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A practical algorithm to detect superexponential behaviour in financial asset price returns

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Abstract

To assist with the detection of bubbles and negative bubbles in financial markets, a criterion is introduced to indicate whether a market is likely to be in a superexponential regime (where growth in such a regime would correspond to an asset price bubble and decline to an negative bubble) as opposed to “normal” exponential behaviour typified by a constant rate of growth or decline. The criterion is founded on the Johansen-Ledoit-Sornette model of asset dynamics in a bubble and is derived from a linear fit to observed data with a non-linear time transformation with parameters distributed uniformly in their permitted ranges. Making use of expected values rather than the underlying distribution, the criterion is straightforward and efficient to compute and can in principle be applied in real time to intra-day markets as well as longer timescales. In some circumstances, the criterion is shown to have certain predictive qualities when applied to a portfolio of stocks, and could be used as input into algorithmic trading strategies. A simple strategy is described which is based on market reversion predictions of a portfolio of stocks and which in back-testing generates notable returns.

1 Introduction

Understanding financial asset-price bubbles and negative bubbles is an important endeavour for finance professionals, theoretical researchers, and public servants alike, and, as a consequence, there is a large body of literature devoted to their study, and, in particular, to their detection and to the prediction of their start and end times. See, for example, Gürkaynak (2008) and Jarrow (2016) for accounts of several
approaches in this field. In this context, and throughout this paper, we consider bubbles to be classified as explained by Zhou & Sornette (2004a,b, 2005). As such, our understanding of the term bubble in this paper is equivalent to the term “positive bubble” as described by these authors. Furthermore, a negative bubble corresponds to the similar use in the papers by Zhou and Sornette.

The valuation of financial assets including stock markets is a subject of controversy and debate, with some of those who support the efficient markets hypothesis arguing that full market knowledge is already implicit in the market price and that consequently bubbles and negative bubbles are either illusory or are ironed out sufficiently quickly by the market so that no effective use can be made of any temporary discrepancy between the asset price and market fundamentals. On the other hand, many researchers and finance professionals regard asset bubbles and negative bubbles as common occurrences, and there have been many critiques of strict application of the efficient markets hypothesis (Malkiel 2003 and Shostak 1997).

Indeed, there is clear evidence that even the long-term behaviour of stock markets can depart from the theoretical models such as a random-walk. For example, Robertson & Wright (1998) give evidence that long-term stock returns might be less uncertain and more predictable than implied by the random-walk model. See Lo & MacKinlay (2002) (and references therein) for a discussion of several distortions in the market which could be exploited, at least in principle.

In Lynch & Mestel (2019), and more generally in Lynch (2020), the standpoint is taken that financial bubbles may be viewed as periods of accelerated superexponential growth driven by herding behaviour in the market. Indeed, recently Ardila-Alvarez et al. (2021) have formulated and tested extensively just such an acceleration effect. In Lynch (2020), this led to the identification of a useful criterion to determine the change-points in a financial time series. The starting point is the seminal Johansen-Ledoit-Sornette stochastic differential equation model of an asset bubble (see Johansen et al. 2000) as elaborated by Cheah & Fry (2015).

The methods in Lynch & Mestel (2019) were built on the assumption that an asset grows exponentially following a geometric Brownian motion process (i.e., “normal growth”). Superexponential growth was modelled by an additional power-law hazard term (Cheah & Fry 2015 and Johansen et al. 2000), switched on by a parameter \( \nu \neq 0 \). After simulating the model using the growth rates and volatility, selected randomly from a previously fitted bi-variate distribution, the fit between the data and the superexponential growth model was tested by calculating a most-likely-value estimate (MLV) from the distribution of \( \nu \) obtained from a maximum-likelihood estimate (based on random simulation of the other parameters of the model).

For the empirical case studied the MLV estimates were found to be well fitted by generalised/skew logistic distributions (for both normal and superexponential growth). Although the two resulting distributions are close they are nevertheless distinct and provided signatures for the two types of growth. These distributions were used as proxies for the probabilities of superexponential and normal growth,
and a threshold was determined to differentiate these two regimes. Using these signature distributions, it was shown, in the case studied, that it was possible (at least in principle) to use past and present prices on each trading day to identify whether superexponential growth or normal growth was more likely to be the governing regime.

Although successful as a theoretical tool, at least in certain markets (see Lynch 2020), the algorithms used in Lynch & Mestel (2019) are computationally expensive which makes them difficult to apply in practice especially in the short time-scales of intra-day markets.

This present paper addresses this deficiency by introducing a proxy criterion based on expectations rather than the distributions themselves. To do this, a new concept, “bubble time”, is introduced and its distribution is studied. Bubble time is essentially a nonlinearly rescaled time based on a power-law approach to criticality in an asset bubble. The concept of a non-linear time transformation has been considered previously in the literature, in particular, in work of Lin & Sornette (2013). In the context of this present work the estimate of the parameter $\nu$ introduced in Lynch & Mestel (2019) can be expressed in terms of bubble time and then a proxy obtained based on expectation. This proxy estimate may be computed at speed and could be applied, for example, to devise an automatic trading algorithm.

In Lynch & Mestel (2019), the likelihood of the existence of superexponential growth was determined by an estimate of the parameter $\nu$ which was compared with a threshold value, $\bar{\nu}_T$, which had been determined by comparing simulations of normal growth and superexponential growth given historical distributions of mean and standard deviation of returns in the chosen market over the entire period of available observations. The new approach in this paper dispenses with the threshold value, and removes the computational overhead of the initial simulations required for each market to determine the appropriate threshold. In consequence, an indicative parameter of the governing regime can be calculated for any point in time, for any set of observed data in a computationally inexpensive manner.

This remainder of this paper is organised as follows: in §2 a maximum log-likelihood analysis of the power-law hazard function is presented and the proxy parameter $\tilde{\nu}'$ is introduced, followed by a discussion of bubble time and its distribution in §3. In §4, the distribution of $\tilde{\nu}'$ is calculated for markets driven by normal growth or decline, while the practical application of the theory is given in §5. The results — applied to an exemplar individual stock and the Nasdaq-100 index — are presented in §6 together with a brief discussion of the application of the theory to automatic trading. A brief conclusion in §7 provides suggestions for future work.
2 Maximum log-likelihood analysis of the power-law hazard function

The parameter $\nu'$ described in Lynch & Mestel (2019) may be described as the log-likelihood maximising growth of the difference in an asset’s log-prices over a fixed period, with the time dimension transformed by a power-law conditioned on the critical time, $t_c$, of an assumed bubble or negative bubble regime.

Consider the observed prices $S = (S_1, S_2, \ldots, S_n)$ of a financial asset observed at times $t = (t_1, t_2, \ldots, t_n)$. Here, and subsequently within the paper, times are expressed with reference to a standard unit of time and are thus dimensionless. In the examples given, typically this time is one trading year, but in applications to intra-day trading this unit might be a trading day. The vector of differences in the log prices is given by $r = (r_1, r_2, \ldots, r_n)$ where $r_i = \log(S_i/S_{i-1})$. We assume that differences of $S$ are log-normally distributed, such that a model $\hat{r}$ of the log returns is the arithmetic Brownian motion process

$$\hat{r} = \Delta t \mu(t) + \sigma \Delta t N(0, 1),$$

where $\mu(t)$ is a time-dependent drift term, $\Delta t$ is the time interval at time $t$, and $\sigma$ is a fixed standard deviation of log returns over the whole observed period, all to be determined. As usual, $N(0, 1)$ is a sample drawn from the standard normal distribution.

Following Cheah & Fry (2015) and Lynch & Mestel (2019), we take $\mu(t) = \mu - \nu h(t)$ where $\mu$ is a constant drift term applicable to the entire observed returns, $h(t) = (t_c - t)^\beta$ is the hazard function and $\nu$ is a constant, which for $\nu \neq 0$ switches on the superexponential behaviour seen in bubbles ($\nu < 0$) and negative bubbles ($\nu > 0$). (Here $t_c$ is the critical time and $\beta$ is the critical exponent. Typically $-1 < \beta < 0$.) Therefore, the model becomes

$$\hat{r} = \Delta t (\mu - \nu (t_c - t)^\beta) + \sigma \Delta t N(0, 1).$$  

In Lynch & Mestel (2019) a simplified probability density function, $\tilde{F}(r, t)$, is derived as

$$\tilde{F}(r_i, t_i) = \frac{1}{\sqrt{2\pi \sigma^2 \Delta t}} \exp \left[ -\frac{(r_i - \Delta t \mu - \nu (t_c - t_i)^\beta)^2}{2\sigma^2 \Delta t} \right].$$

Then, in determining the maximum log-likelihood it was found that the maximising value of $\nu$, i.e., $\nu'$ is given by

$$\nu' = -\frac{1}{\Delta t} \frac{n \sum_{i=1}^{n} (t_i - t_i)^\beta r_i - \bar{r} \sum_{i=1}^{n} (t_i - t_i)^\beta}{n \sum_{i=1}^{n} (t_i - t_i)^{2\beta} - (\sum_{i=1}^{n} (t_i - t_i)^\beta)^2},$$

where $\bar{r}$ is the sample mean of the set of observed log-returns, $r$. Consider the change of variable

$$\bar{T}_i = (t_c - t_i)^\beta.$$
The equation (4) then becomes

$$\nu' = -\frac{1}{\Delta t} \frac{n \left( \sum_{i=1}^{n} \hat{T}_i r_i - \bar{r} \sum_{i=1}^{n} \hat{T}_i \right)}{n \sum_{i=1}^{n} \hat{T}_i^2 - \left( \sum_{i=1}^{n} \hat{T}_i \right)^2} \tag{6}$$

so that

$$\nu' = -\frac{1}{\Delta t} \frac{\text{Cov}(\hat{T}, r)}{\text{Var}(T)} \tag{7}$$

and $\nu'$ can be interpreted as the log-likelihood maximised slope of the linear fit of the observed data against the observation time transformed under the map in equation (5). This suggests that a bubble may be considered as a constant rate of return in what could be described as “bubble time”.

However, each $\hat{T}_i$ is a random variable, being a function of the random variables, $\beta$ and $t_c$. Now, if one writes $\theta_i = t_c - t_i$ so that,

$$\hat{T}_i = \theta_i^\beta \tag{8}$$

then, recalling from Johansen et al. (2000) that $-1 < \beta < 0$ so that the asset price has a finite value but infinite derivative at $t_i = t_c$, one may let $\theta_i$ and $\beta$ be random variables such that

$$\beta \sim U(-1, 0), \quad \theta_i \sim U(s_{i,0}, s_{i,1}) \tag{9}$$

where $s_{i,0}$ is the shortest time period between the critical time, $t_c$, and the observed time $t_i$, and $s_{i,1}$ is the longest. In general, $s_{i,0}$ will be equal to the time between $t_i$ and the trading period following the last observation date of the data set in question, and $s_{i,1}$ will be the value of $s_{i,0}$ plus a fixed length of time that is chosen to limit the critical time beyond the end of the observed data set.

Given that, with suitable assumptions for the values of $s_{i,0}$ and $s_{i,1}$, it is possible to determine the expected values of each $\hat{T}_i$ analytically, this suggests a proxy for $\nu'$ that is derived in terms of $E[\hat{T}]$. Denoting the proxy measure by $\bar{\nu}'$, then it is defined by

$$\bar{\nu}' \triangleq -\frac{1}{\Delta t} \frac{n \left( \sum_{i=1}^{n} E[\hat{T}_i] r_i - \bar{r} \sum_{i=1}^{n} E[\hat{T}_i] \right)}{n \sum_{i=1}^{n} E[\hat{T}_i]^2 - \left( \sum_{i=1}^{n} E[\hat{T}_i] \right)^2} \tag{10}$$

or, expressed more simply as a ratio of sample covariance and sample variance

$$\bar{\nu}' = -\frac{1}{\Delta t} \frac{\text{Cov}(\hat{t}, r)}{\text{Var}(\hat{t})} \tag{11}$$

where $\hat{t} = (E[\hat{T}_1], E[\hat{T}_2], \ldots, E[\hat{T}_n])$. We note that, in testing, this ratio of expectations correlates, at the 97% level, with the most likely value of $\nu'$ in (7), calculated with
simulated values for $\beta$ and $t_c$ in accordance with Lynch & Mestel (2019). As such we are satisfied that the proxy equation, (10), provides a very accurate approximation to the most like value of $\nu'$ obtained by simulation of the dependent random variables distributed by (9).

In the next section we derive an analytical expression for each $E[\hat{T}]$, so that calculation of the parameter $\hat{\nu}'$ becomes straightforward, and the computational expense is greatly reduced.

3 Bubble time distribution

In this section we derive an analytical expression for the expected value $E[\hat{T}]$ of the transformed time, $\hat{T} = \theta^\beta$, given uniform distributions of the random variables $\beta$ and $\theta$. If $s_0$ is the shortest possible time (for example, measured in years for daily prices) between a particular time $t$ and the critical time $t_c$, and $s_1$ is the longest, then there are various cases one needs consider when determining the resulting distribution of the transformed time $\hat{T}$ and in turn, the expected value. These cases depend on the chosen values of $s_0$ and $s_1$, and in particular whether these values are both less than 1, both greater than 1, or span 1. Additionally, one must consider the cases where $s_0 = 1$ and where $s_1 = 1$. For each of these cases, contours of constant value of $T$ are considered for the range of possible values of $\theta$ and $\beta$. The area bounded by these extrema where $\hat{T} \leq T$ defines the probability $Pr(\hat{T} \leq T)$ for the case in question. Once this probability distribution is determined, it may be possible to analytically determine the expected value of the distribution.

In what follows, we shall use properties of the exponential integral and logarithmic functions. We refer the reader to Abramowitz & Stegun (1964) for detailed properties of the functions. For convenience we collect several results that we shall use for reference.

Recall that the exponential integral, $Ei[x]$ is defined for all real $x \neq 0$ by the formula
\[
Ei[x] = \int_{-x}^{\infty} \frac{e^{-t}}{t} dt,
\]
where for $x > 0$, the Cauchy principal value is taken. (In what follows we assume that the Cauchy principal value is taken wherever there is an apparent singularity in the integrand in the range of integration.)

The logarithmic integral
\[
\text{li}[x] = \int_{0}^{x} \frac{dt}{\log t}, \quad x > 0,
\]
is related to $Ei[x]$ by the relation
\[
\text{li}[x] = Ei[\log x], \quad x > 0.
\]
We shall make use of the following formulae:

\[ \text{li}[y] - \text{li}[x] = \int_x^y \frac{dt}{\log t}, \quad 0 < x \leq y, \quad (15) \]

\[ \log |\log y| - \log |\log x| = \int_x^y \frac{dt}{t \log t}, \quad 0 < x \leq y. \quad (16) \]

We now investigate the probability distribution and the expected value of the random variable \( T = \theta^\beta \), where \( \beta \) and \( \theta \) are uniformly distributed with \( \beta \sim U(-1, 0) \) and \( \theta \sim U(s_0, s_1) \) where \( 0 < s_0 < s_1 \). Here \( s_0 \) and \( s_1 \) are non-dimensional since \( \theta \geq 0 \) is a non-dimensional time difference. The distribution and expectation are dependent on the values of \( s_0 \) and \( s_1 \), in particular their values relative to 1. In what follows we shall prove the following result:

**Theorem 3.1.** Let \( T = \theta^\beta \), where \( \beta \) and \( \theta \) are drawn from the distributions \( \beta \sim U(-1, 0) \) and \( \theta \sim U(s_0, s_1) \) where \( 0 < s_0 < s_1 \). Then

\[ \text{E}[T] = \frac{1}{\alpha} (G(s_1) - G(s_0)) \quad (17) \]

where \( G(s) = \text{Ei}[\log s] - \log |\log s| \) and \( \alpha = s_1 - s_0 \).

Note that

\[ G(1) = \lim_{s \to 1} [\text{Ei}[\log s] - \log |\log s|] = \gamma, \quad (18) \]

a result that follows from the relation

\[ \text{Ei}[x] = \log |x| + \gamma + O(x), \quad \text{as} \quad x \to 0. \quad (19) \]

Here \( \gamma \) denotes the Euler-Mascheroni constant. The proof of Theorem 3.1 is somewhat involved and we divide the proof into several cases. In what follows we normalise the areas by dividing by the factor \( \alpha = s_1 - s_0 \) to give related probabilities.

**Case 1:** \( 1 < s_0 < s_1 \). We refer to the diagram in Fig. 1. The curve of \( \theta^\beta = T \) takes the form illustrated in the diagram with the two cases, (a) where \( s_0 < T^{-1} \leq s_1 \), and (b) where \( T^{-1} \leq s_0 \).

We calculate \( \text{Pr}(\theta^\beta \leq T) \), the cumulative distribution function of \( T \). First we note that \( \theta^\beta \leq T \) corresponds to \( \theta \geq T^{1/\beta} \). So we need to calculate the darker shaded region of Fig. 1. For \( T \leq s_1^{-1} \) this area is zero, and for \( s_1^{-1} < T < s_0^{-1} \) the area is given by the integral

\[ \frac{1}{\alpha} \int_{T^{-1}}^{s_1} \left( \frac{\log T}{\log \theta} + 1 \right) d\theta, \quad (20) \]
where \( \alpha = s_1 - s_0 \). For \( s_0^{-1} \leq T < 1 \), the entire shaded area may be conveniently calculated as

\[
\frac{1}{\alpha} \int_{s_0}^{s_1} \left( \frac{\log T}{\log \theta} + 1 \right) d\theta.
\]  

(21)

Finally, for \( T \geq 1 \), the associated area is the full rectangle of area 1.

To calculate the probability distribution function of \( \theta^\beta \) we differentiate with respect to \( T \) using Leibniz’s rule to obtain

\[
f_T(T) = \frac{1}{\alpha T} \left\{ \begin{array}{ll}
\int_{T^{-1}}^{s_1} \frac{d\theta}{T \log \theta} & s_1^{-1} < T < s_0^{-1} \\
\int_{s_0}^{s_1} \frac{d\theta}{T \log \theta} & s_0^{-1} \leq T < 1
\end{array} \right. 
\]  

(22)

and zero for other values of \( T \). These integrals may readily be expressed in terms of \( \text{Ei} \)

\[
f_T(T) = \frac{1}{\alpha T} \left\{ \begin{array}{ll}
\text{Ei}[\log s_1] - \text{Ei}[\log T^{-1}] & s_1^{-1} < T < s_0^{-1} \\
\text{Ei}[\log s_1] - \text{Ei}[\log s_0] & s_0^{-1} \leq T < 1
\end{array} \right. 
\]  

(23)

and zero elsewhere. From this result we may calculate the expectation of the random variable \( T \):

\[
E[T] = \int_{s_1^{-1}}^{s_0^{-1}} T \frac{1}{\alpha T} \int_{T^{-1}}^{s_1} \frac{d\theta}{T \log \theta} dT + \int_{s_0^{-1}}^{1} T \frac{1}{\alpha T} \int_{s_0}^{s_1} \frac{d\theta}{T \log \theta} dT.
\]  

(24)
Figure 2: Contours of $T$ in the two cases: (a) $s_0 < T^{-1} < s_1$, (b) $T^{-1} > s_1$.

Changing the order of integration of the first integral gives

$$E[T] = \frac{1}{\alpha} \left( \int_{s_0}^{s_1} \int_{\theta}^{\theta^{-1}} \frac{1}{\log \theta} dT d\theta + \int_{s_0^{-1}}^{1} \int_{s_0}^{s_1} \frac{d\theta}{\log \theta} dT \right) \quad (25)$$

and so

$$E[T] = \frac{1}{\alpha} \left( \int_{s_0}^{s_1} \left( s_0^{-1} - \theta^{-1} \right) \frac{d\theta}{\log \theta} + \left( 1 - s_0^{-1} \right) \int_{s_0}^{s_1} \frac{d\theta}{\log \theta} \right)$$

$$= \frac{1}{\alpha} \left( \int_{s_0}^{s_1} \frac{d\theta}{\log \theta} - \int_{s_0}^{s_1} \frac{d\theta}{\theta \log \theta} \right). \quad (26)$$

Therefore

$$E[T] = \frac{1}{\alpha} (\text{Ei}[\log s_1] - \text{Ei}[\log s_0] - \log | \log s_1 | + \log | \log s_0 |)$$

$$= \frac{1}{\alpha} (G(s_1) - G(s_0)) \quad (27)$$

as required.

**Case 2:** $s_0 < s_1 < 1$. We refer to the diagram in Fig. 2. Again we calculate $\Pr(\theta^\beta \leq T)$ for $s_0 < T^{-1} < s_1$. Since $\theta^\beta \leq T$ if and only if $\theta \geq T^{1/\beta}$, the relevant area is the shaded region in Fig. 2.
For \( T^{-1} \leq s_0 \), this area is equal to 1, and, for \( s_0 < T^{-1} < s_1 \), the relevant area is
\[
1 - \frac{1}{\alpha} \int_{s_0}^{T^{-1}} \left( \frac{\log T}{\log \theta} + 1 \right) d\theta,
\]
and, for \( s_1 \leq T^{-1} < 1 \), the shaded region has area
\[
\frac{1}{\alpha} \int_{s_0}^{s_1} \left( -\frac{\log T}{\log \theta} \right) d\theta,
\]
while, for \( T^{-1} \geq 1 \), the relevant area is zero. Again differentiating with respect to \( T \) using Leibniz's rule gives the probability density function
\[
f_T(T) = -\frac{1}{\alpha} \left( \int_{s_0}^{T^{-1}} \frac{d\theta}{T \log \theta} \right) s_1^{-1} < T < s_0^{-1},
\]
and zero for all other values of \( T \). Expressing these values in terms of \( \text{Ei} \) gives the p.d.f.
\[
f_T(T) = \frac{1}{\alpha T} \left\{ \begin{array}{ll}
\text{Ei} [\log s_0] - \text{Ei} [\log T^{-1}] & s_1^{-1} < T < s_0^{-1} \\
\text{Ei} [\log s_0] - \text{Ei} [\log s_1] & 1 < T \leq s_1^{-1}.
\end{array} \right.
\]

We now calculate the expectation \( E[T] \) as
\[
E[T] = -\frac{1}{\alpha} \left( \int_{s_1^{-1}}^{s_0^{-1}} \int_{s_0}^{T^{-1}} \frac{d\theta}{\log \theta} dT + \int_{1}^{s_1^{-1}} \int_{s_0}^{s_1} \frac{d\theta}{\log \theta} dT \right).
\]

A calculation similar to the one above gives
\[
E[T] = -\frac{1}{\alpha} \left( \int_{s_0}^{s_1} \int_{s_0}^{s_1^{-1}} \frac{d\theta}{\log \theta} dT + \int_{s_0}^{s_1} \int_{s_0}^{s_0^{-1}} \frac{d\theta}{\log \theta} dT \right)
\]
\[
= -\frac{1}{\alpha} \left( \int_{s_0}^{s_1} (\theta^{-1} - s_1^{-1}) \frac{d\theta}{\log \theta} + (s_1^{-1} - 1) \int_{s_0}^{s_1} \frac{d\theta}{\log \theta} \right)
\]
\[
= -\frac{1}{\alpha} \left( \log |\log s_1| - \log |\log s_0| - \text{Ei} [\log s_1] + \text{Ei} [\log s_0] \right)
\]
\[
= \frac{1}{\alpha} \left( G(s_1) - G(s_0) \right),
\]
as required.

The cases when one of \( s_0, s_1 = 1 \) may be handled taking an appropriate limit. Then the relation \( G(1) = \gamma \) gives, for the case \( 1 < s_0 < s_1 \),
\[
E[T] = \frac{1}{\alpha} [G(s_1) - G(s_0)] = \frac{1}{\alpha} \left( \text{Ei} [\log s_1] - \log |\log s_1| - \gamma \right),
\]
(34)
and, for the case \( s_0 < s_1 = 1 \),

\[
E[T] = \frac{1}{\alpha} [G(s_1) - G(s_0)] = \frac{1}{\alpha} \left( \log |\log s_0| - \text{Ei}[\log s_0] + \gamma \right).
\] (35)

Finally, the case where \( s_0 < 1 < s_1 \) can be deduced from the other cases, by noting that the distribution \( U(s_0, s_1) \) may be written as a weighted average of the distributions \( U(s_0, 1) \) and \( U(1, s_1) \) as follows: \( U(s_0, s_1) = ((1 - s_0)U(s_0, 1) + (s_1 - 1)U(1, s_1))/\alpha \) so that

\[
E[T] = \frac{1}{\alpha} \left[ (1 - s_0) \frac{1}{1 - s_0} [G(1) - G(s_0)] + (s_1 - 1) \frac{1}{s_1 - 1} [G(s_1) - G(1)] \right]
\] (36)

\[
= \frac{1}{\alpha} [G(s_1) - G(s_0)],
\] (37)

as required. This completes the proof of Theorem 3.1.

### 4 Compound distributions and the distribution of \( \bar{\nu}' \) in normal growth regimes

In this section we derive a method to measure to what extent an observed value of \( \bar{\nu}' \) at a particular time indicates superexponential behaviour in the asset price returns at that moment. In order to achieve this, we have considered how values of \( \bar{\nu}' \) should be distributed if the market in question is governed by a normal growth regime. We make the assumption that financial markets have only two governing regimes, normal and superexponential. If one has an expectation of how \( \bar{\nu}' \) should be distributed at a particular point in time, then, given an observed value \( \bar{\nu}' \), it will be possible to determine the likelihood that the market in question is being governed by a normal growth regime. Following our assumption that the alternative can only be a superexponential governing regime, the likelihood of this driving the market is therefore immediately also known.

In what follows, we find that if one is able to make a reasonable approximation of the historical asset price return variance \( \sigma^2 \) as an inverse gamma distribution, then the parameters of this approximate distribution will imply the parameters of a location-scale Student’s t-distribution which should be representative of the distribution of \( \bar{\nu}' \) at a particular moment in time. Note that other distributions for \( \sigma^2 \) are possible; see Afuecheta et al. (2020) for alternative possible models of the variance \( \sigma^2 \) used in the financial modelling literature. However, fitting an inverse gamma distributions has proved a satisfactory approximation for our purposes.

Recall that the proxy parameter \( \bar{\nu}' \) is given by

\[
\bar{\nu}' = -\frac{1}{\Delta t} \frac{\text{Cov}(\bar{t}, r)}{\text{Var}(\bar{t})}
\] (38)
where $\hat{t} = \left( E[\hat{T}_1], E[\hat{T}_2], \ldots, E[\hat{T}_n] \right)$ (obtained from Theorem 3.1 above) and $r = (r_1, \ldots, r_n)$, and where

$$\text{Cov} \left( \hat{t}, r \right) = \frac{1}{n} \sum_{i=1}^{n} (\hat{t}_i - \bar{\hat{t}}) (r_i - \bar{r}), \quad \text{Var} \left( \hat{t} \right) = \frac{1}{n} \sum_{i=1}^{n} (t_i - \bar{\hat{t}})^2, \quad (39)$$

and $\bar{\hat{t}} = \frac{1}{n} \sum_{i=1}^{n} \hat{t}_i$ and $\bar{r} = \frac{1}{n} \sum_{i=1}^{n} r_i$. We model the i.i.d. $r_i \sim N(\mu, \sigma^2)$ so that $\bar{r}$ may be approximated by $\mu$ and so

$$\bar{\nu'} = -\frac{1}{\Delta t \text{Var} \left( \hat{t} \right)} \frac{1}{n} \sum_{i=1}^{n} (\hat{t}_i - \bar{\hat{t}}) \sigma z_i \quad (40)$$

where $z_1, \ldots, z_n$ are i.i.d. $N(0,1)$ variates. Thus, using standard properties of normal distributions,

$$\bar{\nu'} \sim N \left( 0, \frac{\sigma^2}{\frac{1}{n\Delta t^2 \text{Var} \left( \hat{t} \right)}} \right) = N(0, \lambda \sigma^2) \quad (41)$$

where $\lambda = \left( n \Delta t^2 \text{Var} \left( \hat{t} \right) \right)^{-1/2}$, a constant.

Now we assume that $\sigma^2$ is a random variable, with a distribution that approximates the historical volatility in the returns for the asset in question. If $\sigma^2$ is drawn from an inverse gamma distribution with parameters $\alpha$ and $\beta$, then $\lambda \sigma^2$ is also drawn from an inverse gamma distribution with parameters $a = \alpha$ and $b = \lambda \beta$.

From the theory of compound distributions, fitting an inverse gamma distribution for $\lambda \sigma^2 \sim \Gamma^{-1}(a, b)$, compounded with the distribution $\nu' \sim N(0, \lambda \sigma^2)$ results in the compound location-scale Student’s t-distribution $\tilde{\nu}' \sim t_{2a}(0, b/a)$ (Jackman 2009). Recall that a random variable $X$ drawn from the location-scale Student’s t-distribution, $t_{\nu}(\hat{\mu}, \hat{\sigma}^2)$, is derived by setting $X = \hat{\mu} + \hat{\sigma} Y$, where $Y$ is a random variable drawn from a standard Student’s t-distribution with $\nu$ degrees of freedom, and where $\hat{\mu}$ and $\hat{\sigma}$ are location and scale parameters respectively.

Fitting a distribution to the log-return variances can be a computationally expensive exercise, which, for the inverse gamma distribution, at least, can be significantly reduced using approximations such as described by Llera & Beckmann (2016); this enhanced speed of fitting increases the practical application of the methods considered here and opens up the possibility of real-time market analysis, including intra-day markets.

### 5 Practical application of the bubble time theory

A hypothesis underlying this paper is that, during a superexponential growth regime, returns on long positions are driven higher due to participants requiring a better return for what is assumed to be increasing risk of the superexponential period ending
and asset prices reverting to levels that are based more on the fundamental value of the asset in question. Conversely, during a superexponential decline, returns on short positions are driven higher due to the expectation of asset price reversion to higher fundamental levels. With this in mind, a methodology has been arrived at which aims to give a measure of the extent that a market is in a superexponential regime.

The methodology assumes the availability of a time series of $n$ data points of market data comprising of times, $t_i$, and log returns, $r_i$, where $r_i = \log(S_i) - \log(S_{i-1})$, $S$ are (closing) prices, and $i = 1, 2, \ldots, n$.

As described above, a decision must be made regarding the amount of historical data that is used to determine the value of $\bar{\nu}'$ calculated at time $t_i$, denoted by $\bar{\nu}'_i$. For the purposes of this paper, this value is, somewhat arbitrarily, set at $\omega = 50$ data points. We note here that this time window would be optimally placed at the unknown beginning of the current governing regime rather than at an arbitrarily chosen point. However, we leave development of adaptive processes to optimise this time window for future work, and direct the reader to Demos & Sornette (2019, 2017) for presentations of such an adaptive methodology in the context of determining the start of financial bubbles. In order to make use of the bubble-time theory discussed above, one must also make an assumption about the bounds of the critical time, $t_c$, beyond the data point in question. If the $i$th data point is being analysed, then we assume that the lower bound resides at the data point at $t_{i+1}$. Furthermore, we make the arbitrary assumption that the upper bound of $t_c$ is 50 time periods further in the future. Therefore, for each of the $j$ data points up to and including the $i$th data point, $s_{j,0} = t_{i+1} - t_j$ and $s_{j,1} = s_{j,0} + \alpha$ where $\alpha$ is set at 50 time periods.

Now, $\bar{\nu}'_i$ may be calculated for each $i$th data point by taking the returns data $r_i = r_{i-\omega}, \ldots, r_i$ and transforming the times $t_i = t_{i-\omega}, \ldots, t_i$ using Theorem 3.1 with $s_{j,0} = t_{i+1} - t_j$ where $j = i - \omega, \ldots, i$ and $s_{j,1} = s_{j,0} + \alpha$, where $\alpha$ is 50 time periods. This gives the vector of time transformed expectations, $\hat{t}_i$. Then, it is a simple matter to calculate $\bar{\nu}'_i = -\Delta t \text{Cov}(\hat{t}_i, r_i) / \text{Var}(\hat{t}_i)$.

To determine the trading signal at each data point, it is necessary to construct a location-scale Student’s $t$-distribution as described in §4. To facilitate this, a table of daily variances for every $i$th data point is calculated. However, the amount of data that should be used to calculate this daily variance is not known. To mitigate this unknown we take the position that for each data point a number of daily variances should be calculated over randomly chosen time periods ending on the data point in question. The methodology used in this paper determines log return daily variances over 20 randomly chosen time periods from the start of the data set until 20 days prior to the $i$th data point in question, i.e. there will ultimately be 20 observed daily variances for each data point in the data set. Therefore, when the $i$th data point is being considered, all historical daily variances calculated up until $t_i$ are taken into account and an inverse gamma distribution is fitted to these observed variances scaled
by \( \lambda \), as shown in (41), where
\[
\lambda = \frac{1}{\sqrt{i \Delta_t^2 \text{Var}(\hat{t}_i)}} \quad (42)
\]

By using the numerical approximation described in §4, we find the inverse gamma distribution’s parameters, \( a \) and \( b \), at each \( i \)th data point. The found parameters determine the parameters \( \hat{\nu}, \hat{\mu}, \hat{\sigma} \) of the location-scale Student’s t-distribution by setting
\[
\hat{\nu} = 2a, \quad \hat{\mu} = 0, \quad \hat{\sigma}^2 = \frac{b}{a}. \quad (43)
\]

Finally, we calculate \( \psi_i = F(\hat{\nu}_i') \) where \( F(x) \) is the cumulative distribution function of the location-scale Student’s t-distribution function with parameters previously calculated for each \( i \)th data point.

6 Results

By following the method in §5, a signal, \( \psi \in (0, 1) \), where \( \psi = F(\hat{\nu}') \), is given for each trading period, which for the results in this section we take to be a day, expressed as a decimal part of a year. Since we know that superexponential growth should occur when the parameter \( \hat{\nu}' < 0 \) \(^1\), the smaller the value of \( \psi \) the more likely it is that the market is in a bubble regime exhibiting superexponential growth, and conversely, the larger the value of \( \psi \), the more likely it is that the market is in a bubble regime, but experiencing superexponential decline. Between the extremes, the market is likely to be in a normal growth regime typified by geometric Brownian motion. In our analysis we assume that these are the only possible situations. In order to separate out more clearly the cases of superexponential growth and decline, we introduce the following subsidiary signals:
\[
\hat{\psi}_+ = \begin{cases} 
1 - 2\psi & \hat{\nu}' \leq 0 \\
0 & \hat{\nu}' > 0
\end{cases}, \quad \hat{\psi}_- = \begin{cases} 
0 & \hat{\nu}' \leq 0 \\
1 - 2\psi & \hat{\nu}' > 0
\end{cases} \quad (44)
\]

so that as \( \hat{\psi}_+ \) becomes more positive, the higher the probability that the market is in a bubble regime experiencing exponential growth, and as \( \hat{\psi}_- \) becomes more positive, the higher the probability that the market is in a bubble regime experiencing exponential declines.

A fundamental question is whether one should expect a growth bubble regime given by high values of \( \hat{\psi}_+ \) to continue to increase, or one should expect the market to fall in the manner of a market crash. Similarly, do high values of \( \hat{\psi}_- \) lead to an expectation of further declines, or do they indicate the end of a bearish market? To

\(^1\)this is reasonable since the absolute value of \( \nu \) is related to the average crash amplitude
answer these questions, we take an empirical approach and measure the relative information contained within these signals against comparable Monte Carlo simulations.

To demonstrate the applicability of the results, we have considered stock prices of the constituent members of the Nasdaq-100 index with a homogenous choice of time spans and parametrisations. We have investigated two approaches: first, an investigation into the performance of the signals on an individual stock basis, and, second, by considering the signals as they are applied to a portfolio of investments.

6.1 Individual stocks

Let us assume that there is a hypothetical equity trader who is generally willing to risk at most one currency unit on purchasing or selling an asset at the closing price each trading day and reverting the position at the closing price on the following day. In this case, the hypothetical trader should enter into a long or short position in an amount of currency unit given by one of the signals, $\hat{\psi}_+$ or $\hat{\psi}_-$, as calculated on this particular day. Adopting here (and in what follows) the notation, $\hat{\psi}_{i, \pm}$ to denote either $\hat{\psi}_+$ or $\hat{\psi}_-$ at time $i$, it is straightforward to calculate the average daily relative return for the trader over a given period by

$$\bar{r}_\pm = \frac{1}{n} \sum_{i=1}^{n} \frac{S_{i+1} - S_i}{S_i}$$

where $n$ is the number of trading days in the data set being analysed, and $S_i$ is the closing price on the $i$th trading day. It should be noted that we make the assumption that it is possible for the hypothetical trader to be able to enter into frictionless (i.e. costless) trading at the closing price on any day.

Fig. 3 shows this method as applied to the US technology company, Broadcom Inc., over the 2000 trading days prior to 29th July 2022. Visual inspection of the results show that periods of growth are clearly indicated by strongly positive values of $\hat{\psi}_+$, and periods in which the stock price falls are correlated to high values of $\hat{\psi}_-$. However, by looking at the relative price movement between $t_i$ and $t_{i+1}$ for these periods of high signal value, it is not immediately apparent that either of the signals $\hat{\psi}_+$ or $\hat{\psi}_-$ are actually correlated with higher daily profitability.

While a market is signalled to be in a bubble regime experiencing superexponential growth, one might expect, in general, that the prices are rising and, as such, a rational trader should buy the asset. Similarly, one might expect that, when it is signalled that the market is in superexponential decline, market prices should move lower, and the rational trader should sell the asset on these signals. However, rather counterintuitively, we find that in the case of Broadcom Inc. the mean daily return

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\footnote{Although this simple strategy is straightforward to analyse, it does not of course allow the trader to hold on to a position nor vary the amount invested. It does provide however a fair evaluation of usefulness the signal itself, which might be obscured by more sophisticated trading algorithms.}
Figure 3: Broadcom Inc. results. The plots at the top show Broadcom Inc. stock prices between 2016 and 2022. The plot on the left shows a colour representation of $\hat{\psi}_+$ where red represents zero and shades of green represent the value of $\hat{\psi}_+$, as shown in the plot directly below. Similarly, the plots to the right show the same visualisation for $\hat{\psi}_-$. Visual inspection suggests that $\hat{\psi}_+$ describes periods of growth well, and $\hat{\psi}_-$ describes decline. However, when examining the plots of daily returns (bottom plots) with similar colour encoding, it is not apparent that either signal is correlated with positive or negative returns.
for long positions (given by $\hat{\psi}_+$) is $\bar{r}_+ = 0.009\%$ and for long positions (given by $\hat{\psi}_-$) is $\bar{r}_- = 0.068\%$, an order of magnitude greater.

When this profitability measure is applied to long positions in the constituent stocks of Nasdaq-100 index\(^3\), of which Broadcom Inc. is a member, we find that, in general, values of $\hat{\psi}_-$ are more highly correlated to greater returns between $t_i$ and $t_{i+1}$ (see Fig. 4), where just over 65% of the 74 members analysed\(^4\) had higher returns for long positions given by the $\hat{\psi}_-$ signal rather than the $\hat{\psi}_+$ signal.

To test whether any information is contained within these signals, we compare results of two simple trading strategies. We refer to these strategies as the “momentum” strategy, where non-zero values of $\hat{\psi}_{i,+}$ signals are presumed to be buy signals, and non-zero values of $\hat{\psi}_{i,-}$ are presumed to be sell signals. Conversely, the “reversion” strategy is the opposite of the momentum strategy. Therefore, mean daily returns for

\(^3\)The Nasdaq-100 is a US stock market index made up of equity securities issued by 101 of the largest non-financial companies listed on the Nasdaq stock exchange.

\(^4\)In the study, only those stocks which had at least 2,000 trading days’ of data were considered.
the momentum strategy are calculated by

\[
\bar{r}_M = \frac{1}{n} \sum_{i=1}^{n} \left( \hat{\psi}_{i,+} - \hat{\psi}_{i,-} \right) \frac{S_{i+1} - S_i}{S_i},
\]

and for the reversion strategy

\[
\bar{r}_R = \frac{1}{n} \sum_{i=1}^{n} \left( \hat{\psi}_{i,-} - \hat{\psi}_{i,+} \right) \frac{S_{i+1} - S_i}{S_i}.
\]

We compare \(\bar{r}_M\) and \(\bar{r}_R\) to the returns expected from 5000 Monte Carlo simulations of \(\hat{\psi}_{i,\pm}\) for the stock under investigation. The empirical distribution of the average return, \(R\), is calculated for each simulation by

\[
R = \frac{1}{n} \sum_{i=1}^{n} \hat{\Psi}_i \frac{S_{i+1} - S_i}{S_i}
\]

where \(\hat{\Psi}_i \sim U(-1, 1)\), independently for each \(i\) and each simulation and \(n = 1000\).

We obtain an empirical cumulative distribution function, \(F\), of the random variable \(R\) and calculate \(\bar{r}_M\) and \(\bar{r}_R\) from (46) and (47) respectively for the stock under analysis. Applying this to 74 members of the Nasdaq-100 index described above, we find that the median of \(F(\bar{r}_M) = 0.195\), whereas for the median of \(F(\bar{r}_R) = 0.805\). These results suggest that, while both \(\hat{\psi}_+\) and \(\hat{\psi}_-\) contain information, the signal \(\hat{\psi}_-\) is generally a better predictor of upward market movements, and \(\hat{\psi}_+\) is a better predictor of downward movements, and as such the signals can be used more successfully in reversion strategies. The distribution of \(F(\bar{r}_R)\) is shown in Fig. 5.

The signals have some degree of success in generating significant mean daily returns. However, we have found that the \(\hat{\psi}_-\) signal is more successful in generating significant returns than the \(\hat{\psi}_+\) signal when considering long positions. We suggest that this is indicative of it being more likely that the \(\hat{\psi}_+\) signal is giving warning of the imminent end of a period of superexponential growth rather than indicating that superexponential growth should be continuing. Furthermore, we find evidence that the \(\hat{\psi}_-\) signal gives an indication that a period of exponential decline is coming to an end, and the market is about to turn.

For this approach to be useful in devising real-world trading strategies, one must be in a position to decide in advance which stocks will result in significant returns. Clearly, this is not necessarily possible. To remove the need to make a prior judgement regarding which stocks should be considered as suitable candidates, we have investigated a method to apply the signals to a fixed portfolio of stocks.

## 6.2 Portfolio approach

Let us again consider our hypothetical trader but now, rather than investing one currency unit in a particular stock, instead they allocate that one currency unit to a
Figure 5: Nasdaq-100 reversion-strategy returns. For each stock which is a member of the index, an empirical distribution, $F$, is found of the simulated returns, $R$, as calculated by (48). The plot shows the values of $F(\bar{r}_R)$ where $\bar{r}_R$ is the mean daily return achieved as a result of following a reversion strategy. The median (dashed line) of this distribution is $0.805$, which suggests that more than half of the stocks investigated would have achieved returns much greater than would be expected by chance by following the reversion strategy. Since the momentum strategy returns are the negative of the reversion strategy, the distribution of $F(\bar{r}_M)$ is the mirror image of this plot.
portfolio of stocks drawn from the Nasdaq-100 index. In order to make the strategy market neutral in nature, i.e. the strategy is equally effective in both rising and falling markets, the trader decides to allocate one currency unit in purchasing certain stocks, and generates proceeds of one currency unit by selling short other stocks. It is assumed that the trader can implement this strategy efficiently (i.e., we assume that the trader can transact at the same instant, and at the same price upon which the analysis is performed).

Suppose there are \( m \) stocks in the index. Each day, just prior to the market close, the trader follows the algorithmic steps as laid out in §5, and further calculates the signals \( \hat{\psi}_{i,+}^j \) and \( \hat{\psi}_{i,-}^j \) for the \( j \)th stock at time \( t_i \). The signals provide the trader with weights \( \phi_{i,+}^j \) and \( \phi_{i,-}^j \) at time \( t_i \) for the \( j \)th stock given by

\[
\phi_{i,\pm}^j = \frac{\hat{\psi}_{i,\pm}^j}{\sum_{j=1}^{m} \hat{\psi}_{i,\pm}^j} .
\]  

(49)

The trader executes a “market-on-close” purchase order, i.e., a market order that is scheduled to trade at the close, at the most recent trading price, for all the stocks in the index weighted by \( \phi_{i,-}^j \) and a similar market-on-close sell order for all stocks in the index weighted by \( \phi_{i,+}^j \). One can see that by doing so the trader is following the reversion strategy discussed in the previous section. Therefore the daily mean profitability, \( \bar{r} \), of this strategy is given by

\[
\bar{r} = \frac{1}{mn} \sum_{j=1}^{m} \left( \sum_{i=1}^{n} \phi_{i,-}^j \left( \frac{S_{i+1}^j}{S_i^j} - 1 - \tau \right) + \sum_{i=1}^{n} \phi_{i,+}^j \left( 1 - \frac{S_{i+1}^j}{S_i^j} - \tau \right) \right)
\]  

(50)

where \( n \) is the number of trading days in the test data, and \( \tau \) is a estimate of the trading costs for transaction. Here \( S_i^j \) is price of the \( j \)th stock at time \( t_i \). For the purposes of this paper we assume \( \tau \) is zero.

When we consider the 74 stocks of the Nasdaq-100 index that have had sufficient trading data, we have found that daily mean return \( \bar{r} = 0.041\% \) and the development of cumulative returns over the six-year period as shown in Fig. 6. Had this strategy been followed in practice, ignoring any trading costs, it would have resulted in an annualised Sharpe ratio of 1.18 compared to a ratio of 1.03 that would be achieved by being long 74 members of the index over this test period.\(^5\) For this calculation, the risk free rate of return is taken to be zero.

7 Conclusion and further work

In the analysis of returns in financial assets, the theoretical and the practical go hand in hand. Any practical algorithm without some theoretical underpinning lacks legi-

\(^5\)See Sharpe (1994) for more information on the Sharpe ratio
Figure 6: Nasdaq-100 index market-neutral strategy returns – the percentage daily gain or loss of the hypothetical trader following the strategy described in §6.2 accumulates each day, or trading period, if the trader “reinvests” the total daily proceeds of the strategy. Ignoring trading costs, this leads to cumulative gain/losses at time $t_i$ given by $\Pi_{j=1}^i (1 + r_j)$, where $j = 1, \ldots, i$. This is shown as the solid line and corresponds to a six-fold increase in the trader’s wealth over the period between 2016 – 2022, an annual return in excess of 17%.
imacy, while a theoretical model which cannot be readily implemented is necessarily of limited interest.

The quantity \( \bar{\nu}' \) introduced in this paper as a proxy for the parameter \( \nu' \) discussed in Lynch & Mestel (2019) sits in both camps, being simultaneously of theoretical interest and practical applicability. As seen in §3 – §5, \( \bar{\nu}' \) has theoretical foundation and it is also fast to compute, even for real-time analysis. Moreover, as shown in §6 the proxy \( \bar{\nu}' \) can be successfully applied to analyse an important market, and indeed might form the basis of an automated trading algorithm, as explained in §6.2.

The algorithm discussed briefly in §6 is only one possible approach to devising a trading algorithms based on \( \bar{\nu}' \), and an important avenue of future research is to determine not only which markets \( \bar{\nu}' \) is particularly suited to, but also to examine several different automatic trading algorithms based on \( \bar{\nu}' \) to determine which is the most effective.

There are also theoretical aspects of \( \nu' \), and hence of \( \bar{\nu}' \), which need further investigation. The explicit formulae for \( \nu' \) and \( \bar{\nu}' \) given in Lynch & Mestel (2019) and this current paper are dependent on an assumption of constant volatility of returns in respect of determining values of \( \bar{\nu}' \) by analytical methods (as opposed to the variance table which is used to estimate the distribution of \( \bar{\nu}' \) when assuming a normal governing regime). Non constant volatility leads to implicit equations that do not appear to be solvable in closed form. Nevertheless, algorithms based on the implicit numerical solution of these equations can be devised and it remains to be seen whether they provide more accuracy in the distinguishing of exponential and superexponential behaviour.

Let us now turn to the trading strategies that we applied to stocks in the Nasdaq-100 index. In order to generate trading signals we have made the assumption that the market is in either one of two states, namely, normal or superexponential (whether growth or decline,) and that, given an inverse gamma estimation of the distribution of historical variances, we are able to derive a location-scale Student’s t-distribution for values of \( \bar{\nu}' \) that would be expected if the market were to be in a normal regime. This has allowed us to generate trading signals that tell us something about the likelihood that the market is not in a normal growth regime. We have further shown that, at least for the stocks investigated, the signals we derive have an interesting predictive quality in respect of the impending end of a superexponential regime. Clearly, there is much work to do to understand the reasons behind the results presented.

Nevertheless, putting aside the assumptions made in our hypothetical trading strategies, the results obtained over a long period, during which there have been exogenous shocks to the market such as the COVID-19 pandemic, and spanning a wide cross-section of stocks, are, in our opinion, notable, and deserve further investigation. Indeed, if these signals, or others derived from the theory presented here, can be shown to be reasonably widely applicable in the information they carry about the state and future direction of markets, then maybe (at least for a short while) advantageous trading opportunities may exist.
8 Acknowledgement

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References


