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How to cite:

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Version: Poster

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A computer-assisted proof of dynamo growth in the stretch-fold-shear map

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• Our research work is based on a functional linear operator called the Stretch-Fold-Shear (SFS) operator $S$ which arises from a model of dynamo growth.
• A dynamo is the source of the magnetic field of a celestial body such as the Earth. Earth’s magnetic field is generated by a self-exciting dynamo or geo-dynamo.

Introduction

• Existing magnetic field in the inner core + fluid iron in the outer core + electrical current
• Electrical current + Earth’s movement = new magnetic field.

The SFS Operator

• $F = \{c(x)\}$ complex-valued function of a real variable $x \in [-1,1])$
• $S: F \rightarrow F$
• $S c(x) = e^{\frac{ia(x-1)}{2}} c \left(\frac{x-1}{2}\right) - e^{-\frac{ia(x+1)}{2}} c \left(\frac{x+1}{2}\right)$
• $\alpha \geq 0$ (real parameter) is the shear parameter.

When $\alpha = 0$

• We have derived a generating function to generate the eigenvalues and eigenfunctions for $\alpha = 0$:
$$G(x, t) = \sum_{n=0}^{\infty} c_n(x) e^{\frac{t}{m}} = \frac{2i(x-e^{-\frac{t}{2}})}{e^{\frac{t}{2}} - e^{-\frac{t}{2}}}.$$  
• This generating function gives all the eigenfunctions $c_n(x)$ as the coefficients of $e^{\frac{t}{m}}$ with eigenvalue $\lambda_n = \frac{\alpha}{2} x$.

Using Computer-assisted Proof

• For $\alpha > 0$, it may not possible to determine the eigenvalues and eigenfunctions of $S$ exactly.
• This generating function method and perturbation method were tried out to obtain the eigenvalues and eigenfunctions. These have not been successful.
• Finally a computer-assisted proof is used to handle the case $\alpha > 0$.

Mathematical Tools

• Interval arithmetic
• Function ball
• Julia Programming Language

When $\alpha > 0$

• At first the eigenvalue-eigenfunction pairs of the SFS operator have been computed for $\alpha = 0$.
• Using these values, numerical approximations of eigenvalue-eigenfunction pairs have been calculated for dynamically increasing $\alpha$.
• Finally a computer-assisted proof is used to find rigorous bounds around a closed disc function ball of each approximation by implementing interval arithmetic in Julia.

Methodology

Theorem

Let $\alpha \in [0,5]$. Then there exists an eigenvalue-eigenfunction pair $(\lambda, c(x))$ of the operator $S$ satisfying the following:
1. For $\alpha \in [0, \pi/2]$, $|\lambda| < 1$.
2. For $\alpha \in (\pi/2, 5]$, $|\lambda| > 1$.
3. For $\alpha = \pi/2$, $\lambda = -i$.

*It is not claimed that for all $\lambda$, (1) and (2) are true.

Result

The graph (reproducing Andrew Gilbert’s graph) shows that $S$ has an eigenvalue of modulus greater than 1 for all $\alpha$ satisfying $\alpha > \pi/2$.

$k = 1, 2, 3, \ldots$ represent eigenvalues (modulus) corresponding to degree-(2$k + 1$) eigenfunctions.

Graphical Investigation

Conclusion

• To guarantee dynamo growth, we need to prove the existence of an eigenvalue of modulus $> 1$.
• We used a computer-assisted proof to find rigorous bounds on the leading eigenvalue ($k=1$) for $\alpha \in [0,5]$, showing that $S$ has an eigenvalue of modulus greater than 1 for all $\alpha$ satisfying $\pi/2 < \alpha \leq 5$ (above theorem), thereby proving dynamo growth.
• The methods can be readily adapted to other $k = 2, 3 \ldots$ etc, and for larger values of $\alpha$.
• It is notable that, although the eigenvalue-eigenfunction has been approximated numerically, the computer-assisted results found in our work are the first rigorous results to be obtained.

Literature Cited