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Extended Bell and Stirling Numbers From Hypergeometric Exponentiation

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Abstract

Exponentiating the hypergeometric series \( \, _0F_L(1,1,\ldots,1;z) \), \( L = 0,1,2,\ldots \), furnishes a recursion relation for the members of certain integer sequences \( b_L(n) \), \( n = 0,1,2,\ldots \). For \( L > 0 \), the \( b_L(n) \)’s are generalizations of the conventional Bell numbers, \( b_0(n) \). The corresponding associated Stirling numbers of the second kind are also investigated. For \( L = 1 \) one can give a combinatorial interpretation of the numbers \( b_1(n) \) and of some Stirling numbers associated with them. We also consider the \( L \geq 1 \) analogues of Bell numbers for restricted partitions.

The conventional Bell numbers \( b_0(n) \), \( n = 0,1,2,\ldots \), have a well-known exponential generating function

\[ B_0(z) \equiv e^{(e^z - 1)} = \sum_{n=0}^{\infty} b_0(n) \frac{z^n}{n!}, \]  

which can be derived by interpreting \( b_0(n) \) as the number of partitions of a set of \( n \) distinct elements. In this note we obtain recursion relations for related sequences of positive integers, called \( b_L(n) \), \( L = 0,1,2,\ldots \),

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obtained by exponentiating the hypergeometric series \( _0F_L(1,1,\ldots,1;z) \) defined by (2):

\[
_0F_L(1,1,\ldots,1;z) = \sum_{n=0}^{\infty} \frac{z^n}{(n!)^L+1},
\]

(which we shall denote by \( _0F_L(z) \)) and which includes the special cases \( _0F_0(z) \equiv e^z \) and \( _0F_1(z) \equiv I_0(2\sqrt{z}) \), where \( I_0(x) \) is the modified Bessel function of the first kind. For \( L > 1 \), the functions \( _0F_L(z) \) are related to the so-called hyper-Bessel functions \([3],[4],[5]\), which have recently found application in quantum mechanics \([6],[7]\). Thus we are interested in \( b_L(n) \) given by

\[
e^{[n]_0F_L(z)-1} = \sum_{n=0}^{\infty} b_L(n) \frac{z^n}{(n!)^L+1},
\]

to thereby defining a hypergeometric generating function for the numbers \( b_L(n) \). From eq. (4) it follows formally that

\[
b_L(n) = (n!)^L \cdot \left. \frac{d^n}{dz^n} \left(e^{[n]_0F_L(z)-1}\right)\right|_{z=0}.
\]

For \( L = 0 \) the r.h.s of eq. (3) can be evaluated in closed form:

\[
b_0(n) = \frac{1}{n!} \sum_{k=0}^{\infty} \frac{k^n}{k!} = \left\{ \frac{1}{n!} \left[ \left( \frac{d}{dz} \right)^n e^z \right] \right\}_{z=1}.
\]

The first equality in (3) is the celebrated Dobiński formula \([3],[4],[5]\). The second equality in eq. (5) follows from observing that for a power series \( R(z) = \sum_{k=0}^{\infty} A_k z^k \) we have

\[
\left( z \frac{d}{dz} \right)^n R(z) = \sum_{k=0}^{\infty} A_k k^n z^k
\]

and applying eq. (5) to the exponential series \((A_k = (k!)^{-1})\).

The reason for including the divisors \((n!)^L+1\) rather than \(n!\) as in the usual exponential generating function arises from the fact that only by using eq. (2) are the numbers \( b_L(n) \) actually integers. This can be seen from general formulas for exponentiation of a power series \([3]\), which employ the (exponential) Bell polynomials, complicated and rather unwieldy objects. It cannot however be considered as a proof that the \( b_L(n) \) are integers. At this stage we shall use eq. (2) with \( b_L(n) \) real and apply to it an efficient method, described in \([3]\), which will yield the recursion relation for the \( b_L(n) \). (For the proof that the \( b_L(n) \) are integers, see below eq. (10)). To this end we first obtain a result for the multiplication of two power-series of the type (2). Suppose we wish to multiply \( f(x) = \sum_{n=0}^{\infty} a_L(n) \frac{x^n}{(n!)^L+1} \) and \( g(x) = \sum_{n=0}^{\infty} c_L(n) \frac{x^n}{(n!)^{L+1}} \). We get \( f(x) \cdot g(x) = \sum_{n=0}^{\infty} d_L(n) \frac{x^n}{(n!)^{L+1}} \), where

\[
d_L(n) = (n!)^{L+1} \sum_{r+s=n} \frac{a_L(r) c_L(s)}{(r!)^{L+1}(s!)^{L+1}} = \sum_{r=0}^{n} \binom{n}{r}^{L+1} a_L(r) c_L(n-r).
\]

Substitute eq. (9) into eq. (7) and take the logarithm of both sides of eq. (9):

\[
\sum_{n=1}^{\infty} \frac{z^n}{(n!)^{L+1}} = \ln \left( \sum_{n=0}^{\infty} b_L(n) \frac{z^n}{(n!)^L+1} \right).
\]
Now differentiate both sides of eq. (8) and multiply by $z$:

$$
\left( \sum_{n=0}^{\infty} b_L(n) \frac{z^n}{(n!)^{L+1}} \right) \left( \sum_{n=0}^{\infty} n \frac{z^n}{(n!)^{L+1}} \right) = \sum_{n=0}^{\infty} n b_L(n) \frac{z^n}{(n!)^{L+1}},
$$

(9)

which with eq. (5) yields the desired recurrence relation

$$
b_L(n+1) = \frac{1}{n+1} \sum_{k=0}^{n} \binom{n+1}{k} b_L(n+1-k) b_L(k), \quad n = 0, 1, \ldots
$$

(10)

$$
b_L(0) = 1.
$$

(12)

Since eq. (11) involves only positive integers, it follows that the $b_L(n)$ are indeed positive integers. For $L = 0$ one gets the known recurrence relation for the Bell numbers $b_L$:

$$
b_0(n+1) = \sum_{k=0}^{n} \binom{n}{k} b_0(k).
$$

(13)

We have used eq. (11) to calculate some of the $b_L(n)$’s, listed in Table I, for $L = 0, 1, \ldots, 6$. Eq. (11) gives closed form expressions for the $b_L(n)$ directly as a function of $L$ (columns in Table I): $b_L(2) = 1 + 2^L$, $b_L(3) = 1 + 3 \cdot 3^L + (3!)^L$, $b_L(4) = 1 + 4 \cdot 4^L + 3 \cdot 6^L + 6 \cdot 12^L + (4!)^L$, etc. 

The sets of $b_L(n)$ have been checked against the most complete source of integer sequences available \[\text{[10]}\]. Apart from the case $L = 0$ (conventional Bell numbers) only the first non-trivial sequence $L = 1$ is listed\[\text{[1]}\] it turns out that this sequence $b_1(n)$, listed under the heading A023998 in \[\text{[10]}\], can be given a combinatorial interpretation as the number of block permutations on a set of $n$ objects which are uniform, i.e. corresponding blocks have the same size \[\text{[12]}\]. 

Eq. (11) can be generalized by including an additional variable $x$, which will result in “smearing out” the conventional Bell numbers $b_0(n)$ with a set of integers $S_0(n,k)$, such that for $k > n$, $S_0(n,k) = 0$, and $S_0(0,0) = 1$, $S_0(n,0) = 0$. In particular,

$$
B_0(z,x) \equiv e^{x(e^z - 1)} = \sum_{n=0}^{\infty} \left[ \sum_{k=1}^{n} S_0(n,k) x^k \right] \frac{z^n}{n!},
$$

(14)

which leads to the (exponential) generating function of $S_0(n,l)$, the conventional Stirling numbers of the second kind, (see \[\text{[1]}, \text{[8]}\]), in the form

$$
\frac{(e^z - 1)^l}{l!} = \sum_{n=l}^{\infty} \frac{S_0(n,l)}{n!} \frac{z^n}{n!},
$$

(15)

and defines the so-called exponential or Touchard polynomials $l_0^{(0)}(x)$ as

$$
l_0^{(0)}(x) = \sum_{k=1}^{n} S_0(n,k)x^k.
$$

(16)

They satisfy

$$
l_0^{(0)}(1) = b_0(n),
$$

(17)

\(^1\text{(others have since been added)}\)
justifying the term “smearing out” used above.

The appearance of integers in eq. (3) suggests a natural extension with an additional variable \( x \):

\[
B_L(z, x) \equiv e^{aF_L(z) - 1} = \sum_{n=0}^{\infty} \left[ \sum_{k=1}^{n} S_L(n, k) x^k \right] \frac{z^n}{(n!)^{L+1}},
\]

(18)

where we include the right divisors \((n!)^{L+1}\) in the r.h.s of (18).

This in turn defines “hypergeometric” polynomials of type \( L \) and order \( n \) through

\[
l_n^{(L)}(x) = \sum_{k=1}^{n} S_L(n, k)x^k,
\]

(19)

which satisfy

\[
l_n^{(L)}(1) = b_L(n),
\]

(20)

with the \( b_L(n) \) of eq. (14). Thus the polynomials of eq. (14) ”smear out” the \( b_L(n) \) with the generalized Stirling numbers of the second kind, of type \( L \), denoted by \( S_L(n, k) \) (with \( S_L(n, k) = 0 \) if \( k > n \), \( S_L(n, 0) = 0 \) if \( n > 0 \) and \( S_L(0, 0) = 1 \)), which have, from eq. (13) the “hypergeometric” generating function

\[
\left( \frac{aF_L(z) - 1}{L} \right)^l = \sum_{n=0}^{\infty} \frac{S_L(n, l)}{(n!)^{L+1}} z^n, \quad L = 0, 1, 2, \ldots.
\]

(21)

Eq. (21) can be used to derive a recursion relation for the numbers \( S_L(n, k) \), in the same manner as eq. (14) yielded eq. (15). Thus we take the logarithm of both sides of eq. (21), differentiate with respect to \( z \), multiply by \( z \) and obtain:

\[
\left( \sum_{n=0}^{\infty} \frac{S_L(n, l-1)}{(n!)^{L+1}} z^n \right) \left( \sum_{n=0}^{\infty} \frac{n}{(n!)^{L+1}} z^n \right) = \sum_{n=0}^{\infty} \frac{n S_L(n, l)}{(n!)^{L+1}} z^n,
\]

(22)

which, with the help of eq. (14), produces the required recursion relation

\[
S_L(n+1, l) = \sum_{k=l-1}^{n} \binom{n}{k} \binom{n+1}{k}^L S_L(k, l-1),
\]

(23)

\[
S_L(0, 0) = 1, \quad S_L(n, 0) = 0,
\]

(24)

which for \( L = 0 \) is the recursion relation for the conventional Stirling numbers of the second kind [1, 6], and in eq. (24) the appropriate summation range has been inserted. Since the recursions of eq. (23) and eq. (24) involve only integers we conclude that \( S_L(n, l) \) are positive integers.

We have calculated some of the numbers \( S_L(n, l) \) using eq. (21) and have listed them in Tables II and III, for \( L = 1 \) and \( L = 2 \) respectively. Observe that \( S_1(n, 2) = \binom{2n+1}{n+1} - 1 \) and \( S_L(n, n) = (n!)^L, L = 1, 2 \). Also, by fixing \( n \) and \( l \), the individual values of \( S_L(n, l) \) have been calculated as a function of \( L \) with the help of eq. (24), see Table IV, from which we observe

\[
S_L(n, n) = (n!)^L, \quad L = 1, 2, \ldots.
\]

(25)

which is the lowest diagonal in Table IV. We now demonstrate that the repetitive use of eq. (24) permits one to establish closed-form expressions for any supra-diagonal of order \( p \), i.e. the sequence \( S_L(n + p, n) \),
for \( p = 1, 2, 3, \ldots \), if one knows the expression for all \( S_l(n + k, n) \) with \( k < p \). We shall illustrate it here for \( p = 1, 2 \). To this end fix \( l = n \) on both sides of eq. (23). It becomes, upon using eq. (25), and defining \( \alpha_L(n) \equiv S_L(n + 1, n) \), a linear recursion relation

\[
\alpha_L(n) = \frac{n!(n + 1)!^L}{2^L} + (n + 1)^L \alpha_L(n - 1), \quad \alpha_L(0) = 0, \tag{26}
\]

with the solution

\[
\alpha_L(n) = S_L(n + 1, n) = \frac{n(n + 1)}{2} \left[ \frac{(n + 1)!}{2} \right]^L = \left[ \frac{(n + 1)!}{2} \right]^L S_0(n + 1, n), \tag{27}
\]

which gives the second lowest diagonal in Table IV. Observe that for any \( L \), \( S_L(n + 1, n) \) is proportional to \( S_0(n+1, n) = n(n+1)/2 \). The sequence \( S_1(n+1, n) = 1, 9, 72, 600, 5400, 856480, \ldots \) is of particular interest: it represents the sum of inversion numbers of all permutations on \( n \) letters [11]. For more information about this and related sequences see the entry A001809 in [11]. The \( S_L(n + 1, n) \) for \( L > 1 \) do not appear to have a simple combinatorial interpretation. A recurrence equation for \( \beta_L(n) \equiv S_L(n + 2, n) \) is obtained upon substituting eq. (25) and eq. (27) into eq. (23):

\[
\beta_L(n) = \frac{n(n + 1)}{2} \left[ \frac{(n + 2)!}{2} \right]^L \left( \frac{n - 1}{2^L} + \frac{1}{3^L} \right) + (n + 2)^L \beta_L(n - 1), \quad \beta_L(0) = 0. \tag{29}
\]

It has the solution

\[
S_L(n + 2, n) = \frac{n(n + 1)(n + 2)}{3 \cdot 2^L} \left[ \frac{(n + 2)!}{2} \right]^L \left( \frac{3}{2^L}(n - 1) + \frac{4}{3^L} \right) \tag{30}
\]

which is a closed form expression for the second lowest diagonal in Table IV. Clearly, eq. (29) for \( L = 0 \) gives the combinatorial form for the series of conventional Stirling numbers

\[
S_0(n + 2, n) = \frac{n(n + 1)(n + 2)(3n + 1)}{4!}. \tag{31}
\]

In a similar way we obtain

\[
S_L(n + 3, n) = \frac{n(n + 1)(n + 2)(n + 3)}{3 \cdot 2^L} \left[ \frac{(n + 3)!}{3} \right]^L \times \left( n^2 \left( \frac{3}{8} \right)^L + n \left( \frac{1}{4^{L-1}} - \frac{3^{L+1}}{8^L} \right) + \frac{2 + 2 \cdot 3^L}{8^L} - \frac{1}{4^{L-1}} \right) \tag{32}
\]

which for \( L = 0 \) reduces to

\[
S_0(n + 3, n) = \frac{1}{48} n^2(n + 1)^2(n + 2)(n + 3). \tag{33}
\]

Combined with the standard definition [8, 9]

\[
S_0(n, l) = \frac{(-1)^l}{l!} \sum_{k=1}^{l} (-1)^k \binom{l}{k} k^n. \tag{34}
\]
eqs. (28), (31) and (33) give compact expressions for the summation form of $S_0(n + p, n)$. Further, from eq. (28), use of eq. (34) gives the following generating formula

$$S_0(n, l) = \frac{(-1)^l}{l!} \left( \left( \frac{d}{dz} \right)^{n} \left( \sum_{k=1}^{l} (-1)^k \binom{l}{k} z^k \right) \right)_{z=1}^{l}, \quad n \geq l. \quad (35)$$

The formula (1) can be generalized by putting restrictions on the type of resulting partitions. The generating function for the number of partitions of a set of $n$ distinct elements without singleton blocks $b_0(1, n)$ is

$$B_0(1, z) = e^{z-1-z} = \sum_{n=0}^{\infty} b_0(1, n) \frac{z^n}{n!}, \quad (37)$$

or more generally, without singleton, doubleton . . . , $p-$blocks ($p = 0, 1, \ldots$) is

$$B_0(p, z) = e^{z-\sum_{k=0}^{p} \frac{k}{k!}} = \sum_{n=0}^{\infty} b_0(p, n) \frac{z^n}{n!}, \quad (38)$$

with the corresponding associated Stirling numbers defined by analogy with eq. (34) and eq. (22). The numbers $b_0(1, n)$, $b_0(2, n)$, $b_0(3, n)$, $b_0(4, n)$ can be read off from the sequences A000296, A006505, A057837 and A057814 in [10], respectively. For more properties of these numbers see [11].

We carry over this type of extension to eq. (1) and define $b_L(p, n)$ through

$$B_L(p, z) = e^{z_{L=0}^{\infty} \frac{k}{k!}} = \sum_{n=0}^{\infty} b_L(p, n) \frac{z^n}{(n!)^{L+1}}, \quad (39)$$

where $b_L(0, n) = b_L(n)$ from eq. (38). (We know of no combinatorial meaning of $b_L(p, n)$ for $L \geq 1, p > 0$).

The $b_L(p, n)$ satisfy the following recursion relations:

$$b_L(p, n) = \sum_{k=0}^{n-p} \binom{n}{k} \binom{n+1}{k} b_L(p, k), \quad (40)$$
$$b_L(p, 0) = 1, \quad (41)$$
$$b_L(p, 1) = b_L(p, 2) = \cdots = b_L(p, p) = 0, \quad (42)$$
$$b_L(p, p + 1) = 1. \quad (43)$$

That the $b_L(p, n)$ are integers follows from eq. (40). Through eq. (39) additional families of integer Stirling-like numbers $S_{L=0}(n, k)$ can be readily defined and investigated.

The numbers $b_0(p, n)$ are collected in Table V, and Tables VI and VII contain the lowest values of $b_1(p, n)$ and $b_2(p, n)$, respectively.

Formula (1) can be used to express $e$ in terms of $b_0(n)$ in various ways. Two such lowest order (in differentiation) forms are

$$e = 1 + \ln \left( \sum_{n=0}^{\infty} \frac{b_0(n)}{n!} \right) = \ln \left( \sum_{n=0}^{\infty} \frac{b_0(n + 1)}{n!} \right). \quad (44)$$

$$e = 1 + \ln \left( \sum_{n=0}^{\infty} \frac{b_0(n)}{n!} \right) = \ln \left( \sum_{n=0}^{\infty} \frac{b_0(n + 1)}{n!} \right). \quad (45)$$
In the very same way, eq. (3) can be used to express the values of \(_0F_L(z)\) and its derivatives at \(z = 1\) in terms of certain series of \(b_L(n)\)'s. For \(L = 1\), the analogues of eq. (44) and eq. (45) are

\[
I_0(2) = 1 + \ln \left( \sum_{n=0}^{\infty} \frac{b_1(n)}{(n!)^2} \right),
\]

\[
I_0(2) + \ln(I_1(2)) = 1 + \ln \left( \sum_{n=0}^{\infty} \frac{b_1(n+1)}{(n+1)(n!)^2} \right).
\]

and for \(L = 2\) the corresponding formulas are

\[
_0F_2(1,1;1) = 1 + \ln \left( \sum_{n=0}^{\infty} \frac{b_2(n)}{(n!)^3} \right),
\]

\[
_0F_2(1,1;1) + \ln ( _0F_2(2,2;1) ) = 1 + \ln \left( \sum_{n=0}^{\infty} \frac{b_2(n+1)}{(n+1)^2(n!)^3} \right).
\]

By fixing \(z_0\) at values other than \(z_0 = 1\), one can link the numerical values of certain combinations of \(_0F_L(1,1,\ldots; z_0)\), \(_0F_L(2,2,\ldots; z_0)\), \ldots and their logarithms, with other series containing the \(b_L(n)\)'s.

The above considerations can be extended to the exponentiation of the more general hypergeometric functions of type \(_0F_L(k_1,k_2,\ldots,k_L; z)\) where \(k_1, k_2, \ldots, k_L\) are positive integers. We conjecture that for every set of \(k_n\)'s a different set of integers will be generated through an appropriate adaptation of eq. (3).

We quote one simple example of such a series. For

\[
_0F_2(1, 2; z) = \sum_{n=0}^{\infty} \frac{z^n}{(n+1)(n!)^3}
\]

eq. (3) extends to

\[
e^{[_0F_2(1, 2; z) - 1]} = \sum_{n=0}^{\infty} f_2(n) \frac{z^n}{(n+1)(n!)^3}
\]

where the numbers

\[
f_2(n) = (n+1)(n!)^2 \left[ \frac{d^n}{dz^n} e^{[_0F_2(1, 2; z) - 1]} \right]_{z=0}
\]

turn out to be integers: \(f_2(n), n = 0, 1, \ldots, 8\) are: 1, 1, 4, 37, 641, 18276, 798377, 48681011, etc. (A061683).

The analogue of equations (23) and (44) is:

\[
_0F_2(1, 2; 1) = 1 + \ln \left( \sum_{n=0}^{\infty} \frac{f_2(n)}{(n+1)(n!)^3} \right).
\]

**Acknowledgements**

We thank L. Haddad for interesting discussions. We have used Maple\textsuperscript{©} to calculate most of the numbers discussed above.
Table I: Table of $b_L(n)$: $L, n = 0, 1, \ldots, 6$. (The rows give sequences A000110, A023998, A061684–A061688.)

<table>
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<th>$L$</th>
<th>$b_L(0)$</th>
<th>$b_L(1)$</th>
<th>$b_L(2)$</th>
<th>$b_L(3)$</th>
<th>$b_L(4)$</th>
<th>$b_L(5)$</th>
<th>$b_L(6)$</th>
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<td>1</td>
<td>2</td>
<td>5</td>
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<td>3</td>
<td>16</td>
<td>131</td>
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<td>3 464 129 078 126</td>
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Table II: Table of $S_L(n,l)$: for $L = 1$ and $l,n = 1,2,\ldots,8$. (The triangle, read by columns, gives A061691, the rows and diagonals give A017063, A061690, A000142, A001809, A061689.)

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<th>$S_1(3,l)$</th>
<th>$S_1(4,l)$</th>
<th>$S_1(5,l)$</th>
<th>$S_1(6,l)$</th>
<th>$S_1(7,l)$</th>
<th>$S_1(8,l)$</th>
</tr>
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<td>3</td>
<td>6</td>
<td>72</td>
<td>650</td>
<td>5 400</td>
<td>43 757</td>
<td>353 192</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>24</td>
<td>600</td>
<td>10 500</td>
<td>161 700</td>
<td>2 361 016</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>120</td>
<td>5 400</td>
<td>161 700</td>
<td>4 116 000</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>720</td>
<td>5 2920</td>
<td>2 493 120</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>5 040</td>
<td>564 480</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>40 320</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table III: Table of $S_L(n,l)$: for $L = 2$ and $l,n = 1,2,\ldots,8$. (The triangle, read by columns, gives A061692, the rows and diagonals give A061693, A061694, A001044, A061695.)

<table>
<thead>
<tr>
<th>$l$</th>
<th>$S_2(1,l)$</th>
<th>$S_2(2,l)$</th>
<th>$S_2(3,l)$</th>
<th>$S_2(4,l)$</th>
<th>$S_2(5,l)$</th>
<th>$S_2(6,l)$</th>
<th>$S_2(7,l)$</th>
<th>$S_2(8,l)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
<td>27</td>
<td>172</td>
<td>1 125</td>
<td>7 591</td>
<td>52 479</td>
<td>369 580</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>36</td>
<td>864</td>
<td>17 500</td>
<td>351 000</td>
<td>7 197 169</td>
<td>151 633 440</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>576</td>
<td>36 000</td>
<td>1 746 000</td>
<td>80 262 000</td>
<td>3 691 514 176</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>14 400</td>
<td>1 944 000</td>
<td>191 394 000</td>
<td>17 188 416 000</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>518 400</td>
<td>133 358 400</td>
<td>23 866 214 400</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>25 401 600</td>
<td>11 379 916 800</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>8</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>1 625 702 400</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
### Table IV: Table of $S_L(n, l)$: $l, n = 1, 2, \ldots, 6.$

<table>
<thead>
<tr>
<th>$l$</th>
<th>$S_L(1, l)$</th>
<th>$S_L(2, l)$</th>
<th>$S_L(3, l)$</th>
<th>$S_L(4, l)$</th>
<th>$S_L(5, l)$</th>
<th>$S_L(6, l)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>$(2!)^L$</td>
<td>$3 \cdot 3^L$</td>
<td>$4 \cdot 4^L + 3 \cdot 6^L$</td>
<td>$5 \cdot 5^L + 10 \cdot 10^L$</td>
<td>$6 \cdot 6^L + 15 \cdot 15^L + 10 \cdot 20^L$</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>$(3!)^L$</td>
<td>$6 \cdot 12^L$</td>
<td>$10 \cdot 20^L + 15 \cdot 30^L$</td>
<td>$15 \cdot 30^L + 60 \cdot 60^L + 15 \cdot 90^L$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>$(4!)^L$</td>
<td>$10 \cdot 60^L$</td>
<td>$20 \cdot 120^L + 45 \cdot 180^L$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>$(5!)^L$</td>
<td>$15 \cdot 360^L$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>$(6!)^L$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

### Table V: Table of $b_0(p, n)$: $p = 0, 1, 2, 3; \ n = 0, \ldots, 10.$ (The columns give A000110, A000296, A006505, A057837.)

<table>
<thead>
<tr>
<th>$n$</th>
<th>$b_0(0, n)$</th>
<th>$b_0(1, n)$</th>
<th>$b_0(2, n)$</th>
<th>$b_0(3, n)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>5</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>15</td>
<td>4</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>5</td>
<td>52</td>
<td>11</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>6</td>
<td>203</td>
<td>41</td>
<td>11</td>
<td>1</td>
</tr>
<tr>
<td>7</td>
<td>877</td>
<td>162</td>
<td>36</td>
<td>1</td>
</tr>
<tr>
<td>8</td>
<td>4,140</td>
<td>715</td>
<td>92</td>
<td>36</td>
</tr>
<tr>
<td>9</td>
<td>21,147</td>
<td>3,425</td>
<td>491</td>
<td>127</td>
</tr>
<tr>
<td>10</td>
<td>115,975</td>
<td>17,722</td>
<td>2,557</td>
<td>337</td>
</tr>
</tbody>
</table>

### Table VI: Table of $b_1(p, n)$: $p = 0, 1, 2; \ n = 0, \ldots, 9.$ (The columns give A023998, A061696, A061697.)

<table>
<thead>
<tr>
<th>$n$</th>
<th>$b_1(0, n)$</th>
<th>$b_1(1, n)$</th>
<th>$b_1(2, n)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>16</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>4</td>
<td>131</td>
<td>19</td>
<td>1</td>
</tr>
<tr>
<td>5</td>
<td>1,496</td>
<td>101</td>
<td>1</td>
</tr>
<tr>
<td>6</td>
<td>22,482</td>
<td>1,776</td>
<td>201</td>
</tr>
<tr>
<td>7</td>
<td>426,833</td>
<td>23,717</td>
<td>1,226</td>
</tr>
<tr>
<td>8</td>
<td>9,934,563</td>
<td>515,971</td>
<td>5,587</td>
</tr>
<tr>
<td>9</td>
<td>277,006,192</td>
<td>11,893,597</td>
<td>493,333</td>
</tr>
</tbody>
</table>
Table VII: Table of $b_2(p,n)$: $p = 0,1,2$; $n = 0,\ldots,8$. (The columns give A061698–A061700.)

<table>
<thead>
<tr>
<th>$n$</th>
<th>$b_2(0,n)$</th>
<th>$b_2(1,n)$</th>
<th>$b_2(2,n)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
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<td>1</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>5</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>64</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>4</td>
<td>1 613</td>
<td>109</td>
<td>1</td>
</tr>
<tr>
<td>5</td>
<td>69 026</td>
<td>1 001</td>
<td>1</td>
</tr>
<tr>
<td>6</td>
<td>4 566 992</td>
<td>128 876</td>
<td>4 001</td>
</tr>
<tr>
<td>7</td>
<td>437 665 649</td>
<td>4 682 637</td>
<td>42 876</td>
</tr>
<tr>
<td>8</td>
<td>57 903 766 800</td>
<td>792 013 069</td>
<td>347 117</td>
</tr>
</tbody>
</table>

References


(Mentions sequences A000296 A001044 A001809 A006505 A010763 A023998 A057814 A057837 A061683 A061684 A061685 A061686 A061687 A061688 A061689 A061690 A061691 A061692 A061693 A061694 A061695 A061696 A061697 A061698 A061699 A061700.)

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