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Journal Item

How to cite:

Mellish, Steve; Taherzadeh, Shahram and Attenborough, Keith (2020). Use of a Modal Model in Predicting Propagation from a Point Source Over Grooved Ground. The Quarterly Journal of Mechanics and Applied Mathematics, 73(4) pp. 367–382.

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Version: Accepted Manuscript

Link(s) to article on publisher's website:

<http://dx.doi.org/doi:10.1093/qjmam/hbaa018>

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USE OF A MODAL MODEL IN PREDICTING PROPAGATION FROM A POINT SOURCE OVER GROOVED GROUND

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[Received 99 Nonvier 9999. Revise 99 Nonvier 9999]

Summary

Regularly spaced low walls and rectangular lattices on a hard ground have been investigated as a means for reducing noise levels from surface transport. Predictions of the insertion loss of such surfaces has involved the use of computationally intensive numerical methods such as the Boundary Element Method (BEM) or Finite difference techniques (FDTD and PSTD). By considering point-to-point propagation above regularly spaced acoustically-hard grooves with rectangular cross sections, a modal model used hitherto to predict electromagnetic and ultrasonic surface waves is adapted to derive an effective impedance for such a grooved surface. When this effective impedance is used in the classical theory for propagation from a point source above an impedance plane, the modal model enables predictions of excess attenuation spectra that compare closely with those obtained by numerical methods but take a fraction of the time. Also the modal method is extended to predict excess attenuation spectra above phase gradient metasurfaces with porous material in each constituent groove.

1. Introduction

Rough surfaces have been the subject of study in acoustics for some time [1] and tend to be considered as either scattering where the wavelengths involved are comparable to the surface feature sizes leading to scattering, or as homogeneous where the wavelengths are much larger than the surface texture. Many theories have been published to predict the scattering behaviour of particular forms of periodic structure (often with application in electromagnetics), beginning with Rayleigh for sinusoidal surfaces [2] and latterly those such as multiple scattering or boss theory [3–5] and modal decompositions [6–13] which solve for the scattered field. Models of homogeneous ground surfaces tend to yield an effective impedance for the surface from solving for plane wave propagation, such as the defining works of Kirchhoff [14], Biot [15] and Zwicker and Kosten [16] amongst others. Latterly the slit-pore surface impedance model presented by Attenborough *et al.* [17, pp.63-65] has been shown useful for representing many types of typical outdoor ground surface. Since an effective impedance is useful in predicting grazing incidence sound propagation from point or line sources over rough boundaries, this paper solves for the plane wave field using the modal model, uses the result to deduce an effective impedance and uses the effective impedance in a classical model for point to point propagation above a surface to predict the sound field at any given receiver location, which has wider application to practical outdoor acoustic problems. This model is often used in outdoor acoustic noise problems as it applies to many typical situations involving a noise source, intermediate terrain and

a receiving location which is often a inhabited area. Usually the ground is considered as homogeneous, flat and smooth (on the scale of the problem domain), suitable assumptions for most types of ground surface such a fields or grasslands. However, recent studies have shown the benefit of artificial periodic ground surfaces for transport noise attenuation and in particular those consisting of rectangular grooves or lattices [1, 18, 19]. These are effective in part due to their scattering behaviour but cannot be assumed to be homogeneous in structure once the wavelengths involved become comparable to the size of the roughness features. Empirical application of impedance models to rectangular grooved scattering surfaces has been explored in [18].

Our study presents a method to solve for the scattered field by use of an extended modal model [8, 11, 20] and incorporate the scattering behaviour in an effective impedance. This method is significantly less complex and computationally expensive than existing numerical methods such as BEM and so more readily accessible to acoustic engineers.

Three types of rectangular grooved periodic structures are considered, (i) identical acoustically-hard grooves, (ii) grooves with different depths intended to introduce a phase gradient during reflection and (iii) a ‘metasurface’ composed from grooves with different depths containing different thicknesses of a rigid-porous material. Frequency response results obtained using the modal model derived effective impedance for all of the three types of grating are shown to compare favourably with the more computationally expensive method of BEM and the potential benefits of the hybrid structure for broad band noise attenuation are shown.

An $\exp(-i\omega t)$ time convention will be assumed throughout but omitted for brevity. Furthermore, k represents the wave propagation constant in the assumed inviscid air medium with speed of sound c_0 , at rest density ρ_0 and characteristic wave impedance Z_c . Where $Z = p/|\mathbf{v}|$, given that p and $\mathbf{v}$ are the scalar acoustic pressure and vector particle velocity respectively.

2. Point-to-point acoustic propagation above a ground plane

The problem of interest is described by Fig. 1. Sound from the assumed omnidirectional or point source at height z_S travels directly to the receiver at height z_R along path R_i . The specularly reflected path length is $R_A + R_B$. Since the sound spreads spherically or cylindrically from the source (which are equivalent in the x - z plane except for the wavefront spreading loss factor), it is appropriate to use a spherical reflection coefficient denoted by Γ_S , such as that proposed by Chien and Soroka [21]. The rough surface is represented by an effective impedance [16–18], Z_{eff} in the plane $z = 0$. A usual method for calculating the frequency-dependent influence of the ground surface is the Excess Attenuation (EA) defined in decibel (dB), with reference to the free-field where there is no ground surface and as shown in (2.1),

$$EA_{dB} = 20 \log_{10} \left(\left| 1 + \frac{\Gamma_S L(R_A + R_B) \exp(ik(R_A + R_B))}{L(R_i) \exp(ikR_i)} \right| \right). \quad (2.1)$$

The spreading loss L is chosen for the application in hand where 2d problems usually assume a cylindrically spreading line source and 3d (or 2.5d) problems a spherically spreading point source. These are defined respectively by L_C in (2.2) and L_S in (2.3),

$$L_C(R) = \frac{1}{2\pi\sqrt{R}} \quad (2.2)$$

whereas L_S for spherical spreading is,

$$L_S(R) = \frac{1}{4\pi R}. \quad (2.3)$$

This definition represents the point-to-point difference in the total sound level after accounting for wavefront spreading loss.

The point-to-point model of Fig. 1 assumes that the ground is homogeneous and reflection is purely specular given by the combination of the incident p_i and ground reflected p_r pressure components thus,

$$p_{tot}(x_R, z_R) = p_i + p_r. \quad (2.4)$$

However, in the presence of a scattering ground surface where the features are comparable in size to the incident wavelength, such as those under consideration in this work, an additional scattered pressure component, p_{sct} is present and so the field at the receiver becomes,

$$p_{tot}(x_R, z_R) = p_i + p_r + p_{sct}. \quad (2.5)$$

A method for efficiently determining an approximation to the scattered field using a modal model, then incorporating this into a specific Z_{eff} follows henceforth.

3. A modal model for a periodic array of variable depth grooves incorporating porous layers

3.1 The method

A solution to the problem of calculating the electromagnetic field above an infinite array of perfectly reflecting rectangular grooves in 2-dimensions, of the form shown in Fig. 2, was presented by Hessel *et al.* [8] and later applied by Kelders *et al.* [11, 20] to acoustics for the study of ultrasonic surface waves supported by these surface gratings.

The method has recently been applied by the authors [22] to approximate the grooved surface as an effective ground plane impedance to allow the prediction of point-to-point acoustic propagation. In this work the modal model kernel itself is expanded for structures with multiple grooves within each period, having individual dimensions and a hard-backed porous layer of depth l occupying the lower portion of each groove, as shown in Fig. 3.

Such structures are frequently termed in recent literature as metasurfaces or phase gradient structures [23–29]. The model is similar to that for the single depth grooves [8, 11, 20] but the parameters become 2-dimensional with the addition of the index q , relating to the q 'th groove within each period from $q = 0$ to $q = K$ where $K + 1$ is the number of grooves per grating cell period d . The geometry of the scattered field with diffraction order m , depicted in Fig. 2, depends only upon the pitch of the grating structure d , the angle of incidence θ and the incident wavelength so is independent of the surface topography within each cell for a given d . The energy distribution between each diffraction order will however, be dependent upon the cell structure.

The assumptions are made that the surfaces of the grooved structure are acoustically-hard and the grating is infinitely periodic in $\pm x$. Spatial invariance in y is also assumed which reduces the problem to 2-dimensions. Such periodicity allows the problem to be solved for only one individual cell of the structure in the range $0 \leq x \leq d$. Excitation is by homogeneous plane wave with a given angle of incidence θ and propagation constant k

in the x - z plane, within an ideal inviscid air medium where the velocity potential ϕ , obeys the Helmholtz equation (3.1),

$$\nabla^2 \phi + k^2 \phi = 0 \quad (3.1)$$

The domain of the 2d problem is considered as two separate half spaces within the x - z plane, the free field upper space for $z > 0$ and the lower groove space where $z \leq 0$. The acoustic field is defined in the x - z plane by the particle velocity and pressure functions $\mathbf{v}(x, z)$ and $p(x, z)$ respectively, which are solved for continuity on the half space boundary at $z = 0$. As the velocity is a vector quantity it is useful to consider the x and z components separately with the definitions,

$$v_x = \mathbf{v} \cdot \mathbf{i}, \quad v_z = \mathbf{v} \cdot \mathbf{k} \quad (3.2)$$

where $\mathbf{i}$ and $\mathbf{k}$ are the unit vectors for the spatial axes x and z respectively.

Both spaces also exhibit infinite periodicity in x owing to the unbounded extents of the grooved structure, combined with homogeneous plane wave excitation. This constrains the propagation characteristics of each half space region to that of an infinite set of m and n discrete modes for the upper and lower half spaces respectively, with complex amplitude weightings in terms of vertical velocity v_z , of $\hat{V}_m$ (upper space quantities are denoted by hat) and V_n . The incident wave v_z amplitude $\hat{V}_0^i$ is given as an initial condition of the problem and for convenience assigned the value of unity.

Solving the resulting system of equations numerically will yield the groove space mode amplitude set V_n , which along with the given geometry of the problem and $\hat{V}_0^i$, will define the plane wave acoustic field for $z > 0$ via analytic calculation of the $\hat{V}_m$ set.

3.2 Boundary conditions

The boundary conditions of the problem are defined separately for each half space. The free space region $z > 0$ is assumed homogeneous, extending unbounded to infinity, therefore the resulting plane-wave field must also be homogeneous and infinite in extent, so no specific boundary conditions apply. However all wave functions defined in the space must obey the Helmholtz equation of (3.1).

The diagram of Fig. 4 details the boundary elements for each groove feature within a grating period consisting of $K + 1$ grooves. The boundary elements of the structure $\textcircled{A}$ - $\textcircled{C}$ are assumed acoustically-hard with impedance $Z = \infty$. At the groove aperture the impedance condition Z^G is defined from the interaction of the supported wave modes. Z^L is the impedance presented at the air/porous material interface $z = -h_q$ (as defined in Fig. 3).

Given $Z = \infty$ upon the groove walls, the acoustic boundary conditions are as (3.3) and (3.4) for each bounding element type.

$$\textcircled{A} \quad v_z = \frac{\partial p}{\partial z} = 0 \quad (3.3)$$

$$\textcircled{B}, \textcircled{C} \quad v_x = \frac{\partial p}{\partial x} = 0 \quad (3.4)$$

Given (3.3) and (3.4), then each q 'th groove may be considered to behave like a waveguide [32, pp.240,428-432], supporting an infinite set of harmonically related transverse standing

wave mode functions $e_{q,n}(x)$ of harmonic order n and v_z amplitude weighting $V_{q,n}$ where $n \in \mathbf{Z}^+ = \{0, 1, 2, 3, \dots\}$.

The closed end of each groove will be accounted for in the consideration of Z^L . In the free space for $z > 0$, the acoustic velocity and pressure must be continuous at the d periodic vertical boundaries of each grating period as (3.5), in the limit $\delta \rightarrow 0$.

$$\begin{aligned} v_x(\kappa d - \delta x, 0 + \delta z) &= v_x(\kappa d + \delta x, 0 + \delta z) \\ &= \frac{\partial}{\partial x} p(\kappa d - \delta x, 0 + \delta z) = \frac{\partial}{\partial x} p(\kappa d + \delta x, 0 + \delta z) \quad \kappa \in \mathbf{Z} \end{aligned} \quad (3.5)$$

Within the groove space, the unknown groove mode amplitude set $V_{q,n}$ will depend upon the impedance across the groove aperture Z^G (see Fig. 4), defined by the combination of the incident wave and that reflected from the air/porous material boundary at $z = -h_q$, which in turn depends upon the impedance, Z^L , that the hard-backed porous layer presents at the interface. Therefore,

$$Z_{q,n}^G = \frac{Z_c k (-1 + \gamma \exp(ik_{q,n}^z 2h_q))}{k_{q,n} - 1 + \gamma \exp(ik_{q,n}^z 2h_q)} \quad (3.6)$$

where the dimensionless complex reflection coefficient γ , at the interface, is given by (3.7),

$$\gamma = -\frac{1 - (1/Z_{q,n}^L)}{1 + (1/Z_{q,n}^L)} \quad (3.7)$$

and $k_{q,n}^z$ is the z -direction propagation constant for the n 'th order wave mode within the q 'th groove.

The impedance Z^L within the grooves at $z = -h_q$ and which is normalised to Z_c , is calculated using an appropriate effective impedance model for a hard-backed porous layer [16–18, 30] while being mindful of the corresponding assertions the chosen model makes about the physical properties of the material. If a locally reacting model (Z^L is angle of incidence invariant) is chosen all propagation within the layer is assumed to be plane-wave and perpendicular to the surface, so Z^L will not be dependent upon the angle with which the groove mode is incident upon the air/porous material boundary θ_n . Non-locally reacting impedance models able to support transverse modes will have a dependence upon θ_n , thus Z^L will be a function of θ_n .

3.3 Defining the characteristic mode functions

Let us begin by defining the characteristic wave function $e_{q,n}(x)$ in x , within the aperture of each q 'th groove at $z = 0$, which is a standing wave due to the constraints of the groove walls. From (3.4), the pressure gradient in x must vanish upon the walls of each groove at $x'_q = 0$ and $x'_q = a_q$, where x'_q is the q 'th groove local co-ordinate with $x'_q = 0$ being coincident with the left hand q 'th groove wall and is related to x by,

$$x'_q = x - \sum_{j < q} d_j + \frac{d_q - a_q}{2}. \quad (3.8)$$

A solution is an infinite harmonically related set satisfying both the Helmholtz equation (3.1) and the boundary condition (3.4), leading to the standard solution for a waveguide

obtained from separation of variables,

$$\phi(x'_q, z) = \cos\left(\frac{n\pi x'_q}{a_q}\right) \{A_{q,n} \exp(ik_{q,n}^z z) + B_{q,n} \exp(-ik_{q,n}^z z)\} \quad (3.9)$$

where $A_{q,n}$ and $B_{q,n}$ are the amplitudes of the counter propagating modes of order n within the q 'th groove with z -direction wavenumber $k_{q,n}^z$,

$$k_{q,n}^z = \pm \sqrt{k^2 - \left(\frac{n\pi}{a_q}\right)^2}. \quad (3.10)$$

Given that $v_z = \partial\phi/\partial z$ and considering only the x -component of (3.9), we obtain the transverse characteristic wave function,

$$v_z(x'_q) \propto \cos\left(\frac{n\pi x'_q}{a_q}\right). \quad (3.11)$$

For a given angle of propagation within the groove, pressure will be related to v_z by an impedance factor Z and so shares the same proportionality thus,

$$p(x'_q) \propto \cos\left(\frac{n\pi x'_q}{a_q}\right). \quad (3.12)$$

We are concerned with solving for v_z upon $z = 0$ in the single period $0 \leq x \leq d$ and follows the derivation of the characteristic function $e_{q,n}(x)$ for v_z within the aperture of each groove q . Normalising (3.11) such that the self inner product $\int_0^{a_q} e_{q,n}(x'_q)^2 dx = 1$ leads to,

$$\begin{aligned} e_{q,n>0}(x'_q) &= \sqrt{\frac{2}{a_q}} \cos\left(\frac{n\pi x'_q}{a_q}\right) \quad 0 \leq x'_q \leq a_q \\ e_{q,n=0} &= \sqrt{\frac{1}{a_q}} \end{aligned} \quad (3.13)$$

Substituting x'_q with x for $n > 0$ using the relation of (3.8) gives,

$$\begin{aligned} e_{q,n>0}(x) &= \sqrt{\frac{2}{a_q}} \cos\left(\left[\frac{n\pi}{a_q}\right] \left[x - \sum_{j<q} d_j + \frac{d_q - a_q}{2}\right]\right) \\ &\quad \sum_{j<q} d_j + \frac{d_q - a_q}{2} \leq x \leq \sum_{j \leq q} d_j - \frac{d_q - a_q}{2} \end{aligned} \quad (3.14)$$

and by defining the delta function,

$$\delta_e = \begin{bmatrix} 1 & x \geq \sum_{j<q} d_j + \frac{d_q - a_q}{2} \\ 1 & x \leq \sum_{j \leq q} d_j - \frac{d_q - a_q}{2} \\ 0 & \text{else} \end{bmatrix} j \in \mathbf{Z}^+ \quad (3.15)$$

yields the characteristic mode function for v_z ,

$$\begin{aligned} e_{q,n>0}(x) &= \delta_e \sqrt{\frac{2}{a_q}} \cos \left(\left[\frac{n\pi}{a_q} \right] \left[x - \sum_{j<q} d_j + \frac{d_q - a_q}{2} \right] \right) \quad 0 \leq x \leq d \\ e_{q,0} &= \delta_e \sqrt{\frac{1}{a_q}}. \end{aligned} \quad (3.16)$$

The free space field, posed in terms of v_z , is accumulated from an infinite set of $\hat{V}_m$ weighted orthonormal modes where each discrete mode m is characterised upon the half space boundary by the basis function $\hat{e}_m(x, 0 + \delta z)$, which itself satisfies the Helmholtz equation (3.1). The continuity condition of (3.5) requires only that each free space mode is continuous at the boundaries of each cell for velocity and pressure. This admits an infinite set of discrete travelling wave solutions with $2m\pi/d$ periodicity given $m \in \mathbf{Z}$. In addition, the presence of the incident wave introduces a constant phase term $k \sin \theta$. Normalising such that the self inner product $\int_0^d \hat{e}_m(x) \cdot \overline{\hat{e}_m(x)} dx = 1$, where the over-line denotes the complex conjugate, then the characteristic function $\hat{e}_m(x)$ for each discrete mode m is defined upon $z = 0 + \delta z$ thus,

$$\hat{e}_m(x) = d^{-0.5} \exp \left(i \left[k \sin \theta + \frac{2m\pi}{d} \right] x \right). \quad (3.17)$$

From this set of equations the modal model may be constructed into a form allowing numerical solution and the plane wave field subsequently approximated.

3.4 Forming the system of equations

A derivation of the modal model kernel follows henceforth. From the given incident wave characteristics, the geometry of the grooved structure and the resulting boundary conditions, a system of equations is obtained which may be solved numerically to define the acoustic field upon the surface at $z = 0$. The result is then used to calculate the free space field analytically. According to [8], the $\hat{e}_m$ basis function set is orthonormal and hence any $\hat{V}_m$ may expressed in terms the complete $V_{q,n}$ set,

$$\hat{V}_m = \sum_{q=0}^{q=K} \sum_n V_{q,n} \langle e_{q,n}, \hat{e}_m \rangle \quad (3.18)$$

with the inner product defined as,

$$\langle f(x), g(x) \rangle = \int_0^d f(x) \overline{g(x)} dx \quad (3.19)$$

and conversely $V_{q,n}$ may be found from the complete $\hat{V}_m$ set by,

$$V_{q,n} = \sum_m \hat{V}_m \langle e_{q,n}, \overline{\hat{e}_m} \rangle. \quad (3.20)$$

However the $\hat{e}_m$ sets $e_{q,n}$ are not mutually orthonormal and it is this interdependence which limits the model to solution by numeric approximation.

As the problem is posed in terms of z -direction acoustic particle velocity v_z and acoustic pressure p , the impedance relationships which relate p to v_z for each free-space mode m are shown in (3.21). The special case exists for the incident wave impedance $\hat{Z}_0^i$ due to it propagating in the $-z$ direction.

$$\begin{aligned}\hat{Z}_m &= \frac{Z_c k}{\hat{k}_m^z} \\ \hat{Z}_0^i &= -\hat{Z}_0^r = -\hat{Z}_0\end{aligned}\tag{3.21}$$

where $\hat{Z}_0^r$ is the specularly reflected component impedance. The z -axis propagation constant $\hat{k}_m^z$ for the free-space mode m given in (3.22) as the residual of k and the x component propagation constant m .

$$\hat{k}_m^z = \sqrt{k^2 - \left(k \sin \theta + \frac{2m\pi}{d}\right)^2}\tag{3.22}$$

The continuity equations for p and v_z at the boundary of the two half spaces $z = 0$ are then expressed in (3.23) as the matching summation of the constituent modes.

$$\begin{aligned}v_z(x, 0) &= \sum_{q=0}^{q=K} \sum_n V_{q,n} e_{q,n} = \hat{V}_0^i \hat{e}_0 + \hat{V}_0^r \hat{e}_0 + \sum_{m \neq 0} \hat{V}_m \hat{e}_m \quad 0 \leq x \leq d \\ p(x, 0) &= \sum_{q=0}^{q=K} \sum_n Z_{q,n} V_{q,n} e_{q,n} = \hat{Z}_0^i \hat{V}_0^i \hat{e}_0 + \hat{Z}_0^r \hat{V}_0^r \hat{e}_0 + \sum_{m \neq 0} \hat{Z}_m \hat{V}_m \hat{e}_m \quad 0 \leq x \leq d\end{aligned}\tag{3.23}$$

Derivation must take account of acoustic pressure and is related to velocity by $P = ZV$ where P is the acoustic pressure amplitude and V the particle velocity amplitude. The inner product relationship of (3.20) may express P by multiplying V_n by (3.6) and $\hat{V}_m$ by (3.21). This yields the pressure amplitude relationship,

$$P_{q,n} = Z_{q,n}^G V_{q,n} = \hat{Z}_0^i \hat{V}_0^i \langle e_{q,n}, \bar{\hat{e}}_0 \rangle + \hat{Z}_0^r \hat{V}_0^r \langle e_{q,n}, \bar{\hat{e}}_0 \rangle + \sum_{m \neq 0} \hat{Z}_m \hat{V}_m \langle e_{q,n}, \bar{\hat{e}}_m \rangle\tag{3.24}$$

where the $m = 0$ term is initially separated from the major summation due to it encompassing both the incident and specularly reflected modes with opposite signed impedance (see (3.21)). Incorporating the zeroth mode into the major summation to include $m = 0$, gives,

$$Z_{q,n}^G V_{q,n} = 2\hat{Z}_0^i \hat{V}_0^i \langle e_{q,n}, \bar{\hat{e}}_0 \rangle + \sum_m \hat{Z}_m \hat{V}_m \langle e_{q,n}, \bar{\hat{e}}_m \rangle\tag{3.25}$$

Substituting (3.18) into (3.25) to form (3.26), eliminates $\hat{V}_m$ to leave the V_n set as the only unknown.

$$\begin{aligned}Z_{q,n}^G V_{q,n} &= 2\hat{Z}_0^i \hat{V}_0^i \langle e_{q,n}, \bar{\hat{e}}_0 \rangle + \sum_m \hat{Z}_m \left(\sum_{q^\dagger=0}^{q^\dagger=K} \sum_{n^\dagger} V_{q^\dagger, n^\dagger} \langle e_{q^\dagger, n^\dagger}, \hat{e}_m \rangle \right) \langle e_{q,n}, \bar{\hat{e}}_m \rangle \\ &= 2\hat{Z}_0^i \hat{V}_0^i \langle e_{q,n}, \bar{\hat{e}}_0 \rangle + \sum_{q^\dagger=0}^{q^\dagger=K} \sum_{n^\dagger} V_{q^\dagger, n^\dagger} \left(\sum_m \hat{Z}_m \langle e_{q^\dagger, n^\dagger}, \hat{e}_m \rangle \langle e_{q,n}, \bar{\hat{e}}_m \rangle \right)\end{aligned}\tag{3.26}$$

Subtracting $Z_{q,n}^G V_{q,n}$ and $2\hat{Z}_0^i \hat{V}_0^i \langle e_{q,n}, \bar{e}_0 \rangle$ from both sides of (3.26) leads to,

$$-2\hat{Z}_0^i \hat{V}_0^i \langle e_{q,n}, \bar{e}_0 \rangle = \sum_{q\dagger=0}^{q\dagger=K} \sum_{n\dagger} V_{q\dagger,n\dagger} \left(\sum_m \hat{Z}_m \langle e_{q\dagger,n\dagger}, \hat{e}_m \rangle \langle e_{q,n}, \bar{e}_m \rangle \right) - Z_{q,n}^G V_{q,n}. \quad (3.27)$$

Finally, the $Z_{q,n}^G V_{q,n}$ product is absorbed into the summation in $q\dagger, n\dagger$ by use of a Kronecker delta function δ , to obtain the set of equations for the system thus,

$$\sum_{q\dagger=0}^{q\dagger=K} \sum_{n\dagger} V_{q\dagger,n\dagger} \left(\sum_m \left[\hat{Z}_m \langle e_{q\dagger,n\dagger}, \hat{e}_m \rangle \langle e_{q,n}, \bar{e}_m \rangle \right] - \delta_{q,n}^{q\dagger,n\dagger} Z_{q,n} \right) = -2\hat{Z}_0^i \hat{V}_0^i \langle e_{q,n}, \bar{e}_0 \rangle \quad (3.28)$$

$$\delta_{q,n}^{q\dagger,n\dagger} = \begin{bmatrix} 1 & a = a\dagger, q = q\dagger \\ 0 & else \end{bmatrix}.$$

By solving the system of $K(N \times N)$ equations in (3.28) and truncating m to $\pm M$ and n to N , the set of V_n modal amplitudes for the groove space are obtained. From (3.18) the set of free space modal amplitudes $\hat{V}_m$ are calculated analytically. Expanding (2.5) to form (3.29), the homogeneous plane wave pressure field in free space p_{tot} is therefore given by modal summation.

$$p_{tot}(x_R, z_R) = \hat{Z}_0^i \hat{V}_0^i \hat{e}_0 \exp(-i\hat{k}_0^z z) + \hat{Z}_0^r \hat{V}_0^r \hat{e}_0 \exp(i\hat{k}_0^z z) + \sum_{m \neq 0} \hat{Z}_m \hat{V}_m \hat{e}_m \exp(i\hat{k}_m^z z) \quad (3.29)$$

While the homogeneous plane wave field is itself of interest, it may differ significantly from the field due to a point source at location (x_S, z_S) . Obtaining an approximation to the field due to a point source using the modal solution will be considered henceforth.

4. Deriving an effective impedance from the homogeneous modal model field

In this section the modal model solution to the homogeneous plane wave field will be used to deduce an effective impedance Z_{eff} for the periodically grooved surface. This approach is advantageous over simply taking the plane wave field directly, which is not appropriate for line or point source excitation. Firstly, Z_{eff} is approximately wave type independent allowing direct application to the point-to-point model with a line or point source. Furthermore, the plane wave summation does not generally exhibit a surface wave, but it is implicit in Z_{eff} as a property of the surface itself, so will be apparent upon application to a spherical or cylindrical reflection coefficient Γ_S [21, 31].

The plane wave field at the (x, z) receiver location R , defined in Fig. 1, is given by the accumulation of the free-space modes as in (3.29). To improve similarity with practical outdoor acoustic problems, a geometric scheme is introduced [22] to limit the extent of the grooved surface to between the points x_1 and x_2 defined in Fig. 5.

Modes which would not geometrically intersect the receiver are omitted from the summation of (3.29). Evanescent modes possessing an imaginary k_m^z are also omitted because they are surface modes present only to solve the boundary conditions in the presence of the homogeneous incident plane wave, not a true surface wave [10]. With reference to

Fig. 5, this is achieved by defining the Kronecker delta function of,

$$\delta_{\theta_2}^{\theta_1}(\theta_m) = \begin{bmatrix} 1 & \theta_2 \geq \theta_m \geq \theta_1 \\ 0 & \Im(\theta_m) > 0 \\ 0 & \text{else} \end{bmatrix} \quad (4.1)$$

Applying this to (3.29) to ensure that modes which do not satisfy (4.1) are not included in the field, leads to the revised field summation of,

$$p_{tot}(x_R, z_R) = \hat{Z}_0^i \hat{V}_0^i \hat{e}_0 \exp(-i\hat{k}_0^z z) + \hat{Z}_0^r \hat{V}_0^r \hat{e}_0 \exp(i\hat{k}_0^z z) + \sum_{m \neq 0} \delta_{\theta_2}^{\theta_1}(\theta_m) \hat{Z}_m \hat{V}_m \hat{e}_m \exp(i\hat{k}_m^z z) \quad (4.2)$$

Although x_1 and x_2 may be set arbitrarily, they will be assumed for this work to coincide horizontally with x_S and x_R respectively so that the grating structure occupies the space between the source and receiver.

From the field at point R and including an approximate phase constant to remove the plane wave path length difference l_{PD} between the incident and reflected components $l_{PD} \approx (R_A + R_B) - R_i$, the plane wave reflection coefficient Γ_P of the grooved surface may be calculated as follows,

$$\Gamma_P = \frac{p_r(x_R, z_R) + p_{sct}(x_R, z_R)}{p_i(x_R, z_R) \exp(-ikl_{PD})} \quad (4.3)$$

Rearranging (4.3) to the form of (4.4), yields the effective impedance for the grooved surface, Z_{eff} , given that $Z_{\text{eff}} = p/v_z$ upon the surface at the point of specular reflection which is the origin O in Fig. 1.

$$Z_{\text{eff}} = \frac{(1 + \Gamma_P) \sec \theta}{1 - \Gamma_P} \quad (4.4)$$

The grooved surface is now represented by an equivalent impedance, valid only for the angle of incidence θ for which it was calculated.

5. Results

The results obtained from the methods presented are compared with a Boundary Element Method (BEM) analysis of equivalent geometry, with the source and receiver 0.1 m above the surface and separated by 2 m (see Fig. 1). The slit-pore impedance model [17, pp.63-65] will be used to represent the impedance of the porous layer Z_L as this model has been shown to provide good agreement with a variety of typical porous materials [18]. A flow resistivity of $5.43 \times 10^3 \text{ Pas m}^{-2}$ will be assumed with a porosity of 0.5. This is typical of a gravel which might be a practical material for filling the grooves. Results are presented as excess-attenuation spectra referenced to the free-field, with 200 points and in the frequency band 100 to 2000 Hz, which is where most man-made nuisance noise is focused [18, 19]. Consistent with previous works [11, 22], truncations for the modal model are $N = 5$ and $M = 25$. The process for obtaining the results is to numerically solve the system of modal model equations (3.28), accumulate the plane wave field using (4.2), from this calculate Z_{eff} with (4.4) and obtain the EA spectrum using (2.1).

A trio of results are presented, beginning with Fig. 6, a single grooved grating, followed by a periodic structure in which each period contains three grooves of different depths and

no porous layer in Fig. 7 and the same but with a porous layer in each of the three grooves in Fig. 8.

It is apparent that the extended modal model agrees well with the BEM predictions but with a significantly shorter computation time, approximately 2 hours for Fig. 7 with BEM versus a few minutes for the modal result.

The multiple destructive interference minima of Fig. 7 are evident due to the multiple depth grooves whereas just a single such minimum is observed in Fig. 6 for the single groove. Adding the porous layer then flattens the response to a wide band attenuation of the noise level in Fig. 8. This characteristic is desirable for noise attenuating structures and future work will attempt to lower the frequency at which this attenuation occurs to further improve the practical applicability. Addition of the porous layer also reduces the magnitude of the surface wave which will in part, be due to thermo-viscous losses in the porous layer. However the slope around the first minimum is shallowed and attenuation in this region is less than for the non-porous layered structures.

A plot of the matrix condition number with respect to inversion for the single grooved case of Fig. 6 is shown in Fig. 9. The curve exhibits three distinct peaks, 'A', 'B' and 'C'. At 'A' the incident wavelength is close to $2h$ and h at 'C', suggesting they are related to the physical phenomenon of wavelength-multiple resonances within the groove. At 'B' the incident wavelength is approximately $2a$ suggesting the spatial frequencies of the free space and groove modes are closely aligned and liable to mutual interaction resonance which is a spurious effect of the numerical method. However there is little sign these effects have adversely affected the results in this particular case.

6. Conclusions

The noise attenuating properties of periodic ground structures have been the topic of significant study and those consisting of rectangular cross sectioned grooves particularly suitable for practical application due to the ease with which they may be constructed. Hitherto prediction of sound propagation above such structures has however been computationally expensive. An efficient modal model has been extended and successfully applied to obtain an effective impedance for rectangularly groove ground surfaces. Predictions of excess attenuation spectra using the modal model deduced effective impedance are in good agreement with those obtained using BEM which accounts for the detailed surface topography, at much lower computational cost. Periodic variable depth grooves with a porous material infill are predicted to be capable of broadband excess attenuation during near grazing sound propagation.

7. References

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Fig. 1 Point-to-point acoustic propagation model.

Fig. 2 Rectangular groove definition and resulting scattered field.

Fig. 3 Multiple grooved periodic surface with porous layers.

Fig. 4 Boundary conditions for each q 'th groove element.

Fig. 5 Rectangular groove definition and resulting scattered field.

Fig. 6 Single rectangular groove, $d=0.15$, $a=0.12$, $h=0.2$ (Solid=Modal (Z_{eff}), broken=BEM).

Fig. 7 Three hard bottomed rectangular grooves per period without porous layer, $d_q=[0.05,0.05,0.05]$, $a_q=[0.04,0.04,0.04]$, $h_q=[0.2, 0.15, 0.1]$ (Solid=Modal (Z_{eff}), broken=BEM).

Fig. 8 Three rectangular grooves per period with porous layer, $d_q=[0.05,0.05,0.05]$, $a_q=[0.04,0.04,0.04]$, $h_q=[0.05, 0.05, 0.05]$, $l_q=[0.15, 0.1, 0.05]$ (Solid=Modal (Z_{eff}), broken=BEM).

Fig. 9 Condition number (wrt inversion), of modal matrix for $d=0.15$, $a=0.12$, $h=0.2$.