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Journal Item

You Cannot *Patch* Active Plasma and Collisionless Sheath

Raoul N. Franklin

***Abstract*—The fluid equations for the plasma-sheath are examined over the range of collisionality from collisionless to fully collisional. The method of "patching" plasma to sheath is examined critically in comparison with matched asymptotic approximations and solutions computed without approximation. It is concluded that it is necessary to include a transition layer in order to smoothly join plasma and collisionless sheath—and then the Bohm Criterion applies. When the sheath is collisional, the orderings are different and there is no transitional layer but equally there is no collisionally modified Bohm Criterion. Nevertheless, a method is given for calculating the ion flux to the wall under conditions where the sheath is collisional.**

***Index Terms*—Bohm Criterion, matched asymptotic approximations, plasma sheaths.**

Nomenclature

C_s	Ion sound speed.
c_{Bm}	Modified Bohm speed.
e	Electron charge.
E	Electric field.
E_p	Electric field at the patching point.
E_s	Electric field at the sheath edge.
k	Boltzmann's constant.
L	Plasma transverse dimension.
M	Ion mass.
n_i	Ion density.
n_e	Electron density.
n_{e0}	Electron density at center.
n_s	Plasma density at the sheath edge.
N	Normalized charged particle density.
r_p	Probe radius.
T_e	Electron temperature.
U	Normalized ion speed.
v_i	Ion speed.
v_e	Electron speed.
v_{is}	Ion speed at the sheath edge.
V	Electrostatic potential.
x	Spatial coordinate.
X	Normalized spatial coordinate.
Y	Normalized spatial coordinate.
Z	Ionization rate.
ϕ	Normalized potential.
η	Normalized potential, measured relative to the plasma edge.
λ_D	Electron Debye length.
λ_{D0}	Electron Debye length at center.
ν_i	Ion collision frequency.
λ_{Ds}	Electron Debye length at sheath edge.
λ_i	Ion mean free path.

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I. Introduction

In an effort to simplify consideration of the plasma-wall boundary region, Godyak and Sternberg [1] proposed that the plasma (quasi-neutral) solution and the sheath (zero generation) solution be joined at a point where the electric field had a specified value, namely $kT_e/e\lambda_D$, where T_e is the electron temperature and λ_D the local Debye length.

There is no doubt that in the plasma-sheath the electric field passes through this value as it increases monotonically from virtually zero deep within the plasma to several times $kT_e/e\lambda_D$—even when the wall is only at floating potential.

The purpose of this note is to examine critically the patching process and the developments of it, because that process denies the existence of the region that is necessary to join plasma and collisionless sheath smoothly. Such a region was the natural result of the analyses of the probe-plasma situation given by Lam [2], [3], and for the plasma-wall sheath by Franklin and Ockendon [4]. The analyses referred to, involved the use of the method of matched asymptotic approximations which had by then transformed the field of fluid mechanics and gas dynamics [5]. Its use in plasma physics has been less intensive, but the list of references [2]–[4], [6]–[25], [33], [34], shows that it has been applied in a variety of situations with a significant increase in recent years.

We will keep our considerations here confined to the basics of the problem because part of the difficulty in assimilating the technique into the natural "armoury" of plasma physicists has been its apparent mathematical complexity. This contrasts with the situation in fluid mechanics where in the latter half of last century it was the "playground" of applied mathematicians.

II. Basic Equations

We will work essentially in the collisionless fluid approximation, as did Godyak and Sternberg, when the relevant equations are for charged particle generation and loss in the volume in one dimension

$$\frac{d}{dx}(n_i v_i) = \frac{d}{dx}(n_e v_e) \quad (1)$$
$$= Z n_e. \quad (2)$$

The collisionless ion fluid momentum equation is

$$v_i \frac{dv_i}{dx} + \frac{Zn_e}{n_i} v_i = \frac{e}{M}\frac{dV}{dx}. \quad (3)$$

While the electrons obey a Boltzmann relation

$$n_e = n_{e0} \exp\left(\frac{eV}{kT_e}\right). \quad (4)$$

Normalizing these equations with $N = n_i/n_{e0} = n_e/n_{e0}$, $U = v_i/c_s$, $X = xZ/c_s$ where $c_s^2 = kT_e/M$ leads to equations for the plasma

$$\frac{dX}{dU} = \frac{1-U^2}{1+U^2}, N = \frac{1}{1+U^2} = \exp(-\phi)$$

where $\phi = -eV/kT_e$. The equations are singular (i.e., they have infinite spatial derivatives) at the point where $U = 1$.

From these, one can determine parameters where the field attains the value suggested for patching viz. $kT_e/e\lambda_{Ds}$. Writing $U = 1-u$ where u is small compared to 1, one readily finds that $u = (\lambda_{Ds}/L) \cdot (ZL/c_s)$ where $x = L$ at the plasma boundary and $ZL/c_s = \pi/2 - 1$ [26, p. 28]. Normally, $\lambda_{Ds}/L \sim 10^{-3}$ in gas discharge plasmas and, thus, the ion speed is very close to the Bohm speed c_s and the point in question is less than a Debye length from the singular point.

Now the sheath solution can be written as

$$\frac{d^2\eta}{dY^2} = (1+2\eta)^{-1/2} - \exp(-\eta) \quad (5)$$

where η is the normalized potential relative to the singular point and $Y = -x/\lambda_{Ds}$ where λ_{Ds} is the Debye length corresponding to $n_{es} = n_{e0}/2$. $d^2\eta/dY^2 \to 0$ as $\eta \to 0$. Equation (5) can readily be integrated to give

$$\frac{1}{2}\left(\frac{d\eta}{dY}\right)^2 = (1+2\eta)^{1/2} - 2 + \exp(-\eta). \quad (6)$$

Thus, we can find the value of η where $d\eta/dY$ has the requisite value for patching. It is very close to 2.4 (2.4031), and thus we are confronted with a serious dilemma. Continuity of electric field requires a major discontinuity in potential and consequent discontinuities in both electron and ion densities as well as ion speed. The ion flux is virtually continuous, but that is because there is next to no charged particle generation in the sheath.

If one avoids the discontinuities in potential and electron density by redefining the reference potential, one is still left with a discontinuity in ion density and ion speed, and this is intimately linked with the fact that while the field is continuous its spatial derivative is not.

Thus, we see that it is not possible to join plasma and sheath *smoothly* by "cutting and pasting." The reason for this lies in the structure of the equations being approximated to by considering only the plasma solution and the sheath solution.

Fig. 1 shows both solutions for the electron and ion densities and the smooth joining found both by matched asymptotic approximations and by computation of the full plasma-sheath equations. Fig. 2 concentrates on potential and electric field and shows again all three solutions for the electric field expressed in Debye lengths based on the plasma edge density $n_s = n_0/2$ as a function of the potential. The field where the transition layer breaks away from the plasma solution agrees with Riemann's estimate [15] of $(\lambda_{Ds}/L)^{3/5} = 0.0631$ and where it joins the collisionless space charge sheath is ~ 1.

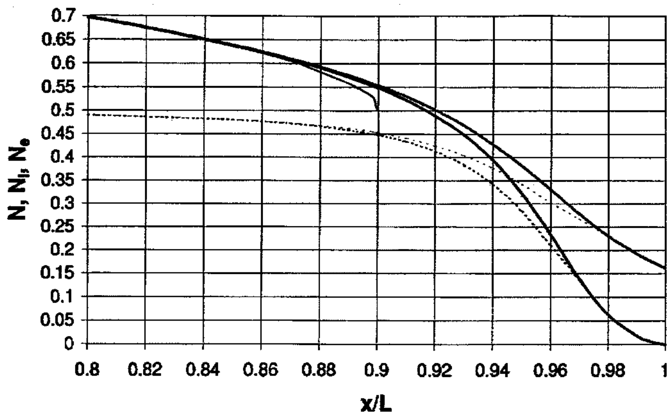

Fig. 1. The regions near the wall for a plane collisionless plasma and collisionless sheath for the case where $\lambda_{D0}/L = 0.7071 \times 10^{-2}$, λ_{D0} is the central Debye length and L the plasma dimension. The fine line is the plasma solution (N) ending at the singular point. The dotted lines are the ion (N_i) and electron (N_e) densities corresponding to the collisionless sheath solution. The full lines are the computed solutions for the densities. They are indistinguishable from the asymptotic solutions in the interval 0.87–0.97.

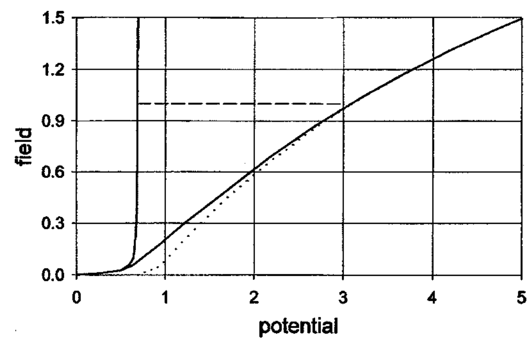

Fig. 2. The variation of electric field E in units of $kT_e/e\lambda_{Ds}$ as a function of the potential in units of kT_e/e again for $\lambda_{D0}/L = 0.7071 \times 10^{-2}$. The fine lines correspond to (a) the plasma solution which is singular where $\phi =$ ln 2 = 0.693, and (b) the collisionless sheath solution which has $\psi = 0$ where $\phi = 0.693$. The full line is the asymptotic/computed solution joining them smoothly. The dotted line corresponds to patching where $E = kT_e/\lambda_{Ds}$.

The need for and the purpose of, the transition layer is clear and the claim of Godyak and Sternberg that patching leads to a smooth joining of plasma and sheath is seen to be false. A particular conclusion from the representation in Fig. 2 is the importance of the transition layer in the growth of the electric field and the electrostatic potential.

The *disjunction* between "plasma" and "sheath" is made manifest.

What patching does is effectively to remove the transition layer that is necessary to join plasma and sheath smoothly. This is not surprising because Godyak and Sternberg in their approach explicitly rejected the concept of a transition layer. In their words "Several attempts have been made to remove these discontinuities by introducing a transition layer. Here we avoid considerations of transition layers by an appropriate choice of boundary condition at the plasma-sheath interface." They were mistaken in their assertion that there were discontinuities as can be seen from the curves giving the full solutions in Figs. 1 and

2. Discontinuities arise only from limiting consideration to the plasma solutions and sheath solutions alone, as they did.

III. Concept of a Collisionally Modified Bohm Criterion

They then went on to propose the concept of a collisionally modified Bohm Criterion suggesting that it was given by

$$c_{Bm} = c_s \left(1 + \frac{\pi}{2}\frac{\lambda_{Ds}}{\lambda_i}\right)^{-1/2} \tag{7}$$

at the patching point and this has been taken up elsewhere [27] and become part of the "received wisdom."

When the sheath is collisional, Blank [28] showed that there is a smooth joining of active plasma and space charge sheath without the need to introduce a transitional layer and the ion speed in the sheath is much less than the Bohm speed. However, the claim by Godyak and Sternberg that there is a collisionally modified Bohm speed c_{Bm} cannot be sustained. The significance of the Bohm speed as a Criterion derives from the disjunction between plasma and collisionless sheath and the transition layer is the region where the ion speed passes through the sonic or Bohm speed. Thus, there is no characteristic speed when there is no transition layer.

At the point within the sheath where the field reaches the value $kT_e/e\lambda_{Ds}$ the ion speed is given by

$$\frac{Mv_i^2}{\lambda_i} = eE = \frac{kT_e}{\lambda_{Ds}}$$

or

$$v_i = c_s \left(\frac{\lambda_i}{\lambda_{Ds}}\right)^{1/2}$$

which apart from a numerical factor is (7) above when $\lambda_i \ll \lambda_{Ds}$, but since the sheath joins the plasma smoothly under collisional conditions this point has not the same physical significance.

We conclude that there is no such thing as a collisionally modified Bohm Criterion. That this is so has now been comprehensively demonstrated by computational solutions of the full plasma-sheath equations [29], [30].

The thought that the Bohm Criterion might be modified by collisions was an idea that needed to be explored (see Valentini [31] and Riemann's refutation [15]). However, the time has now come to recognize that it was a "blind alley" and move on.

For a number of years, it seemed that the Bohm Criterion was a relation that held only in the limiting case where $\lambda_i/L \to 0$. However, it is now clear that its region of validity extends to points where the ion mean free path must be compared with both λ_{D0} and L. It is still valid when $\lambda_i < L$ and $\lambda_{D0} < \lambda_i$, but not if $\lambda_i < \lambda_{D0}$ and the limit to its validity is given approximately by $\lambda_i = \lambda_{D0}^{4/5} L^{1/5}$ [22], [29], [30], [32]. The theoretical and experimental evidence is now in place.

The reasoning behind Godyak and Sternberg's choice of $E_p = kT_e/e\lambda_{Ds}$ for the field value at which to patch plasma and sheath lay in the fact that for a collisionless sheath the potential across the sheath is of the order of kT_e/e and its length of the order of λ_{Ds}. However, Blank [28] showed that for the constant collision frequency for momentum transfer model that the collisional sheath has dimension $L^{1/3}\lambda_{D0}^{2/3}$ where λ_{D0} is the central Debye length and the plasma solution varies linearly near the wall. This was consistent with earlier work [33], [34] which considered the sheath around an electrostatic probe of radius r_p and found that such a collisional sheath had dimensions $r_p^{1/3}\lambda_D^{2/3}$. Thus the field E_s at the sheath edge is

$$E_s = -\left(\frac{dV}{dx}\right)_s = -\frac{kT_e}{e}\frac{1}{n_s}\left(\frac{dn}{dx}\right)_s = \frac{kT_e}{e}\frac{\pi}{2}L^{-1/3}\lambda_{D0}^{-2/3}$$

and the local Debye length

$$\lambda_{Ds} = \lambda_{D0}\left(\frac{n_0}{n_s}\right)^{1/2} = \lambda_{D0}\left(\frac{L}{\lambda_{D0}}\right)^{1/3}\left(\frac{2}{\pi}\right)^{1/2}$$

so that

$$E_p = \frac{kT_e}{e}\left(\frac{\pi}{2}\right)^{1/2} L^{-1/3}\lambda_{D0}^{-2/3}$$

and, thus, the field at entry to the sheath $E_s = E_p(\pi/2)^{1/2}$ or $E_p \sim E_s$. However, from the equation of motion we have

$$Mv_{is}\nu_i = eE_s = kT_e\frac{\pi}{2}L^{-1/3}\lambda_{D0}^{-2/3}$$

or

$$v_{is} = \frac{c_s^2}{\nu_i}\frac{\pi}{2}\left(\frac{\lambda_{D0}}{L}\right)^{1/3}\frac{1}{\lambda_{D0}}$$

giving

$$\frac{v_{is}}{c_s} = \left(\frac{\pi}{2}\right)^{3/2}\frac{\lambda_i}{\lambda_{Ds}} \tag{8}$$

with $\lambda_i = c_s/\nu_i$.

The corresponding matched asymptotic approximation treatment with a constant mean free path has been given recently [25]. The collisional sheath thickness has dimension $L^{1/5}\lambda_{D0}^{4/5}$ and the plasma density varies parabolically near the wall. Following the same reasoning as previously, we again find that $E_s \sim E_p$ but the ion equation of motion gives

$$\frac{v_{is}^2}{c_s^2} \sim \frac{\lambda_i}{\lambda_{D0}}\left(\frac{\lambda_{D0}}{L}\right)^{1/5} \sim \frac{\lambda_i}{\lambda_{Ds}}. \tag{9}$$

Thus, (8) and (9) may be compared with (7) above and show that the ion speed at "entry" to a collisional sheath is in both models related to λ_i/λ_{Ds} but this does not imply that there is a modified Bohm Criterion. What it does mean is that the ion flux to the wall can be calculated using such expressions provided that they are coupled with an appropriate value of the ion density. This assertion can be demonstrated using the relevant plasma equations because in the sheath there is no generation and the ion flux is constant.

The fact that when both plasma and sheath are collisional the ion equation of motion is the same in plasma and sheath is what ensures that plasma and sheath join smoothly. Thus, patching under those circumstances introduces minimal discontinuity of the variables provided that they are defined locally. Indeed there is no difference between the two, except that patching occurs at an apparently arbitrary point. So far as the local Debye length is concerned, this means knowing how the electron density varies in the plasma solution near the wall and the scale length of the sheath.

However, stating that the ions are moving in equilibrium with the local electric field does not amount to a modified Bohm Criterion. The criterion and the transition layer were necessary for a collisionless sheath to join smoothly to a plasma (collisional or collisionless) and the criterion fails once the ion collision length becomes less than the scale of the transition layer (and the layer no longer exists). For this reason there is no criterion when the outer layer(s) are sufficiently collisional.

The ion speed at the patching point for the collisionless space charge sheath is much greater ($\sim$ 2.4096 times) than the Bohm speed, but is given by the expressions (8), (9) above when the sheath is collisional and a smooth variation occurs in the range $\lambda_i \sim \lambda_{Ds}$ but (7) does not describe that transition.

The only physical quantity that has a finite nonzero value at the "plasma edge"and, therefore, through the space charge sheath is the ion flux and that can be simply calculated from the plasma solution. Since

$$n_e = n_{e0} \exp \frac{eV}{kT_e} = n_{e0} \cos \frac{\pi}{2} \frac{x}{L}$$

$$\frac{e}{kT_e} \frac{dV}{dx} = \frac{-\pi}{2L} \frac{\sin \frac{\pi}{2} \frac{x}{L}}{\cos \frac{\pi}{2} \frac{x}{L}}$$

but

$$MV_i \nu_i = -e \frac{dV}{dx}$$

therefore

$$n_i v_i \rightarrow \frac{kT_e}{M} \frac{\pi}{2L} \frac{n_{e0}}{\nu_i} = n_{e0} \frac{\pi}{2} \frac{\lambda}{L} c_s$$

as $x \rightarrow L$ and this is precisely the value given by a convoluted calculation determining self consistently the ion density and speed at the point where the field is given by

$$eE = \frac{kT_e}{\lambda_{Ds}}.$$

IV. Conclusion

Thus, we conclude that this patching point introduced by Godyak and Sternberg has no significance when the sheath is collisionless, but corresponds approximately to the beginning of the space charge collisional sheath at high pressures. Then, patching is not appropriate since plasma and sheath join smoothly.

The Bohm Criterion derives from the requirements on (3) and its integral as clearly shown in Franklin and Ockendon [4]. The collisional space charge sheath joins the plasma solution smoothly without there being an additional requirement on the ion speed, and thus the introduction of the concept of a collisionally modified Bohm criterion was misguided.

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