} collisions, approaches a finite limit as $\mathcal{N} \rightarrow \infty$. The typical radius of the microscopic droplets in clouds (which form on condensation nuclei such as salt particles from evaporated sea spray) is approximately $10 \mu\text{m}$, and the liquid water content of a dense cloud is approximately 1 gm^{-3} [1, 2]. Using these data, the rate of collision of droplets is found to be very small. The mean time for the first collision, τ_1 , depends upon the dispersion of the droplet sizes, but it is much less than one collision per hour. A raindrop may have a radius as large as 1 mm. Because most collisions of the growing droplet are with microscopic droplets, approximately 10^6 collisions are required to create a raindrop. It is common experience that newly formed clouds can generate rain showers quite rapidly, on a timescale of less than half an hour. Given the slow rate of the first few collisions, and the very large number of collisions that are required, it hard to understand how rainfall can happen so quickly.

The Lifshitz-Slyozov analysis of Ostwald ripening [4, 5] does not offer a resolution of this problem because it acts too slowly [6, 7]. It has also been pointed out that turbulence in cumulus clouds will enhance the collision rate [8], but extensive studies, reviewed in [9], indicate that this effect is not sufficient, except possibly in the most unstable atmospheric conditions. The most satisfying solution to this problem is the ‘lucky droplet’ concept, originally proposed by Telford [10] and Twomey [11]. These authors pointed out that it is those microscopic droplets which undergo their first few collisions exceptionally quickly that will grow into raindrops, and absorb a large number of other microscopic droplets in the process. In fact, the fastest growing droplets could absorb all the remaining microscopic water droplets on their way to becoming raindrops, so that the occurrence of a rain shower is determined by the growth history of the (approximately) one-in-a-million fastest growing droplets.

Subsequently, Kostinski and Shaw [12] formulated a model for the time taken for droplets to grow by collision, and presented some simulations indicating that it could yield a satisfactory solution of the problem. It is desirable to have a simple and maximally transparent expression for this ‘lucky droplet’ principle. In [13], it was proposed that large deviation theory [14, 15] is the relevant tool. When T is small, the probability $P(T)$ that a microscopic droplet undergoes runaway growth to become a rain droplet within time T is exponentially small: write

$$P(T) = \exp[-\mathcal{J}(T)] \quad (2)$$

where $\mathcal{J}(T)$ is closely related to the *entropy function* or *rate function* in large deviation theory. If a rain droplet is the result of coalescence of \mathcal{N} microscopic droplets, then the fraction of the microscopic droplets which are removed after time T is greater than $\mathcal{N}P(T)$, provided the number of microscopic droplets has not yet been significantly reduced by collisions. The onset of the rain shower is at a time T^* when a significant fraction of the microscopic water droplets have been removed by coalescence into falling rain drops. It will be argued that this time T^* is estimated by writing $\mathcal{N} \exp[-\mathcal{J}(T^*)] \sim 1$. The time T^* at which a rain shower occurs is therefore estimated by solving the equation

$$\mathcal{J}(T^*) = \ln \mathcal{N}. \quad (3)$$

The function $\mathcal{J}(t)$ diverges as $t \rightarrow 0$. It is, therefore, expected that, in the limit as $\mathcal{N} \rightarrow \infty$, the ratios $T^*/\langle T \rangle$ and T^*/τ_1 become small. This implies that, if \mathcal{N} is sufficiently large, a

rain shower is expected to occur after a time which is small compared to the mean time for the first collision.

Equation (3) offers a surprisingly simple criterion for the onset of rain showers. If the brief arguments presented above are considered more carefully, as is done in Sect. 2 below, it can be argued that solving a variant of Eq. (3) gives an upper bound on the time taken to produce a rain shower, in the context of a model for a homogeneous atmosphere.

In order to make practical use of Eq. (3), $\mathcal{J}(T)$ should be expressed as a functional of the set of mean collision times τ_n . This is addressed in Sect. 3. Both of the earlier works which quantify the lucky droplet model, [12, 13], assumed that τ_n has a power-law dependence upon n , writing $\tau_n = \tau_1 n^{-\gamma}$, but it is desirable to determine $\mathcal{J}(T)$ in a more general case. Because the terminal velocity of a small sphere is proportional to the square of its radius a , and the collision cross section also increases as a^2 , the collision rate is proportional to a^4 for droplets which have undergone a large number of collisions, but which have a terminal velocity with small Reynolds number. The volume of a droplet is proportional to the number of collisions that it has undergone, implying that $a^3 \sim n$, so that the exponent is $\gamma = 4/3$. However, this choice might not give a good description of the time between the first few collisions (\bar{n} , say), which may occur at a different rate [1, 2] (for example, collisions of very small droplets may be suppressed by lubrication effects). Accordingly, the following calculations emphasise the more general case where

$$\tau_n = \tau_1 n^{-\gamma} [1 + Q(n/\bar{n})] \tag{4}$$

where $Q(x)$ is a positive function which approaches zero when $x \gg 1$, and which may be divergent as $x \rightarrow 0$. The objective is to give useful approximations to $\mathcal{J}(T)$ which are applicable in this more general case. The numerical illustrations will use

$$Q(x) = x^{-\delta} \exp(-x) \tag{5}$$

with $\gamma = 4/3$ and $\delta > 0$ (which implies that the first few collisions occur more slowly than predicted by the $\tau_n \sim n^{-4/3}$ relation). It will be shown that the time taken to initiate a rain shower is surprisingly insensitive to the first few values of the mean collision times τ_n .

It is interesting to consider the growth history of the fastest growing droplets. This is addressed in Sect. 4. Surprisingly, it is found that their first few collisions occur at approximately equally spaced time intervals, regardless of the form of dependence of τ_n upon n . To avoid unhelpful complication of the notation, Sect. 4 considers the growth history of droplets where the collision times are a simple power-law, $\tau_n = n^{-\gamma}$, rather than the more general case described by (4).

The formulae developed in Sects. 3 and 4 are compared with numerical simulations. Section 5 is a concluding summary.

2 Modelling Rain Showers

Assume that there is, initially, a uniform density of microscopic water droplets in a homogeneous atmosphere. These undergo collisions, such that the density of droplets $\rho(m, t)\delta m$ in a small interval of mass, $[m, m + \delta m]$, varies as a function of time. Collision of a droplet of mass in the small interval $[m_1, m_1 + \delta m_1]$ with another droplet having mass in the small interval $[m_2, m_2 + \delta m_2]$ occurs with rate $K(m_1, m_2)\rho(m_1, t)\rho(m_2, t)\delta m_1\delta m_2$, where $K(m_1, m_2)$ is termed the collision kernel. The density of droplets $\rho(m, t)$ with mass m after time t is then

specified by the Smoluchowski equation [1–3, 16, 17]:

$$\begin{aligned} \frac{\partial \rho(m, t)}{\partial t} = & \frac{1}{2} \int_0^m dm' K(m - m', m') \rho(m - m', t) \rho(m', t) \\ & - \int_0^\infty dm' K(m, m') \rho(m, t) \rho(m', t). \end{aligned} \quad (6)$$

The collision kernel for small droplets falling under gravity has the form

$$K(m_1, m_2) = A \varepsilon(m_1, m_2) (m_1^{1/3} + m_2^{1/3})^2 |m_1^{2/3} - m_2^{2/3}| \quad (7)$$

where A is a constant and $\varepsilon(m_1, m_2) = \varepsilon(m_2, m_1)$ is a collision efficiency, which is assumed to approach unity after the first few collisions, but which may be small in the initial stages of droplet growth [1–3].

If the dispersion of the initial masses is small, the size of a droplet can be described by the number n of droplets from which it has been formed by coalescence. It is this simplified picture which will be used in the following discussions, in which the initial un-collided droplets will be referred to as monomers, and the larger droplets as n -mers. The density of n -mers, $\tilde{\rho}(n, t)$, obeys a simplified Smoluchowski equation

$$\begin{aligned} \frac{\partial \tilde{\rho}(n, t)}{\partial t} = & \frac{1}{2} \sum_{m=1}^{n-1} \tilde{K}(n - m, m) \tilde{\rho}(n - m, t) \tilde{\rho}(m, t) \\ & - \sum_{m=1}^{\infty} \tilde{K}(n, m) \tilde{\rho}(n, t) \tilde{\rho}(m, t). \end{aligned} \quad (8)$$

Because the collision kernel, Eq. (7), increases sufficiently rapidly as a function of the masses, the mean time to reach an infinite cluster size is finite: in the literature of the Smoluchowski equation, this runaway growth is termed a gelation transition [17]. If the clusters are allowed to grow to infinite size, the Smoluchowski equation predicts that the gelation transition for the kernel (7) occurs in zero time [17–20]. This is a consequence of the fact that the Smoluchowski equation is a mean-field approximation: in an infinite volume, there is one cluster that grows arbitrarily quickly, and this will absorb all of the other particles.

In the application to modelling clouds, it will be assumed that the droplets grow to a finite size, achieved after $\mathcal{N} \approx 10^6$ microscopic droplets have coalesced. The size limit could be a consequence of the finite depth of the cloud, or because raindrops fragment due to aerodynamic forces above a certain size.

If this upper cutoff is imposed, it is possible in principle solve the Smoluchowski equation to calculate the probability $P(T)$ that a given droplet has grown by coalescence to size \mathcal{N} after time T . Because the raindrop has absorbed \mathcal{N} microscopic droplets in this process, if $\mathcal{N}P(T) = \mu$, with $0 < \mu < 1$, then at time T a fraction of the liquid water content of the cloud greater than or equal to μ has become macroscopic rain droplets. It is, therefore, possible to estimate the time T^* for the onset of a rain shower by solving the equation $P(T^*) = \mu/\mathcal{N}$. Under typical circumstances a rain shower converts a few percent of the liquid water content into rain, so taking $\mu > 0.1$ (say) would give a reasonable upper bound on the time taken to produce a rain shower.

In the context of this approach to estimating T^* , the fact that the Smoluchowski equation is a mean-field equation is not a deficiency, because the period between collisions is sufficiently large that the system is ‘well-mixed’. However, it is extremely difficult to determine useful analytical expressions for solutions of the Smoluchowski equation, and even if these were available, it would be difficult to use them to determine $P(T)$.

An alternative approach is to consider the statistics of the time T directly. A cluster of droplets grows to size M by a sequence of collision events. The time between collision j and the next collision will be denoted by t_j , and the time taken to undergo M collisions is denoted by T_M . The collision with index j is with a particle of size n_j . The collisions are sufficiently infrequent that they may be regarded as independent events. Size \mathcal{N} is reached at a time T when the the following equations are satisfied:

$$\begin{aligned}
 T &= \sum_{j=1}^M t_j \\
 \mathcal{N} &\geq \sum_{j=1}^M n_j .
 \end{aligned}
 \tag{9}$$

The size n_j of the droplet which is absorbed at the j^{th} collision, and the time t_j for this collision, are random variables. Their statistics are determined by proposing different random values of t_j for each possible n_j . These will be denoted by $t^\dagger(n_j)$, and t_j is determined by picking the minimum of these $t^\dagger(n_j)$ as the time until the next collision. The $t^\dagger(n_j)$ are all independent variables with a Poisson distribution, with PDF $p(t^\dagger) = \exp(-t^\dagger/\langle t^\dagger \rangle)/\langle t^\dagger \rangle$. If the collision with index j occurs at time T_j when the cluster size is M , then the mean values are

$$\langle t^\dagger(n_j) \rangle = \frac{1}{\tilde{K}(M, n_j)\tilde{\rho}(n_j, T_j)} .
 \tag{10}$$

Determining T from Eqs. (9), (10) is still a very complex task, and a further simplification will be applied. If only those droplet growth events involving collisions with monomers are included, then the growth to size \mathcal{N} will be slower, giving an upper bound of T of the form

$$\begin{aligned}
 \hat{T} &= \sum_{j=1}^{\mathcal{N}} t_j \\
 p(t_j) &= R_j \exp(-\tilde{R}_j t_j) \quad \tilde{R}_j = \tilde{K}(j, 1)\tilde{\rho}(1, T_j)
 \end{aligned}
 \tag{11}$$

where T_j is the time to reach size j . This quantity is still difficult to calculate, because it is necessary to solve the Smoluchowski equation to determine the time dependence of the density of monomers, $\tilde{\rho}(1, t)$. An easier approach is to assume that the density of monomers does not decrease to less than some fraction ν of its initial value before time \hat{T} , that is $\tilde{\rho}(1, t) \geq \nu\tilde{\rho}(1, 0)$, with $0 < \nu < 1$, for all $t \in [0, \hat{T}]$. The time T is then bounded above by \hat{T}/ν , where \hat{T} is given by (11), with $R_n = \tilde{K}(n, 1)\tilde{\rho}(1, 0)$.

To summarise, the time taken to produce a rain shower is less than the time T^* which solves

$$\mathcal{N}P(T^* \times \nu) = \mu
 \tag{12}$$

where μ is the lower bound on the fraction of of the liquid water content which is precipitated, and $1 - \nu$ is the upper bound on the fraction of microscopic water droplets which undergo collision. (Note that $\mu + \nu < 1$). If $P(T) \sim \exp[-\mathcal{J}(T)]$, the time for onset of a shower is bounded by the solution of

$$\ln \mathcal{N} - \ln \mu = \mathcal{J}(T^* \times \nu)
 \tag{13}$$

which is closely related to Eq. (3). The function $\mathcal{J}(T)$ must diverge as $T \rightarrow 0$, so that when \mathcal{N} is very large, the value of T^* decreases when μ is decreased.

The number \mathcal{N} is very large, typically approximately 10^6 . In the limit as $\mathcal{N} \rightarrow \infty$, and for μ, ν fixed numbers of order unity, the solution of (13) for T^* approaches zero. This indicates that T^* can be much less than the mean time for the first collision, in which case $1 - \nu$ will indeed be a small quantity.

The problem of bounding the time taken to form a rain shower is, therefore, addressed by calculating the function $\mathcal{J}(T)$.

3 Determining the Entropy Function

3.1 General Approach

The following calculation uses the same approach as [13], but it treats the more general case described in the introduction. The probability $P(T)$ that the sum (11) is less than T cannot be determined explicitly, but it can be related to the *cumulant generating function*, $\lambda(k)$, defined by writing

$$\exp[-\lambda(k)] \equiv \langle \exp(-kT) \rangle = \int_0^\infty dT p(T) \exp(-kT) \tag{14}$$

where $p(T) = dP/dT$ is the probability density function (PDF) of T . The cumulant generating function $\lambda(k)$ can be determined exactly as a summation: noting that the t_n are independent

$$\lambda(k) = \sum_{n=1}^{\mathcal{N}} \ln(1 + k\tau_n) . \tag{15}$$

Note that $\exp[-\lambda(k)]$ is the Laplace transform of $p(T)$, which can be inverted by means of the Bromwich integral:

$$p(T) = \frac{1}{2\pi i} \int_C dz \exp[zT - \lambda(z)] \tag{16}$$

where C is a path in the complex plane with $z = \mathcal{R} + i\mu$, with μ a real number running from $-\infty$ to ∞ , and $\mathcal{R} > -1/\tau_1$. This integral is estimated using the saddle point method: there is a saddle point at $z = k^*$ satisfying

$$T = \sum_{n=1}^{\mathcal{N}} \frac{\tau_n}{1 + k^*\tau_n} . \tag{17}$$

The PDF of T is then approximated by

$$p(T) \sim \frac{1}{\sqrt{2\pi|\lambda''(k^*)|}} \exp[-J(T)] \tag{18}$$

where $J(T)$ is a Legendre transform of $\lambda(k)$:

$$J(T) = \lambda(k^*) - Tk^* . \tag{19}$$

Note that Eq. (18) is very similar in form to (2), implying that the Legendre transform function $J(T)$ is closely related to the large deviation entropy $\mathcal{J}(T)$ (defined by (2)) that is required.

Noting that $P(T) = \exp[-\mathcal{J}(T)]$ is the integral of $p(t)$ from 0 to T , and that within the region of integration $\exp[-J(T - \tau)]$ has its maximum at $\tau = 0$, applying Watson’s lemma to make an asymptotic estimate yields

$$\begin{aligned}
 P(T) &= \int_0^T dt p(t) \\
 &\sim \int_0^T d\tau \frac{1}{\sqrt{2\pi|\lambda''(k^*(T))|}} \exp[-J(T - \tau)] \\
 &\sim \frac{1}{\sqrt{2\pi|\lambda''(k^*(T))||J'(T)|}} \exp[-J(T)].
 \end{aligned}
 \tag{20}$$

By computing the function $J(T)$ Eq. (20) can be used to determine $\mathcal{J}(T)$.

3.2 Asymptotic Approximation for the Entropy Function

In order to write down explicit approximations for $\mathcal{J}(T)$, it is necessary to approximate the cumulant generating function by means of analytic functions. This can be achieved by approximating the summation of (15) by an integral. The case in which $\tau_n = n^{-\gamma}$ is a foundation for the other examples. In [13], the cumulant generating function for $\tau_n = n^{-\gamma}$ was obtained in the limit of as $k \rightarrow \infty$ and $\mathcal{N} \rightarrow \infty$: the result is

$$\lambda(k) \sim \gamma A(\gamma)k^{1/\gamma} - \frac{1}{2} \ln k - \gamma C + O(1/k)
 \tag{21}$$

where

$$A(\gamma) = \frac{1}{\gamma} \int_0^\infty dx \frac{x^{-(\gamma-1)/\gamma}}{1+x}
 \tag{22}$$

and where the remainder term $C = 0.918966\dots$ can be expressed in terms of an infinite sum. The coefficient $A(\gamma)$ can be expressed in terms of the Euler beta function. For the important case where $\gamma = 4/3$, $A(4/3) = 3\pi/(2\sqrt{2})$.

In this work Eq. (21) will be extended to account for \mathcal{N} being finite. Also, the formula will be extended to the case where the first few collisions occur with rates that do not conform to a power law, and which are given by Eqs. (4) and (5).

The correction due to finite \mathcal{N} is easily obtained. Equation (21) was obtained by taking the upper limit of the summation in (15) to infinity. Subtracting the sum from \mathcal{N} to ∞ from (21), assuming that $k\mathcal{N}^{-\gamma} \ll 1$, and approximating the resulting summation by an integral, the correction $\Delta\lambda(k)$ for finite \mathcal{N} is

$$\begin{aligned}
 \Delta\lambda(k) &= \sum_{n=\mathcal{N}+1}^\infty \ln [1 + k\tau_n] \sim \sum_{n=\mathcal{N}+1}^\infty \ln [1 + kn^{-\gamma}] \\
 &\sim k \sum_{n=\mathcal{N}+1}^\infty n^{-\gamma} \sim \int_{n=\mathcal{N}+1}^\infty dn n^{-\gamma} \\
 &\sim k \frac{\mathcal{N}^{-(\gamma-1)}}{\gamma - 1}.
 \end{aligned}
 \tag{23}$$

This is to be subtracted from (21). This correction is most significant in the case where $\gamma - 1$ is small. It is straightforward to modify this calculation to deal with cases where the final stages

of droplet growth conforms to a different relation from $\tau_n \sim n^{-\gamma}$, provided the summations in (23) remains convergent as $n \rightarrow \infty$.

Next consider the correction due to replacing $\tau_n = n^{-\gamma}$ with Eqs. (4) and (5). The consequent change in the cumulant generating function $\lambda(k)$ is

$$\begin{aligned} \Delta\lambda(k) &= \sum_{n=1}^{\infty} \ln \left[\frac{1 + kn^{-\gamma} [1 + Q(n/\tilde{n})]}{1 + kn^{-\gamma}} \right] \\ &= \sum_{n=1}^{\infty} \ln \left[1 + \frac{Q(n/\tilde{n})}{1 + n^\gamma/k} \right] \\ &\sim \sum_{n=1}^{\infty} \ln \left[1 + Q(n/\tilde{n}) \left(1 - \frac{n^\gamma}{k} \right) \right] \\ &\sim \sum_{n=1}^{\infty} \ln[1 + Q(n/\tilde{n})] - \frac{1}{k} \sum_{n=1}^{\infty} \frac{Q(n/\tilde{n})n^\gamma}{1 + Q(n/\tilde{n})}. \end{aligned} \tag{24}$$

Combining these gives a more refined version of Eq. (21):

$$\lambda(k) \sim \gamma A(\gamma)k^{1/\gamma} - \frac{1}{2} \ln k - \gamma C - \frac{k\mathcal{N}^{-(\gamma-1)}}{\gamma - 1} + \Sigma_1 - \frac{\Sigma_2}{k} \tag{25}$$

where

$$\Sigma_1 = \sum_{n=1}^{\infty} \ln [1 + Q(n/\tilde{n})], \quad \Sigma_2 = \sum_{n=1}^{\infty} \frac{Q(n/\tilde{n})n^\gamma}{1 + Q(n/\tilde{n})}. \tag{26}$$

In order to obtain the PDF of T , it is necessary to determine the Legendre transform of (25). This requires the solution of (17) to obtain $k^*(T)$. Using (25) to approximate the sum in (17), k^* satisfies

$$\tilde{T} \equiv T + \frac{\mathcal{N}^{-(\gamma-1)}}{\gamma - 1} = Ak^{-(\gamma-1)/\gamma} - \frac{1}{2k} + \frac{\Sigma_2}{k^2}. \tag{27}$$

The solution for small T is

$$k^*(T) = \left(\frac{\tilde{T}}{A} \right)^{-\gamma/(\gamma-1)} \left[1 - \frac{\gamma \tilde{T}^{1/(\gamma-1)}}{2(\gamma - 1)A^{\gamma/(\gamma-1)}} + \dots \right] \tag{28}$$

where \tilde{T} is defined by (27). This yields an asymptotic expression for the probability density $p(T)$:

$$p(T) \sim K \tilde{T}^{-\frac{(3\gamma-1)}{2(\gamma-1)}} \exp \left(-(\gamma - 1)b\tilde{T}^{-1/(\gamma-1)} \right) \tag{29}$$

where

$$b = A^{\gamma/(\gamma-1)}, \quad K = \sqrt{\frac{\gamma}{2\pi(\gamma - 1)}} \exp(\gamma C - \Sigma_1) \tag{30}$$

and using Eq. (20), the probability of undergoing \mathcal{N} collisions in time less than T is

$$P(T) \sim \frac{K}{b\sqrt{\tilde{T}}} \exp \left(-(\gamma - 1)b\tilde{T}^{-1/(\gamma-1)} \right). \tag{31}$$

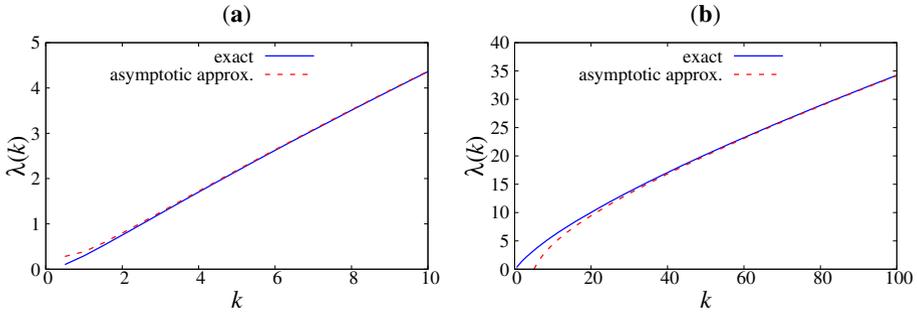


Fig. 1 Cumulant generating function $\lambda(k)$, comparing exact value obtained by summation Eq. (15) with asymptotic approximation, (25). **a** $\tau_n = n^{-4/3}$, $\mathcal{N} = 10^4$. **b** $\tau_n = n^{-4/3}[1 + (n/\tilde{n})^{-2/3} \exp(-n/\tilde{n})]$, $\tilde{n} = 3$, $\mathcal{N} = 10^4$

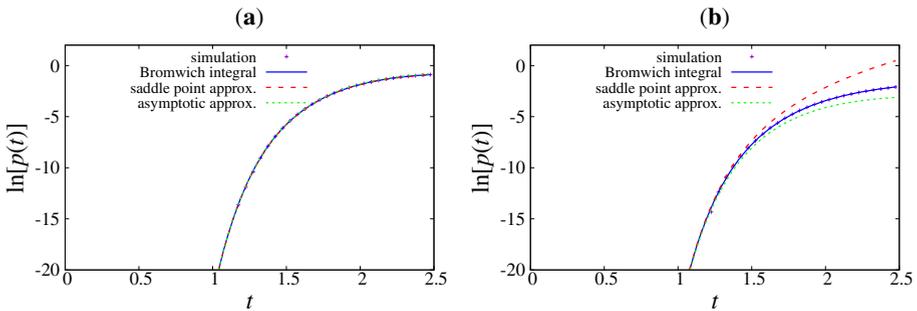


Fig. 2 Comparing different approximations for the PDF $p(t)$ of the time taken for \mathcal{N} collisions, according to the model (1). Simulations and the Bromwich integral (15) agree precisely. The saddle-point approximation (18) and Eq. (29) are in excellent agreement when $\tau_n = n^{-4/3}$, $\mathcal{N} = 10^4$ (case (a)), and asymptotic to these other curves as $t \rightarrow 0$ when $\tau_n = n^{-4/3}[1 + (n/\tilde{n})^{-2/3} \exp(-n/\tilde{n})]$, $\tilde{n} = 3$, $\mathcal{N} = 10^4$ (case (b))

This is an asymptotic formula which is valid in the limit as $T \rightarrow 0$, where $P(T)$ approaches zero very rapidly as T decreases. Note that, because $\mathcal{J}(t) \equiv -\ln[P(t)]$, Eq. (31) gives an explicit approximation for the function $\mathcal{J}(t)$ in Eq. (3).

3.3 Numerical Investigations

The final result of the calculation, Eq. (31), depends upon a sequence of asymptotic approximations, which may, or may not, work well in practice. These were tested by numerical experiments.

Figure 1 compares the exact expression of $\lambda(k)$, Eq. (15), with its asymptotic approximation, Eq. (25). Figure 1a shows results for the case where $\tau_n = n^{-\gamma}$, with $\gamma = 4/3$ and $\mathcal{N} = 10^4$. Figure 1b displays results for τ_n given by (4) and (5), with $\gamma = 4/3$, $\delta = 2/3$, $\tilde{n} = 3$ and $\mathcal{N} = 10^4$. There is excellent agreement as $k \rightarrow \infty$ in both cases, showing that the approximation of $\lambda(k)$ by elementary analytic functions is very accurate. Correspondingly, the asymptotic approximation to $p(t)$ which was obtained from (25) should be very accurate in the limit as $t \rightarrow 0$.

Figure 2 compares different approaches to determining the PDF, $p(T)$, of the time taken to reach size \mathcal{N} . These are

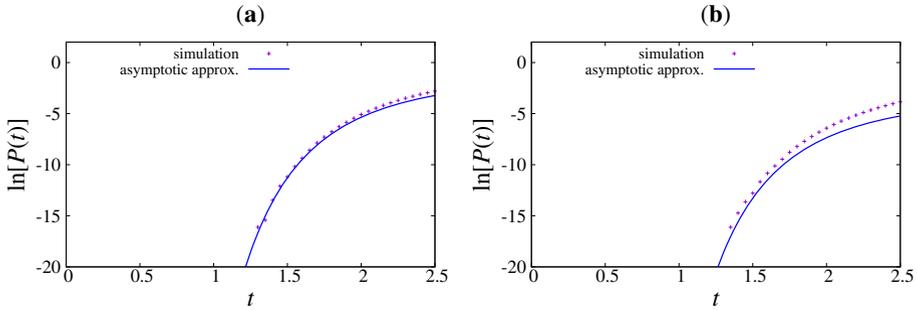


Fig. 3 Comparing a simulation the probability $P(t)$ for a droplet to reach \mathcal{N} collisions after time t , with its asymptotic approximation, (31): $\tau_n = n^{-4/3}[1 + (n/\tilde{n})^{-2/3} \exp(-n/\tilde{n})]$, with **a**: $\tilde{n} = 2, \mathcal{N} = 10^5$, **b** $\tilde{n} = 4, \mathcal{N} = 10^5$

1. Evaluation by simulation using $\mathcal{M} = 10^8$ realisations of Eq. (1).
2. Numerical evaluation of the Bromwich integral, Eqs. (15), (16).
3. Saddle point approximation, Eq. (18), with the cumulant generating function $\lambda(k)$ evaluated using the asymptotic formula (25). The saddle-point Eq. (17) is solved numerically to determine k^* .
4. Asymptotic approximation, using Eq. (29).

The two panels show the same cases as Fig. 1. The first two methods should yield the same results, apart from fluctuations due to finite sample size of the simulation. The results of using the saddle-point approximation and Eq. (29) are both in excellent agreement in the limit as $t \rightarrow 0$.

Figure 3 compares estimates of $P(t)$ obtained by simulation (using 10^7 realisations of (1)) with the result of the asymptotic formula, (31), for a combination of parameter values which differs from Figs. 1 and 2. Again, there is good agreement as $t \rightarrow 0$. A notable feature of the Figs. 2 and 3 is that changing the rates of the initial collisions has very little effect. For example, comparing Figs. 3b and 2a, the mean time for the first collision, τ_1 , is increased by a factor of $1 + 4^{2/3} = 3.519\dots$, and the total number of collisions is increased to $\mathcal{N} = 10^5$ from $\mathcal{N} = 10^4$, but the intercept on the t axis is only shifted by a small factor.

3.4 Implications for Rainfall

Section 2 gave an Eq. (13), which gives an upper bound on the time T^* to produce a rain shower, which removes a fraction greater than μ of the liquid water content of a cloud, subject to the constraint that the fraction of un-collided droplets is always greater than ν up until time T^* . Making use of this upper bound in the general case would require a complicated calculation, but there is a limiting case where (13) can be replaced by a simpler relation, Eq. (3). This is realised if the time T^* is sufficiently small that the fraction of un-collided droplets remains close to unity up to time T^* . At short times, the first collision of a droplet is almost always with another un-collided droplet, so that the fraction of un-collided droplets at time t is approximated by

$$\nu(t) = 1 - \frac{t}{\tau_1} + O(t^2) \tag{32}$$

so that if the solution of (3) satisfies $T^*/\tau_1 \ll 1$, then the use of this simplified Eq. (3) to estimate T^* is justified.

Now that the large-deviation rate function is available from Eq. (31), it is possible to estimate the ratio T^*/τ_1 and establish when the use of the simplified Eq. (3) is indeed justifiable (implying that a shower can occur in a timescale which is much shorter than the mean time for the first collision). From (31), the rate function may be written in the form

$$\mathcal{J}(T) = \alpha(\gamma) + \Sigma_1 + \frac{1}{2} \ln \left(\frac{\tilde{T}}{T_0} \right) + \left(\frac{\tilde{T}}{T_0} \right)^{-\frac{1}{\gamma-1}} \tag{33}$$

where \tilde{T} is defined by (27), $\alpha(\gamma)$ is independent of \tilde{T} and

$$T_0(\gamma) = [A(\gamma)]^\gamma (\gamma - 1)^{\gamma-1} . \tag{34}$$

In the important case where $\gamma = 4/3$, $A(4/3) = 3\pi/2\sqrt{2}$, $T_0(4/3) = 3.4508\dots$ and $\alpha(4/3) = -0.38018\dots$. Another case, for comparison, is $\gamma = 2$, where $A(2) = \pi/2$, $T_0(2) = 2.4674\dots$ and $\alpha(2) = -0.81398\dots$

Now consider using these results to estimate T^* by approximating the solution of (3). An instructive approach is to approximate the rate function by $\mathcal{J}(t) \approx \Sigma_1 + (t/T_0)^{-1/(\gamma-1)}$, so that the solution to (3) is approximated by

$$T^* \sim T_0 [\ln \mathcal{N} - \Sigma_1]^{-(\gamma-1)} \tag{35}$$

where $\mathcal{N} \approx 10^6$ is the number of microscopic droplets which coalesce to form a raindrop, and where Σ_1 is defined by Eq. (26). Note that this estimate for T^* is rather insensitive to the rates of the first few collisions (which only enter through the quantity Σ_1). This estimate for T^* is to be compared with the timescale for the first collision, τ_1 , as specified by Eq. (4), so that

$$\frac{T^*}{\tau_1} \approx \frac{T_0}{1 + Q(1/\tilde{n})} [\ln \mathcal{N} - \Sigma_1]^{-(\gamma-1)} . \tag{36}$$

In the case where the collision times follow a simple power-law, $\tau_n = n^{-\gamma}$, and $Q(x) = \Sigma_1 = 0$, the ratio T^*/τ_1 is not a small number when $\gamma = 4/3$ and $\mathcal{N} = 10^6$: in that case Eq. (36) gives $\frac{T^*}{\tau_1} \approx 1.438$, and a numerical solution of (3) yields $\frac{T^*}{\tau_1} \approx 1.381$. If, however, the first few collisions are much slower, this has a small effect on T^* , while τ_1 can be greatly increased. For example, if $\mathcal{N} = 10^6$ and $\tau_n = n^{-4/3} [1 + (n/\tilde{n})^{-2/3} \exp(-n/\tilde{n})]$, with $\tilde{n} = 5$, then $\Sigma_1 = 4.3671\dots$ and Eq. (36) gives $\frac{T^*}{\tau_1} \approx 0.428$, which is a fair approximation to the ratio obtained by a numerical solution of (3), which yields $\frac{T^*}{\tau_1} \approx 0.460$. Because this ratio is less than unity, using the simplified condition, Eq. (3) should give a fair approximation to the time taken to produce a rain shower. While it would be possible to make a more accurate estimate, the uncertainties arising from this estimate are less than those which are inherent in the model.

4 Distribution of Time for Passing Through Size N

A cluster that undergoes runaway growth in time T must pass through size N (with $N < \mathcal{N}$), which is reached at some intermediate time $t_N < T$. The probability to undergo runaway

growth in time T may be written

$$P(T) = \int_0^T dt_1 \int_0^{t_1} dt_2 p_N^<(t_2) p_N^>(t_1 - t_2) \tag{37}$$

where $p_N^<(t)$ is the PDF for a cluster to grow from size 1 to size N in time t and $p_N^>(t)$ is the PDF to grow from N to \mathcal{N} in time t . Thus

$$p(t) = \int_0^t dt' p_N^<(t') p_N^>(t - t') \tag{38}$$

is the probability density to have undergone runaway growth in time t . When t is small, both of the probability densities $p_N^>$ and $p_N^<$ are very small, and can be addressed by large deviation theory: write

$$p_N^<(t) = \exp[-\mathcal{J}_N^<(t)] , \quad p_N^>(t) = \exp[-\mathcal{J}_N^>(t)] . \tag{39}$$

In this limit it is, therefore, expected that the distribution $p_N^<(t') p_N^>(t - t')$ is very sharply peaked, with the maximum with respect to t' at a point t_N^* . It is expected that those clusters which reach size \mathcal{N} in a short time T pass through size N at a time which is close to t_N^* . This time t_N^* is well approximated by the position of the minimum of

$$F_N(t) = \mathcal{J}_N^<(t) + \mathcal{J}_N^>(T - t) . \tag{40}$$

Now consider, in succession, the problem of determining the functions $\mathcal{J}_N^>$ and $\mathcal{J}^<$, before estimating t_N^* .

To avoid over-complicated notation, this calculation will be carried out when the τ_n are a simple power-law: $\tau_n = n^{-\gamma}$.

4.1 Runaway Growth: N to \mathcal{N}

For power-law mean collision time $\tau_n = n^{-\gamma}$, the cumulant generating function for growth from N to \mathcal{N} is

$$\lambda_N^>(k) = \sum_{n=N}^{\mathcal{N}} \ln(1 + kn^{-\gamma}) . \tag{41}$$

If $N = 1$, this is well approximated by Eq. (21), combined with (23). Assuming that $kN^{-\gamma} \gg 1$, subtracting the sum from 1 to $N - 1$ gives

$$\lambda_N^>(k) \sim \gamma A(\gamma) k^{1/\gamma} - \left(N - \frac{1}{2}\right) \ln k - \gamma C - \sigma_N - k \frac{\mathcal{N}^{-(\gamma-1)}}{\gamma - 1} \tag{42}$$

with

$$\sigma_N = \sum_{n=1}^{N-1} \ln \tau_n . \tag{43}$$

Following the approach described in Sect. 3, inverting the Laplace transform using a Bromwich integral, and approximating this using a saddle point (Laplace) approximation gives

$$p_N^>(t) = \frac{1}{\sqrt{2\pi |\lambda_N^>''(k^*)|}} \exp[-J_N^>(k^*)] . \tag{44}$$

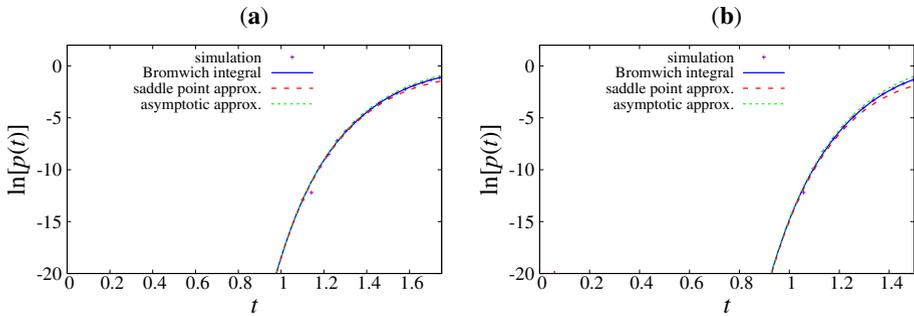


Fig. 4 Comparing Eq. (48) with simulation, the Bromwich integral (15), and saddle-point approximation (18). For both panels, $\tau_n = n^{-4/3}$, $\mathcal{N} = 10^4$. **a** $N = 2$, **b** $N = 3$

The saddle point k^* satisfies

$$t = \frac{d\lambda^>}{dk}(k^*), \quad \frac{d}{dk} \lambda_N^>(k) = Ak^{(1-\gamma)/\gamma} - \frac{N - \frac{1}{2}}{k} - \frac{\mathcal{N}^{-(\gamma-1)}}{\gamma - 1} \tag{45}$$

and $J_N^>(k) = \lambda_N^>(k) - k\lambda_N^>(k)$. The solution is, at leading order in $1/k$,

$$k^* \sim b\tilde{t}^{-1/(\gamma-1)} \left(1 - \frac{\gamma(N - \frac{1}{2})}{(\gamma - 1)b} \tilde{t}^{1/(\gamma-1)} \right), \quad \tilde{t} = t + \frac{\mathcal{N}^{-(\gamma-1)}}{\gamma - 1} \tag{46}$$

and b is given by (30). Noting that $Ab^{1/\gamma} = b$, and expressing J_N as a function of t by writing $J_N(t) = J_N(k^*(t))$ gives

$$J_N^>(\tilde{t}) = (\gamma - 1)b\tilde{t}^{-1/(\gamma-1)} + \frac{\gamma}{\gamma - 1} \left(N - \frac{1}{2} \right) \ln \tilde{t} - \left(N - \frac{1}{2} \right) \ln b - \gamma\mathcal{C} - \sigma_N \tag{47}$$

and hence

$$p_N^>(\tilde{t}) \sim \sqrt{\frac{\gamma}{2\pi(\gamma - 1)}} b^N \exp[\gamma\mathcal{C} + \sigma_N] \tilde{t}^{-\frac{\gamma(2N+1)-1}{2(\gamma-1)}} \exp\left(-(\gamma - 1)b\tilde{t}^{-1/(\gamma-1)}\right). \tag{48}$$

Upon setting $N = 1$, this result is in agreement with (29) when $\Sigma_1 = 0$. Figure 4 compares Eq. (48) with exact evaluation of $p_N^>(t)$ (simulation and evaluation of the Bromwich integral), and with the saddle point approximation, with $\gamma = 4/3$ and $\mathcal{N} = 10^4$, for $N = 2$ (a) and $N = 3$ (b).

4.2 Initial Growth: 1 to $N - 1$

Now estimate the probability density $p_N^<(t)$ to undergo the first $N - 1$ collisions in time t , using the same approach. The cumulant generating function is

$$\lambda_N^<(k) = \sum_{n=1}^{N-1} \ln(1 + kn^{-\gamma}). \tag{49}$$

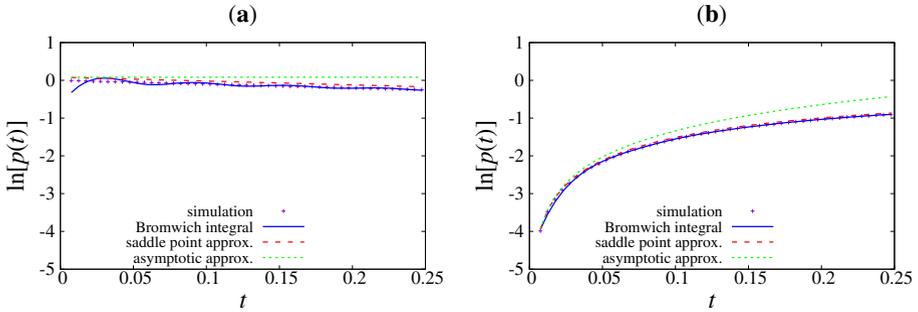


Fig. 5 Comparing Eq. (54) with simulation, the Bromwich integral (15), and saddle-point approximation (18). For both panels, $\tau_n = n^{-4/3}$, $\mathcal{N} = 10^4$. **a** $N = 2$, **b** $N = 3$

To leading order in k^{-1} , this is

$$\lambda_N^<(k) \sim (N - 1) \ln k - \sigma_N \tag{50}$$

where σ_N was defined in (43). The saddle point satisfies

$$t = \frac{d\lambda_N^<}{dk}(k^*) \tag{51}$$

so that

$$k^* = \frac{N - 1}{t} \tag{52}$$

and hence $\frac{d^2\lambda_N^<}{dk^2}(k^*) = -t^2/(N - 1)$. As a function of t , at leading order,

$$J_N^<(t) \sim -(N - 1) \ln t + (N - 1) [\ln(N - 1) - 1] + \sigma_N \tag{53}$$

so that

$$p_N^<(t) \sim \sqrt{\frac{N - 1}{2\pi}} \exp[-\sigma_N - (N - 1) (\ln(N - 1) - 1)] t^{N-2} . \tag{54}$$

Figure 5 compares Eq. (54) with exact evaluation on $p_N^<(t)$ (simulation and evaluation of the Bromwich integral), and with the saddle point approximation, with $\gamma = 4/3$ and $\mathcal{N} = 10^4$, for $N = 2$ (a) and $N = 3$ (b).

4.3 Growth History of Lucky Drops

The forms of the functions $\mathcal{J}_N^<$ and $\mathcal{J}_N^>$ can now be identified. Ignoring irrelevant constant terms, the function defined in (40) is

$$F_N(\tilde{t}) = -(N - 2) \ln \left(\tilde{t} - \frac{\mathcal{N}^{-(\gamma-1)}}{\gamma - 1} \right) + b(\gamma - 1)(T - \tilde{t})^{-1/(\gamma-1)} + \frac{\gamma(2N + 1) - 1}{2(\gamma - 1)} \ln(T - \tilde{t}) \tag{55}$$

where \tilde{t} was defined in (46). Note that there are two terms which are functions of $T - \tilde{t}$, and that in the limit as $T \rightarrow 0$, the power-law term is dominant over the logarithmic term.

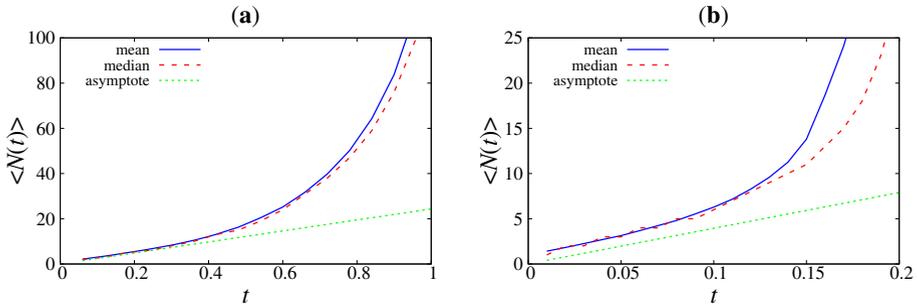


Fig. 6 Plot of mean particle size $\langle N(t) \rangle$ for the fraction $K/\mathcal{M} \approx P(T)$ of fastest growing particles, which reach size \mathcal{N} in time less than T . **a** $\gamma = 4/3, T = 1.5, P(T) = 1.884 \dots \times 10^{-4}, \mathcal{N} = 10^4$. **b** $\gamma = 2, T = 0.25, P(T) = 3.681 \dots \times 10^{-4}, \mathcal{N} = 10^4$. In both cases the number of realisations of Eq. (1) was $\mathcal{M} = 10^6$

Ignoring the term in $\ln(T - \tilde{t})$, and differentiating the remaining terms with respect to \tilde{t}

$$F'_N(\tilde{t}) \approx b(T - \tilde{t})^{-\gamma/(\gamma-1)} - \frac{(N - 2)}{\tilde{t} - \mathcal{N}^{-(\gamma-1)}/(\gamma - 1)} \tag{56}$$

so that, if $t_N/T \ll 1$, the minimum of $F_N(t)$ is at

$$t_N^* \approx \frac{N - 2}{b} T^{\gamma/(\gamma-1)} + \frac{\mathcal{N}^{-(\gamma-1)}}{\gamma - 1} . \tag{57}$$

This implies that, for those ‘lucky’ droplets that do undergo runaway growth, the typical value t_N^* of the time when they have undergone N collisions is initially increasing linearly with N , regardless of the value of the exponent γ . This is a surprising result. It is, however, consistent with the observations made in Sect. 3, that the growth time of the fast-growing droplets is not strongly influenced by the first few values of the τ_n .

The approximation (57) breaks down when the predicted value of t_N^* is no longer small compared to T . This occurs after N^* collisions, where $N^* \sim bT^{-1/(\gamma-1)}$.

The history of the fastest growing droplets was investigated numerically. The growth process represented by (1) was simulated for \mathcal{M} realisations, and the random number seeds for the K realisations where the sum (1) is less than T were recorded. These K cases were run again, and their growth history $N(t)$ was recorded (where N is the size of the cluster at time t). Figure 6 plots the mean size $\langle N(t) \rangle$ (and also the median size) of these K clusters, which represent the K/\mathcal{M} fraction which exhibits the fastest growth. Simulations with $\tau_n = n^{-\gamma}$, for two different values of γ , both show a linear initial growth. The mean value $\langle N(t) \rangle$ is compared with the prediction

$$\langle N(t) \rangle \approx bT^{-\gamma/(\gamma-1)}t + \text{const.} . \tag{58}$$

obtained from Eq. (57), and good agreement is observed.

5 Conclusions

The slow rate of collision between microscopic droplets in ice-free (‘warm’) clouds appears to impose a severe difficulty in explaining the rapid onset of rain showers. However, this difficulty *may* be overcome by recognising that, because a raindrop is the result of coalescence

of roughly a million microscopic droplets, a rain shower is created by the one-in-a-million fraction of droplets which have the fastest growth.

Reference [13] proposed a surprisingly simple Eq. (3), for determining the time taken for the onset of a rain shower. This work has developed the arguments supporting Eq. (3) in greater detail. It has also shown how the function $\mathcal{J}(T)$ can be approximated in cases where the mean collision times τ_n have a much more general dependence on n than the simple power-law considered in [13]. The probability $P(t)$ for undergoing \mathcal{N} collisions in time t is given by an explicit asymptotic expression, Eq. (31), which is valid for a rather general form of the collision times τ_n . This formula implies that the timescale for the onset of a rain shower *can* indeed be short compared to the mean time for even the first of the million collisions.

Equation (31) allows a quantitative application to understanding rainfall from a homogeneous atmosphere. However, its principal significance for meteorology may be qualitative rather than quantitative, in that it implies that the apparent kinetic barrier to making rainfall from warm clouds is of little significance, even for a homogeneous atmosphere. In practice, the atmosphere may contain a sufficiently large fraction of unusual nucleation centres, which nucleate atypically large microscopic droplets. It is these larger droplets which are likely to become raindrops. The implication is that, if a cloud has sufficient liquid water content to create a shower, then there is no significant kinetic barrier to precipitation.

While not readily experimentally observable, it is of interest to understand the growth history of those rare, fast-growing droplets that do become raindrops. Rather surprisingly, they are found to initially grow linearly as a function of time, irrespective of the n -dependence of the mean collision times, τ_n .

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