Extended Bell and Stirling numbers from hypergeometric exponentiation

How to cite:

For guidance on citations see FAQs.

© 2001 The Authors

Version: [not recorded]

Link(s) to article on publisher’s website:
Extended Bell and Stirling Numbers From Hypergeometric Exponentiation

J.-M. Sixdeniers
K. A. Penson
A. I. Solomon

Université Pierre et Marie Curie, Laboratoire de Physique Théorique des Liquides,
Tour 16, 5ième étage, 4 place Jussieu, 75252 Paris Cedex 05, France

Email addresses: sixdeniers@lptl.jussieu.fr, penson@lptl.jussieu.fr and a.i.solomon@open.ac.uk

Abstract

Exponentiating the hypergeometric series \( _0F_L(1, 1, \ldots, 1; z) \), \( L = 0, 1, 2, \ldots \), furnishes a recursion relation for the members of certain integer sequences \( b_L(n) \), \( n = 0, 1, 2, \ldots \). For \( L > 0 \), the \( b_L(n) \)'s are generalizations of the conventional Bell numbers, \( b_0(n) \). The corresponding associated Stirling numbers of the second kind are also investigated. For \( L = 1 \) one can give a combinatorial interpretation of the numbers \( b_1(n) \) and of some Stirling numbers associated with them. We also consider the \( L \geq 1 \) analogues of Bell numbers for restricted partitions.

The conventional Bell numbers \( b_0(n) \), \( n = 0, 1, 2, \ldots \), have a well-known exponential generating function

\[
B_0(z) \equiv e^{(e^z - 1)} = \sum_{n=0}^{\infty} b_0(n) \frac{z^n}{n!},
\]

which can be derived by interpreting \( b_0(n) \) as the number of partitions of a set of \( n \) distinct elements. In this note we obtain recursion relations for related sequences of positive integers, called \( b_L(n) \), \( L = 0, 1, 2, \ldots \),

\[1 \] Permanent address: Quantum Processes Group, Open University, Milton Keynes, MK7 6AA, United Kingdom.
obtained by exponentiating the hypergeometric series \( aF_L(1,1,\ldots,1;z) \) defined by (2):

\[
aF_L(1,1,\ldots,1;z) = \sum_{n=0}^{\infty} \frac{z^n}{(n!)^{L+1}},
\]

(which we shall denote by \( aF_L(z) \)) and which includes the special cases \( aF_0(z) \equiv e^z \) and \( aF_1(z) \equiv I_0(2\sqrt{z}) \), where \( I_0(x) \) is the modified Bessel function of the first kind. For \( L > 1 \), the functions \( aF_L(z) \) are related to the so-called hyper-Bessel functions \( \beta \), \( \gamma \), \( \delta \), which have recently found application in quantum mechanics \( \alpha \), \( \beta \). Thus we are interested in \( b_L(n) \) given by

\[
e^{[aF_L(z)-1]} = \sum_{n=0}^{\infty} b_L(n) \frac{z^n}{(n!)^{L+1}},
\]

thereby defining a hypergeometric generating function for the numbers \( b_L(n) \). From eq. (1) it follows formally that

\[
b_L(n) = (n!)^L \cdot \frac{d^n}{dz^n} \left(e^{[aF_L(z)-1]}\right) \bigg|_{z=0}.
\]

For \( L = 0 \) the r.h.s of eq. (1) can be evaluated in closed form:

\[
b_0(n) = \frac{1}{e} \sum_{k=0}^{\infty} \frac{k^n}{k!} = \left\{ \frac{1}{e^z} \left( -\frac{d}{dz} \right)^n e^z \right\}_{z=1}.
\]

The first equality in (1) is the celebrated Dobiński formula \( \alpha \), \( \beta \), \( \gamma \). The second equality in eq. (1) follows from observing that for a power series \( R(z) = \sum_{k=0}^{\infty} A_k z^k \) we have

\[
\left( -\frac{d}{dz} \right)^n R(z) = \sum_{k=0}^{\infty} A_k k^n z^k
\]

and applying eq. (1) to the exponential series \( (A_k = (k!)^{-1}) \).

The reason for including the divisors \( (n!)^{L+1} \) rather than \( n! \) as in the usual exponential generating function arises from the fact that only by using eq. (1) are the numbers \( b_L(n) \) actually integers. This can be seen from general formulas for exponentiation of a power series \( \delta \), which employ the (exponential) Bell polynomials, complicated and rather unwieldy objects. It cannot however be considered as a proof that the \( b_L(n) \) are integers. At this stage we shall use eq. (1) with \( b_L(n) \) real and apply to it an efficient method, described in \( \gamma \), which will yield the recursion relation for the \( b_L(n) \). (For the proof that the \( b_L(n) \) are integers, see below eq. (1)). To this end we first obtain a result for the multiplication of two power-series of the type (1). Suppose we wish to multiply \( f(x) = \sum_{n=0}^{\infty} a_L(n) (\frac{x^n}{(n!)^{L+1}} \) and \( g(x) = \sum_{n=0}^{\infty} c_L(n) (\frac{x^n}{(n!)^{L+1}} \). We get \( f(x) \cdot g(x) = \sum_{n=0}^{\infty} d_L(n) (\frac{x^n}{(n!)^{L+1}} \), where

\[
d_L(n) = (n!)^{L+1} \sum_{r+s=n} \frac{a_L(r)c_L(s)}{r!s!(n!)^{L+1}} = \sum_{r=0}^{\infty} \binom{n}{r}^{L+1} a_L(r) c_L(n-r).
\]

Substitute eq. (1) into eq. (1) and take the logarithm of both sides of eq. (1):

\[
\sum_{n=1}^{\infty} \frac{z^n}{(n!)^{L+1}} = \ln \left( \sum_{n=0}^{\infty} b_L(n) \frac{z^n}{(n!)^{L+1}} \right).
\]
Now differentiate both sides of eq. (8) and multiply by $z$:

$$
\left( \sum_{n=0}^{\infty} b_L(n) \frac{z^n}{(n!)^{L+1}} \right) \left( \sum_{n=0}^{\infty} n \frac{z^n}{(n!)^{L+1}} \right) = \sum_{n=0}^{\infty} n b_L(n) \frac{z^n}{(n!)^{L+1}} ,
$$

(9)

which with eq. (9) yields the desired recurrence relation

$$
b_L(n+1) = \frac{1}{n+1} \sum_{k=0}^{n} \binom{n+1}{k} (n+1-k) b_L(k) , \quad n = 0, 1, \ldots
$$

(10)

and

$$
b_L(0) = 1 .
$$

(12)

Since eq. (11) involves only positive integers, it follows that the $b_L(n)$ are indeed positive integers. For $L = 0$ one gets the known recurrence relation for the Bell numbers $b_L$:

$$
b_0(n+1) = \sum_{k=0}^{n} \binom{n}{k} b_0(k) .
$$

(13)

We have used eq. (11) to calculate some of the $b_L(n)$’s, listed in Table I, for $L = 0, 1, \ldots, 6$. Eq. (11), for $n$ fixed, gives closed form expressions for the $b_L(n)$ directly as a function of $L$ (columns in Table I):

- $b_L(2) = 1 + 2^L ,
- b_L(3) = 1 + 3 \cdot 3^L + (3!)^L ,
- b_L(4) = 1 + 4 \cdot 4^L + 3 \cdot 6^L + 6 \cdot 12^L + (4!)^L ,

etc.

The sets of $b_L(n)$ have been checked against the most complete source of integer sequences available [10]. Apart from the case $L = 0$ (conventional Bell numbers) only the first non-trivial sequence $L = 1$ is listed. It turns out that this sequence $b_1(n)$, listed under the heading A023998 in [10], can be given a combinatorial interpretation as the number of block permutations on a set of $n$ objects which are uniform, i.e. corresponding blocks have the same size.

Eq. (11) can be generalized by including an additional variable $x$, which will result in “smearing out” the conventional Bell numbers $b_0(n)$ with a set of integers $S_0(n,k)$, such that for $k > n$, $S_0(n,k) = 0$, and $S_0(0,0) = 1$, $S_0(n,0) = 0$. In particular,

$$
B_0(z,x) \equiv e^{x(e^z - 1)} = \sum_{n=0}^{\infty} \left[ \sum_{k=1}^{n} S_0(n,k) x^k \right] \frac{z^n}{n!} ,
$$

(14)

which leads to the (exponential) generating function of $S_0(n,l)$, the conventional Stirling numbers of the second kind, (see [1], [8]), in the form

$$
\frac{(e^z - 1)^l}{l!} = \sum_{n=l}^{\infty} \frac{S_0(n,l)}{n!} z^n ,
$$

(15)

and defines the so-called exponential or Touchard polynomials $l_0^{(0)}(x)$ as

$$
l_0^{(0)}(x) = \sum_{k=1}^{n} S_0(n,k)x^k .
$$

(16)

They satisfy

$$
l_0^{(0)}(1) = b_0(n) ,
$$

(17)

1 (others have since been added)
justifying the term “smearing out” used above.

The appearance of integers in eq. (3) suggests a natural extension with an additional variable \( x \):

\[
B_L(z, x) \equiv e^{x [a F_L(z)] - 1} = \sum_{n=0}^{\infty} \left( \sum_{k=1}^{n} S_L(n, k) x^k \right) \frac{z^n}{(n!)^{L+1}},
\]

where we include the right divisors \((n!)^{L+1}\) in the r.h.s of (18).

This in turn defines “hypergeometric” polynomials of type \( L \) and order \( n \) through

\[
l_L^{(L)}(x) = \sum_{k=1}^{n} S_L(n, k) x^k, \tag{19}
\]

which satisfy

\[
l_L^{(L)}(1) = b_L(n), \tag{20}
\]

with the \( b_L(n) \) of eq. (11). Thus the polynomials of eq. (14) ”smear out” the \( b_L(n) \) with the generalized Stirling numbers of the second kind, of type \( L \), denoted by \( S_L(n, k) \) (with \( S_L(n, k) = 0 \) if \( k > n \), \( S_L(n, 0) = 0 \) if \( n > 0 \) and \( S_L(0, 0) = 1 \)), which have, from eq. (13) the “hypergeometric” generating function

\[
\left( \frac{a F_L(z) - 1}{L+1} \right) = \sum_{n=0}^{\infty} \frac{n S_L(n, l)}{(n!)^{L+1}} z^n, \quad L = 0, 1, 2, \ldots. \tag{21}
\]

Eq. (21) can be used to derive a recursion relation for the numbers \( S_L(n, k) \), in the same manner as eq. (3) yielded eq. (14). Thus we take the logarithm of both sides of eq. (21), differentiate with respect to \( z \), multiply by \( z \) and obtain:

\[
\left( \sum_{n=0}^{\infty} \frac{S_L(n, l - 1)}{(n!)^{L+1}} z^n \right) \left( \sum_{n=0}^{\infty} \frac{n}{(n!)^{L+1}} z^n \right) = \sum_{n=0}^{\infty} \frac{n S_L(n, l)}{(n!)^{L+1}} z^n, \tag{22}
\]

which, with the help of eq. (11), produces the required recursion relation

\[
S_L(n + 1, l) = \sum_{k=l-1}^{n} \binom{n}{k} \binom{n+1}{k} L S_L(k, l - 1), \tag{23}
\]

\[
S_L(0, 0) = 1, \quad S_L(n, 0) = 0, \tag{24}
\]

which for \( L = 0 \) is the recursion relation for the conventional Stirling numbers of the second kind \([1], [2]\), and in eq. (24) the appropriate summation range has been inserted. Since the recursions of eq. (23) and eq. (24) involve only integers we conclude that \( S_L(n, l) \) are positive integers.

We have calculated some of the numbers \( S_L(n, l) \) using eq. (21) and have listed them in Tables II and III, for \( L = 1 \) and \( L = 2 \) respectively. Observe that \( S_1(n, 2) = \binom{2n+1}{n+1} - 1 \) and \( S_L(n, n) = (n!)^L, L = 1, 2 \). Also, by fixing \( n \) and \( l \), the individual values of \( S_L(n, l) \) have been calculated as a function of \( L \) with the help of eq. (24), see Table IV, from which we observe

\[
S_L(n, n) = (n!)^L, \quad L = 1, 2, \ldots. \tag{25}
\]

which is the lowest diagonal in Table IV. We now demonstrate that the repetitive use of eq. (24) permits one to establish closed-form expressions for any supra-diagonal of order \( p \), i.e. the sequence \( S_L(n + p, n) \),
for $p = 1, 2, 3, \ldots$, if one knows the expression for all $S_L(n + k, n)$ with $k < p$. We shall illustrate it here for $p = 1, 2$. To this end fix $l = n$ on both sides of eq. (23). It becomes, upon using eq. (25), and defining $\alpha_L(n) \equiv S_L(n + 1, n)$, a linear recursion relation

$$\alpha_L(n) = \frac{n[(n + 1)!]^L}{2^L} + (n + 1)^L \alpha_L(n - 1), \quad \alpha_L(0) = 0,$$

with the solution

$$\alpha_L(n) = S_L(n + 1, n) = \frac{n(n + 1)}{2} \left[ \frac{(n + 1)!}{2} \right]^L$$

$$= \left[ \frac{(n + 1)!}{2} \right]^L S_0(n + 1, n),$$

which gives the second lowest diagonal in Table IV. Observe that for any $L$, $S_L(n + 1, n)$ is proportional to $S_0(n + 1, n) = n(n + 1)/2$. The sequence $S_1(n + 1, n) = 1, 9, 72, 600, 5400, 856480, \ldots$ is of particular interest: it represents the sum of inversion numbers of all permutations on $n$ letters [10]. For more information about this and related sequences see the entry A001809 in [10]. The $S_L(n + 1, n)$ for $L > 1$ do not appear to have a simple combinatorial interpretation. A recurrence equation for $\beta_L(n) \equiv S_L(n + 2, n)$ is obtained upon substituting eq. (25) and eq. (27) into eq. (23):

$$\beta_L(n) = \frac{n(n + 1)}{2} \left[ \frac{(n + 2)!}{2} \right]^L \left( \frac{n - 1}{2^L} + \frac{1}{3^L} \right) + (n + 2)^L \beta_L(n - 1), \quad \beta_L(0) = 0.$$ 

It has the solution

$$S_L(n + 2, n) = \frac{n(n + 1)(n + 2)}{3 \cdot 2^L} \left[ \frac{(n + 2)!}{2} \right]^L \left( \frac{3}{2^L}(n - 1) + \frac{4}{3^L} \right)$$

which is a closed form expression for the second lowest diagonal in Table IV. Clearly, eq. (29) for $L = 0$ gives the combinatorial form for the series of conventional Stirling numbers

$$S_0(n + 2, n) = \frac{n(n + 1)(n + 2)(3n + 1)}{4!}.$$ 

In a similar way we obtain

$$S_L(n + 3, n) = \frac{n(n + 1)(n + 2)(n + 3)}{3 \cdot 2^L} \left[ \frac{(n + 3)!}{3} \right]^L \times \left( \frac{3}{8} \right)^L + n \left( \frac{1}{4^{L-1}} - \frac{3^{L+1}}{8^L} \right) + \frac{2 + 2 \cdot 3^L - 1}{4^{L-1}}$$

which for $L = 0$ reduces to

$$S_0(n + 3, n) = \frac{1}{48} n^2(n + 1)^2(n + 2)(n + 3).$$

Combined with the standard definition [8], [9]

$$S_0(n, l) = \frac{(-1)^l}{l!} \sum_{k=1}^{l} (-1)^k \binom{l}{k} k^n.$$ 

(34)
eqs. (28), (31), and (33) give compact expressions for the summation form of $S_0(n+p,n)$. Further, from eq. (34), use of eq. (6) gives the following generating formula

$$S_0(n,l) = \left( -\frac{1}{l!} \right)^l \left[ \left( \frac{d}{dz} \right)^n \left( \sum_{k=1}^{l} (-1)^k \binom{l}{k} z^k \right) \right]_{z=1}$$

(35)

$$= \left( -\frac{1}{l!} \right)^l \left[ \left( \frac{d}{dz} \right)^n [(1-z)^l - 1] \right]_{z=1}, \quad n \geq l.$$  

(36)

The formula (1) can be generalized by putting restrictions on the type of resulting partitions. The generating function for the number of partitions of a set of $n$ distinct elements without singleton blocks $b_0(1,n)$ is [14],

$$B_0(1,z) = e^{e^z - 1 - z} = \sum_{n=0}^{\infty} b_0(1,n) \frac{z^n}{n!},$$

(37)

or more generally, without singleton, doubleton ..., $p-$blocks ($p = 0, 1, \ldots$) is [14]

$$B_0(p,z) = e^{e^z - \sum_{k=0}^{p} \frac{k^k}{k!}} = \sum_{n=0}^{\infty} b_0(p,n) \frac{z^n}{n!},$$

(38)

with the corresponding associated Stirling numbers defined by analogy with eq. (14) and eq. (22). The numbers $b_0(1,n), b_0(2,n), b_0(3,n), b_0(4,n)$ can be read off from the sequences A000296, A006505, A057837 and A057814 in [14], respectively. For more properties of these numbers see [11].

We carry over this type of extension to eq. (3) and define $b_L(p,n)$ through

$$B_L(p,z) \equiv e^{e^{F_L(z)} - \sum_{k=0}^{L} \frac{k^k}{(n)^{k}} = \sum_{n=0}^{\infty} b_L(p,n) \frac{z^n}{(n!)^{L+1}},$$

(39)

where $b_L(0,n) = b_L(n)$ from eq. (3). (We know of no combinatorial meaning of $b_L(p,n)$ for $L \geq 1$, $p > 0$). The $b_L(p,n)$ satisfy the following recursion relations:

$$b_L(p,n) = \sum_{k=0}^{n-p} \binom{n}{k} \binom{n+1}{k}^L b_L(p,k),$$

(40)

$$b_L(p,0) = 1,$$  

(41)

$$b_L(p,1) = b_L(p,2) = \cdots = b_L(p,p) = 0,$$  

(42)

$$b_L(p,p+1) = 1.$$  

(43)

That the $b_L(p,n)$ are integers follows from eq. (40). Through eq. (39) additional families of integer Stirling-like numbers $S_L,p(n,k)$ can be readily defined and investigated.

The numbers $b_0(p,n)$ are collected in Table V, and Tables VI and VII contain the lowest values of $b_1(p,n)$ and $b_2(p,n)$, respectively.

Formula (1) can be used to express $e$ in terms of $b_0(n)$ in various ways. Two such lowest order (in differentiation) forms are

$$e = 1 + \ln \left( \sum_{n=0}^{\infty} \frac{b_0(n)}{n!} \right) =$$

$$\ln \left( \sum_{n=0}^{\infty} \frac{b_0(n+1)}{n!} \right).$$

(45)
In the very same way, eq. (3) can be used to express the values of \( _0F_L(z) \) and its derivatives at \( z = 1 \) in terms of certain series of \( b_L(n) \)'s. For \( L = 1 \), the analogues of eq. (44) and eq. (45) are

\[
I_0(2) = 1 + \ln \left( \sum_{n=0}^{\infty} \frac{b_1(n)}{(n!)^2} \right),
\]

\[
I_0(2) + \ln(I_1(2)) = 1 + \ln \left( \sum_{n=0}^{\infty} \frac{b_1(n+1)}{(n+1)(n!)^2} \right)
\]

and for \( L = 2 \) the corresponding formulas are

\[
_0F_2(1,1;1) = 1 + \ln \left( \sum_{n=0}^{\infty} \frac{b_2(n)}{(n!)^2} \right),
\]

\[
_0F_2(1,1;1) + \ln( _0F_2(2,2;1) ) = 1 + \ln \left( \sum_{n=0}^{\infty} \frac{b_2(n+1)}{(n+1)^2(n!)^2} \right).
\]

By fixing \( z_0 \) at values other than \( z_0 = 1 \), one can link the numerical values of certain combinations of \( _0F_L(1,1,\ldots;z_0) \), \( _0F_L(2,2,\ldots;z_0) \),... and their logarithms, with other series containing the \( b_L(n) \)'s.

The above considerations can be extended to the exponentiation of the more general hypergeometric functions of type \( _0F_L(k_1,k_2,\ldots,k_L;z) \) where \( k_1, k_2, \ldots, k_L \) are positive integers. We conjecture that for every set of \( k_n \)'s a different set of integers will be generated through an appropriate adaptation of eq. (3).

We quote one simple example of such a series. For

\[
_0F_2(1,2;z) = \sum_{n=0}^{\infty} \frac{z^n}{(n+1)(n!)^3}
\]

eq. (3) extends to

\[
e^{[_0F_2(1,2;z)]-1} = \sum_{n=0}^{\infty} f_2(n) \frac{z^n}{(n+1)(n!)^3}
\]

where the numbers

\[
f_2(n) = (n+1)(n!)^2 \left[ \frac{d^n}{dz^n} e^{[_0F_2(1,2;z)]-1} \right]_{z=0}
\]

turn out to be integers: \( f_2(n), n = 0, 1, \ldots, 8 \) are: 1, 1, 4, 37, 641, 18276, 789377, 48681011, etc. (A061683).

The analogue of equations (23) and (44) is:

\[
_0F_2(1,2;1) = 1 + \ln \left( \sum_{n=0}^{\infty} \frac{f_2(n)}{(n+1)(n!)^3} \right).
\]

Acknowledgements

We thank L. Haddad for interesting discussions. We have used Maple© to calculate most of the numbers discussed above.
Table I: Table of $b_L(n)$: $L, n = 0, 1, \ldots, 6$. (The rows give sequences A000110, A023998, A061684–A061688.)

<table>
<thead>
<tr>
<th>$L$</th>
<th>$b_L(0)$</th>
<th>$b_L(1)$</th>
<th>$b_L(2)$</th>
<th>$b_L(3)$</th>
<th>$b_L(4)$</th>
<th>$b_L(5)$</th>
<th>$b_L(6)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>5</td>
<td>15</td>
<td>52</td>
<td>203</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>3</td>
<td>16</td>
<td>131</td>
<td>1 496</td>
<td>22 482</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>1</td>
<td>5</td>
<td>64</td>
<td>1 613</td>
<td>69 026</td>
<td>4 566 992</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>1</td>
<td>9</td>
<td>298</td>
<td>25 097</td>
<td>4 383 626</td>
<td>1 394 519 922</td>
</tr>
<tr>
<td>4</td>
<td>1</td>
<td>1</td>
<td>17</td>
<td>1 540</td>
<td>461 105</td>
<td>350 813 126</td>
<td>573 843 627 152</td>
</tr>
<tr>
<td>5</td>
<td>1</td>
<td>1</td>
<td>33</td>
<td>8 506</td>
<td>9 483 041</td>
<td>33 056 715 626</td>
<td>293 327 384 637 282</td>
</tr>
<tr>
<td>6</td>
<td>1</td>
<td>1</td>
<td>65</td>
<td>48 844</td>
<td>209 175 233</td>
<td>3 464 129 078 126</td>
<td>173 566 857 025 139 312</td>
</tr>
</tbody>
</table>

Table II: Table of $S_L(n, l)$: for $L = 1$ and $l, n = 1, 2, \ldots, 8$. (The triangle, read by columns, gives A061691, the rows and diagonals give A017063, A061690, A000142, A001809, A061689.)

<table>
<thead>
<tr>
<th>$l$</th>
<th>$S_1(1, l)$</th>
<th>$S_1(2, l)$</th>
<th>$S_1(3, l)$</th>
<th>$S_1(4, l)$</th>
<th>$S_1(5, l)$</th>
<th>$S_1(6, l)$</th>
<th>$S_1(7, l)$</th>
<th>$S_1(8, l)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>9</td>
<td>34</td>
<td>125</td>
<td>461</td>
<td>1 715</td>
<td>6 434</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>6</td>
<td>72</td>
<td>650</td>
<td>5 400</td>
<td>43 757</td>
<td>353 192</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>24</td>
<td>600</td>
<td>10 500</td>
<td>161 700</td>
<td>2 361 016</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>120</td>
<td>5 400</td>
<td>161 700</td>
<td>4 116 000</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>720</td>
<td>52 920</td>
<td>2 493 120</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>5 040</td>
<td>564 480</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>8</td>
<td></td>
<td>40 320</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table III: Table of $S_L(n, l)$: for $L = 2$ and $l, n = 1, 2, \ldots, 8$. (The triangle, read by columns, gives A061692, the rows and diagonals give A061693, A061694, A001044, A061695.)

<table>
<thead>
<tr>
<th>$l$</th>
<th>$S_2(1, l)$</th>
<th>$S_2(2, l)$</th>
<th>$S_2(3, l)$</th>
<th>$S_2(4, l)$</th>
<th>$S_2(5, l)$</th>
<th>$S_2(6, l)$</th>
<th>$S_2(7, l)$</th>
<th>$S_2(8, l)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
<td>27</td>
<td>172</td>
<td>1 125</td>
<td>7 591</td>
<td>52 479</td>
<td>369 580</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>36</td>
<td>864</td>
<td>17 500</td>
<td>351 000</td>
<td>7 197 169</td>
<td>151 633 440</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>576</td>
<td>36 000</td>
<td>1 746 000</td>
<td>80 262 000</td>
<td>3 691 514 176</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>14 400</td>
<td>1 944 000</td>
<td>191 394 000</td>
<td>17 188 416 000</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td></td>
<td>518 400</td>
<td>133 358 400</td>
<td>23 866 214 400</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>7</td>
<td></td>
<td>25 401 600</td>
<td>11 379 916 800</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>8</td>
<td></td>
<td></td>
<td>1 625 702 400</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Table IV: Table of $S_L(n,l)$: $l, n = 1, 2, \ldots, 6$.

<table>
<thead>
<tr>
<th>$l$</th>
<th>$S_L(1,l)$</th>
<th>$S_L(2,l)$</th>
<th>$S_L(3,l)$</th>
<th>$S_L(4,l)$</th>
<th>$S_L(5,l)$</th>
<th>$S_L(6,l)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>$(2!)^L$</td>
<td>$3 \cdot 3^L$</td>
<td>$4 \cdot 4^L + 3 \cdot 6^L$</td>
<td>$5 \cdot 5^L + 10 \cdot 10^L$</td>
<td>$6 \cdot 6^L + 15 \cdot 15^L + 10 \cdot 20^L$</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>$(3!)^L$</td>
<td>$6 \cdot 12^L$</td>
<td>$10 \cdot 20^L + 15 \cdot 30^L$</td>
<td>$15 \cdot 30^L + 60 \cdot 60^L + 15 \cdot 90^L$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>$(4!)^L$</td>
<td>$10 \cdot 60^L$</td>
<td>$20 \cdot 120^L + 45 \cdot 180^L$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>$(5!)^L$</td>
<td>$15 \cdot 360^L$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>$(6!)^L$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table V: Table of $b_0(p,n)$: $p = 0, 1, 2, 3$; $n = 0, \ldots, 10$. (The columns give A000110, A000296, A006505, A057837.)

<table>
<thead>
<tr>
<th>$n$</th>
<th>$b_0(0,n)$</th>
<th>$b_0(1,n)$</th>
<th>$b_0(2,n)$</th>
<th>$b_0(3,n)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>5</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>15</td>
<td>4</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>5</td>
<td>52</td>
<td>11</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>6</td>
<td>203</td>
<td>41</td>
<td>11</td>
<td>1</td>
</tr>
<tr>
<td>7</td>
<td>877</td>
<td>162</td>
<td>36</td>
<td>1</td>
</tr>
<tr>
<td>8</td>
<td>4140</td>
<td>715</td>
<td>92</td>
<td>36</td>
</tr>
<tr>
<td>9</td>
<td>21147</td>
<td>3425</td>
<td>491</td>
<td>127</td>
</tr>
<tr>
<td>10</td>
<td>115975</td>
<td>17722</td>
<td>2557</td>
<td>337</td>
</tr>
</tbody>
</table>

Table VI: Table of $b_1(p,n)$: $p = 0, 1, 2$; $n = 0, \ldots, 9$. (The columns give A023998, A061696, A061697.)

<table>
<thead>
<tr>
<th>$n$</th>
<th>$b_1(0,n)$</th>
<th>$b_1(1,n)$</th>
<th>$b_1(2,n)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>16</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>4</td>
<td>131</td>
<td>19</td>
<td>1</td>
</tr>
<tr>
<td>5</td>
<td>1496</td>
<td>101</td>
<td>1</td>
</tr>
<tr>
<td>6</td>
<td>22482</td>
<td>1776</td>
<td>201</td>
</tr>
<tr>
<td>7</td>
<td>426833</td>
<td>23717</td>
<td>1226</td>
</tr>
<tr>
<td>8</td>
<td>9334563</td>
<td>515971</td>
<td>5587</td>
</tr>
<tr>
<td>9</td>
<td>277006192</td>
<td>11893597</td>
<td>493333</td>
</tr>
</tbody>
</table>
Table VII: Table of $b_2(p, n)$: $p = 0, 1, 2$; $n = 0, \ldots, 8$. (The columns give A061698–A061700.)

<table>
<thead>
<tr>
<th>$n$</th>
<th>$b_2(0, n)$</th>
<th>$b_2(1, n)$</th>
<th>$b_2(2, n)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>5</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>64</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>4</td>
<td>1613</td>
<td>109</td>
<td>1</td>
</tr>
<tr>
<td>5</td>
<td>69 026</td>
<td>1 001</td>
<td>1</td>
</tr>
<tr>
<td>6</td>
<td>4 566 992</td>
<td>128 876</td>
<td>4 001</td>
</tr>
<tr>
<td>7</td>
<td>437 665 649</td>
<td>4 682 637</td>
<td>42 876</td>
</tr>
<tr>
<td>8</td>
<td>57 903 766 800</td>
<td>792 013 069</td>
<td>347 117</td>
</tr>
</tbody>
</table>

References


(Mentions sequences A000296 A001044 A001809 A006505 A010763 A023998 A057814 A057837 A061683 A061684 A061685 A061686 A061687 A061688 A061689 A061690 A061691 A061692 A061693 A061694 A061695 A061696 A061697 A061698 A061699 A061700.)

Received April 5, 2001; published in Journal of Integer Sequences, June 22, 2001.

Return to Journal of Integer Sequences home page.