Discrete tomography of Penrose model sets

Journal Item

DISCRETE TOMOGRAPHY OF PENROSE MODEL SETS

MICHAEL BAAKE AND CHRISTIAN HUCK

Abstract. Various theoretical and algorithmic aspects of inverse problems in discrete tomography of planar Penrose model sets are discussed. These are motivated by the demand of materials science for the reconstruction of quasicrystalline structures from a small number of images produced by quantitative high resolution transmission electron microscopy.

1. Introduction

Discrete tomography is concerned with the inverse problem of retrieving information about some discrete object from (generally noisy) information about its incidences with certain query sets. A typical example is the *reconstruction* of a finite planar point set from its line sums in a small number of directions. More precisely, for a *direction* $u \in \mathbb{S}^1$ (the unit circle), the (*discrete parallel*) *X-ray* $X_u F$ of a finite subset F of the Euclidean plane $\mathbb{R}^2$ in direction u gives the number of points of the set on each line in $\mathbb{R}^2$ parallel to u, i.e., $X_u F$ is the function $X_u F : \mathcal{L}_u \longrightarrow \mathbb{N}_0 := \mathbb{N} \cup \{0\}$, defined by

$$X_u F(\ell) := \operatorname{card}(F \cap \ell\,) = \sum_{x \in \ell} \mathbb{1}_F(x)\,,$$

where $\mathcal{L}_u$ is the set of lines in direction u in $\mathbb{R}^2$, with obvious generalization to higher dimensions. In the classical setting, motivated by crystals, the positions to be determined form a subset of a translate of the square lattice $\mathbb{Z}^2$ or, more generally, of arbitrary lattices L in $\mathbb{R}^d$, where $d \geq 2$. The cases $d = 2$ and $d = 3$ are practically relevant. In fact, many of the problems in discrete tomography have been studied on $\mathbb{Z}^2$, the 'classical planar setting' of discrete tomography; see [1, 2, 3, 4].

In the longer run, by also having other structures than perfect crystals in mind, one has to take into account wider classes of sets, or at least significant deviations from the lattice structure. As an intermediate step between periodic and random (or amorphous) *Delone sets* (defined below), we consider systems of *aperiodic order*, more precisely, of so-called *model sets* (or *mathematical quasicrystals*), which are commonly accepted as a reasonable mathematical model for quasicrystalline structures in nature [5].

The main motivation for our interest in the discrete tomography of model sets comes from the question how to reconstruct three-dimensional (quasi)crystals, or planar layers of them, from their images under quantitative *high resolution transmission electron microscopy* (HRTEM) in a small number of directions. In fact, in [6, 7], a technique called QUANTITEM (quantitative analysis of the information coming from transmission electron microscopy) is described, based on HRTEM, which can effectively measure the number of atoms lying on lines parallel to certain directions. In particular, with the growing importance of surface science, there is also increasing interest in additional methods for the reconstruction of planar structures which can supplement STM approaches. At present, the measurement of the number of atoms lying on a line can only be achieved for some crystals; see [6, 7]. However, it is reasonable to expect that future developments in technology will improve this situation. It seems thus timely to investigate the mathematical foundations now.

Here, we restrict ourselves to an example, namely to the well-known class of planar model sets $\varLambda_{\mathrm{P}} \subset \mathbb{R}^2$ that are associated with the well-known *Penrose tiling*, and present some results on the discrete tomography of these sets, with an emphasis both on reconstruction and uniqueness problems. Note that proofs are omitted; details and extensions will appear in [8, 9].

2. Penrose Model Sets

We always let $\zeta_5 = e^{2\pi i/5}$, a primitive 5th root of unity in $\mathbb{C}$. Then, $\mathbb{Q}(\zeta_5)$ is the corresponding cyclotomic field, an algebraic number field of degree 4 over $\mathbb{Q}$, and $\mathbb{Z}[\zeta_5]$ is its subring of cyclotomic integers.

Remark 1. Let C_5 denote the cyclic group of order 5, i.e., $\mathsf{C}_5 = \mathbb{Z}/5\mathbb{Z}$. Moreover, C_5 is understood to be supplied with the discrete topology. It is well-known that every $z \in \mathbb{Z}[\zeta_5]$ can uniquely be written as $z = \sum_{j=0}^{3} a_j(z)\zeta_5^j$, with $a_j(z) \in \mathbb{Z}$. Let σ_2 be the automorphism of the Galois extension $\mathbb{Q}(\zeta_5)/\mathbb{Q}$ that is given by $\zeta_5 \longmapsto \zeta_5^2$. Identifying $\mathbb{R}^2$ and $\mathbb{C}$ in the canonical way, σ_2 gives rise to a map

$$.\tilde{} \;:\; \mathbb{Z}[\zeta_5] \longrightarrow \mathbb{R}^2 \times (\mathbb{R}^2 \times \mathsf{C}_5)\,,$$

defined by $z \longmapsto \Big(z, \big(\sigma_2(z), \sum_{j=0}^{3}(a_j(z) \ (\mathrm{mod}\, 5))\big)\Big)$. Via projection on the second factor, this induces a map $.^\star \;:\; \mathbb{Z}[\zeta_5] \longrightarrow \mathbb{R}^2 \times \mathsf{C}_5$, defined by $z \longmapsto \big(\sigma_2(z), \sum_{j=0}^{3}(a_j(z) \ (\mathrm{mod}\, 5))\big)$. Then, $\mathbb{Z}[\zeta_5]^{\sim}$ is a lattice in $\mathbb{R}^2 \times (\mathbb{R}^2 \times \mathsf{C}_5)$, i.e., a co-compact discrete subgroup. In fact, $\mathbb{Z}[\zeta_5]^{\sim}$ is the $\mathbb{Z}$-span of the set $\{1^{\sim}, (\zeta_5)^{\sim}, (\zeta_5^2)^{\sim}, (\zeta_5^3)^{\sim}\}$. Finally, note that $\mathbb{Z}[\zeta_5]^\star$ is dense in $\mathbb{R}^2 \times \mathsf{C}_5$; see [10].

It is well known by now that *model sets* arise from so-called *cut and project schemes*, compare [12, 13]. In particular, the class of *Penrose model sets* (PMS) arises from the following *cut and project scheme*; cf. [14]:

(1)

$$\begin{array}{ccccc}
\mathbb{R}^2 & \stackrel{\pi}{\longleftarrow} & \mathbb{R}^2 \times (\mathbb{R}^2 \times \mathsf{C}_5) & \stackrel{\pi_{\mathrm{int}}}{\longrightarrow} & \mathbb{R}^2 \times \mathsf{C}_5 \\
\cup \ \text{dense} & & \cup \ \text{lattice} & & \cup \ \text{dense} \\
\mathbb{Z}[\zeta_5] & \stackrel{1\text{–}1}{\longleftrightarrow} & \underbrace{\Big\{\Big(z = \sigma_1(z), \underbrace{\big(\sigma_2(z), \sum_{j=0}^{3}(a_j(z) \ (\mathrm{mod}\, 5))\big)}_{=z^\star}\Big) \mid z \in \mathbb{Z}[\zeta_5]\Big\}}_{=\mathbb{Z}[\zeta_5]^{\sim}} & \stackrel{1\text{–}1}{\longleftrightarrow} & \mathbb{Z}[\zeta_5]^\star
\end{array}$$

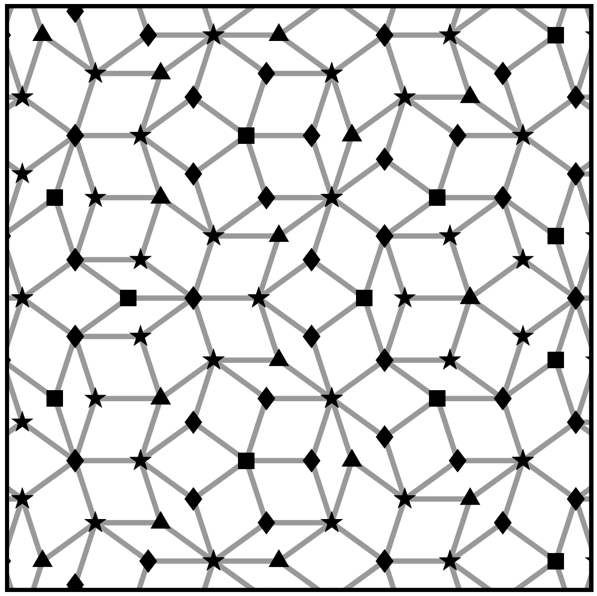

FIGURE 1. A patch of the fivefold symmetric Penrose tiling, with the four distinct classes of vertices indicated.

Given any subset $W \subset \mathbb{R}^2 \times \mathsf{C}_5$ with $\varnothing \neq W^\circ \subset W \subset \overline{W^\circ}$ and $\overline{W^\circ}$ compact, a so-called *window*, and any $t \in \mathbb{R}^2$, we obtain a planar model set $\varLambda(t, W) := t + \varLambda(W)$ relative to the above cut and project scheme (1) by setting $\varLambda(W) := \{z \in \mathbb{Z}[\zeta_5] \mid z^\star \in W\}$.

Let P be the convex hull of the 5th roots of unity, which is a regular pentagon centred at the origin. Set $W^{(1)} := P$, $W^{(2)} := -P$, $W^{(3)} := \tau P$ and $W^{(4)} := -\tau P$, with $\tau = (1+\sqrt{5})/2$ the golden ratio, and

$$W_P := \bigcup_{j=1}^{4} \Big(W^{(j)} \times \{j \ (\mathrm{mod}\, 5)\} \Big) \ \subset \ \mathbb{R}^2 \times \mathsf{C}_5 \, .$$

Moreover, for $u \in \mathbb{R}^2$, set $W_P^u := (u, 0 \ (\mathrm{mod}\, 5)) + W_P$, $(W_P^u)^{(j)} := (u, 0 \ (\mathrm{mod}\, 5)) + \big(W^{(j)} \times \{j \ (\mathrm{mod}\, 5)\}\big)$ and $\varLambda_{\mathrm{P}}^u := \varLambda(W_P^u)$. If $\varLambda_{\mathrm{P}}^u$ is *generic*, i.e., if one has $W_P^u \cap \mathbb{Z}[\zeta_5]^\star = \varnothing$, then all translates of $\varLambda_{\mathrm{P}}^u$, meaning the sets $t + \varLambda_{\mathrm{P}}^u$ with $t \in \mathbb{R}^2$, are called *Penrose model sets*. Note that this formulation avoids the usual ambiguities from non-minimal embeddings into 5-space.

Remark 2. $\varLambda_{\mathrm{P}}^0$ is not generic, while generic examples are obtained by shifting the window, i.e., $\varLambda_{\mathrm{P}}^u$ is generic for almost all $u \in \mathbb{R}^2$. Joining any two points with distance 1 in a generic $\varLambda_{\mathrm{P}}^u$ by edges results in a *Penrose tiling*, which is a tiling with two types of rhombi. See Figure 1 for a generic example; different generic choices of u result in locally indistinguishable (LI) Penrose tilings. Note that Penrose model sets $\varLambda_{\mathrm{P}} \subset \mathbb{R}^2$ are aperiodic, meaning that they have no translational symmetries. Further, Penrose model sets are *Delone sets*, i.e., they are uniformly discrete and relatively dense; cf. [12].

3. Problems in Discrete Tomography of Penrose Model Sets

Let $\varLambda_{\mathrm{P}}$ be a PMS, $k \in \mathbb{N}$ and $R > 0$. A finite subset C of $\varLambda_{\mathrm{P}}$ is called a *convex set in* $\varLambda_{\mathrm{P}}$ when its convex hull contains no new points of $\varLambda_{\mathrm{P}}$, i.e., when one has $C = \mathrm{conv}(C) \cap \varLambda_{\mathrm{P}}$. We denote by $\mathcal{F}(\varLambda_{\mathrm{P}})$, $\mathcal{F}_{\le k}(\varLambda_{\mathrm{P}})$, $\mathcal{D}_{<R}(\varLambda_{\mathrm{P}})$ and $\mathcal{C}(\varLambda_{\mathrm{P}})$ the set of finite subsets of $\varLambda_{\mathrm{P}}$, the set of finite subsets of $\varLambda_{\mathrm{P}}$ having cardinality $\le k$, the set of subsets of $\varLambda_{\mathrm{P}}$ with diameter less than R and the set of convex subsets of $\varLambda_{\mathrm{P}}$, respectively.

Remark 3. The uniform discreteness of Penrose model sets $\varLambda_{\mathrm{P}}$ immediately implies the inclusion $\mathcal{D}_{<R}(\varLambda_{\mathrm{P}}) \subset \mathcal{F}(\varLambda_{\mathrm{P}})$.

Clearly, in order to obtain electron microscopy images of high resolution, one should allow only directions which yield dense lines in Penrose model sets. These directions are clearly contained in the set of all directions, called $\varLambda_{\mathrm{P}}$-*directions*, which are parallel to a non-zero element of the difference set of $\varLambda_{\mathrm{P}}$, i.e.,

$$\varLambda_{\mathrm{P}} - \varLambda_{\mathrm{P}} := \{\lambda - \lambda' \mid \lambda, \lambda' \in \varLambda_{\mathrm{P}}\} \ \subset \ \mathbb{Z}[\zeta_5]\,.$$

Calling a direction $u \in \mathbb{S}^1$ a $\mathbb{Z}[\zeta_5]$-*direction* when it is parallel to an element of $\mathbb{Z}[\zeta_5] \setminus \{0\}$, one has the following result.

Proposition 1. *If $\varLambda_{\mathrm{P}}$ is a* PMS*, the set of $\varLambda_{\mathrm{P}}$-directions equals the set of $\mathbb{Z}[\zeta_5]$-directions.*

Let us indicate the main *algorithmic problems* in discrete tomography of Penrose model sets. For a direction $u \in \mathbb{S}^1$, we use $\mathcal{L}_u^{\mathbb{Z}[\zeta_5]}$ to denote the set of elements ℓ_u of $\mathcal{L}_u$ that pass through a point of $\mathbb{Z}[\zeta_5]$. Let $u_1, \dots, u_m \in \mathbb{S}^1$ be $m \ge 2$ pairwise non-parallel $\mathbb{Z}[\zeta_5]$-directions. The corresponding *consistency*, *reconstruction* and *uniqueness problems* are defined as follows.

> Consistency.
> Given functions $p_{u_i} : \mathcal{L}_{u_i} \longrightarrow \mathbb{N}_0$, $i \in \{1, \dots, m\}$, whose supports are finite and satisfy $\mathrm{supp}(p_{u_i}) \subset \mathcal{L}_{u_i}^{\mathbb{Z}[\zeta_5]}$, decide whether there is a finite set F which is contained in a PMS and satisfies $X_{u_i}F = p_{u_i}$, for all $i \in \{1, \dots, m\}$.
>
> Reconstruction.
> Given functions $p_{u_i} : \mathcal{L}_{u_i} \longrightarrow \mathbb{N}_0$, $i \in \{1, \dots, m\}$, whose supports are finite and satisfy $\mathrm{supp}(p_{u_i}) \subset \mathcal{L}_{u_i}^{\mathbb{Z}[\zeta_5]}$; in the case that Consistency is satisfied, construct a finite set F which is contained in a PMS and satisfies $X_{u_i}F = p_{u_i}$, for all $i \in \{1, \dots, m\}$.
>
> Uniqueness.
> Given a finite subset F of a PMS, decide whether there is a different finite set F' that is also a subset of a PMS and satisfies $X_{u_i}F = X_{u_i}F'$, for all $i \in \{1, \dots, m\}$.

In general, the above problem Reconstruction can have many solutions of rather different shape. Therefore, one is also interested in *uniqueness results*, e.g., the (unique) *determination* of the set

$$\bigcup_{\varLambda_{\mathrm{P}}\ \mathrm{PMS}} \mathcal{F}(\varLambda_{\mathrm{P}})\,,$$

or suitable subsets thereof by the X-rays in a small number of $\mathbb{Z}[\zeta_5]$-directions. More precisely, we define the concept of *determination* and the interactive concept of *successive determination* as follows. Let $\mathcal{E}$ be a collection of finite subsets of $\mathbb{R}^2$ and let $U \subset \mathbb{S}^1$ be a finite set of directions. We say that $\mathcal{E}$ is *determined* by the X-rays in the directions of U if, for all $F, F' \in \mathcal{E}$, one has

$$(X_u F = X_u F', \ \ \forall u \in U) \implies F = F' .$$

We say that $\mathcal{E}$ is *successively determined* by the X-rays in the directions of $U = \{u_1, \dots, u_m\}$, if, for a given $F \in \mathcal{E}$, these can be chosen inductively (i.e., the choice of u_j depends on all $X_{u_k} F$ with $k \in \{1, \dots, j-1\}$) such that, for all $F' \in \mathcal{E}$, one has

$$(X_u F' = X_u F, \ \ \forall u \in U) \implies F' = F .$$

We say that $\mathcal{E}$ is *determined* (resp., *successively determined*) by m X-rays if there is a set U of m pairwise non-parallel directions such that $\mathcal{E}$ is determined (resp., successively determined) by the X-rays in the directions of U.

4. Computational Complexity and Uniqueness Results

Let us begin with a result on computational complexity, where we apply the real RAM-model of computation, see [15]. Here, each of the standard elementary operations on reals counts only with unit cost. This leads to the following tractability result.

Theorem 1. *When restricted to two $\mathbb{Z}[\zeta_5]$-directions, the problems* Consistency, Reconstruction *and* Uniqueness *can be solved in polynomial time in the real RAM-model.*

Remark 4. It seems to be rather obvious from the results in [3] that one cannot expect a generalization of Theorem 1 to the case of three or more $\mathbb{Z}[\zeta_5]$-directions. More precisely, we expect that, when restricted to three or more $\mathbb{Z}[\zeta_5]$-directions, the problems Consistency, Reconstruction and Uniqueness are $\mathbb{NP}$-hard.

Let us now present results dealing with the (successive) determination of finite subsets of Penrose model sets. Though we are not interested in non-$\mathbb{Z}[\zeta_5]$-directions for practical reasons, we begin with the following observation.

Proposition 2. *If $\varLambda_{\rm P}$ is a* PMS *and $u \in \mathbb{S}^1$ is a non-$\mathbb{Z}[\zeta_5]$-direction, the class of finite subsets $\mathcal{F}(\varLambda_{\rm P})$ is determined by the single X-ray in direction u.*

This last result immediately follows from the observation that, for all PMS $\varLambda_{\rm P}$, each line in the plane in a non-$\mathbb{Z}[\zeta_5]$-direction passes through at most one point of $\varLambda_{\rm P}$, the latter being the reason for the practical irrelevance of this result. On the other hand, the next result shows that any fixed finite number of X-rays in $\mathbb{Z}[\zeta_5]$-directions does not suffice to determine the whole class of finite subsets of a fixed PMS $\varLambda_{\rm P}$.

Proposition 3. *Let $\varLambda_{\rm P}$ be a* PMS *and $U \subset \mathbb{S}^1$ an arbitrary, but fixed finite set of pairwise non-parallel $\mathbb{Z}[\zeta_5]$-directions. Then, the set $\mathcal{F}(\varLambda_{\rm P})$ is not determined by the X-rays in the directions of U.*

In order to obtain results on uniqueness, one has to restrict the class of finite sets under consideration. Within the class of finite subsets of a fixed PMS $\varLambda_{\rm P}$ with bounded cardinality, there is the following result.

Proposition 4. *Let $\varLambda_{\mathrm{P}}$ be a* PMS *and $k \in \mathbb{N}$. Then, the set $\mathcal{F}_{\leq k}(\varLambda_{\mathrm{P}})$ is determined by any set of $k+1$ pairwise non-parallel $\mathbb{Z}[\zeta_5]$-directions, while any set of $1 + \lfloor \log_2 k \rfloor$ pairwise non-parallel X-rays in $\mathbb{Z}[\zeta_5]$-directions is insufficient for this purpose.*

This last result is once again of limited relevance in practice, because typical atomic structures to be determined comprise about 10^6 to 10^9 atoms, and, in order not to damage or even destroy the examined structures, one has to make sure that one uses at most 4 or 5 X-rays.

Proposition 5. *Let $\varLambda_{\mathrm{P}}$ be a* PMS *and $R > 0$. Then, the set $\mathcal{D}_{<R}(\varLambda_{\mathrm{P}})$ is determined by two X-rays in $\mathbb{Z}[\zeta_5]$-directions.*

Though the last result seems to be more satisfactory, it is probably still of restricted use in practice. Here, the reason is that, in general, the second $\mathbb{Z}[\zeta_5]$-direction cannot be chosen in such a way that it yields dense lines in Penrose model sets $\varLambda_{\mathrm{P}}$, in other words, one would have to deal with images of poor resolution. A deeper result is the following, which deals with the class of convex subsets of a fixed PMS $\varLambda_{\mathrm{P}}$.

Theorem 2. *There is a set $U \subset \mathbb{S}^1$ of four pairwise non-parallel $\mathbb{Z}[\zeta_5]$-directions such that, for all* PMS *$\varLambda_{\mathrm{P}}$, the set $\mathcal{C}(\varLambda_{\mathrm{P}})$ is determined by the X-rays in the directions of U, while, for all* PMS *$\varLambda_{\mathrm{P}}$ and any set $U \subset \mathbb{S}^1$ of three or less pairwise non-parallel $\mathbb{Z}[\zeta_5]$-directions, the set $\mathcal{C}(\varLambda_{\mathrm{P}})$ is not determined by the X-rays in the directions of U.*

For example, the set of $\mathbb{Z}[\zeta_5]$-directions parallel to the elements of the following set U has the desired property to determine $\mathcal{C}(\varLambda_{\mathrm{P}})$ by the X-rays in its directions,

$$U := \{(1+\tau) + \zeta_5, (\tau - 1) + \zeta_5, -\tau + \zeta_5, 2\tau - \zeta_5\} \,. \tag{2}$$

Remark 5. By a result of Pleasants [16], these directions can yield dense lines in Penrose model sets. It follows that, in the practice of quantitative HRTEM, the resolution coming from these directions is rather high, which makes Theorem 2 look promising.

Above, we restricted the class of finite subsets of a fixed Penrose model set $\varLambda_{\mathrm{P}}$ under consideration. In order to obtain positive uniqueness results, a second option is to consider the interactive technique of successive determination. One has the following positive results.

Theorem 3. *If $\varLambda_{\mathrm{P}}$ is a* PMS, *the set $\mathcal{F}(\varLambda_{\mathrm{P}})$ is successively determined by two X-rays in $\mathbb{Z}[\zeta_5]$-directions, while the set*

$$\bigcup_{\varLambda_{\mathrm{P}} \text{ PMS}} \mathcal{F}(\varLambda_{\mathrm{P}})$$

is successively determined by three X-rays in $\mathbb{Z}[\zeta_5]$-directions.

Unfortunately, this result is again somewhat limited in practice because, in general, one cannot make sure that all the $\mathbb{Z}[\zeta_5]$-directions which are used match dense lines in Penrose model sets.

Final Remark

For further details in this spirit, we refer to [2, 3, 4, 18] for the lattice case and [17, 18] for cyclotomic model sets which also provide a systematic generalization of the setting explained here for the PMS.

Acknowledgements

It is our pleasure to thank U. Grimm, P. Gritzmann, B. Langfeld and B. Sing for helpful discussions. The authors were supported by the German Research Council (Deutsche Forschungsgemeinschaft), within the Collaborative Research Centre (Sonderforschungsbereich) 701.

References

[1] G. T. Herman and A. Kuba (Editors), *Discrete Tomography: Foundations, Algorithms, and Applications* (Birkhäuser, Boston, 1999).

[2] P. Gritzmann, *On the reconstruction of finite lattice sets from their X-rays*, in E. Ahronovitz and C. Fiorio (Editors), Lecture Notes on Computer Science (Springer, London 1997), pp. 19–32.

[3] R. J. Gardner, P. Gritzmann and D. Prangenberg, *On the computational complexity of reconstructing lattice sets from their X-rays*, Discrete Math. **202** (1999), 45–71.

[4] R. J. Gardner and P. Gritzmann, *Discrete tomography: determination of finite sets by X-rays*, Trans. Amer. Math. Soc. **349** (1997), 2271–2295.

[5] W. Steurer, *Twenty years of structure research on quasicrystals. Part I. Pentagonal, octagonal, decagonal and dodecagonal quasicrystals*, Z. Kristallogr. **219** (2004), 391–446.

[6] C. Kisielowski, P. Schwander, F. H. Baumann, *et al.*, *An approach to quantitative high-resolution transmission electron microscopy of crystalline materials*, Ultramicroscopy **58** (1995), 131–155.

[7] P. Schwander, C. Kisielowski, M. Seibt, *et al.*, *Mapping projected potential, interfacial roughness, and composition in general crystalline solids by quantitative transmission electron microscopy*, Phys. Rev. Lett. **71** (1993), 4150–4153.

[8] C. Huck, *Discrete tomography of three-dimensional model sets*, in preparation.

[9] C. Huck, Ph. D. thesis, in preparation.

[10] P. A. B. Pleasants, *Designer quasicrystals: cut-and-project sets with pre-assigned properties*, in [11], pp. 95–141.

[11] M. Baake and R. V. Moody (Editors), *Directions in Mathematical Quasicrystals* (CRM Monograph Series, vol. **13**, AMS, Providence, RI, 2000).

[12] R. V. Moody, *Model sets: a survey*, in F. Axel, F. Dénoyer and J.-P. Gazeau (Editors), *From Quasicrystals to More Complex Systems* (EDP Sciences, Les Ulis, and Springer, Berlin, 2000), pp. 145–166. `math.MG/0002020`

[13] M. Baake, *A guide to mathematical quasicrystals*, in J.-B. Suck, M. Schreiber and P. Häussler (Editors), *Quasicrystals. An Introduction to Structure, Physical Properties, and Applications* (Springer, Berlin, 2002), pp. 17–48. `math-ph/9901014`

[14] M. Baake and R. V. Moody, *Multi-component model sets and invariant densities*, in M. de Boissieu and J.-L. Verger-Gaugry (Editors), *Aperiodic '97* (World Scientific, Singapore, 1998), pp. 9–20. `math-ph/9809005`

[15] F. P. Preparata and M. I. Shamos, *Computational Geometry: An Introduciton* (Springer, New York, 1985).

[16] P. A. B. Pleasants, *Lines and planes in* 2- *and* 3-*dimensional quasicrystals*, in P. Kramer and Z. Papadopolos (Editors), *Coverings of Discrete Quasiperiodic Sets* (Springer Tracts in Modern Physics, vol. **180**, Springer, Berlin, 2003), pp. 185–225.

[17] M. Baake, P. Gritzmann, C. Huck, *et al.*, *Discrete tomography of planar model sets*, Acta Cryst. A**62** (2006), 419–433. `math.MG/0609393`

[18] C. Huck, *Uniqueness in discrete tomography of planar model sets*, preprint (2006).

Fakultät für Mathematik, Universität Bielefeld, Postfach 10 01 31, 33501 Bielefeld, Germany
E-mail address: `{mbaake,huck}@math.uni-bielefeld.de`
URL: `http://www.math.uni-bielefeld.de/baake/`