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# Absence of absolutely continuous diffraction spectrum for certain S-adic tilings

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## Abstract

Quasiperiodic tilings are often considered as structure models of quasicrystals. In this context, it is important to study the nature of the diffraction measures for tilings. In this article, we investigate the diffraction measures for S-adic tilings in  $\mathbb{R}^d$ , which are constructed from a family of geometric substitution rules. In particular, we firstly give a sufficient condition for the absolutely continuous component of the diffraction measure for an S-adic tiling to be zero. Next, we prove this sufficient condition for ‘almost all’ binary block-substitution cases and thus prove the absence of the absolutely continuous diffraction spectrum for most of S-adic tilings from a family of binary block substitutions.

Keywords: tiling, diffraction, S-adic sequence, Lyapunov exponent, renormalisation

Mathematics Subject Classification numbers: 52C23, 52C22.

## 1. Introduction

A tiling is a cover of  $\mathbb{R}^d$  by its countably many subsets (tiles)  $T$  with the property that  $T = \overline{T^\circ}$  (i.e. each tile is the closure of its interior). There exist tilings  $\mathcal{T}$  that are non-periodic (meaning that  $\mathcal{T} + x = \mathcal{T}$  holds for  $x = 0$  only) but still admit repetitions of patterns: for

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example,  $\mathcal{T}$  may be repetitive [6, definition 5.8], or almost periodic in a sense, such as in [7, chapter 5] and in [21], to name a few. For this reason, such tilings are often considered as structure models of quasicrystals. The diffraction measures defined for these tilings then correspond to physical diffraction patterns. In this context, it is important to study the nature of diffraction measures for tilings. Especially, it is interesting to know when a diffraction measure is pure point (a sum of point or Dirac measures), or its absolutely continuous part with respect to the Lebesgue measure is zero, since these conditions are considered to be related to the long-range order of the original tiling. The long-range order of a tiling is not a well-defined term, but at least related to its diffraction measure. To clarify the situation on long-range order, it is worthwhile to investigate when the diffraction measure satisfies the above conditions.

There are several ways to construct interesting non-periodic tilings. One of the most common approaches is via substitution (or inflation) rules. (There are ‘symbolic’ substitution rules and ‘geometric’ ones, the spectrum of which are related [18], but in this article we only deal with ‘geometric’ ones.) Given a substitution rule  $\rho$  in  $\mathbb{R}^d$ , it gives rise to self-affine tilings, which are often repetitive and almost-periodic. The class of self-affine tilings is included in the class of S-adic tilings, which are tilings that are generated by a family of substitution rules. Similarly to self-affine tilings, the S-adic tilings possess self-similar structures or inverse-limit structures, which make such tilings interesting. The class of S-adic tilings is far larger than the class of self-affine tilings. In this article, we investigate this larger class of tilings. In particular, we investigate the relations between the inverse-limit structure and one form of order of the tiling, that is, the absence of the absolutely continuous part of its diffraction spectrum.

The spectral properties of self-affine tilings have been intensively studied. A key conjecture is the Pisot substitution conjecture, which states that self-affine tilings obtained from substitution rules of Pisot type are pure point diffractive, that is, their diffraction measures are pure point. This is still an open problem, but there are several partial positive answers. Here, we just mention that the binary one-dimensional case, in which there are only two tiles up to translation, is solved [28]. The definition of Pisot type for substitution includes irreducibility, but for some reducible cases, in the setting where the substitution is binary block-substitution, Mañibo [22, 23] and Baake–Grimm [8] proved the absence of absolutely continuous components in the diffraction pattern.

On the symbolic side, S-adic words, each of which is generated by a family of symbolic substitutions, have been studied. The simplest class of S-adic words is one of words via finite families of substitutions. Such S-adic words are relevant in the context of continued fractions and interval exchange transformations (see for example [11] and references therein) and of Schrödinger operators [17]. The study of the spectral properties of S-adic words is undertaken in the literature and the Pisot conjecture is proved for special cases [12, 13].

It is natural to study the spectral properties of the geometric analogues for S-adic words, namely S-adic tilings. As in the study of S-adic words, we should start with the simplest situation where we consider only finitely many geometric substitutions. In this article, we study the diffraction spectrum for S-adic tilings in  $\mathbb{R}^d$  defined via a finite family of (geometric) substitutions, which generalises the single substitution (self-affine) case [8, 22, 23]. We generalise the method from [5, 23] to prove that (I) an inequality for Fourier matrices is sufficient for the absence of an absolutely continuous component in the diffraction measure, for quite a general class of S-adic tilings (including, but not only, the binary case), and (II) the sufficient condition in (I) is satisfied for ‘almost all’ binary block-substitution cases, and so, for such an S-adic tiling, the absolutely continuous part of the diffraction measure is zero. The precise statement for claim (I) is found in theorem 3.2, in the setting specified in setting 3.1. The special case for claim (II) is elaborated on below, and the precise statement for claim (II) is theorem 3.30, where the setting for this result is detailed in setting 3.19. The key ingredients

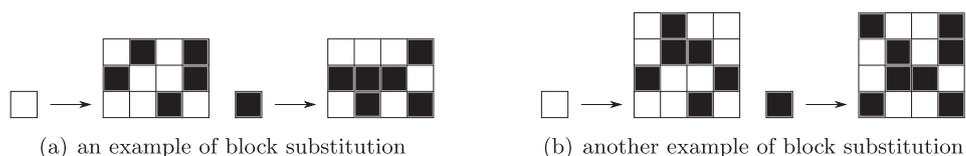


Figure 1. Examples of block substitutions.

are renormalisation technique developed by [3, 4, 8, 9, 22, 23] and Furstenberg–Kesten and Oseledets theorems.

To elaborate on the claim (II), let us consider two substitutions,  $\rho_1$  in figure 1(a) and  $\rho_2$  in figure 1(b). Such substitutions (one with prototiles with support  $[0, 1]^d$ ) are called block substitutions. For arbitrary sequence  $i_1, i_2, \dots$  in  $\{1, 2\}^{\mathbb{N}}$ , by choosing an appropriate increasing sequence  $n_1 < n_2 < \dots$  and appropriate patches  $\mathcal{P}_k, k = 1, 2, \dots$ , we have a convergence

$$\mathcal{T} = \lim_{k \rightarrow \infty} \rho_{i_1} \circ \rho_{i_2} \circ \dots \circ \rho_{i_{n_k}}(\mathcal{P}_k)$$

and  $\mathcal{T}$  is a tiling. (For details, see page 7969.) Such  $\mathcal{T}$  is called an S-adic tiling belonging to the directive sequence  $(i_n)_n$  for  $\rho_1, \rho_2$ . The special case for the main result of this paper (theorem 3.30) is as follows.

**Theorem 1.1. (A special case of theorem 3.30).** *Let  $p_1, p_2$  be two positive real numbers with  $p_1 + p_2 = 1$ . Endow  $\{1, 2\}^{\mathbb{N}}$  the product probability measure  $\mu$  for the probability measure on  $\{1, 2\}$  defined by  $(p_1, p_2)$ . Then, for  $\mu$ -almost all  $(i_n)_n \in \{1, 2\}^{\mathbb{N}}$ , the S-adic tilings belonging to  $(i_n)_n$  for  $\rho_1, \rho_2$  have zero absolutely continuous diffraction spectrum, that is, the absolutely continuous part of the diffraction measure is zero.*

We can replace  $\{1, 2\}^{\mathbb{N}}$  with its subshift  $X$ , as follows:

**Theorem 1.2. (A special case of theorem 3.30).** *Let  $X$  be a subshift of  $\{1, 2\}^{\mathbb{N}}$  which admits shift-invariant ergodic Borel probability measure  $\mu_X$ . Assume the shift map on  $X$  is surjective. Then, for  $\mu_X$ -almost all  $(i_n)_n \in X$ , the S-adic tilings belonging to  $(i_n)_n$  for  $\rho_1, \rho_2$  have zero absolutely continuous diffraction spectrum.*

Note that this is not included in theorem 1.1 because  $\mu(X)$  might be zero.

We can replace  $\rho_1, \rho_2$  with arbitrary finite family of binary block substitutions, with a mild assumption on substitution matrices, with possibly different expansion maps. Furthermore, the dimension can be arbitrary: for any  $d = 1, 2, 3, \dots$  and binary block substitutions in  $\mathbb{R}^d$ , we have similar results.

The above treatment of S-adic tiling deals with them as ‘randomisations’ of single-substitution cases (see for example [11] for the argument on the symbolic side). There is another way of randomisation of substitution rules, namely random substitutions (see for example [20]). We consider classes of S-adic tilings as a different sort of randomisations of substitutions. By taking this point of view, we can take advantage of ergodic theory to study S-adic tilings.

Note that sometimes in the literature for S-adic words, only directive sequences that are given by graphs are considered, but in this paper we can deal with almost all directive sequences from arbitrary subshifts with a mild assumption.

This paper is organised as follows. In section 2, we introduce our notation and some necessary background. Section 3 contains our main results; in particular, we state and prove

claims (I) and (II) given above. The first claim is proved in section 3.1, while the second claim is proved in section 3.2. We defer the proofs of some of our claims in section 3.2 to an appendix.

## 2. General background

### 2.1. Notations

For a finite set  $F$ , we denote its cardinality by  $\#F$ . In this article,  $\mu_L$  denotes the Lebesgue measure on  $\mathbb{R}^d$ . The symbol  $\mathbb{T}$  refers to the one-dimensional torus  $\{z \in \mathbb{C} \mid |z| = 1\}$ . For a natural number  $n$ , we will identify  $\mathbb{T}^n$  measure-theoretically with  $[0, 1)^n$ , on which the Lebesgue measure  $\mu_L$  is the complete rotation-invariant probability measure. Let  $\pi : \mathbb{R}^n \rightarrow \mathbb{T}^n$  be defined via  $\pi(s_1, s_2, \dots, s_n) = (e^{2\pi i s_1}, e^{2\pi i s_2}, \dots, e^{2\pi i s_n})$ . In  $\mathbb{R}^d$ , for  $x \in \mathbb{R}^d$  and  $R > 0$ , the closed ball  $\{y \in \mathbb{R}^d \mid \|x - y\| \leq R\}$  is denoted by  $B(x, R)$ . If  $x = 0$ , we use the symbol  $B_R$  for  $B(0, R)$ .

### 2.2. A generality for tilings and substitutions

In this section, we sketch a generality for the theory of tilings. For a detailed exposition, we refer to [6]. Let  $d$  be a natural number and we consider tilings in  $\mathbb{R}^d$ .

Let  $L$  be a finite set. A *labelled tile* is a pair  $T = (S, \ell)$  consisting of a compact set  $S$  in  $\mathbb{R}^d$  with  $\overline{S^\circ} = S$  (the closure of the interior coincides with the original  $S$ ) and an element  $\ell \in L$ . The set  $S$  is called the *support* of  $T$  and denoted by  $\text{supp } T$ . The element  $\ell$  is called the *label* of  $T$ .

Alternatively, we can consider ‘unlabelled’ tiles, that is, a compact subset  $T$  of  $\mathbb{R}^d$  such that  $\overline{T^\circ} = T$ . For an unlabelled tile  $T$ , we denote the space it covers (that is,  $T$  itself) by  $\text{supp } T$ , called the support of  $T$ , in order to cover the theory for labelled and for unlabelled tiles by the same notation. Both labelled tiles and unlabelled tiles are called tiles. We deal with both cases simultaneously by the above notation. Note that each of the cases are required because (1) we often have to distinguish two tiles with the same support by assigning them different labels, as in example 2.1, and because (2) we often meet situations where tiles have different support and labels are hence redundant, as in example 2.2.

A set  $\mathcal{P}$  of tiles in  $\mathbb{R}^d$  is called a *patch* if  $(\text{supp } T)^\circ \cap (\text{supp } S)^\circ = \emptyset$  for each distinct  $S$  and  $T$  in  $\mathcal{P}$ . The support of a patch  $\mathcal{P}$  is the subset  $\bigcup_{T \in \mathcal{P}} \text{supp } T$  of  $\mathbb{R}^d$  and is denoted by  $\text{supp } \mathcal{P}$ . (Sometimes we take the closure after taking the union in this definition, but in this article we only deal with situations where the union is already a closed set. We use the same notation as the support of a tile, but there is no possibility of confusion.) A patch  $\mathcal{P}$  is called a *tiling* if  $\text{supp } \mathcal{P} = \mathbb{R}^d$ .

For an unlabelled tile  $S$ ,  $S + x$  denotes the usual translation. For a labelled tile  $T = (S, \ell)$  and  $x \in \mathbb{R}^d$ , we set  $T + x = (S + x, \ell)$ . For a patch  $\mathcal{P}$  (with either labelled or unlabelled tiles) and  $x \in \mathbb{R}^d$ , we define the translate of  $\mathcal{P}$  by  $x$  via

$$\mathcal{P} + x = \{T + x \mid T \in \mathcal{P}\}.$$

A tiling  $\mathcal{T}$  is said to be *non-periodic* if  $x = 0$  is the only element in  $\mathbb{R}^d$  that satisfies  $\mathcal{T} + x = \mathcal{T}$ . In this article, we are mainly interested in non-periodic tilings.

There are several ways to construct interesting non-periodic tilings. In this article, we consider tilings constructed via substitution rules. First, for a finite set  $\mathcal{A}$  of tiles in  $\mathbb{R}^d$ , let  $\mathcal{A}^*$  be the set of all patches of which tiles are translates of elements of  $\mathcal{A}$ . A *substitution rule* (or an *inflation rule*) is a triple  $\sigma = (\mathcal{A}, \phi, \rho)$  where

- $\mathcal{A}$  is a finite set of tiles, called the *alphabet* of  $\sigma$ ,
- $\phi : \mathbb{R}^d \rightarrow \mathbb{R}^d$  is a linear map with  $\min_{\|v\|=1} \|\phi(v)\| > 1$ , called the *expansion map*, and
- $\rho$  is a map  $\mathcal{A} \rightarrow \mathcal{A}^*$  such that

$$\text{supp } \rho(T) = \phi(\text{supp } T)$$

holds for each  $T \in \mathcal{A}$ .

The map  $\rho$  itself is also often referred to as a substitution (or inflation) rule. The third condition (on the supports) means that the map  $\rho$  gives the result of first expanding the tile  $T$  by the expansion map  $\phi$  and then subdividing it to obtain a patch  $\rho(T)$ . The following examples will illustrate this point.

**Example 2.1.** Let us consider the case where  $d = 1$ . Let  $T_1 = ([0, 1], 1)$  and  $T_2 = ([0, 1], 2)$ . The Thue–Morse substitution is a substitution  $\rho_{\text{TM}}$  of which alphabet is  $\{T_1, T_2\}$ , expansion map is  $\mathbb{R} \ni x \mapsto 2x \in \mathbb{R}$  and the rule is given by

$$\begin{aligned} \rho_{\text{TM}}(T_1) &= \{T_1, T_2 + 1\} \\ \rho_{\text{TM}}(T_2) &= \{T_2, T_1 + 1\}. \end{aligned}$$

The final condition in the definition of a substitution rule is indeed satisfied for this rule, since  $\text{supp } \rho_{\text{TM}}(T_i) = [0, 2]$  and  $2 \text{supp } T_i = [0, 2]$  for  $i = 1, 2$ .

**Example 2.2.** Again, consider the case where  $d = 1$ . Set  $\tau = \frac{1+\sqrt{5}}{2}$ , the golden ratio. Let  $\mathcal{A} = \{T_a, T_b\}$ , where  $T_a = [0, \tau]$  and  $T_b = [0, 1]$ . The Fibonacci substitution is the map  $\rho_{\text{F}}$

$$\begin{aligned} \rho_{\text{F}}(T_a) &= \{T_a, T_b + \tau\}, \\ \rho_{\text{F}}(T_b) &= \{T_a\}. \end{aligned}$$

Again, with an expansion map  $\mathbb{R} \ni x \mapsto \tau x \in \mathbb{R}$ , the final condition in the definition of a substitution rule is satisfied since  $\tau^2 = \tau + 1$ .

**Example 2.3.** A substitution rule such that the supports of elements of the alphabet are all  $[0, 1]^d$  and the expansion map is a diagonal matrix with natural numbers greater than 1 as diagonal entries is called a block substitution. For example, figures 1(a) and (b) are block substitutions with a common alphabet  $\mathcal{A} = \{([0, 1]^2, B), ([0, 1]^2, W)\}$  and an expansion map defined by

$$\begin{pmatrix} 4 & 0 \\ 0 & 3 \end{pmatrix},$$

and one by  $4I$  ( $I$  being the identity matrix), respectively.

**Example 2.4.** It is more difficult to find multiple substitutions with a shared alphabet with more complicated supports than to find ones with block-supported proto-tiles. In [19], the authors found multiple substitutions with the shared alphabet consisting of triangles. We can also obtain several substitutions that share the common alphabet with possibly non-block supports by taking one substitution and defining another substitution by exchanging the positions of some tiles in the supertile, or by exchanging some tiles in the supertile with another ones.

For a substitution rule  $\rho$ , one can define a displacement matrix and Fourier matrix, which will play important roles in the study of diffraction, as follows. Let  $\mathcal{A} = \{T_1, T_2, \dots, T_{n_a}\}$  be the alphabet for the substitution  $\rho$  in  $\mathbb{R}^d$ . For each  $i$  and  $j$ , there is a *digit set*  $T_{i,j} \subset \mathbb{R}^d$  for  $\rho$ , which is determined by

$$\rho(T_j) = \{T_i + x \mid i \in \{1, 2, \dots, n_a\}, x \in T_{i,j}\}. \tag{1}$$

The *substitution matrix* of  $\rho$  is the matrix whose  $(i, j)$ -element is  $\#T_{i,j}$ . We then define the *Fourier matrix*,  $B$ , which is a  $n_a \times n_a$  matrix function on  $\mathbb{R}^d$ . We need to specify its value  $B(t)$  for each  $t \in \mathbb{R}^d$ . We define the  $(i, j)$  component of  $B(t)$ , denoted by  $B_{i,j}(t)$ , as

$$B_{i,j}(t) = \sum_{s \in T_{i,j}} e^{2\pi i \langle s, t \rangle},$$

where  $\langle \cdot, \cdot \rangle$  is the standard inner product in  $\mathbb{R}^d$ . Let us consider an explicit example to illustrate these definitions.

**Example 2.5.** For the Thue–Morse substitution  $\rho_{\text{TM}}$  from example 2.1, the Fourier matrix is

$$B(t) = \begin{pmatrix} 1 & e^{2\pi i t} \\ e^{2\pi i t} & 1 \end{pmatrix}.$$

Given a geometric substitution rule  $\rho$  in  $\mathbb{R}^d$ , one can construct a tiling in  $\mathbb{R}^d$  by iterating the map  $\rho$ . To be more precise, for a given geometric substitution rule  $\rho$ , we can define a map  $\rho : \mathcal{A}^* \rightarrow \mathcal{A}^*$  (denoted by the same symbol), as follows. First, for  $T \in \mathcal{A}$  and  $x \in \mathbb{R}^d$ , set  $\rho(T + x) = \rho(T) + \phi(x)$  ( $\phi$  being the expansion map). Then we define  $\rho(\mathcal{P})$ , where  $\mathcal{P}$  is a patch consisting of translates of elements of  $\mathcal{A}$  (that is, an element of  $\mathcal{A}^*$ ), via

$$\rho(\mathcal{P}) = \bigcup_{T \in \mathcal{P}} \rho(T).$$

Since now the domain and the range of the new map  $\rho$  are the same, we can iterate it. We can often take the limit

$$\lim_{n \rightarrow \infty} \rho^{kn}(\mathcal{P}) \tag{2}$$

to obtain a tiling, for a suitable  $k > 0$  and an initial patch  $\mathcal{P}$ . The convergence in equation (2) is with respect to the *local matching topology*, in which two patches  $\mathcal{P}$  and  $\mathcal{Q}$  are ‘close’ if there are small displacements  $x, y \in \mathbb{R}^d$  such that  $\mathcal{P} + x$  and  $\mathcal{Q} + y$  agree inside  $B_R$  for some large  $R > 0$ . (See, for example, [6, p 129].)

**Example 2.6.** For the Thue–Morse substitution  $\rho_{\text{TM}}$ , define  $\mathcal{P}$  via

$$\mathcal{P} = \{T_1 - 1, T_0\}.$$

Then,  $\rho_{\text{TM}}^2(\mathcal{P})$  is (if we write it symbolically) 1001.0110, where  $\cdot$  denotes the place of origin. We obtain  $\rho_{\text{TM}}^2(\mathcal{P}) \supset \mathcal{P}$ , and this in turn means that  $\rho_{\text{TM}}^2(\mathcal{P}) \subset \rho_{\text{TM}}^4(\mathcal{P}) \subset \rho_{\text{TM}}^6(\mathcal{P}) \cdots$ . The patches obtained by iteration ‘grow’ in  $\mathbb{R}$ , and in the limit they form a tiling

$$\lim_{n \rightarrow \infty} \rho_{\text{TM}}^{2n}(\mathcal{P}) = \bigcup_{n > 0} \rho_{\text{TM}}^{2n}(\mathcal{P}),$$

which is called a *Thue–Morse tiling*.

An *S-adic tiling* is a tiling obtained by replacing each of  $kn$   $\rho$ 's in  $\rho^{kn}(\mathcal{P})$  in equation (2) with a substitution rule from a finite set of geometric substitution rules. To be precise, consider a finite set  $\{\rho_1, \rho_2, \dots, \rho_{m_a}\}$  of geometric substitution rules in  $\mathbb{R}^d$  that share the same alphabet  $\mathcal{A}$  but do not necessarily share the same expansion map. We call sequences  $i_1, i_2, \dots$  of elements of  $\{1, 2, \dots, m_a\}$  *directive sequences*. Given a directive sequence  $i_1, i_2, \dots$ , any tiling  $\mathcal{T}$  of the form

$$\mathcal{T} = \lim_{l \rightarrow \infty} \rho_{i_1} \circ \rho_{i_2} \circ \dots \circ \rho_{i_{n_l}}(\mathcal{P}_l), \tag{3}$$

where  $n_1 < n_2 < \dots$  and where the  $\mathcal{P}_l$  are patches that are included in some  $\rho_{j_1} \circ \rho_{j_2} \circ \dots \circ \rho_{j_m}(\mathcal{P})$  ( $m > 0, j_1, j_2, \dots, j_m \in \{1, 2, \dots, m_a\}$  and  $\mathcal{P} \in \mathcal{A}$ ), is called an *S-adic tiling belonging to the directive sequence  $i_1, i_2, \dots$  for the family  $\{\rho_1, \rho_2, \dots, \rho_{m_a}\}$* . This is a geometric version of the symbolic S-adic sequences (see for example [11]) and the order of  $\rho_{i_j}$  in (3) coincides with one for the symbolic counterpart. The convergence in (3) is assured by the following finiteness condition. In general, the family  $\{\rho_1, \rho_2, \dots, \rho_{m_a}\}$  is said to have *finite local complexity (FLC)* if for each compact  $K \subset \mathbb{R}^d$  the set

$$\begin{aligned} & \{\rho_{j_1} \circ \rho_{j_2} \circ \dots \circ \rho_{j_n}(\mathcal{P}) \cap (K + x) \mid \mathcal{P} \in \mathcal{A}, \\ & n > 0, j_1, j_2, \dots, j_n \in \{1, 2, \dots, m_a\}, x \in \mathbb{R}^d\} \end{aligned}$$

is finite up to translation, where the symbol  $\cap$  is defined via

$$\mathcal{P} \cap S = \{T \in \mathcal{P} \mid \text{supp } T \cap S \neq \emptyset\}$$

for a patch  $\mathcal{P}$  in  $\mathbb{R}^d$  and an  $S \subset \mathbb{R}^d$ . If the set of substitutions  $\{\rho_1, \rho_2, \dots, \rho_{m_a}\}$  have FLC, given an arbitrary directive sequence  $i_1, i_2, \dots$ , we can find some  $n_1 < n_2 < \dots$  and some patches  $\mathcal{P}_l$  such that the limit in (3) converges, because the patches after the  $\lim$  symbol in (3) are included in a compact set. This is seen by the fact that a space  $X$  of patches in  $\mathbb{R}^d$  such that for each compact  $K \subset \mathbb{R}^d$

$$\{\mathcal{P} \cap (K + x) \mid \mathcal{P} \in X, x \in \mathbb{R}^d\}$$

is finite up to translation is relatively compact, by the standard diagonalisation argument ([26, theorem 1.1] or [24, corollary 3.20 and Lemma 3.24]). We can start with a sequence  $(Q_n)_n$  of patches and the sequence

$$\rho_{i_1} \circ \rho_{i_2} \circ \dots \circ \rho_{i_n}(Q_n), \quad n = 1, 2, \dots$$

admits a convergent subsequence.

In the discussion of *symbolic* S-adic sequences, we can consider cases where the symbolic substitution rules do not share the same alphabet, but in this article we only deal with the case where substitutions are geometric and share a common alphabet. This is a strong assumption but all the block substitutions, which we mainly deal with in this paper, and ones in example 2.4 are included in our scope. Often, given a directive sequence  $i_1, i_2, \dots$ , we use the notation

$$\rho_{i[k,l]} = \rho_{i_k} \circ \rho_{i_{k+1}} \circ \dots \circ \rho_{i_{l-1}}, \tag{4}$$

for two positive integers  $k < l$ .

Given an S-adic tiling of the form (3), the sequence  $(\rho_{i[2,n_l]}(\mathcal{P}_l))_{l > 0}$  admits a convergent subsequence, again by a diagonalisation argument as above. We can take a subsequence

$(n_l^{(2)}, \mathcal{P}_l^{(2)})_l$  of the sequence  $(n_l, \mathcal{P}_l)$  so that that the limit  $\lim_l \rho_{[2, n_l^{(2)}]}(\mathcal{P}_l^{(2)})$  converges. We can further take a subsequence  $(n_l^{(3)}, \mathcal{P}_l^{(3)})_l$  of  $(n_l^{(2)}, \mathcal{P}_l^{(2)})_l$  such that the limit  $\lim_l \rho_{[3, n_l^{(3)}]}(\mathcal{P}_l^{(3)})$  converges. Proceeding in this way, we can take nested subsequences  $(n_l^{(k)}, \mathcal{P}_l^{(k)})_l$  for  $k = 1, 2, 3, \dots$ . We set  $m_l = n_l^{(l)}$  and  $\mathcal{Q}_l = \mathcal{P}_l^{(l)}$  for  $l = 1, 2, \dots$ . Then, we have convergences

$$\mathcal{T}^{(k)} = \lim_{l \rightarrow \infty} \rho_{[l, m_l]}(\mathcal{Q}_l)$$

for each  $k > 0$  with common  $(m_l)_l$  and  $(\mathcal{Q}_l)_l$ . This implies that, for each  $k$ , we have  $\rho_k(\mathcal{T}^{(k+1)}) = \mathcal{T}^{(k)}$ . These ‘de-substituted tilings’  $\mathcal{T}^{(2)}, \mathcal{T}^{(3)}, \dots$  of the given  $\mathcal{T} = \mathcal{T}^{(1)}$  give us a hierarchical structure for  $\mathcal{T}^{(1)}$ , which is similar to self-affine tilings, and will be useful later; such an inverse-limit structure enables us to construct renormalisation scheme, by which we can use ergodic theory to study the diffraction spectrum for  $\mathcal{T}^{(1)}$ .

### 2.3. Patch frequencies

In order to discuss the diffraction of tilings, we use the concept of the frequency of patches. In general, if  $\mathcal{T}$  is a tiling in  $\mathbb{R}^d$ , if  $\mathcal{P}$  is a (usually finite) non-empty patch and if the limit

$$\lim_{R \rightarrow \infty} \frac{1}{\mu_L(B_R)} \# \{t \in B_R \mid \mathcal{P} + t \subset \mathcal{T}\}$$

converges, this limit is called the *frequency of  $\mathcal{P}$  in  $\mathcal{T}$*  and denoted by  $\text{freq}_{\mathcal{T}} \mathcal{P}$  or  $\text{freq } \mathcal{P}$ . (Here, we consider averaging with respect to  $\{B_R \mid R > 0\}$ , but we can also consider averaging along van Hove sequences.) If  $\mathcal{T}$  is an S-adic tiling, often the following *uniform patch frequency* holds.

**Theorem 2.7.** *Let  $\rho_1, \rho_2, \dots, \rho_{m_a}$  be (geometric) substitution rules in  $\mathbb{R}^d$  that share a common alphabet. Let  $A_i$  be the substitution matrix for  $\rho_i$ . Take a directive sequence  $i_1, i_2, \dots \in \{1, 2, \dots, m_a\}$  and an S-adic tiling  $\mathcal{T}$  belonging to this directive sequence. Assume the following four conditions:*

- (a) *there are  $n_0 > 0$  and  $i_{0,1}, i_{0,2}, \dots, i_{0,n_0} \in \{1, 2, \dots, m_a\}$  such that all entries in the product matrix*

$$A_{i_{0,1}} A_{i_{0,2}} \cdots A_{i_{0,n_0}}$$

*are greater than 0;*

- (b) *for any  $n > 0$  there is  $k > n$  such that*

$$i_k = i_{0,1}, \quad i_{k+1} = i_{0,2}, \quad \dots, \quad i_{k+n_0-1} = i_{0,n_0},$$

- (c) *for each  $i$ , every row in  $A_i$  is non-zero, and*
- (d) *for each  $P \in \mathcal{A}$ , the sequence  $(\phi_{i_1} \circ \phi_{i_2} \circ \dots \circ \phi_{i_n}(\text{supp } P))_n$  has the van Hove property.*

*Then, for any finite non-empty patch  $\mathcal{P}$ , there is  $c_{\mathcal{P}} \in \mathbb{R}$  such that*

$$\lim_{R \rightarrow \infty} \frac{1}{\mu_L(B_R)} \# \{t \in B_R \mid \mathcal{P} + t \subset \mathcal{S}\} = c_{\mathcal{P}}$$

*converges uniformly for  $\mathcal{S} \in \{\mathcal{T} + t \mid t \in \mathbb{R}^d\}$ .*

**Sketch of proof.** This is the ‘geometric’ version of the argument in [11, section 5.2] and the proof is similar.  $\square$

Note that the uniform convergence on the orbit  $\{\mathcal{T} + t | t \in \mathbb{R}^d\}$  implies the uniform convergence on the continuous hull, the closure of the orbit with respect to the local matching topology. Note also that for the single substitution case ( $m_a = 1$ ), the above conditions (a)–(c) for the convergence of patch frequency are satisfied if the substitution matrix is primitive.

2.4. *Fourier transform, diffraction and the Lebesgue decomposition*

The diffraction measures associated with tilings are physically important. They model the results of diffraction experiments. Mathematically, the diffraction measure of a tiling is the Fourier transform of the autocorrelation measure associated with the tiling, described as follows.

In what follows, we have to deal with objects such as  $\sum_{t \in D} c_t \delta_t$ , where  $D \subset \mathbb{R}^d$ ,  $c_t \in \mathbb{C}$  and  $\delta_t$  is the Dirac (point) measure at  $t$ . We consider them as complex measures in the sense of [14] and call them Radon measures. For a Radon measure  $\mu$  on  $\mathbb{R}^d$  and a function  $\varphi \in L^1(\mu)$ , we use a notation

$$\langle \varphi, \mu \rangle = \int_{\mathbb{R}^d} \varphi \, d\mu.$$

Let  $C_c(\mathbb{R}^d)$  denote the vector space of all complex-valued, continuous, compactly supported functions on  $\mathbb{R}^d$ . According to [2], a Radon measure  $\mu$  on  $\mathbb{R}^d$  is said to be *Fourier transformable* if there is another Radon measure  $\nu$  on  $\mathbb{R}^d$  such that, for each  $\varphi, \psi \in C_c(\mathbb{R}^d)$ , the inverse Fourier transform  $\widetilde{\varphi * \psi}$  of the convolution of  $\varphi$  and  $\psi$  is in  $L^1(\nu)$  and

$$\langle \varphi * \psi, \mu \rangle = \langle \widetilde{\varphi * \psi}, \nu \rangle$$

If such a  $\nu$  exists, it is unique, called the *Fourier transform* of  $\mu$  and denoted by  $\hat{\mu}$ . It is known that if  $\mu$  is *positive definite*, that is, if for each  $\varphi \in C_c(\mathbb{R}^d)$  we have

$$\langle \mu, \varphi * \bar{\varphi} \rangle \geq 0,$$

then  $\mu$  is Fourier transformable and the Fourier transform  $\hat{\mu}$  is positive [7, theorem 4.11.5].

Given a Radon measure  $\mu$ , we define its *diffraction measure* as follows. First, assume the following limit, the *autocorrelation measure*, exists:

$$\mu \otimes \tilde{\mu} = \lim_{R \rightarrow \infty} \frac{\mu|_{B_R} * \tilde{\mu}|_{B_R}}{\mu_L(B_R)}, \tag{5}$$

where, for a Radon measure  $\mu$  and a subset  $S \subset \mathbb{R}^d$ , the restriction  $\mu|_S$  is a Radon measure that sends  $\varphi \in C_c(\mathbb{R}^d)$  to  $\int_S \varphi u \, d\mu$ .  $\tilde{\mu}$  is defined via  $\langle \tilde{\mu}, \varphi \rangle = \overline{\langle \mu, \bar{\varphi} \rangle}$ , where  $\bar{\varphi}(t) = \overline{\varphi(-t)}$  for each  $t \in \mathbb{R}^d$ . The limit (5) is nothing but a Radon measure that sends  $\varphi$  to

$$\lim_{R \rightarrow \infty} \frac{1}{\mu_L(B_R)} \int_{B_R} \int_{B_R} \varphi(s - t) u(s) \overline{u(-t)} \, d\mu(s) \, d\mu(t),$$

By construction, this limit  $\mu \otimes \tilde{\mu}$  is positive definite, and so its Fourier transform exists and is positive. We call this Fourier transform the *diffraction measure* for  $\mu$  [6, definition 9.2].

In general, given a finite set  $\mathcal{A} = \{T_1, T_2, \dots, T_{n_a}\}$  of tiles and a tiling  $\mathcal{T}$  in  $\mathbb{R}^d$  whose tiles are translates of elements of  $\mathcal{A}$ , we set  $D_i$  via

$$D_i = \{t \in \mathbb{R}^d \mid T_i + t \in \mathcal{T}\}.$$

We then take complex numbers  $w_1, w_2, \dots, w_{n_a}$ , and consider a Radon measure

$$\mu_{\mathcal{T}} = \sum_{i=1}^{n_a} w_i \sum_{t \in D_i} \delta_t.$$

The diffraction measure for  $\mu_{\mathcal{T}}$  is called the *diffraction measure for  $\mathcal{T}$* . It is easy to prove that the autocorrelation measure is

$$\mu_{\mathcal{T}} \otimes \widetilde{\mu_{\mathcal{T}}} = \sum_{i,j=1}^{n_a} w_i \overline{w_j} \sum_{z \in \mathbb{R}^d} \text{freq}_{\mathcal{T}}\{T_j, T_i + z\} \delta_z. \tag{6}$$

The *Lebesgue decomposition* [15, section 5] of a Radon measure is fundamental in the theory of diffraction. In general, a Radon measure  $\mu$  on  $\mathbb{R}^d$  is *pure point* if its total variation  $|\mu|$  is pure point, that is, a sum of Dirac measures. If  $|\mu|(\{x\}) = 0$  for each  $x \in \mathbb{R}^d$ , then  $\mu$  is said to be *continuous*. Any Radon measure  $\mu$  is uniquely decomposed into its pure point part  $\mu_{\text{pp}}$  and continuous part  $\mu_{\text{c}}$ . The continuous part is further decomposed into the singular continuous component  $\mu_{\text{sc}}$ , which is mutually singular with the Lebesgue measure on  $\mathbb{R}^d$ , and absolutely continuous component  $\mu_{\text{ac}}$ , which is absolutely continuous with respect to the Lebesgue measure. Thus we have a decomposition

$$\mu = \mu_{\text{pp}} + \mu_{\text{sc}} + \mu_{\text{ac}}. \tag{7}$$

This decomposition of  $\mu$  into its pure point, its continuous and singular, and its continuous and absolutely continuous part is unique.

That  $\mu_{\text{ac}}$  is absolutely continuous means that there is a locally integrable function  $f \in L^1_{\text{loc}}(\mathbb{R}^d)$  such that

$$\langle \mu_{\text{ac}}, \varphi \rangle = \int \varphi f \, d\mu_{\text{L}}$$

holds for each  $\varphi \in C_c(\mathbb{R}^d)$ , where the right-hand side is the integral with respect to the Lebesgue measure  $\mu_{\text{L}}$ . This  $f$  is called the *Radon–Nikodym derivative* of the Radon measure  $\mu$ .

### 3. Main results

#### 3.1. A relation between the asymptotic behavior of Fourier matrices and absolutely continuous spectrum for S-adic tilings

In this section, we prove a sufficient condition for the absence of the absolutely continuous part of the diffraction measure for S-adic tilings. The following setting is assumed for the whole section.

**Setting 3.1.** *In this section, we take finite set  $\{\rho_1, \rho_2, \dots, \rho_{m_a}\}$  of substitution rules in  $\mathbb{R}^d$  that share the same arbitrary (not necessarily  $[0, 1]^d$ -supported) alphabet  $\mathcal{A} = \{T_1, T_2, \dots, T_{n_a}\}$ . We assume the family  $\{\rho_1, \rho_2, \dots, \rho_{m_a}\}$  has FLC (page 7969). (Note that each substitution here is a ‘geometric’ one and not a ‘symbolic’ one.) The existence of such common tiles and*

alphabet is the assumption which we start with. Let  $\phi_i$  be the expansion map for the substitution  $\rho_i$ . (For different  $i$  and  $j$ , the maps  $\phi_i$  and  $\phi_j$  may be different.) The Fourier matrix for  $\rho_i$  is denoted by  $B^{(i)}$ , where  $B^{(i)}(t) = (B_{k,j}^{(i)}(t))_{k,j}$ .

We consider a directive sequence  $(i_j)_{j=1,2,\dots}$  in  $\{1, 2, \dots, m_a\}^{\mathbb{N}}$  and let  $\mathcal{T}^{(1)}$  be an S-adic tiling that belongs to  $(i_j)_j$ . As we have seen on page 7969–7970, we have an increasing sequence  $n_1 < n_2 < \dots$  of natural numbers and patches  $\mathcal{P}_l$  consisting of translates of alphabets such that

$$\begin{aligned} \mathcal{T}^{(k)} &= \lim_{l \rightarrow \infty} \rho_{i_k} \circ \rho_{i_{k+1}} \circ \dots \circ \rho_{i_{n_l}}(\mathcal{P}_l) \\ &= \lim_{l \rightarrow \infty} \rho_{i_{[k,n_l]}}(\mathcal{P}_l) \end{aligned}$$

converges for each  $k = 1, 2, \dots$ . Note that we do not assume recognisability here, but we do assume that, for each  $\mathcal{T}^{(k)}$ , the patch frequencies converge.

Since each  $\rho_i$ , regarded as a map that sends a patch  $\mathcal{P}$  to another patch  $\rho_i(\mathcal{P})$ , is continuous with respect to the local matching topology, we see that

$$\rho_{i_k}(\mathcal{T}^{(k+1)}) = \mathcal{T}^{(k)}$$

for each  $k = 1, 2, \dots$ . This can be used to ‘compare’ the autocorrelation measure for  $\mathcal{T}^{(k)}$  and one for  $\mathcal{T}^{(k+1)}$ .

The fundamental idea to study the diffraction spectrum is to use renormalisation equations [3, 4, 8, 9, 22, 23]. The above ‘de-substitution’ or inverse-limit structure gives us a renormalisation scheme, which in turn gives us a sufficient condition for zero absolutely continuous spectrum in terms of an asymptotic of norms of Fourier matrices (theorem 3.2). Such an asymptotic behavior can be checked by ergodic theory as in section 3.2. The special case of theorem 3.2 for the substitution case (the case where  $m_a = 1$ ) was proved by Mañibo [22, 23]. Below we adapt Mañibo’s idea to the general S-adic case. The main difficulty is that, whereas for a self-affine tiling  $\mathcal{T}$ , its de-substituted tiling is  $\mathcal{T}$  itself, for the S-adic situation, the de-substituted tilings may all differ. This problem is overcome by lemma 3.16.

The goal of this section is to prove the following theorem.

**Theorem 3.2.** *If there is  $\varepsilon > 0$  such that*

$$\begin{aligned} \liminf_{k \rightarrow \infty} &\left( \frac{1}{2k} \log \det \phi_{i_1} \det \phi_{i_2} \dots \det \phi_{i_k} \right. \\ &\left. - \frac{1}{k} \log \left\| B^{(i_1)}(t) B^{(i_2)}(\phi_{i_1}^*(t)) \dots B^{(i_k)}(\phi_{i_{k-1}}^* \circ \phi_{i_{k-2}}^* \circ \dots \circ \phi_{i_1}^*(t)) \right\| \right) > \varepsilon \end{aligned} \quad (8)$$

for Lebesgue-a.e.  $t \in \mathbb{R}$ , where  $*$  denotes the adjoint, then the diffraction spectrum of  $\mathcal{T}^{(1)}$  has zero absolutely continuous part.

**Remark 3.3.** There is a similar result for the one-dimensional case for the corresponding dynamical spectrum in a recent paper by Bufetov and Solomyak [16, corollary 4.5]. They proved a sufficient condition for the absence of absolutely continuous dynamical spectrum for suspension flows for S-adic sequences. This covers some cases which theorem 3.2 does not cover, but theorem 3.2 deals with some cases which Bufetov and Solomyak did not. The sufficient condition in [16] is similar to theorem 3.2, but they replace  $\frac{1}{2k} \log \lambda_{i_1} \lambda_{i_2} \dots \lambda_{i_k}$  with  $\frac{1}{2k} \log \|A_{i_1} A_{i_2} \dots A_{i_k}\|$  ( $A_i$  is the substitution matrix for  $\rho_i$ ) and assume that the limit of these as  $k \rightarrow \infty$  is convergent. Moreover, they assume the recognisability for the directive sequence

$i_1, i_2, \dots$ . Therefore, theorem 3.2 covers some cases that Bufetov and Solomyak did not cover, since the dimension  $d$  of the tiling, the shapes of proto-tiles and the directive sequence are arbitrary in this theorem. On the other hand, Bufetov and Solomyak deal with arbitrary suspension flows for S-adic sequences, whereas in this paper, for the one-dimensional cases, we only deal with tile lengths that come from Perron–Frobenius eigenvector.

For the rest of this section we will prove theorem 3.2. The readers may skip the proof for the first reading and move to an application of this theorem in section 3.2.

**Definition 3.4.** Let  $D_i^{(k)}$  be defined via

$$D_i^{(k)} = \{t \in \mathbb{R}^d \mid T_i + t \in \mathcal{T}^{(k)}\}.$$

The density of each  $D_i^{(k)}$  is defined via

$$\text{dens } D_i^{(k)} = \lim_{R \rightarrow \infty} \frac{1}{\mu_L(B_R)} \# D_i^{(k)} \cap B_R,$$

where the limit is convergent since the patch frequencies converge.

For each  $k = 1, 2, \dots$ , each  $i, j = 1, 2, \dots, n_a$  and each  $z \in \mathbb{R}^d$ , we set

$$\begin{aligned} \nu_{i,j}^{(k)}(z) &= \text{freq}_{\mathcal{T}^{(k)}}\{T_i, T_j + z\} \\ &= \lim_{R \rightarrow \infty} \frac{1}{\mu_L(B_R)} \# \{t \in B_R \mid T_i + t, T_j + t + z \in \mathcal{T}^{(k)}\}. \end{aligned}$$

**Lemma 3.5.** For each  $k$ , we have the following equation:

$$\frac{1}{\det \phi_{i_k}} \sum_{m,n=1}^{n_a} \sum_{x \in T_{i,m}^{(i_k)}} \sum_{y \in T_{j,n}^{(i_k)}} \nu_{m,n}^{(k+1)}(\phi_{i_k}^{-1}(z + x - y)) = \nu_{i,j}^{(k)}(z). \tag{9}$$

**Proof.** The proof is essentially the same as the substitution case (that is, the case where  $m_a = 1$ ) in [23], but since the proof in [23] appears to use recognisability, we here give an outline of the proof without using recognisability.

Take  $i, j, k$  and  $z \in D_j^{(k)} - D_i^{(k)}$ . For each  $t \in D_i^{(k)}$  such that  $t + z \in D_j^{(k)}$ , since  $T_i + t \in \mathcal{T}^{(k)}$  and  $T_j + t + z \in \mathcal{T}^{(k)}$ , there are  $m, n \in \{1, 2, \dots, n_a\}$ ,  $s_m \in D_m^{(k+1)}$  and  $s_n \in D_n^{(k+1)}$  such that  $T_i + t \in \rho_{i_k}(T_m + s_m)$  and  $T_j + t + z \in \rho_{i_k}(T_n + s_n)$ . (These  $m, n, s_m, s_n$  are unique.) By computation, it follows that

$$s_n - s_m = \phi_{i_k}^{-1}(z + x - y)$$

for some  $x \in T_{i,m}^{(i_k)}$  and  $y \in T_{j,n}^{(i_k)}$ , and so we obtain a map that sends

$$t \in \{t \in \mathbb{R}^d \mid T_i + t, T_j + t + z \in \mathcal{T}^{(k)}\} \tag{10}$$

to

$$s_m \in \bigsqcup_{m,n \leq n_a} \bigsqcup_{x \in T_{i,m}^{(i_k)}} \bigsqcup_{y \in T_{j,n}^{(i_k)}} \{s \in \mathbb{R}^d \mid T_m + s \in \mathcal{T}^{(k+1)}, T_n + s + \phi_{i_k}^{-1}(z + x - y) \in \mathcal{T}^{(k+1)}\}, \tag{11}$$

where  $\bigsqcup$  means taking the union while regarding the sets as disjoint. We can show that this map is a bijection.

We then see that

$$\begin{aligned} & \sum_{m,n \leq n_a} \sum_{x \in T_{i,m}^{(i_k)}} \sum_{y \in T_{j,n}^{(i_k)}} \# \{s \in \phi_{i_k}^{-1}(B_{R-C'}) | T_m + s \in \mathcal{T}^{(k+1)}, T_n + s + \phi_{i_k}^{-1}(x - y + z) \in \mathcal{T}^{(k+1)}\} \\ & \leq \# \{t \in B_R | T_i + t, T_j + t + z \in \mathcal{T}^{(k)}\} \\ & \leq \sum_{m,n \leq n_a} \sum_{x \in T_{i,m}^{(i_k)}} \sum_{y \in T_{j,n}^{(i_k)}} \# \{s \in \phi_{i_k}^{-1}(B_R) + B_C | T_m + s \in \mathcal{T}^{(k+1)}, T_n + s + \phi_{i_k}^{-1}(z + x - y) \in \mathcal{T}^{(k+1)}\}, \end{aligned}$$

where  $C'$  is the maximal norm of vectors in all digits for  $\rho_1, \rho_2, \dots, \rho_{m_a}$  and  $C$  is the maximal diameter for the elements of the alphabet. By dividing these by  $\mu_L(B_R)$  and taking the limit as  $R \rightarrow \infty$ , we obtain the desired equation.  $\square$

**Definition 3.6.** For each  $k = 1, 2, \dots, i, j = 1, 2, \dots, n_a$ , we set

$$\Upsilon_{i,j}^{(k)} = \sum_{z \in D_j^{(k)} - D_i^{(k)}} \nu_{i,j}^{(k)}(z) \delta_z.$$

By FLC, for each compact set  $K$  the set  $K \cap (D_j^{(k)} - D_i^{(k)})$  is a finite set, and so the infinite sum  $\sum_{z \in D_j^{(k)} - D_i^{(k)}}$  (with respect to the vague topology) is well-defined.

Note that, by equation (6), it suffices to investigate these  $\Upsilon_{i,j}^{(k)}$  in order to understand the diffraction measure for  $\mathcal{T}^{(k)}$ . The Fourier transforms of the  $\Upsilon_{i,j}^{(k)}$  will generate the diffraction measures for  $\mathcal{T}^{(k)}$ . In order to investigate the nature of these diffraction measures, it is useful to use the relation between  $\Upsilon_{i,j}^{(k)}$  ( $i, j = 1, 2, \dots, n_a$ ) and  $\Upsilon_{m,n}^{(k+1)}$  ( $m, n = 1, 2, \dots, n_a$ ) stated in proposition 3.8 below. In order to give a statement, we first introduce two symbols.

**Definition 3.7.** If a Radon measure  $\mu$  on  $\mathbb{R}^d$  and a homeomorphism  $g : \mathbb{R}^d \rightarrow \mathbb{R}^d$  are given, define another Radon measure  $g.\mu$  via

$$\langle g.\mu, \varphi \rangle = \langle \mu, \varphi \circ g \rangle$$

for each  $\varphi \in C_c(\mathbb{R}^d)$ .

Let  $\mu$  be a Radon measure on  $\mathbb{R}^d$  and let  $\xi$  be a continuous function on  $\mathbb{R}^d$ . The new Radon measure  $\xi\mu$  is defined via

$$\langle \xi\mu, \varphi \rangle = \langle \mu, \xi\varphi \rangle$$

for each  $\varphi \in C_c(\mathbb{R}^d)$ , where  $\xi\varphi$  is the pointwise multiplication.

In what follows, we use *convolutions* of Radon measures [7, definition 4.9.18].

**Proposition 3.8. (Renormalisation equation).** For each  $k, i, j$ , we have

$$\Upsilon_{i,j}^{(k)} = \frac{1}{\det \phi_{i_k}} \sum_{m,n=1}^{n_a} \sum_{x \in T_{i,m}^{(i_k)}} \sum_{y \in T_{j,n}^{(i_k)}} \delta_{y-x} * (\phi_{i_k} \cdot \Upsilon_{m,n}^{(k+1)}). \tag{12}$$

**Proof.** This can be proved using lemma 3.5 by a direct computation.  $\square$

This renormalisation equation gives rise to an equation between Fourier transforms, as in proposition 3.9. In what follows, for each  $t \in \mathbb{R}^d$ , the symbol  $\exp_t$  denotes the exponential function defined via  $\exp_t(s) = e^{2\pi i \langle s, t \rangle}$  for each  $s \in \mathbb{R}^d$ , where  $\langle \cdot, \cdot \rangle$  is the standard inner product.

**Proposition 3.9.** *Each  $\Upsilon_{i,j}^{(k)}$  is Fourier transformable and we have*

$$\widehat{\Upsilon_{i,j}^{(k)}} = \frac{1}{(\det \phi_{i_k})^2} \sum_{m,n=1}^{n_a} \sum_{x \in T_{i_m}^{(i_k)}} \sum_{y \in T_{j_n}^{(i_k)}} \exp_{x-y}(\phi_{i_k}^*)^{-1} \cdot \widehat{\Upsilon_{m,n}^{(k+1)}}. \tag{13}$$

where  $*$  denotes the adjoint map.

**Proof.** To prove that each  $\Upsilon_{i,j}^{(k)}$  is Fourier transformable, fix  $k$  and set  $\omega_i = \sum_{x \in D_i^{(k)}} \delta_x$  for each  $i = 1, 2, \dots, n_a$ . By computation we have

$$\Upsilon_{i,j}^{(k)} = \omega_i \circledast \widetilde{\omega}_j,$$

which is the sum of four positive definite Radon measures by the polarisation identity. Since each positive definite Radon measure is Fourier transformable, the sum  $\Upsilon_{i,j}^{(k)}$  is also Fourier transformable. The formula (13) is a direct consequence of proposition 3.8.  $\square$

We can decompose each  $\Upsilon_{i,j}^{(k)}$  into its pure point, absolutely continuous and singular continuous component (with respect to the Lebesgue measure, see page 7972):

$$\Upsilon_{i,j}^{(k)} = (\Upsilon_{i,j}^{(k)})_{pp} + (\Upsilon_{i,j}^{(k)})_{ac} + (\Upsilon_{i,j}^{(k)})_{sc}.$$

The Radon–Nikodym derivative of the absolutely continuous part  $(\Upsilon_{i,j}^{(k)})_{ac}$  is denoted by  $h_{i,j}^{(k)} \in L^1_{loc}(\mathbb{R}^d)$ .

By the uniqueness of the Lebesgue decomposition and by proposition 3.9, we have the following result.

**Proposition 3.10.** *The Radon–Nikodym derivative  $h_{i,j}^{(k)}$  satisfies*

$$h_{i,j}^{(k)}(t) = \frac{1}{\det \phi_{i_k}} \sum_{m,n=1}^{n_a} \sum_{x \in T_{i_m}^{(i_k)}} \sum_{y \in T_{j_n}^{(i_k)}} \exp_{x-y}(t) h_{m,n}^{(k+1)}(\phi_{i_k}^*(t)) \tag{14}$$

for Lebesgue-a.e.  $t \in \mathbb{R}^d$ .

We will investigate when  $h_{i,j}^{(1)}$  are zero and so the absolutely continuous spectrum for  $\mathcal{T}^{(1)}$  is zero. It is convenient to define the following Radon–Nikodym matrix.

**Definition 3.11.** For each  $k$ , define the Radon–Nikodym matrix  $H^{(k)}$  for the tiling  $\mathcal{T}^{(k)}$ , which is a matrix-valued function on  $\mathbb{R}^d$ , via

$$H^{(k)}(t) = (h_{i,j}^{(k)}(t))_{i,j}.$$

Then, proposition 3.10 now translates into the following result.

**Proposition 3.12.** *For each  $k$ , we have*

$$H^{(k)}(t) = \frac{1}{\det \phi_{i_k}} B^{(i_k)}(t) H^{(k+1)}(\phi_{i_k}^*(t)) B^{(i_k)}(t)^*. \tag{15}$$

In section 3.2, we will prove that these  $H^{(k)}(t)$  are in fact zero for certain choices of  $\{\rho_1, \rho_2, \dots, \rho_{m_a}\}$ . For this purpose, the following lemma is useful, because a positive definite matrix is zero if its trace is zero.

**Lemma 3.13.** *For each  $k$ , the matrix  $H^{(k)}(t)$  is a positive definite matrix for Lebesgue-a.e.  $t \in \mathbb{R}^d$ .*

**Proof.** By the definition of a positive definite matrix, we have to prove that, whenever we take  $w_1, w_2, \dots, w_{n_a} \in \mathbb{C}$ , we have

$$\sum_{i,j} w_i \overline{w_j} h_{i,j}^{(k)}(t) \geq 0. \tag{16}$$

For  $w_j$  which are arbitrarily taken from  $\mathbb{C}$ , set

$$\gamma = \sum w_j \sum_{t \in D_j^{(k)}} \delta_t.$$

By (6), the autocorrelation measure for  $\gamma$  is positive definite and coincides with

$$\sum_{i,j} w_i \overline{w_j} \Upsilon_{j,i}^{(k)}.$$

Since the Fourier transform of this Radon charge is positive, so is its Radon–Nikodym derivative, which is

$$\sum_{i,j} w_i \overline{w_j} h_{j,i}^{(k)},$$

by the uniqueness of the Lebesgue decomposition. □

**Definition 3.14.** For each  $k \geq 1$ , set

$$\lambda^{(k)} = \det \phi_{i_1} \det \phi_{i_2} \cdots \det \phi_{i_k},$$

and

$$\phi^{(k)} = \phi_{i_1} \circ \phi_{i_2} \circ \cdots \circ \phi_{i_k}.$$

and set  $\lambda^{(0)} = 1$ . For each  $k \geq 1$  and  $t \in \mathbb{R}^d$ , define

$$\overline{B}^{(k)}(t) = B^{(i_1)}(t) B^{(i_2)}(\phi_{i_1}^*(t)) \cdots B^{(i_k)}(\phi_{i_{k-1}}^*(t)).$$

By iterating the equation (15), we have:

**Proposition 3.15.** *For each  $k \geq 1$ , we have*

$$H^{(1)}(t) = \frac{1}{\lambda^{(k)}} \overline{B}^{(k)}(t) H^{(k+1)}(\phi_{i_k}^*(t)) \overline{B}^{(k)}(t)^* \tag{17}$$

for Lebesgue-a.e.  $t \in \mathbb{R}$ . Hence we have the following inequality for the traces:

$$\text{Tr}(H^{(1)}(t)) \leq \frac{\|\overline{B}^{(k)}(t)\|^2}{\lambda^{(k)}} \text{Tr}(H^{(k+1)}(\phi_{i_k}^*(t))). \tag{18}$$

By proposition 3.15, we have the following strategy to prove that  $\text{Tr}(H^{(1)}(t)) = 0$  and hence  $H^{(1)}(t) = 0$  ( $H^{(1)}(t)$  is a positive definite matrix) and the absence of an absolutely continuous component of the diffraction measure for  $\mathcal{T}^{(1)}$ .

First, we prove that (i) the traces  $\text{Tr}(H^{(k+1)}(\phi_{(k)}^* t))$  are, in a sense, ‘bounded from above’. Next, we prove that (ii) the convergence

$$\frac{\|\overline{B}^{(k)}(t)\|^2}{\lambda^{(k)}} \rightarrow 0 \quad \text{as } k \rightarrow \infty \tag{19}$$

implies that  $\text{Tr}(H^{(1)}(t)) = 0$ , using (i) and the inequality (18).

We now proceed to prove (i) (lemma 3.16) and (ii) (theorem 3.2). The remaining part, condition (19), is proved for a special class of S-adic tilings in section 3.2.

**Lemma 3.16.** *There are  $A > 0, k_0 > 0, R_0 > 0$  such that, for any natural number  $k \geq k_0$  and any real number  $R > R_0$ , we have*

$$\int_{[-R,R]^d} \text{Tr}(H^{(k)}(\phi_{(k-1)}^*(t))) dt \leq AR^d. \tag{20}$$

**Proof.** By [7, corollary 4.9.12], there is an  $f \in C_c(\mathbb{R}^d)$  such that  $\widetilde{f * \tilde{f}} \geq 1_{[0,1]^d}$ , where  $1_S$  for  $S \subset \mathbb{R}^d$  is the characteristic function for  $S$ . By computation, we have

$$1_{\phi_{(k-1)}^*([0,1]^d+t)} \leq (T_t(\widetilde{f * \tilde{f}})) \circ (\phi_{(k-1)}^*)^{-1},$$

where  $T_t$  denotes the translation by  $t$ .

If  $k$  is large enough,  $z = 0$  is the only element  $z \in D_i^{(k)} - D_i^{(k)}$  such that  $\phi_{(k-1)}(z) \in \text{supp } f * \tilde{f}$ . For any  $k > 0$  which is large enough in this sense,  $i = 1, 2, \dots, n_a$  and  $t \in \mathbb{R}^d$ , we have

$$\begin{aligned} \langle 1_{\phi_{(k-1)}^*([0,1]^d+t)}, \widehat{\Upsilon}_{i,i}^{(k)} \rangle &\leq \det \phi_{(k-1)} \langle \exp_{-t}(f * \tilde{f}) \circ \phi_{(k-1)}, \Upsilon_{i,i}^{(k)} \rangle \\ &\leq \det \phi_{(k-1)} f * \tilde{f}(0) \nu_{i,i}^{(k)}(0). \end{aligned}$$

Note that  $\nu_{i,i}^{(k)}(0) = \text{dens } D_i^{(k)}$ . Set  $B = \sup_{k,i} \text{dens } D_i^{(k)}$ . This is finite since  $\mathcal{A}$  is fixed. For an arbitrary choice of  $R > 0$ , if  $n$  is a natural number with  $n \leq R < n + 1$ , we have:

$$\begin{aligned} \int_{[-R,R]^d} \text{Tr}(H^{(k)}(\phi_{(k-1)}^*(t))) dt &\leq \sum_{i=1}^{n_a} \sum_{v \in \mathbb{Z}^d \cap [-n-1,n]^d} \frac{1}{\det \phi_{(k-1)}} \langle \widehat{\Upsilon}_{i,i}^{(k)}, 1_{\phi_{(k-1)}^*([0,1]^d+v)} \rangle \\ &\leq n_a(2R + 2)^d f * \tilde{f}(0)B. \end{aligned}$$

By taking  $A > 2^d n_a f * \tilde{f}(0)B$ , we have the conclusion. □

We now prove a sufficient condition for  $H^{(1)}$  to be zero. As we stated before, the condition is the exponential convergence of  $\frac{\|\overline{B}^{(k)}(t)\|^2}{\lambda^{(k)}} \rightarrow 0$  as  $k \rightarrow \infty$ . We will prove that this sufficient condition is actually satisfied for a class of S-adic tilings, in section 3.2, and thus prove the absence of absolutely continuous part of the diffraction spectrum for this class.

**Theorem 3.17.** *If there is  $\varepsilon > 0$  such that*

$$\liminf_{k \rightarrow \infty} \left( \frac{1}{2k} \log \lambda^{(k)} - \frac{1}{k} \log \|\overline{B}^{(k)}(t)\| \right) > \varepsilon \tag{21}$$

for Lebesgue-a.e.  $t \in \mathbb{R}$ , then we have  $H^{(1)} = 0$  for Lebesgue-a.e.  $t \in \mathbb{R}$ , where  $H^{(1)}$  is the Radon–Nikodym matrix for  $\mathcal{T}^{(1)}$  as defined in definition 3.11.

**Proof.** Take an arbitrary positive real number  $R$ . For each  $k \geq 1$ , set

$$E_k = \left\{ t \in [-R, R]^d \mid \inf_{l \geq k} \left( \frac{1}{2l} \log \lambda^{(l)} - \frac{1}{l} \log \|\bar{B}^{(l)}(t)\| \right) \geq \frac{\varepsilon}{2} \right\}.$$

The sets  $E_1, E_2, \dots$  are increasing and  $\bigcup_{k \geq 1} E_k$  has full measure  $(2R)^d$ .

For each  $k \geq 1$  and  $t \in E_k$ , we have

$$\frac{\|\bar{B}^{(k)}(t)\|^2}{\lambda^{(k)}} \leq e^{-k\varepsilon},$$

and

$$\begin{aligned} \int_{[-R,R]^d} \text{Tr}(H^{(1)}(t)) dt &= \int_{E_k} \text{Tr}(H^{(1)}(t)) dt + \int_{[-R,R]^d \setminus E_k} \text{Tr}(H^{(1)}(t)) dt \\ &\leq \int_{E_k} \frac{\|\bar{B}^{(k)}(t)\|^2}{\lambda^{(k)}} \text{Tr}(H^{(k+1)}(\phi_{(k)}^*(t))) dt + \int_{[-R,R]^d \setminus E_k} \text{Tr}(H^{(1)}(t)) dt. \end{aligned}$$

Here, if  $R$  is large enough, the first term tends to zero as  $k \rightarrow \infty$ , since there exists  $A > 0$  as in lemma 3.16 and since

$$\int_{E_k} \frac{\|\bar{B}^{(k)}(t)\|^2}{\lambda^{(k)}} \text{Tr}(H^{(k+1)}(\phi_{(k)}^*(t))) dt \leq e^{-k\varepsilon} \int_{[-R,R]^d} \text{Tr}(H^{(k+1)}(\phi_{(k)}^*(t))) dt \leq e^{-k\varepsilon} AR^d.$$

The second term also tends to zero because  $\bigcup_{k \geq 1} E_k$  has full measure  $(2R)^d$  and

$$\int_{[-R,R]^d} \text{Tr}(H^{(1)}(t)) dt < \infty.$$

We see that

$$\int_{[-R,R]^d} \text{Tr}(H^{(1)}(t)) dt = 0,$$

which, by lemma 3.13, implies that  $\text{Tr}(H^{(1)}(t)) = 0$  a.e. which in turn implies that  $H^{(1)}(t) = 0$  a.e. since all the eigenvalues are zero and  $H^{(1)}(t)$  is diagonalisable.  $\square$

**The proof for theorem 3.2.** The absence of the absolutely continuous part of the diffraction spectrum for  $\mathcal{T}^{(1)}$  follows from equation (6).  $\square$

**Remark 3.18.** If the patch frequency converges uniformly, the corresponding dynamical system for  $\mathcal{T}^{(1)}$  is uniquely ergodic, and so by [10, theorem 5], the absence of absolutely continuous diffraction spectrum holds for any tiling in the continuous hull of  $\mathcal{T}^{(1)}$ .

### 3.2. The absence of absolutely continuous spectrum for binary block-substitution case

In this section, we deal with S-adic tilings with binary block substitution rules. The following setting is assumed for the whole section.

**Setting 3.19.** In this section, let  $\mathcal{A} = \{T_1, T_2\}$  consist of two tiles in  $\mathbb{R}^d$  with  $\text{supp } T_j = [0, 1]^d$  for each  $j$ .

We take a finite family of substitutions  $\rho_1, \rho_2, \dots, \rho_{m_a}$  with alphabet  $\mathcal{A}$ . We assume these are block substitutions. In other words, the expansion map  $\phi_i$  for  $\rho_i$  is defined by a diagonal matrix with integer diagonal entries.  $B^{(i)}(t)$  is the Fourier matrix for  $\rho_i$ .

We assume that, for each  $i$ , there is a  $t \in \mathbb{R}^d$  such that the Fourier matrix  $B^{(i)}(t)$  is not singular. This means that  $B^{(i)}(t)$  is not singular for almost every  $t \in \mathbb{R}^d$ , since the determinant is a sum of exponential functions.

Next, we take a measure-preserving system  $(X, \mathcal{B}, \mu, S_0)$ , where  $(X, \mathcal{B}, \mu)$  is a standard probability space and  $S_0 : X \rightarrow X$  is a measure-preserving transformation. We assume that the map  $S_0$  is ergodic and surjective.

We choose a decomposition

$$X = \bigcup_{j=1}^{m_a} E_j$$

of  $X$  into pairwise disjoint subsets  $E_j \in \mathcal{B}$ ,  $j = 1, 2, \dots, m_a$ .

The goal of this section is to prove theorem 3.30. In theorem 3.30, we will show the absence of the absolutely continuous component in the diffraction spectrum for S-adic tilings constructed from  $\rho_1, \rho_2, \dots, \rho_{m_a}$  obtained from ‘almost all’ directive sequences given by the coding of the system  $(X, S_0)$ , by using theorem 3.2. Specifically, we prove that there is some  $E \subset X$  with measure 1 such that, for each  $x \in E$ , the directive sequence  $i_1, i_2, \dots$  defined by

$$S_0^{n-1}(x) \in E_{i_n},$$

for each  $n$ , gives rise to an S-adic tiling with zero absolutely continuous diffraction spectrum. (We prove that any S-adic tiling belonging to such a directive sequence has this property.) For example, if  $X \subset \{1, 2, \dots, m_a\}^{\mathbb{N}}$  is a subshift with surjective shift-map and  $E_j = \{x \in X \mid x_1 = j\}$ , then the directive sequence defined by  $x \in X$  is  $x$  itself. The theorem says that for almost all  $x \in X$ , the S-adic tilings belonging to  $x$  have zero absolutely continuous spectrum. Special cases for the theorem are stated in Introduction. The readers can skip the detail of this section and jump to theorem 3.30 and example 3.31 for the first reading.

Note that since each  $\rho_i$  is a block substitution, the digit  $T_{k,l}^{(i)}$  for  $\rho_i$  is inside  $\mathbb{Z}^d$ , and moreover for each  $k$  and  $l$ ,

$$T_{1,k}^{(i)} \cup T_{2,k}^{(i)} = \phi_i([0, 1)^d) \cap \mathbb{Z}^d.$$

We denote this set by  $F_i$ .

Theorem 3.2 applies to the Fourier matrices  $B^{(i)}(t)$ , but we will replace  $B^{(i)}(t)$  with the following matrices  $C^{(i)}(z)$ , where  $z \in \mathbb{T}^d$ : the latter has the advantage of a compact domain.

**Definition 3.20.** For each  $i = 1, 2, \dots, m_a$  and  $t \in \mathbb{R}^d$ , set

$$C^{(i)}(\pi(t)) = B^{(i)}(t).$$

( $\pi : \mathbb{R}^d \rightarrow \mathbb{T}^d$  is the quotient map.) Since the elements in the digits are all in  $\mathbb{Z}^d$ , this is well-defined.

For a technical reason, we replace the system  $(X, \mathcal{B}, \mu, S_0)$  with its natural extension. Let  $(Y, \mathcal{C}, \nu, S_1)$  be a natural extension of  $(X, \mathcal{B}, \mu, S_0)$ . In other words,  $(Y, \mathcal{C}, \nu)$  is a separable and complete probability space and  $S_1 : Y \rightarrow Y$  is an invertible measure-preserving transformation, and there exists a factor map  $f : Y \rightarrow X$ . Such a system and a factor map exist since  $S_0$  is surjective, and  $S_1$  is ergodic [25, section 1.6.3].

The main tool to prove theorem 3.30 is Furstenberg–Kesten theorem and Oseledets ergodic theorem. For these theorems, see for example [29]. We will use these theorems to the following cocycle and obtain an estimate (8). We define a map  $C : \mathbb{T}^d \times Y \rightarrow GL_2(\mathbb{C})$  via

$$C(z, x) = \begin{cases} C^{(i)}(z)^{-1} & \text{if } x \in f^{-1}(E_i) \text{ and } C^{(i)}(z) \text{ is invertible,} \\ I & \text{otherwise.} \end{cases}$$

( $I$  is the identity matrix, but this does not matter since  $C^{(i)}(z)$  is almost surely invertible.)

We set  $R : \mathbb{T}^d \times Y \rightarrow \mathbb{T}^d \times Y$  via

$$R(\pi(t), x) = (\pi(\phi_i(t)), S_1(x))$$

for  $t \in \mathbb{R}^d$  and  $x \in f^{-1}(E_i)$ , for each  $i = 1, 2, \dots, m_a$ .

The following lemma will be useful when we use ergodic theorems later. Here,  $\mu_L$  is the Lebesgue measure on  $\mathbb{T}^d$ , which is measure theoretically identified with  $[0, 1]^d$ . It is in the proof of this lemma where we use the fact that  $S_1$  is invertible and ergodic.

**Lemma 3.21.** *The transformation  $R$  is  $\mu_L \times \nu$ -preserving and ergodic.*

**Proof.** This will be proved in the appendix. □

We apply the Furstenberg–Kesten theorem and the multiplicative ergodic theorem by Oseledets to  $\mathbb{T}^d \times Y$ ,  $R$  and  $C$ . In order to invoke these theorem, we first have to confirm the following lemma.

**Lemma 3.22.** *The two maps that send  $(z, x) \in \mathbb{T}^d \times Y$  to  $\log^+ \|C(z, x)\|$  and to  $\log^+ \|C(z, x)^{-1}\|$ , respectively, are both integrable with respect to  $\mu_L \times \nu$  (the product measure of  $\mu_L$  and  $\nu$ ).*

**Proof.** It suffices to prove that the maps that send  $z \in \mathbb{T}^d$  to  $\log \|C^{(i)}(z)^{\pm 1}\|$  are integrable for each  $i$ .

We take  $i$  and fix it. Since  $\|C^{(i)}(z)\|$  is bounded for  $z \in \mathbb{T}^d$ , we see that  $\log^+ \|C^{(i)}(z)\|$  is integrable. To prove the integrability of  $\log^+ \|C^{(i)}(z)^{-1}\|$ , note that  $C^{(i)}(z)^{-1} = \frac{1}{|\det C^{(i)}(z)|} C^{(i)}(z)^{\text{ad}}$ , where ad denotes the adjugate matrix. It suffices to show that  $\log |\det C^{(i)}(z)|$  is integrable, and this is proved by [27, p 223, lemma 2]. □

In what follows, we set

$$C_n(\omega) = C(R^{n-1}(\omega)) C(R^{n-2}(\omega)) \cdots C(\omega),$$

for each  $\omega \in \mathbb{T}^d \times Y$  and  $n \in \mathbb{Z}_{>0}$ . (It is customary to denote this by  $C^n$ , but to clearly distinguish this from  $C^{(i)}$ , we prefer to use this notation.)

**Proposition 3.23.** *There are  $\chi_+, \chi_- \in \mathbb{R}$  such that*

$$\begin{aligned} \chi_+ &= \lim_{n \rightarrow \infty} \frac{1}{n} \log \|C_n(\omega)\|, \\ \chi_- &= \lim_{n \rightarrow \infty} \frac{1}{n} \log \|C_n(\omega)^{-1}\|^{-1}, \end{aligned}$$

for almost all  $\omega \in \mathbb{T}^d \times Y$ .

We also have

$$\chi_+ + \chi_- = \lim_{n \rightarrow \infty} \frac{1}{n} \log |\det C_n(\omega)| \tag{22}$$

for almost all  $\omega \in \mathbb{T}^d \times Y$ .

For each  $v \in \mathbb{R}^2 \setminus \{0\}$ , the limit

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \|C_n(\omega)v\|$$

converges for almost all  $\omega$  and the limit is either  $\chi_+$  or  $\chi_-$ .

**Proof.** This is a direct consequence of the Furstenberg–Kesten theorem [29, theorem 3.12] and the Oseledets theorem [29, theorem 4.1 and section 4.3.3], if we modify the latter to the case of flags inside  $\mathbb{C}^2$ , not  $\mathbb{R}^2$ .  $\square$

We then analyse  $\chi_+$  and  $\chi_-$  to obtain the estimate (8). These are related to logarithmic Mahler measures for certain polynomials, as follows. (The logarithmic Mahler measure  $m(p)$  for a complex-coefficient,  $d$ -variable polynomial  $p$  is defined via

$$m(p) = \int_{\mathbb{T}^d} \log |p(z_1, z_2, \dots, z_d)| dz_1 dz_2 \cdots dz_d$$

where the integral is with respect to the (normalised) Lebesgue measure. For Mahler measures, see for example [27, p 224].) First, we define polynomials associated with each substitution  $\rho_1, \rho_2, \dots, \rho_{m_a}$ .

**Definition 3.24.** For each  $i = 1, 2, \dots, m_a$  and  $k, l = 1, 2$ , we set

$$S_{k,l}^{(i)} = T_{k,1}^{(i)} \cap T_{l,2}^{(i)}.$$

In other words,  $S_{k,l}^{(i)}$  is the set of all places in  $F_i = \phi_i([0, 1)^d) \cap \mathbb{Z}^d$  where in the image of  $\phi_i(T_1)$  there is a  $T_k$  and in the image of  $\phi_i(T_2)$  there is a  $T_l$ .

Define polynomials  $q_{k,l}^{(i)}$  (for  $i = 1, 2, \dots, m_a$  and  $k, l = 1, 2$ ) as follows. First set

$$z^f = (z_1^{f_1}, z_2^{f_2}, \dots, z_d^{f_d})$$

for  $z = (z_1, z_2, \dots, z_d) \in \mathbb{T}^d$  and  $f = (f_1, f_2, \dots, f_d) \in \mathbb{Z}^d$ . Next, set

$$q_{k,l}^{(i)}(z) = \sum_{f \in S_{k,l}^{(i)}} z^f$$

for each  $z \in \mathbb{T}^d$ .

**Example 3.25.** If  $\rho_i$  is the Thue–Morse substitution, if we write it symbolically, it sends 1 to 12 and 2 to 21. We have

$$S_{1,1}^{(i)} = \emptyset, \quad S_{1,2}^{(i)} = \{0\}, \quad S_{2,1}^{(i)} = \{1\}, \quad S_{2,2}^{(i)} = \emptyset,$$

and the polynomials are

$$q_{1,1}^{(i)}(z) = 0, \quad q_{1,2}^{(i)}(z) = 1, \quad q_{2,1}^{(i)}(z) = z, \quad q_{2,2}^{(i)}(z) = 0.$$

**Remark 3.26.** It follows that for each  $i \in \{1, 2, \dots, m_a\}$ , we have

$$C^{(i)}(z) = \begin{pmatrix} q_{1,1}^{(i)}(z) + q_{1,2}^{(i)}(z), & q_{1,1}^{(i)}(z) + q_{2,1}^{(i)}(z) \\ q_{2,1}^{(i)}(z) + q_{2,2}^{(i)}(z), & q_{1,2}^{(i)}(z) + q_{2,2}^{(i)}(z) \end{pmatrix}.$$

Then, we obtain the following results, lemmas 3.27–3.29, which are modifications of results from [22, 23].

**Lemma 3.27.** For each  $i$  and  $z \in \mathbb{T}^d$ , we have

$$\det C^{(i)}(z) = \left( q_{1,2}^{(i)}(z) - q_{2,1}^{(i)}(z) \right) \sum_{f \in F_i} z^f.$$

Moreover, for each  $i$  and  $z$ , the vector  $(1, -1)^\top$  is an eigenvector of  $C^{(i)}(z)$  with eigenvalue  $q_{1,2}^{(i)}(z) - q_{2,1}^{(i)}(z)$ .

Note that the assumption (setting 3.19) of non-singularity of the  $B^{(i)}(t)$ , and hence of the  $C^{(i)}(z)$ , implies that, for all  $i$ , the polynomials  $q_{1,2}^{(i)}(z) - q_{2,1}^{(i)}(z)$  are not zero.

To analyse  $\chi_+$  and  $\chi_-$ , we first prove the following.

**Lemma 3.28.** We have  $\chi_+ = 0$  and  $\chi_- = -\sum_{i=1}^{m_a} \mu(E_i) \mathfrak{m}(q_{1,2}^{(i)} - q_{2,1}^{(i)})$ .

**Proof.** By equation (22), lemma 3.21 and Birkhoff’s ergodic theorem, for almost all  $\omega \in \mathbb{T}^d \times Y$ , we have

$$\begin{aligned} \chi_+ + \chi_- &= \int \log |\det C| d\mu_L \times \nu \\ &= -\sum_{i=1}^{m_a} \mu(E_i) \mathfrak{m}(q_{1,2}^{(i)} - q_{2,1}^{(i)}). \end{aligned} \tag{23}$$

Note that we used the fact that the logarithmic Mahler measure for  $\sum_{f \in F_i} z^f$  vanishes by Jensen’s formula [1, p 207].

On the other hand, by lemma 3.27, setting

$$v = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

we have

$$\frac{1}{n} \log \| C_n(z, x)v \| \rightarrow -\sum_{i=1}^{m_a} \mu(E_i) \mathfrak{m}(q_{1,2}^{(i)} - q_{2,1}^{(i)})$$

again by Birkhoff’s ergodic theorem. By proposition 3.23, the last value is either  $\chi_+$  or  $\chi_-$ . Since  $\chi_+ \geq \chi_-$  and  $\mathfrak{m}(q_{1,2}^{(i)} - q_{2,1}^{(i)}) \geq 0$  by [27, p 228, lemma 3], by using equation (23), we have the claim.  $\square$

**Lemma 3.29.** For each  $i = 1, 2, \dots, m_a$ , we have

$$\mathfrak{m}(q_{1,2}^{(i)} - q_{2,1}^{(i)}) < \log \sqrt{\det \phi_i}.$$

**Proof.** By Jensen’s inequality and Hölder’s inequality, for each  $i$ ,

$$\exp \mathfrak{m}(q_{1,2}^{(i)} - q_{2,1}^{(i)}) \leq \sqrt{\det \phi_i}.$$

If the polynomial  $q_{1,2}^{(i)} - q_{2,1}^{(i)}$  is not a monomial, its absolute value is not a constant function and so the Jensen’s inequality is strict. If the polynomial  $q_{1,2}^{(i)} - q_{2,1}^{(i)}$  is a monomial we have also a strict inequality.  $\square$

We now prove the main result of this section. We consider S-adic tilings belonging to a directive sequence  $i_1, i_2, \dots$  that is obtained by a coding of  $S_0$  starting at some  $x \in X$ . In other words,  $i_1, i_2, \dots$  is the sequence of  $\{1, 2, \dots, m_a\}$  that satisfies

$$S_0^{n-1}(x) \in E_{i_n}$$

for each  $n = 1, 2, \dots$ .

**Theorem 3.30.** *Suppose that the substitution matrix  $A_i$  for each  $\rho_i$  has only strictly positive entries, or more generally, suppose that*

(a) *there are  $i_{0,1}, i_{0,2}, \dots, i_{0,n_0} \in \{1, 2, \dots, m_a\}$  such that the substitution matrix*

$$A_{i_{0,1}} A_{i_{0,2}} \cdots A_{i_{0,n_0}}$$

*for  $\rho_{i_{0,1}} \circ \rho_{i_{0,2}} \cdots \circ \rho_{i_{0,n_0}}$  has only strictly positive entries, and that*

(b)

$$\mu \left( \bigcap_{j=1}^{n_0} S_0^{-(j-1)} E_{i_{0,j}} \right) > 0. \tag{24}$$

*Then, there is a set  $E \in \mathcal{B}$  of full measure such that, if we take  $x \in E$  and define a directive sequence  $i_1, i_2, \dots$  as the coding of  $S_0$  starting at  $x$  with respect to the decomposition  $X = \bigcup E_i$ , and if  $\mathcal{T}$  is an S-adic tiling which belongs to this directive sequence for the family  $\{\rho_1, \rho_2, \dots, \rho_{m_a}\}$ , then  $\mathcal{T}$  has zero absolutely continuous diffraction spectrum.*

**Proof.** By Birkhoff’s ergodic theorem, for almost all  $x \in X$  we have

$$\lim_{n \rightarrow \infty} \frac{1}{2n} \log(\det \phi_{i_1} \det \phi_{i_2} \cdots \det \phi_{i_n}) = \frac{1}{2} \sum_{i=1}^{m_a} \mu(E_i) \log \det \phi_i.$$

(Here,  $i_1, i_2, \dots$  are dependent on  $x$ .) By proposition 3.23 and lemma 3.28, for almost all  $y \in Y$  and  $t \in \mathbb{R}^d$ , if  $i_1, i_2, \dots$  are codings of  $S_0$  starting at  $f(y)$ , then we have

$$\begin{aligned} & \lim_{n \rightarrow \infty} \frac{1}{n} \log \| B^{(i_1)}(t) B^{(i_2)}(\phi_{i_1}(t)) \cdots B^{(i_n)}(\phi_{i_{n-1}} \circ \phi_{i_{n-2}} \circ \cdots \circ \phi_{i_1}(t)) \| \\ &= \lim_{n \rightarrow \infty} \frac{1}{n} \log \| C^{(i_1)}(\pi(t)) C^{(i_2)}(\pi(\phi_{i_1}(t))) \cdots C^{(i_n)}(\pi(\phi_{i_{n-1}} \phi_{i_{n-2}} \cdots \phi_{i_1}(t))) \| \\ &= \lim_{n \rightarrow \infty} \frac{1}{n} \log \| C(\pi(t), y)^{-1} C(R(\pi(t), y))^{-1} \cdots C(R^{n-1}(\pi(t), y))^{-1} \| \\ &= \sum_{i=1}^{m_a} \mu(E_i) m(q_{1,2}^{(i)} - q_{2,1}^{(i)}). \end{aligned} \tag{25}$$

By lemma 3.29, there is  $\varepsilon > 0$  such that the number (25) is equal to

$$\frac{1}{2} \sum_{i=1}^{m_a} (\mu(E_i) \log(\det \phi_i)) - \varepsilon.$$

These computations show that the inequality (8) is satisfied. (Note that  $\phi_i^* = \phi_i$ .)

In setting 3.1, under which theorem 3.2 is proved, we assumed the patch frequencies for each  $\mathcal{T}^{(k)}$  to be convergent. In order to use theorem 3.2, we have to prove this assumption for the present case. By (24), ergodicity of  $S_0$  and Poincaré recurrence, for almost all  $x \in X$ , there are infinitely many  $n \in \mathbb{Z}_{>0}$  such that  $S_0^n(x) \in \bigcap_{j=1}^{n_0} S_0^{-(j-1)} E_{i_{0,j}}$ . (We used [30, theorem 1.5 (iii)].) This means that there are infinitely many  $n$  such that

$$i_n i_{n+1} \cdots i_{n+n_0-1} = i_{0,1} i_{0,2} \cdots i_{0,n_0}.$$

(These are equal as words.) The patch frequencies converge by theorem 2.7.

The statement, the absence of absolutely continuous spectrum, is proved by using theorem 3.2. □

**Example 3.31.** Let  $\rho_1$  be the period-doubling substitution and  $\rho_2$  be the Thue–Morse substitution. Let  $X = \{1, 2\}^{\mathbb{N}}$  and  $(X, \mathcal{B}, \mu, S_0)$  be the Bernoulli shift for some probability vector  $(p_1, p_2)$  with  $p_1 p_2 \neq 0$ . Then, for almost all  $x \in X$ , the S-adic tilings belonging to the sequence  $x_1, x_2, \dots$  have zero absolutely continuous spectrum. We can replace  $(X, \mathcal{B}, \mu, S_0)$  with an arbitrary surjective ergodic measure-preserving system on a standard probability space and get the same conclusion. For example, we take some Sturmian sequences as a directive sequence and obtain the same conclusion, by replacing  $(X, \mathcal{B}, \mu, S_0)$  with an irrational rotation on  $\mathbb{T}$ . The same conclusion holds when we replace  $\rho_1$  and  $\rho_2$  with arbitrary two binary substitution rules such that the supports of the prototiles are all  $[0, 1]$  and the substitution matrices have only strictly positive entries. The expansion factors can be different. In other words, if we mix two binary constant-length substitutions, we always get a tiling with zero absolutely continuous diffraction spectrum.

We can also apply theorem 3.30 for higher-dimensional block substitutions. We can replace the above  $\rho_1$  and  $\rho_2$  with arbitrary two block substitutions in  $\mathbb{R}^d$  with substitution matrices with only strictly positive entries. For example, for  $d = 2$ , if  $\rho_1, \rho_2$  are ones that were depicted in figure 1(a) and (b), we have the same conclusion as in the first paragraph in this example.

Moreover, the number of substitutions does not have to be 2. Let  $\rho_1, \rho_2, \dots, \rho_{m_a}$  be a finite family of block substitutions in  $\mathbb{R}^d$ . Take a space  $\{1, 2, \dots, m_a\}^{\mathbb{N}}$  and endow it the product probability measure for the probability measure on  $\{1, 2, \dots, m_a\}$  given by a probability vector  $(p_1, p_2, \dots, p_{m_a})$  with  $p_1 p_2 \cdots p_{m_a} \neq 0$ . We can also relax the assumption on the substitution matrices and the substitution matrices may have 0 as entries, but we assume there are  $i_{0,1}, i_{0,2}, \dots, i_{0,n_0} \in \{1, 2, \dots, m_a\}$  such that the product  $A_{i_{0,1}} A_{i_{0,2}} \cdots A_{i_{0,n_0}}$  has only strictly positive entries. Then, for almost all  $x \in \{1, 2, \dots, m_a\}^{\mathbb{N}}$ , the S-adic tilings constructed by  $\rho_1, \rho_2, \dots, \rho_{m_a}$  belonging to the directive sequence  $x$  has zero absolutely continuous diffraction spectrum. We can also take other subshifts of  $\{1, 2, \dots, m_a\}^{\mathbb{N}}$  and obtain the same conclusion. In this case, we assume the existence of  $i_{0,1} i_{0,2} \cdots i_{0,n_0}$  in the language with the same condition. For example, let  $X$  be the hull of a  $m_a$ -letter primitive symbolic substitution. Endow  $X$  with a shift-invariant probability measure defined by frequencies. Then, with respect to that measure, for almost all  $x \in X$ , the S-adic tilings belonging to  $x$  has zero absolutely continuous diffraction spectrum. The space  $X$  and the shift may be one with positive entropy, such as a shift of finite type. In any case, if we mix several substitutions, we almost surely obtain an S-adic tiling with zero absolutely continuous spectrum.

**Remark 3.32.** In theorem 3.30, the author could not prove the absence of absolutely continuous part for an arbitrary directive sequence, since this theorem is an ‘almost sure’ result. For example, when  $m_a = 2$ , we do not know what happens if we take the Fibonacci sequence

as a directive sequence. It would be interesting to decide whether this theorem holds for arbitrary directive sequences, or whether for some directive sequences there may be non-vanishing absolutely continuous components in the diffraction spectrum.

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### Appendix. The ergodicity of $R$ in section 3.2

Here, we prove the ergodicity of the transformation  $R$  in section 3.2. In particular, let  $(Y, \mathcal{C}, \nu)$  be a probability space and  $Y = \bigcup_{i=1}^{m_a} E'_i$  be a decomposition of  $Y$  into pairwise disjoint  $E'_i = f^{-1}(E_i) \in \mathcal{C}$ , for  $i = 1, 2, \dots, m_a$ . Let  $S = S_1 : Y \rightarrow Y$  be an invertible, measure-preserving and ergodic transformation. For each  $i$ , take integers  $n_1^{(i)}, n_2^{(i)}, \dots, n_d^{(i)}$  greater than 1 and let  $\phi_i$  be the linear map defined by the diagonal matrix with diagonal entries  $n_1^{(i)}, n_2^{(i)}, \dots, n_d^{(i)}$ . Define a skew product  $R : \mathbb{T}^d \times Y \rightarrow \mathbb{T}^d \times Y$  via

$$R(\pi(t), x) = (\pi(\phi_i(t)), S(x)),$$

for each  $t \in \mathbb{R}^d$ ,  $x \in E'_i$  and  $i = 1, 2, \dots, m_a$ . We aim to prove the following result, where  $\mu_{\mathbb{L}}$  is the Lebesgue measure on  $[0, 1)^d$ .

**Lemma 3.33.** *The transformation  $R$  is measure preserving and ergodic with respect to the product measure  $\mu_{\mathbb{L}} \times \nu$ .*

We divide the proof into several lemmas. First, we define a notion as follows.

**Definition 3.34.** For each  $n = (n_1, n_2, \dots, n_d) \in \mathbb{Z}_{>0}^d$ , define a map  $T_n$  on  $[0, 1)^d$  via

$$T_n(t_1, t_2, \dots, t_d) = (n_1 t_1 - [n_1 t_1], n_2 t_2 - [n_2 t_2], \dots, n_d t_d - [n_d t_d])$$

where, for each  $s \in \mathbb{R}$ ,  $[s]$  denotes the largest integer smaller than or equal to  $s$ .

For each  $n = (n_1, n_2, \dots, n_d) \in \mathbb{Z}_{>0}^d$ , set

$$\mathcal{I}_n = \left\{ \left[ \frac{k_1}{n_1}, \frac{k_1 + 1}{n_1} \right) \times \left[ \frac{k_2}{n_2}, \frac{k_2 + 1}{n_2} \right) \times \dots \times \left[ \frac{k_d}{n_d}, \frac{k_d + 1}{n_d} \right) \mid k_j = 0, 1, \dots, n_j - 1 \right\}.$$

Note that for each  $I \in \mathcal{I}_n$ , the restriction  $T_n|_I : I \rightarrow [0, 1)$  is bijective and expands the Lebesgue measure by  $n_1 n_2 \dots n_d$ .

The set  $\mathbb{Z}_{>0}^d$  is a directed set. Any real-valued map with domain  $\mathbb{Z}_{>0}^d$  can be regarded as a net and the limit of such a map makes sense. In other words, a map  $f : \mathbb{Z}_{>0}^d \rightarrow \mathbb{R}$  converges to an  $\alpha \in \mathbb{R}$  if for any  $\varepsilon > 0$ , there are  $(n_1, n_2, \dots, n_d) \in \mathbb{Z}_{>0}^d$  such that  $m = (m_1, m_2, \dots, m_d) \in \mathbb{Z}_{>0}^d$  and  $m_1 > n_1, m_2 > n_2, \dots, m_d > n_d$  imply  $|f(m) - \alpha| < \varepsilon$ .

The proof of the following lemma is straightforward.

**Lemma 3.35.** *If  $I_0 \subset [0, 1]^d$  is a product of intervals, then we have*

$$\lim_{n=(n_1, n_2, \dots, n_d) \rightarrow \infty} \frac{1}{n_1 n_2 \dots n_d} \#\{I \in \mathcal{I}_n \mid I \subset I_0\} = \mu_L(I_0).$$

By using lemma 3.35, we prove the following lemma.

**Lemma 3.36.** *For all Borel subsets  $E, F$  of  $[0, 1)$ , we have*

$$\lim_{n \rightarrow \infty} \mu_L(T_n^{-1}(E) \cap F) = \mu_L(E) \mu_L(F).$$

**Proof.** It suffices to prove the claim for the case where  $E$  and  $F$  are products of intervals, since any general  $E$  and  $F$  are approximated by finite disjoint unions of products of intervals. Assume that  $E$  and  $F$  are products of intervals. For each  $n \in \mathbb{Z}_{>0}^d$  and  $I \in \mathcal{I}_n$ , we have either (1)  $I \subset F$ , (2)  $I \cap F \neq \emptyset$  and  $I \not\subset F$ , or (3)  $I \cap F = \emptyset$ . In case (1), we have

$$\mu_L(T_n|_I^{-1}(E) \cap F) = \frac{1}{n} \mu_L(E).$$

For each  $n$ , the number of  $I$  in  $\mathcal{I}_n$  with case (2) is at most

$$2 \sum_{j=1}^d n_1 n_2 \dots n_{j-1} n_{j+1} \dots n_d.$$

For each such  $I$ , the measure  $\mu_L(T_n|_I^{-1}(E) \cap F)$  is at most  $\frac{1}{n_1 n_2 \dots n_d}$ . Finally for  $I$  with case (3), we have  $T_n|_I^{-1}(E) \cap F = \emptyset$ .

By using these observations and lemma 3.35, we see that

$$\begin{aligned} \lim_{n \rightarrow \infty} \mu_L(T_n^{-1}(E) \cap F) &= \lim \sum_{I \in \mathcal{I}_n, I \subset F} \mu_L(T_n|_I^{-1}(E) \cap F) \\ &= \mu_L(E) \mu_L(F). \end{aligned} \quad \square$$

**Definition 3.37.** For each  $n \in \mathbb{Z}_{>0}$ , define  $\mathcal{J}(n) = \{1, 2, \dots, m_a\}^n$ . For each element  $\mathbf{i} = (i_1, i_2, \dots, i_n) \in \mathcal{J}(n)$ , set

$$E_{\mathbf{i}}' = S^n(E'_{i_1}) \cap S^{n-1}(E'_{i_2}) \cap \dots \cap S(E'_{i_n}).$$

We also define a map  $T_{\mathbf{i}} : \mathbb{T}^d \rightarrow \mathbb{T}^d$  for  $\mathbf{i} = (i_1, i_2, \dots, i_n)$ , via

$$T_{\mathbf{i}}(\pi(t)) = \pi(\phi_{i_n} \circ \phi_{i_{n-1}} \circ \dots \circ \phi_{i_1}(t))$$

for each  $t \in \mathbb{R}^d$ .

**Lemma 3.38.** *Let  $n \in \mathbb{Z}_{>0}$  and  $\mathbf{i} \in \mathcal{J}(n)$ . For a Borel subset  $F \subset \mathbb{T}^d$  and  $E \in \mathcal{C}$  with  $E \subset E_{\mathbf{i}}'$ , we have*

$$R^{-n}(F \times E) = (T_{\mathbf{i}} \times S^n)^{-1}(F \times E).$$

**Proof.** For a  $(\pi(t), x) \in \mathbb{T}^d \times Y$ , if we have  $R^n(\pi(t), x) \in F \times E$ , then  $S^n(x) \in E_{\mathbf{i}}'$ , and

$$x \in E'_{i_1}, \quad S(x) \in E'_{i_2}, \quad \dots, \quad S^{n-1}(x) \in E'_{i_n},$$

where  $i = (i_1, i_2, \dots, i_n)$ . This implies that

$$R(\pi(t), x) = (\pi(\phi_{i_1}(t)), S(x)), \quad R^2(\pi(t), x) = (\pi(\phi_{i_2} \circ \phi_{i_1}(t)), S^2(x)), \dots,$$

and

$$\begin{aligned} R^n(\pi(t), x) &= (\pi(\phi_{i_n} \circ \phi_{i_{n-1}} \circ \dots \circ \phi_{i_1}(t)), S^n(x)) \\ &= (T_i(\pi(t)), S^n(x)). \end{aligned}$$

This proves that  $R^{-n}(F \times E) \subset (T_i \times S^n)^{-1}(F \times E)$ . The reverse inclusion is proved in a similar way.  $\square$

**Lemma 3.39.** *The transformation  $R$  is measure preserving with respect to the product measure  $\mu_L \times \nu$ .*

**Proof.** For  $E \in \mathcal{C}$  and a Borel  $F \subset \mathbb{T}^d$ , by lemma 3.38 we have

$$\begin{aligned} \mu_L \times \nu(R^{-1}(F \times E)) &= \sum_{i=1}^{m_\alpha} \mu_L \times \nu(R^{-1}(F \times (E \cap E'_i))) \\ &= \sum_{i=1}^{m_\alpha} \mu_L \times \nu((T_i \times S)^{-1}(F \times (E \cap E'_i))) \\ &= \mu_L \times \nu(F \times E). \end{aligned} \quad \square$$

**Lemma 3.40.**  *$R$  is ergodic with respect to  $\mu_L \times \nu$ .*

**Proof.** For Borel sets  $F_1, F_2 \subset \mathbb{T}^d$ ,  $G_1, G_2 \in \mathcal{C}$  and  $n \in \mathbb{Z}_{>0}$ , we have

$$\begin{aligned} \mu_L \times \nu(R^{-n}(F_1 \times G_1) \cap (F_2 \times G_2)) &= \sum_{i \in \mathcal{J}(n)} \mu_L \times \nu(R^{-n}(F_1 \times (G_1 \cap E'_i)) \cap (F_2 \times G_2)) \\ &= \sum_{i \in \mathcal{J}(n)} \mu_L(T_i^{-1}(F_1) \cap F_2) \nu(S^{-n}(G_1 \cap E'_i) \cap G_2), \end{aligned}$$

where we used lemma 3.38 for the second equality.

By lemma 3.36, for arbitrary  $\varepsilon > 0$ , there is  $n_0 > 0$  such that, if  $n \geq n_0$  and  $i \in \mathcal{J}(n)$ , the number

$$\varepsilon_i = \mu_L(T_i^{-1}(F_1) \cap F_2) - \mu_L(F_1)\mu_L(F_2)$$

has an estimate

$$|\varepsilon_i| < \varepsilon.$$

We therefore have that

$$\mu_L \times \nu(R^{-n}(F_1 \times G_1) \cap (F_2 \times G_2)) = \sum_{i \in \mathcal{J}(n)} \mu_L(F_1) \mu_L(F_2) \nu(S^{-n}(G_1 \cap E'_i) \cap G_2) + \varepsilon_n,$$

where

$$\varepsilon_n = \sum_{i \in \mathcal{J}(n)} \varepsilon_i \nu(S^{-n}(G_1 \cap E'_i) \cap G_2).$$

Since the  $E'_i$  (with  $i \in \mathcal{J}(n)$ ) give a partition of  $X$ , we firstly see that  $|\varepsilon_n| < \varepsilon$ , and secondly conclude that

$$\sum_{i \in \mathcal{J}(n)} \nu(S^{-n}(G_1 \cap E'_i) \cap G_2) = \nu(S^{-n}(G_1) \cap G_2).$$

For each  $N > 0$ , we have

$$\frac{1}{N} \sum_{n=0}^{N-1} \mu_L \times \nu(R^{-n}(F_1 \times G_1) \cap (F_2 \times G_2)) = \mu_L(F_1) \mu_L(F_2) \frac{1}{N} \sum_{n=0}^{N-1} (\nu(S^{-n}(G_1) \cap G_2) + \varepsilon_n).$$

By ergodicity of  $S$ , as  $N \rightarrow \infty$  this converges to

$$\mu_L(F_1) \mu_L(F_2) \nu(G_1) \nu(G_2) = \mu_L \times \nu(F_1 \times G_1) \mu_L \times \nu(F_2 \times G_2),$$

up to an error term of absolute value less than  $\varepsilon$ . Since  $\varepsilon$  was arbitrary, we see that

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=0}^{N-1} \mu_L \times \nu(R^{-n}(F_1 \times G_1) \cap (F_2 \times G_2)) = \mu_L \times \nu(F_1 \times G_1) \mu_L \times \nu(F_2 \times G_2).$$

Since  $F_1, F_2, G_1$  and  $G_2$  are arbitrary, this proves the ergodicity of  $R$ . □

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