Extended Bell and Stirling Numbers From Hypergeometric Exponentiation

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Abstract

Exponentiating the hypergeometric series $\binom{0}{F_L}(1,1,\ldots,1;z)$, $L = 0,1,2,\ldots$, furnishes a recursion relation for the members of certain integer sequences $b_L(n)$, $n = 0,1,2,\ldots$. For $L > 0$, the $b_L(n)$'s are generalizations of the conventional Bell numbers, $b_0(n)$. The corresponding associated Stirling numbers of the second kind are also investigated. For $L = 1$ one can give a combinatorial interpretation of the numbers $b_1(n)$ and of some Stirling numbers associated with them. We also consider the $L \geq 1$ analogues of Bell numbers for restricted partitions.

The conventional Bell numbers $b_0(n)$, $n = 0,1,2,\ldots$, have a well-known exponential generating function

$$B_0(z) \equiv e^{(e^z-1)} = \sum_{n=0}^{\infty} b_0(n) \frac{z^n}{n!},$$

which can be derived by interpreting $b_0(n)$ as the number of partitions of a set of $n$ distinct elements. In this note we obtain recursion relations for related sequences of positive integers, called $b_L(n)$, $L = 0,1,2,\ldots$,
obtained by exponentiating the hypergeometric series \( {}_0F_L(1, 1, \ldots, 1; z) \) defined by (2):

\[
{}_0F_L(1, 1, \ldots, 1; z) = \sum_{n=0}^{\infty} \frac{z^n}{(n!)^{L+1}},
\]

(which we shall denote by \( {}_0F_L(z) \)) and which includes the special cases \( {}_0F_0(z) \equiv e^z \) and \( {}_0F_1(z) \equiv I_0(2\sqrt{z}) \), where \( I_0(x) \) is the modified Bessel function of the first kind. For \( L > 1 \), the functions \( {}_0F_L(z) \) are related to the so-called hyper-Bessel functions \( [3], [4], [5] \), which have recently found application in quantum mechanics \( [6], [7] \). Thus we are interested in \( b_L(n) \) given by

\[
e^{[{}_0F_L(z)-1]} = \sum_{n=0}^{\infty} b_L(n) \frac{z^n}{(n!)^{L+1}},
\]

thereby defining a hypergeometric generating function for the numbers \( b_L(n) \). From eq. (3) it follows formally that

\[
b_L(n) = (n!)^L \cdot \frac{d^n}{dz^n} \left. \left( e^{[{}_0F_L(z)-1]} \right) \right|_{z=0}.
\]

For \( L = 0 \) the r.h.s of eq. (4) can be evaluated in closed form:

\[
b_0(n) = \frac{1}{e} \sum_{k=0}^{\infty} \frac{k^n}{k!} = \frac{1}{e^z} \left\{ \left( \frac{d}{dz} \right)^n e^z \right\}_{z=1}.
\]

The first equality in (5) is the celebrated Dobiński formula \( [3], [4], [5] \). The second equality in eq. (5) follows from observing that for a power series \( R(z) = \sum_{k=0}^{\infty} A_k z^k \) we have

\[
\left( z \frac{d}{dz} \right)^n R(z) = \sum_{k=0}^{\infty} A_k k^n z^k
\]

and applying eq. (5) to the exponential series \( (A_k = (k!)^{-1}) \).

The reason for including the divisors \( (n!)^{L+1} \) rather than \( n! \) as in the usual exponential generating function arises from the fact that only by using eq. (3) are the numbers \( b_L(n) \) actually integers. This can be seen from general formulas for exponentiation of a power series \( [3] \), which employ the (exponential) Bell polynomials, complicated and rather unwieldy objects. It cannot however be considered as a proof that the \( b_L(n) \) are integers. At this stage we shall use eq. (3) with \( b_L(n) \) real and apply to it an efficient method, described in \( [3] \), which will yield the recursion relation for the \( b_L(n) \). (For the proof that the \( b_L(n) \) are integers, see below eq. (7)). To this end we first obtain a result for the multiplication of two power-series of the type \( [3] \). Suppose we wish to multiply \( f(x) = \sum_{n=0}^{\infty} a_L(n) \frac{z^n}{(n!)^{L+1}} \) and \( g(x) = \sum_{n=0}^{\infty} c_L(n) \frac{x^n}{(n!)^{L+1}} \). We get \( f(x) \cdot g(x) = \sum_{n=0}^{\infty} d_L(n) \frac{x^n}{(n!)^{L+1}}, \) where

\[
d_L(n) = (n!)^{L+1} \sum_{r+s=n}^{\infty} \frac{a_L(r)c_L(s)}{(r!)^{L+1}(s!)^{L+1}} = \sum_{r=0}^{n} \binom{n}{r} L+1 \cdot a_L(r) c_L(n-r).
\]

Substitute eq. (6) into eq. (7) and take the logarithm of both sides of eq. (7):

\[
\sum_{n=1}^{\infty} \frac{z^n}{(n!)^{L+1}} = \ln \left( \sum_{n=0}^{\infty} b_L(n) \frac{z^n}{(n!)^{L+1}} \right).
\]
Now differentiate both sides of eq. (8) and multiply by \( z \):
\[
\left( \sum_{n=0}^{\infty} b_L(n) \frac{z^n}{(n!)^{L+1}} \right) \left( \sum_{n=0}^{\infty} n \frac{z^n}{(n!)^{L+1}} \right) = \sum_{n=0}^{\infty} n b_L(n) \frac{z^n}{(n!)^{L+1}},
\]
which with eq. (8) yields the desired recurrence relation
\[
b_L(n+1) = \frac{1}{n+1} \sum_{k=0}^{n} \binom{n+1}{k} (n+1-k) b_L(k), \quad n = 0, 1, \ldots
\]
\[
= \sum_{k=0}^{n} \binom{n}{k} \left( \binom{n+1}{k} \right)^L b_L(k),
\]
\[
b_L(0) = 1.
\]
Since eq. (10) involves only positive integers, it follows that the \( b_L(n) \) are indeed positive integers. For \( L = 0 \) one gets the known recurrence relation for the Bell numbers:
\[
b_0(n+1) = \sum_{k=0}^{n} \binom{n}{k} b_0(k).
\]

We have used eq. (10) to calculate some of the \( b_L(n) \)'s, listed in Table I, for \( L = 0, 1, \ldots, 6 \). Eq. (10), for \( n \) fixed, gives closed form expressions for \( b_L(n) \) directly as a function of \( L \) (columns in Table I): 
\( b_L(2) = 1 + 2^L \), \( b_L(3) = 1 + 3 \cdot 3^L + (3!)^L \), \( b_L(4) = 1 + 4 \cdot 4^L + 3 \cdot 6^L + 6 \cdot 12^L + (4!)^L \), etc.

The sets of \( b_L(n) \) have been checked against the most complete source of integer sequences available \[10\]. Apart from the case \( L = 0 \) (conventional Bell numbers) only the first non-trivial sequence \( L = 1 \) is listed\[11\]; it turns out that this sequence \( b_1(n) \), listed under the heading A023998 in \[10\], can be given a combinatorial interpretation as the number of block permutations on a set of \( n \) objects which are uniform, i.e. corresponding blocks have the same size \[12\].

Eq. (10) can be generalized by including an additional variable \( x \), which will result in “smearing out” the conventional Bell numbers \( b_0(n) \) with a set of integers \( S_0(n,k) \), such that for \( k > n \), \( S_0(n,k) = 0 \), and \( S_0(0,0) = 1 \), \( S_0(n,0) = 0 \). In particular,
\[
B_0(z,x) \equiv e^{x(e^z-1)} = \sum_{n=0}^{\infty} \left[ \sum_{k=1}^{n} S_0(n,k) x^k \right] \frac{z^n}{n!},
\]
which leads to the (exponential) generating function of \( S_0(n,l) \), the conventional Stirling numbers of the second kind, (see \[1, 8\]), in the form
\[
\frac{(e^z-1)^l}{l!} = \sum_{n=l}^{\infty} S_0(n,l) \frac{z^n}{n!},
\]
and defines the so-called exponential or Touchard polynomials \( l_n^{(0)}(x) \) as
\[
l_n^{(0)}(x) = \sum_{k=1}^{n} S_0(n,k)x^k.
\]
They satisfy
\[
l_n^{(0)}(1) = b_0(n),
\]
\[1\] (others have since been added)
justifying the term “smearing out” used above.

The appearance of integers in eq. (3) suggests a natural extension with an additional variable $x$:

$$B_L(z, x) \equiv e^{[a F_L(z) - 1]} = \sum_{n=0}^{\infty} \left[ \sum_{k=1}^{n} S_L(n, k) x^k \right] \frac{z^n}{(n!)^{L+1}},$$

where we include the right divisors $(n!)^{L+1}$ in the r.h.s of (18).

This in turn defines “hypergeometric” polynomials of type $L$ and order $n$ through

$$B_L(z, x) \equiv e^{[a F_L(z) - 1]} = \sum_{n=0}^{\infty} \left[ \sum_{k=1}^{n} S_L(n, k) x^k \right] \frac{z^n}{(n!)^{L+1}},$$

where we include the right divisors $(n!)^{L+1}$ in the r.h.s of (18).

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$$l_n^{(L)}(x) = \sum_{k=1}^{n} S_L(n, k) x^k,$$

which satisfy

$$l_n^{(L)}(1) = b_L(n),$$

with the $b_L(n)$ of eq. (14). Thus the polynomials of eq. (14) ”smear out” the $b_L(n)$ with the generalized Stirling numbers of the second kind, of type $L$, denoted by $S_L(n, k)$ (with $S_L(n, k) = 0$, if $k > n$, $S_L(n, 0) = 0$ if $n > 0$ and $S_L(0, 0) = 1$), which have, from eq. (15) the “hypergeometric” generating function

$$\left( \frac{a F_L(z) - 1}{L+1} \right) = \sum_{n=0}^{\infty} S_L(n, l) \frac{z^n}{(n!)^{L+1}}, \quad L = 0, 1, 2, \ldots.$$

Eq. (21) can be used to derive a recursion relation for the numbers $S_L(n, k)$, in the same manner as eq. (1) yielded eq. (14). Thus we take the logarithm of both sides of eq. (21), differentiate with respect to $z$, multiply by $z$ and obtain:

$$\left( \sum_{n=0}^{\infty} S_L(n, l - 1) \frac{z^n}{(n!)^{L+1}} \right) \left( \sum_{n=0}^{\infty} \frac{n z^n}{(n!)^{L+1}} \right) = \sum_{n=0}^{\infty} \frac{n S_L(n, l)}{(n!)^{L+1}} z^n,$$

which, with the help of eq. (1), produces the required recursion relation

$$S_L(n + 1, l) = \sum_{k=l-1}^{n} \binom{n}{k} \binom{n + 1}{k} S_L(k, l - 1),$$

$$S_L(0, 0) = 1, \quad S_L(n, 0) = 0,$$

which for $L = 0$ is the recursion relation for the conventional Stirling numbers of the second kind [1], [8], and in eq. (24) the appropriate summation range has been inserted. Since the recursions of eq. (23) and eq. (24) involve only integers we conclude that $S_L(n, l)$ are positive integers.

We have calculated some of the numbers $S_L(n, l)$ using eq. (21) and have listed them in Tables II and III, for $L = 1$ and $L = 2$ respectively. Observe that $S_1(n, 2) = \binom{2n + 1}{n + 1} - 1$ and $S_L(n, n) = (n!)^{L}, L = 1, 2$. Also, by fixing $n$ and $l$, the individual values of $S_L(n, l)$ have been calculated as a function of $L$ with the help of eq. (23), see Table IV, from which we observe

$$S_L(n, n) = (n!)^{L}, \quad L = 1, 2, \ldots$$

which is the lowest diagonal in Table IV. We now demonstrate that the repetitive use of eq. (23) permits one to establish closed-form expressions for any supra-diagonal of order $p$, i.e. the sequence $S_L(n + p, n)$,
for \( p = 1, 2, 3, \ldots \), if one knows the expression for all \( S_L(n+k,n) \) with \( k < p \). We shall illustrate it here for \( p = 1, 2 \). To this end fix \( l = n \) on both sides of eq. (23). It becomes, upon using eq. (25), and defining \( \alpha_L(n) \equiv S_L(n+1,n) \), a linear recursion relation

\[
\alpha_L(n) = \frac{n[(n+1)!]^L}{2^L} + (n+1)^L \alpha_L(n-1), \quad \alpha_L(0) = 0, \tag{26}
\]

with the solution

\[
\alpha_L(n) = S_L(n+1,n) = \frac{n(n+1)}{2} \left[ \frac{(n+1)!}{2} \right]^L = \left[ \frac{(n+1)!}{2} \right]^L S_0(n+1,n), \tag{27}
\]

which gives the second lowest diagonal in Table IV. Observe that for any \( L \), \( S_L(n+1,n) \) is proportional to \( S_0(n+1,n) = n(n+1)/2 \). The sequence \( S_1(n+1,n) = 1, 9, 72, 600, 5400, 856480, \ldots \) is of particular interest: it represents the sum of inversion numbers of all permutations on \( n \) letters \([10]\). For more information about this and related sequences see the entry A001809 in \([10]\). The \( S_L(n+1,n) \) for \( L > 1 \) do not appear to have a simple combinatorial interpretation. A recurrence equation for \( \beta_L(n) \equiv S_L(n+2,n) \) is obtained upon substituting eq. (25) and eq. (27) into eq. (23):

\[
\beta_L(n) = \frac{n(n+1)}{2!} \left[ \frac{(n+2)!}{2!} \right]^L \left( \frac{n-1}{2^L} + \frac{1}{3^L} \right) + (n+2)^L \beta_L(n-1), \quad \beta_L(0) = 0. \tag{29}
\]

It has the solution

\[
S_L(n+2,n) = \frac{n(n+1)(n+2)}{3 \cdot 2^L} \left[ \frac{(n+2)!}{2} \right]^L \left( \frac{3}{2^L(n-1)} + \frac{4}{3^L} \right) \tag{30}
\]

which is a closed form expression for the second lowest diagonal in Table IV. Clearly, eq. (29) for \( L = 0 \) gives the combinatorial form for the series of conventional Stirling numbers

\[
S_0(n+2,n) = \frac{n(n+1)(n+2)(3n+1)}{4!}. \tag{31}
\]

In a similar way we obtain

\[
S_L(n+3,n) = \frac{n(n+1)(n+2)(n+3)}{3 \cdot 2^L} \left[ \frac{(n+3)!}{3} \right]^L \times \left( n^2 \left( \frac{3}{8} \right)^L + n \left( \frac{1}{4^{L-1}} - \frac{3^{L+1}}{8^L} \right) + \frac{2 + 2 \cdot 3^L}{8^L} - \frac{1}{4^{L-1}} \right) \tag{32}
\]

which for \( L = 0 \) reduces to

\[
S_0(n+3,n) = \frac{1}{48} n^2 (n+1)^2(n+2)(n+3). \tag{33}
\]

Combined with the standard definition \([8], [9]\)

\[
S_0(n,l) = \frac{(-1)^l}{l!} \sum_{k=1}^{l} (-1)^k \binom{l}{k} k^n. \tag{34}
\]
eqs. (28), (31) and (33) give compact expressions for the summation form of \( S_0(n+p, n) \). Further, from eq. (34), use of eq. (6) gives the following generating formula:

\[
S_0(n, l) = (-1)^l \frac{l!}{l!} \left[ \left( \frac{d}{dz} \right)^n \left( \sum_{k=1}^{l} (-1)^k \binom{l}{k} z^k \right) \right]_{z=1}, \quad n \geq l.
\]  

The formula (34) can be generalized by putting restrictions on the type of resulting partitions. The generating function for the number of partitions of a set of \( n \) distinct elements without singleton blocks is given by:

\[
B_0(1, z) = e^{e^{z-1}} = \sum_{n=0}^{\infty} b_0(1, n) \frac{z^n}{n!},
\]  

or more generally, without singleton, doubleton \ldots, \( p \)-blocks \((p = 0, 1, \ldots)\) is:

\[
B_0(p, z) = e^{e^{z-\sum_{k=0}^{p} \frac{z^k}{k!}}} = \sum_{n=0}^{\infty} b_0(p, n) \frac{z^n}{n!},
\]  

with the corresponding associated Stirling numbers defined by analogy with eq. (14) and eq. (22). The numbers \( b_0(1, n), b_0(2, n), b_0(3, n), b_0(4, n) \) can be read off from the sequences A000296, A006505, A057837 and A057814 in [10], respectively. For more properties of these numbers see [11].

We carry over this type of extension to eq. (34) and define \( B_L(p, n) \) through:

\[
B_L(p, z) = e^{\frac{\partial}{\partial z} \left( \frac{z}{e^{z-\sum_{k=0}^{p} \frac{z^k}{k!}}} \right)} = \sum_{n=0}^{\infty} b_L(p, n) \frac{z^n}{(n!)^{L+1}},
\]

where \( b_L(0, n) = b_L(n) \) from eq. (34). (We know of no combinatorial meaning of \( b_L(p, n) \) for \( L \geq 1, p > 0 \).) The \( b_L(p, n) \) satisfy the following recursion relations:

\[
b_L(p, n) = \sum_{k=0}^{n-p} \binom{n}{k} \binom{n+1}{k} b_L(p, k),
\]

\[
b_L(p, 0) = 1,
\]

\[
b_L(p, 1) = b_L(p, 2) = \cdots = b_L(p, p) = 0,
\]

\[
b_L(p, p+1) = 1.
\]

That the \( b_L(p, n) \) are integers follows from eq. (40). Through eq. (34) additional families of integer Stirling-like numbers \( S_{L,p}(n, k) \) can be readily defined and investigated.

The numbers \( b_0(p, n) \) are collected in Table V, and Tables VI and VII contain the lowest values of \( b_1(p, n) \) and \( b_2(p, n) \), respectively.

Formula (31) can be used to express \( e \) in terms of \( b_0(n) \) in various ways. Two such lowest order (in differentiation) forms are:

\[
e = 1 + \ln \left( \sum_{n=0}^{\infty} \frac{b_0(n)}{n!} \right) = \ln \left( \sum_{n=0}^{\infty} \frac{b_0(n+1)}{n!} \right).
\]
In the very same way, eq. (3) can be used to express the values of \( \text{\(_0F_L(z)\)} \) and its derivatives at \( z = 1 \) in terms of certain series of \( b_L(n) \)'s. For \( L = 1 \), the analogues of eq. (44) and eq. (45) are

\[
I_0(2) = 1 + \ln \left( \sum_{n=0}^{\infty} \frac{b_1(n)}{(n!)^2} \right),
\]

\[
I_0(2) + \ln(I_1(2)) = 1 + \ln \left( \sum_{n=0}^{\infty} \frac{b_1(n+1)}{(n+1)(n!)^2} \right).
\]

and for \( L = 2 \) the corresponding formulas are

\[
\text{\(_0F_2(1,1;1)\)} = 1 + \ln \left( \sum_{n=0}^{\infty} \frac{b_2(n)}{(n!)^3} \right),
\]

\[
\text{\(_0F_2(1,1;1)\)} + \ln \left( \text{\(_0F_2(2,2;1)\)} \right) = 1 + \ln \left( \sum_{n=0}^{\infty} \frac{b_2(n+1)}{(n+1)^2(n!)^3} \right).
\]

By fixing \( z_0 \) at values other than \( z_0 = 1 \), one can link the numerical values of certain combinations of \( \text{\(_0F_L(1,1,\ldots;z_0)\) , \(_0F_L(2,2,\ldots;z_0)\),...} \) and their logarithms, with other series containing the \( b_L(n) \)'s.

The above considerations can be extended to the exponentiation of the more general hypergeometric functions of type \( \text{\(_0F_L(k_1,k_2,\ldots,k_L;z)\)} \) where \( k_1, k_2, \ldots, k_L \) are positive integers. We conjecture that for every set of \( k_n \)'s a different set of integers will be generated through an appropriate adaptation of eq. (3).

We quote one simple example of such a series. For

\[
\text{\(_0F_2(1,2;z)\)} = \sum_{n=0}^{\infty} \frac{z^n}{(n+1)(n!)^3}
\]

eq. (3) extends to

\[
e^{[\text{\(_0F_2(1,2;z)\)}-1]} = \sum_{n=0}^{\infty} f_2(n) \frac{z^n}{(n+1)(n!)^3}
\]

where the numbers

\[
f_2(n) = (n+1)(n!)^2 \left[ \frac{d^n}{dz^n} e^{[\text{\(_0F_2(1,2;z)\)}-1]} \right]_{z=0}
\]

turn out to be integers: \( f_2(n) \), \( n = 0, 1, \ldots, 8 \) are: 1, 1, 4, 37, 641, 18276, 789377, 48681011, etc. (A061683).

The analogue of equations (23) and (44) is:

\[
\text{\(_0F_2(1,2;1)\)} = 1 + \ln \left( \sum_{n=0}^{\infty} \frac{f_2(n)}{(n+1)(n!)^3} \right).
\]

**Acknowledgements**

We thank L. Haddad for interesting discussions. We have used Maple\textsuperscript{©} to calculate most of the numbers discussed above.
Table I: Table of $b_L(n)$: $L, n = 0, 1, \ldots, 6$. (The rows give sequences A000110, A023998, A061684–A061688.)

<table>
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<th>$b_L(0)$</th>
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<th>$b_L(2)$</th>
<th>$b_L(3)$</th>
<th>$b_L(4)$</th>
<th>$b_L(5)$</th>
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Table II: Table of $S_L(n,l)$: for $L = 1$ and $l,n = 1,2,\ldots, 8$. (The triangle, read by columns, gives A061691, the rows and diagonals give A017063, A061690, A000142, A001809, A061689.)

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<td>4</td>
<td>24</td>
<td>600</td>
<td>10 500</td>
<td>161 700</td>
<td>2 361 016</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>120</td>
<td>5 400</td>
<td>161 700</td>
<td>4 116 000</td>
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<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>720</td>
<td>52 920</td>
<td>2 493 120</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>5 040</td>
<td>564 480</td>
<td></td>
<td></td>
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<td></td>
</tr>
<tr>
<td>8</td>
<td></td>
<td>40 320</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table III: Table of $S_L(n,l)$: for $L = 2$ and $l,n = 1,2,\ldots, 8$. (The triangle, read by columns, gives A061692, the rows and diagonals give A061693, A061694, A001044, A061695.)

<table>
<thead>
<tr>
<th>$l$</th>
<th>$S_2(1,l)$</th>
<th>$S_2(2,l)$</th>
<th>$S_2(3,l)$</th>
<th>$S_2(4,l)$</th>
<th>$S_2(5,l)$</th>
<th>$S_2(6,l)$</th>
<th>$S_2(7,l)$</th>
<th>$S_2(8,l)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
<td>27</td>
<td>172</td>
<td>1 125</td>
<td>7 591</td>
<td>52 479</td>
<td>369 580</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>36</td>
<td>864</td>
<td>17 500</td>
<td>351 000</td>
<td>7 197 169</td>
<td>151 633 440</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>576</td>
<td>36 000</td>
<td>1 746 000</td>
<td>80 262 000</td>
<td>3 691 514 176</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>14 400</td>
<td>1 944 000</td>
<td>191 394 000</td>
<td>17 188 416 000</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>518 400</td>
<td>133 358 400</td>
<td>23 866 214 400</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>25 401 600</td>
<td>11 379 916 800</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>8</td>
<td></td>
<td>1 625 702 400</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Table IV: Table of $S_L(n, l)$: $l, n = 1, 2, \ldots, 6$.

<table>
<thead>
<tr>
<th>$l$</th>
<th>$S_L(1, l)$</th>
<th>$S_L(2, l)$</th>
<th>$S_L(3, l)$</th>
<th>$S_L(4, l)$</th>
<th>$S_L(5, l)$</th>
<th>$S_L(6, l)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>$(2!)^L$</td>
<td>$3 \cdot 3^L$</td>
<td>$4 \cdot 4^L + 3 \cdot 6^L$</td>
<td>$5 \cdot 5^L + 10 \cdot 10^L$</td>
<td>$6 \cdot 6^L + 15 \cdot 15^L + 10 \cdot 20^L$</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>$(3!)^L$</td>
<td>$6 \cdot 12^L$</td>
<td>$10 \cdot 20^L + 15 \cdot 30^L$</td>
<td>$15 \cdot 30^L + 60 \cdot 60^L + 15 \cdot 90^L$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>$(4!)^L$</td>
<td>$10 \cdot 60^L$</td>
<td>$20 \cdot 120^L + 45 \cdot 180^L$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>$(5!)^L$</td>
<td>$15 \cdot 360^L$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>$(6!)^L$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table V: Table of $b_0(p, n)$: $p = 0, 1, 2, 3$; $n = 0, \ldots, 10$. (The columns give A000110, A000296, A006505, A057837.)

<table>
<thead>
<tr>
<th>$n$</th>
<th>$b_0(0, n)$</th>
<th>$b_0(1, n)$</th>
<th>$b_0(2, n)$</th>
<th>$b_0(3, n)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>5</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>15</td>
<td>4</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>5</td>
<td>52</td>
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<td>1</td>
<td>1</td>
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<td>6</td>
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<td>41</td>
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<td>7</td>
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<td>36</td>
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<td>17722</td>
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<td>337</td>
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</tbody>
</table>

Table VI: Table of $b_1(p, n)$: $p = 0, 1, 2$; $n = 0, \ldots, 9$. (The columns give A023998, A061696, A061697.)

<table>
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<tr>
<th>$n$</th>
<th>$b_1(0, n)$</th>
<th>$b_1(1, n)$</th>
<th>$b_1(2, n)$</th>
</tr>
</thead>
<tbody>
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<td>0</td>
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<td>1</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
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<td>0</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>1</td>
<td>0</td>
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<tr>
<td>3</td>
<td>16</td>
<td>1</td>
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</tr>
<tr>
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<td>131</td>
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<td>101</td>
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<td>1776</td>
<td>201</td>
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<td>7</td>
<td>426833</td>
<td>23717</td>
<td>1226</td>
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<td>8</td>
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<td>515971</td>
<td>5587</td>
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<td>9</td>
<td>277006192</td>
<td>11893597</td>
<td>493333</td>
</tr>
</tbody>
</table>
Table VII: Table of $b_2(p, n)$: $p = 0, 1, 2$; $n = 0, \ldots, 8$. (The columns give A061698–A061700.)

<table>
<thead>
<tr>
<th>$n$</th>
<th>$b_2(0, n)$</th>
<th>$b_2(1, n)$</th>
<th>$b_2(2, n)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
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<td>1</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>5</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
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<td>4</td>
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<td>109</td>
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<td>69026</td>
<td>1001</td>
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<td>4566992</td>
<td>128876</td>
<td>4001</td>
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<td>57903766800</td>
<td>792013069</td>
<td>347117</td>
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References


(Mentions sequences A000296 A001044 A001809 A006505 A010763 A023998 A057814 A057837 A061683 A061684 A061685 A061686 A061687 A061688 A061689 A061690 A061691 A061692 A061693 A061694 A061695 A061696 A061697 A061698 A061699 A061700.)

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