Open Research Online

The Open University's repository of research publications and other research outputs

Orientably-regular maps on twisted linear fractional groups

Book Section

How to cite:

Erskine, Grahame; Hriňáková, Katarína and Širáň, Jozef (2020). Orientably-regular maps on twisted linear fractional groups. In: Jones, Gareth A; Ponomarenko, Ilya and Širáň, Jozef eds. Isomorphisms, Symmetry and Computations in Algebraic Graph Theory. PROMS, 305. Springer, Cham, pp. 1–35.

For guidance on citations see [FAQs](#).

© 2020 Springer Nature Switzerland AG

<https://creativecommons.org/licenses/by-nc-nd/4.0/>

Version: Accepted Manuscript

Link(s) to article on publisher's website:

http://dx.doi.org/doi:10.1007/978-3-030-32808-5_1

Copyright and Moral Rights for the articles on this site are retained by the individual authors and/or other copyright owners. For more information on Open Research Online's data [policy](#) on reuse of materials please consult the policies page.

oro.open.ac.uk

Orientably-regular maps on twisted linear fractional groups

Grahame Erskine

Open University, Milton Keynes, U.K.

Katarína Hriňáková

Slovak University of Technology, Bratislava, Slovakia

Jozef Širáň

Open University, Milton Keynes, U.K., and
Slovak University of Technology, Bratislava, Slovakia

Abstract

We present an enumeration of orientably-regular maps with automorphism group isomorphic to the twisted linear fractional group $M(q^2)$ for any odd prime power q .

Keywords: Orientably-regular map; Automorphism group; Twisted linear fractional group.

1 Introduction

A (finite) orientably-regular map $\mathcal{M}$ is a cellular embedding of a connected graph in a compact, oriented surface, such that the group $\text{Aut}^+(\mathcal{M})$ of all orientation-preserving automorphisms of the embedding is transitive, and hence regular, on arcs of the embedded graph. In a regular map, all vertices have the same valency, say k , and all faces are bounded by closed walks of the same length, say, ℓ ; the map is then said to be of type (k, ℓ) . The group $\mathcal{A} = \text{Aut}^+(\mathcal{M})$ is generated by two elements x and y of order k and ℓ such that x acts as a clockwise rotation of $\mathcal{M}$ about a vertex by $2\pi/k$ and y acts as a clockwise rotation by $2\pi/\ell$ about the centre of a face incident with the vertex. The product xy is then a rotation of $\mathcal{M}$ about the centre of an edge that is incident to both the vertex and the face.

Orientably-regular maps can be viewed as maps having the ‘highest level’ of orientation-preserving symmetry among general maps (i.e., cellular embeddings of graphs). Regularity of $\mathcal{A}$ on the arc set of the embedded graph enables one to identify the map $\mathcal{M}$ with the triple $(\mathcal{A}, x, y)$ in such a way that arcs, edges, vertices and faces correspond to (say, left) cosets of the trivial group (that is, to elements of $\mathcal{A}$) and of the subgroups $\langle xy \rangle$, $\langle x \rangle$ and $\langle y \rangle$ of $\mathcal{A}$. Incidence between arcs, edges, vertices and faces is given by non-empty intersection of the corresponding cosets, and the action of $\mathcal{A}$ on the cosets is simply given by left multiplication.

It follows that orientably-regular maps are, up to isomorphism, in a one-to-one correspondence with equivalence classes of triples (G, x, y) , where G is a finite group admitting a presentation of the form $G = \langle x, y; x^k = y^\ell = (xy)^2 = \dots = 1 \rangle$, with two triples (G_1, x_1, y_1) and (G_2, x_2, y_2) being equivalent if there is a group isomorphism $G_1 \rightarrow G_2$ taking x_1 onto x_2 and y_1 onto y_2 . This way, investigation of orientably-regular maps can be reduced to purely group-theoretic considerations. The corresponding algebraic theory has been developed in depth in the influential paper [15].

The study of orientably-regular maps has rich history and, save Platonic solids and Kepler's polyhedra, has roots in the late 19th century; for historical information and surveys see e.g. [15, 16, 19, 21]. For an excellent introduction into fascinating connections between orientably-regular maps, Dyck's triangle groups, Riemann surfaces and Galois groups we recommend [16]. In particular, classification of orientably-regular maps has implications in classification of Riemann surfaces, cf. [15, 16].

Since the concept of an orientably-regular map includes the underlying graph, the carrier surface and the supporting automorphism group, classification attempts for such maps in most cases follow one of these three directions. A number of influential results have been obtained in classification of orientably-regular maps in the first two directions; we refer to [21] for the most recent survey. In this paper we focus on the third direction, that is, classification of orientably-regular maps by their automorphism groups, in which results are much less abundant.

Leaving the trivial case of Abelian groups aside, classification of orientably-regular maps with a given automorphism group has been completed for the following types of groups, ranked by complexity of their structure:

- groups of nilpotency class two and three [1];
- products of two cyclic groups one of which acts regularly on vertices of the map [7];
- groups with cyclic odd-order and dihedral even-order Sylow subgroups [4];
- $\text{PSL}(2, q)$ and $\text{PGL}(2, q)$, where q is an arbitrary prime power [18, 22, 5];
- Ree groups ${}^2G_2(3^n)$ for odd $n > 1$, restricted to maps of type (k, ℓ) for $\ell = 3$ and $k = 7, 9$ and all prime $k \equiv 11 \pmod{12}$ [13];
- Suzuki group $\text{Sz}(2^n)$ for odd $n > 1$, restricted to maps of type $(5, 4)$ [14].

By this list, the only family of *simple* groups for which a classification of the corresponding orientably-regular maps is known are the groups $\text{PSL}(2, q)$ for any prime power $q > 3$. Classification of orientably-regular maps with automorphism group $\text{PGL}(2, q)$, the obvious degree-two extension of $\text{PSL}(2, q)$, can be extracted from the corresponding classification for $\text{PSL}(2, q^2)$ through the well-understood inclusion $\text{PGL}(2, q) < \text{PSL}(2, q^2)$, cf. [22, 5].

For odd q , however, the simple group $\text{PSL}(2, q^2)$ admits another interesting extension of degree two, namely, the group $M(q^2)$, also known as a *twisted* linear fractional group.

By a classical result of Zassenhaus [23], the groups $\mathrm{PGL}(2, q)$ and $M(q^2)$ are the only finite sharply 3-transitive groups (of degree $q + 1$ and $q^2 + 1$, respectively). This motivates the question of classification of orientably-regular maps with automorphism group isomorphic to $M(q^2)$.

In this paper we present a complete enumeration of (isomorphism classes of) orientably-regular maps with automorphism group isomorphic to $M(q^2)$. The results are strikingly different from those for the groups $\mathrm{PGL}(2, q)$ in many ways. To give three examples, we note that (a) all the orientably-regular maps for $\mathrm{PGL}(2, q)$ are reflexible, while this is not the case for $M(q^2)$; (b) the groups $\mathrm{PGL}(2, q)$ are also automorphism groups of non-orientable regular maps while the groups $M(q^2)$ are not; and (c) for any even $k, \ell \geq 4$ not both equal to 4 there are orientably-regular maps of type (k, ℓ) with automorphism group $\mathrm{PGL}(2, q)$ for infinitely many values of q , while for infinitely many such pairs (k, ℓ) there are no orientably-regular maps for $M(q^2)$ of that type for any q .

By our outline and the algebraic theory of [15], enumeration of orientably-regular maps with a given automorphism group G reduces to enumeration of all triples (G, x, y) with $G = \langle x, y; x^k = y^\ell = (xy)^2 = \dots = 1 \rangle$ up to conjugation by elements of $\mathrm{Aut}(G)$, that is, by considering triples (G, x, y) and (G, x', y') equivalent if there is an automorphism of G taking (x, y) onto (x', y') . We do this systematically for the twisted linear groups $G = M(q^2)$. In sections 2 and 3 we introduce the group $M(q^2)$ and study its subgroups. Sections 4, 5 and 6 deal with identifying ‘canonical’ forms of elements of G and study their conjugacy in depth. In sections 7, 8 and 9 we develop arguments for counting ‘canonical’ pairs of elements of G . All the auxiliary facts are then processed in section 10 to produce our main result:

Theorem *Let $q = p^f$ be an odd prime power, with $f = 2^\alpha o$ where o is odd. The number of orientably-regular maps $\mathcal{M}$ with $\mathrm{Aut}^+(\mathcal{M}) \cong M(q^2)$ is, up to isomorphism, equal to*

$$\frac{1}{f} \sum_{d|o} \mu(o/d) h(2^\alpha d) ,$$

where $h(x) = (p^{2x} - 1)(p^{2x} - 2)/8$ and μ is the Möbius function.

We note that this result may be interpreted as counting generating pairs (x, y) of $G = M(q^2)$ such that $(xy)^2 = 1$, up to conjugacy in $\mathrm{Aut}(G)$, which may be of independent interest to specialists in group theory. The final section 12 contains related results and remarks.

2 The twisted linear groups $M(q^2)$

For a finite field F let $S(F)$ and $N(F)$ be the set of non-zero squares and non-squares of F . The general linear group $\mathrm{GL}(2, F)$ is the group of all non-singular 2×2 matrices with entries in F ; restriction to matrices with determinant 1 gives the special linear group

$\mathrm{SL}(2, F)$. The groups $\mathrm{PGL}(2, F)$ and $\mathrm{PSL}(2, F)$, the quotients of $\mathrm{GL}(2, F)$ and $\mathrm{SL}(2, F)$ by the corresponding centres, are known as the linear fractional groups. They can equivalently be described as groups of all transformations $z \mapsto (az + b)/(cz + d)$ of the set $F \cup \{\infty\}$ (with the obvious rules for calculations with ∞), with $ad - bc \neq 0$ and $ad - bc \in S(F)$ for $\mathrm{PGL}(2, F)$ and $\mathrm{PSL}(2, F)$, respectively. The group $\mathrm{PSL}(2, F)$ is an index 2 subgroup of $\mathrm{PGL}(2, F)$ unless F has characteristic 2, in which case the two groups are the same.

Suppose now that F admits an automorphism σ of order 2, which happens if and only if $|F| = q^2$ for some prime power q , and σ is then given by $x \mapsto x^q$ for every $x \in F$. If, in addition, q is odd, then one may ‘twist’ the transformations described above by considering the permutations of $F \cup \{\infty\}$ defined by $z \mapsto (az + b)/(zc + d)$ if $ad - bc \in S(F)$ and $z \mapsto (az^\sigma + b)/(cz^\sigma + d)$ if $ad - bc \in N(F)$. These transformations form a group under composition, denoted $M(F)$ or $M(q^2)$, and called the *twisted fractional linear group*. Observe that $\mathrm{PSL}(2, F)$ is a subgroup of $M(F)$ of index two, again. By a well-known result due to Zassenhaus [23], the groups $\mathrm{PGL}(2, F)$ for an arbitrary finite field F , and $M(F)$ for fields of order q^2 for an odd prime power q , are precisely the finite, sharply 3-transitive permutation groups (on the set $F \cup \{\infty\}$ in both cases).

In this paper we will focus on the twisted fractional linear groups, with the goal to classify the orientably-regular maps they support. For our purposes, however, it will be useful to work with a different representation of these groups. From this point on, let $F = \mathrm{GF}(q^2)$ for some odd prime power q and let $F_0 \cong \mathrm{GF}(q)$ be its unique subfield of order q ; let F^* and F_0^* be the corresponding multiplicative groups. Further, let σ be the unique automorphism of F of order 2; we have $x^\sigma = x^q$ for any $x \in F$, and $x^\sigma = x$ if and only if $x \in F_0$. If $A \in \mathrm{GL}(2, F)$, by A^σ we denote the matrix in $\mathrm{GL}(2, F)$ obtained by applying σ to every entry of A .

Let $J = \mathrm{GL}(2, F) \rtimes Z_2$, where multiplication in the semidirect product is defined by $(A, i)(B, j) = (AB^{\sigma^i}, i+j)$; equivalently, J is an extension of $\mathrm{GL}(2, F)$ by the automorphism σ . To introduce a ‘twisted’ subgroup of J , for every $A \in \mathrm{GL}(2, F)$ we define $\iota_A \in Z_2 = \{0, 1\}$ by letting $\iota_A = 0$ if $\det(A) \in S(F)$ and $\iota_A = 1$ if $\det(A) \in N(F)$. We now let $K = \{(A, \iota_A); A \in \mathrm{GL}(2, F)\}$; multiplication in K is, of course, given by $(A, \iota_A)(B, \iota_B) = (AB^{\sigma^{\iota_A}}, \iota_A + \iota_B)$ for any $A, B \in \mathrm{GL}(2, F)$. The group K and its quotient groups will be of principal importance in what follows.

Let $K_0 = \{(A, 0); A \in \mathrm{GL}(2, F), \iota_A = 0\}$ be the subgroup of K index 2 of K . The centre L of K_0 consists of elements of the form $(D, 0)$, where $D \in \mathrm{GL}(F)$ is a scalar matrix; obviously L is also a normal subgroup of both K and J . It can be checked that the factor group $G = K/L$ is isomorphic to $M(q^2)$, and since K has index 2 in J , the group $G = M(q^2)$ is (isomorphic to) a subgroup of index 2 of $\overline{G} = J/L$. The group $\overline{G}$ can alternatively be described as $G\langle\sigma\rangle$, the split extension of G by $\langle\sigma\rangle \simeq Z_2$. Observe also that the factor group $G_0 = K_0/L$ is isomorphic to $\mathrm{PSL}(2, F)$, and if q is a prime, the group J/L is isomorphic to $\mathrm{P}\Gamma\mathrm{L}(2, q^2)$.

Elements $(A, i)L$, that is, cosets $\{(\delta A, i); \delta \in F^*\}$, of the factor groups $G = K/L$ and $\overline{G} = J/L$ will throughout be denoted $[A, i]$; they will be called *untwisted* if $i = 0$ and *twisted* if $i = 1$.

For our final enumeration it will be necessary to determine the automorphism group of $M(q^2)$. While the result appears to be ‘obvious’ we provide a simple proof based on an fact which may be folklore to group-theorists.

Lemma 1 *Let U be a characteristic subgroup of a group $\tilde{U}$ of index 2. Suppose that the centre of U is trivial and every automorphism of U extends to an automorphism of $\tilde{U}$. Then $\text{Aut}(U) \cong \text{Aut}(\tilde{U})$.*

Proof. The assumption of U being characteristic in $\tilde{U}$ implies that every $h \in \text{Aut}(\tilde{U})$ restricts to an $h_U \in \text{Aut}(U)$. Since each automorphism of U extends to an automorphism of $\tilde{U}$, the assignment $h \mapsto h_U$ is a group epimorphism $\vartheta : \text{Aut}(\tilde{U}) \rightarrow \text{Aut}(U)$. Suppose that $h \in \text{Aut}(\tilde{U})$ is in the kernel of ϑ , so that h_U is the identity mapping on U . For every $x \in U$ and every $y \in \tilde{U} \setminus U$ we have $y^{-1}xy \in U$ and hence $y^{-1}xy = h(y^{-1}xy) = h(y)^{-1}xh(y)$, which implies that $h(y)y^{-1}$ commutes with x for all $x \in U$. Observe that $h(y)y^{-1} \in U$, since U was assumed to have index 2 in $\tilde{U}$. By triviality of the centre of U we have $h(y)y^{-1} = 1$ and as this is valid for all $y \in \tilde{U} \setminus U$ we conclude that h is the identity on $\tilde{U}$. It follows that the kernel of ϑ is trivial and so $\text{Aut}(U) \cong \text{Aut}(\tilde{U})$. $\square$

We now apply Lemma 1 to $U = G_0 = \text{PSL}(2, q^2)$ and $\tilde{U} = G = M(q^2)$ for $q = p^f$, where p is an odd prime and f a positive integer. Being a simple subgroup of $M(q^2)$, the group G_0 is characteristic (and of index two) in G . It is well known (see e.g. [11]) that $\text{Aut}(G_0) \cong \text{P}\Gamma\text{L}(2, q^2) \simeq \text{PGL}(2, q^2) \rtimes Z_{2f}$, with an element $(C, \varphi) \in \text{PGL}(2, q^2) \rtimes Z_{2f}$ acting on G_0 by $X \mapsto (C^{-1}XC)^\varphi$. Now, any (C, φ) is easily seen to extend to G by $[X, \iota_X] \mapsto ([C, 0]^{-1}[X, \iota_X][C, 0])^\varphi$. By Lemma 1 we now obtain:

Proposition 1 *The automorphism group of $M(q^2)$ is isomorphic to $\text{P}\Gamma\text{L}(2, q^2)$.* $\square$

3 Twisted subgroups of $M(q^2)$

Let $q = p^f$ for an odd prime p and a positive integer f ; these will be fixed throughout. In this section we will focus on the *twisted* subgroups of $M(p^{2f})$, that is, those isomorphic to $M(p^{2e})$ for suitable $e \leq f$. From now on we will use the notation $F_m = \text{GF}(p^m)$ for a Galois field of order p^m for $m \leq f$ but keep letting $F = \text{GF}(p^{2f})$. We begin by identifying the possible values of e .

Lemma 2 *A group $M(p^{2e})$ is isomorphic to a subgroup of $M(p^{2f})$ if and only if e is a divisor of f such that f/e is odd.*

Proof. If e divides f , then $p^{2f} - 1 = (p^{2e} - 1)Q$ for $Q = p^{2e(d-1)} + p^{2e(d-2)} + \dots + p^{2e} + 1$ and $d = f/e$. Assume that d is odd. If ξ is a primitive element of F , then ξ^Q is a primitive element of $F_{2e} < F$. Further, the restriction of the assignment $x \mapsto x^{p^f}$ for $x \in F$ onto F_{2e} coincides with the mapping $x \mapsto x^{p^e}$ for $x \in F_{2e}$. To see this, it is clearly sufficient

to consider the effect on taking the powers of $x = \xi^Q$, for which we have $x^{p^e} = x^{p^f}$ if and only if $Qp^e \equiv Qp^f \pmod{(p^{2f} - 1)}$. The congruence is further equivalent to $p^e \equiv p^f \pmod{(p^{2e} - 1)}$ and also to $p^{2e} - 1$ dividing $p^{f-e} - 1$, which is true if and only if $2e$ is a divisor of $f - e = (d - 1)e$, and the last condition is satisfied because d is odd. This shows that the restriction of the automorphism $\sigma : x \mapsto x^{p^f}$ of F onto F_{2e} coincides with the automorphism $\sigma' : x \mapsto x^{p^e}$ of F_{2e} . By construction of the twisted linear groups introduced in section 2 it is now clear that $M(p^{2f})$ contains a copy of $M(p^{2e})$.

To prove the reverse implication, assume that $H \cong M(p^{2e})$ is a subgroup of $G = M(p^{2f})$. From the order of H dividing the order of G one sees that e divides f , and our goal is to show that f/e is odd. We prove this by induction on f/e . The case when $f/e = 1$ is obvious and so we assume that $f > e$. Let $\tilde{H}$ be a maximal subgroup of G containing H . From Theorem 1.5 of [11] it follows that $\tilde{H}$ is isomorphic to the normaliser of $\text{PSL}(2, p^{2g})$ in G for some positive divisor g of f such that f/g is an odd prime. As before, let G_0 be the (normal) subgroup of index 2 in G isomorphic to $\text{PSL}(2, p^{2f})$. The group $\tilde{H} \cap G_0$ is contained in a maximal subgroup of G_0 containing $\text{PSL}(2, p^{2g})$. However, the group $\text{PSL}(2, p^{2g})$ itself is a maximal subgroup of G as f/g is odd, cf. [11] again. It follows that $\tilde{H} \cap G_0$ must be isomorphic to $\text{PSL}(2, p^{2g})$, and therefore $\tilde{H}$ must be isomorphic to $M(p^{2g})$. But $\tilde{H}$ contains also $M(p^{2e}) \cong H$ and therefore e divides g . As $g/e < f/e$, we may apply our induction hypothesis by which g/e is odd. Since f/g was an odd prime we conclude that f/e is odd as well, completing the induction step. $\square$

If f/e is odd, a particularly important copy of $M(p^{2e})$ in $M(p^{2f})$ is formed by all the pairs $[X, \iota_X]$ with $X \in \text{GL}(2, p^{2e})$ such that all entries of X lie in the subfield F_{2e} of F ; this copy will be called *canonical*. The copy of $\text{PSL}(2, p^{2e})$ in $M(p^{2f})$ formed by all the pairs $[X, 0]$ with $X \in \text{SL}(2, F_{2e})$ will be called *canonical* as well. We now prove a useful auxiliary result on canonical subgroups.

Proposition 2 *Let f/e be an odd integer and let $H \cong M(p^{2e})$ be a subgroup of $G = M(p^{2f})$ such that H contains the canonical copy of $\text{PSL}(2, p^{2e})$. Then, H is equal to the canonical copy of $M(p^{2e})$ in G .*

Proof. Let H be a copy of $M(p^{2e})$ in G such that $H_0 = H \cap \text{PSL}(2, p^{2f})$ is equal to the canonical copy of $\text{PSL}(2, p^{2e})$ in G . Obviously, H_0 is a normal subgroup of H of index two. Let $[A, 1]$ be an element of $H \setminus H_0$, where A is the 2×2 matrix with rows (a, b) and (c, d) for some $a, b, c, d \in F$ with $\delta = ad - bc \in N(F)$. We may assume that the entry c in the lower left corner of A is non-zero. Indeed, if $c = 0$ and $b \neq 0$, letting D be an off-diagonal matrix with entries -1 and 1 we may replace $[A, 1]$ with the product $[D, 0][A, 1] \in H \setminus H_0$, and if A is a diagonal matrix we may replace $[A, 1]$ with the product $[D', 0][A, 1] \in H \setminus H_0$ for a matrix D' with rows $(1, 0)$ and $(1, 1)$. Then, since we are working with projective groups, we may assume that $c = 1$, so that $\delta = ad - b$.

By our assumption the group H_0 also contains the element $[C, 0]$ with C having rows $(1, 1)$ and $(0, 1)$. Normality of H_0 in H implies that $[A, 1][C, 0][A, 1]^{-1} = [ACA^{-1}, 0] \in H_0$

and also $[A, 1]^{-1}[C, 0][A, 1] = [(A^\sigma)^{-1}CA^\sigma, 0] \in H_0$. Evaluating the products we obtain

$$\begin{aligned}\varepsilon ACA^{-1} &= \begin{pmatrix} a & b \\ 1 & d \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} d & -b \\ -1 & a \end{pmatrix} = \begin{pmatrix} \delta - a & a^2 \\ -1 & \delta + a \end{pmatrix}, \text{ and} \\ \varepsilon'(A^\sigma)^{-1}CA^\sigma &= \begin{pmatrix} d^\sigma & -b^\sigma \\ -1 & a^\sigma \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a^\sigma & b^\sigma \\ 1 & d^\sigma \end{pmatrix} = \begin{pmatrix} \delta^\sigma + d^\sigma & (d^\sigma)^2 \\ -1 & \delta^\sigma - d^\sigma \end{pmatrix}\end{aligned}$$

for some $\varepsilon, \varepsilon' \in F^*$. Since H_0 is assumed to be equal to the canonical copy $\text{PSL}(2, p^{2e})$ in G , all the remaining entries of the two matrices on the right-hand sides above must lie in F_{2e} . This readily implies that both $a, \delta, d^\sigma \in F_{2e}$, and since F_{2e} is setwise preserved by σ we also have $d \in F_{2e}$ and so $b = ad - \delta \in F_{2e}$ as well. We conclude that $A \in \text{GL}(2, p^{2e})$ and hence the subgroup H generated by $[A, 1]$ and H_0 is identical with the canonical copy of $M(p^{2e})$ in G . $\square$

As a consequence we prove that all twisted subgroups of G are conjugate. Recall that G_0 denotes the (unique) subgroup of G isomorphic to $\text{PSL}(2, p^{2f})$.

Proposition 3 *If f/e is an odd integer, then all subgroups of $G = M(p^{2f})$ isomorphic to $M(p^{2e})$ are conjugate in G_0 .*

Proof. Let $H \cong M(p^{2e})$ be a subgroup of G ; letting $H_0 = H \cap G$ we obviously have $H_0 \cong \text{PSL}(2, p^{2e})$. The classification of subgroups of $\text{PSL}(2, p^{2f})$, nicely displayed in [20], tells us that in the case when f/e is odd, all subgroups of G_0 isomorphic to $\text{PSL}(2, p^{2e})$ are conjugate in G_0 . It follows that there is an inner automorphism π of G_0 such that $\pi(H_0)$ is equal to the canonical copy of $\text{PSL}(2, p^{2e})$ in G_0 . The group $\pi(H)$ is then isomorphic to $M(p^{2e})$ and contains the canonical copy of $\text{PSL}(2, p^{2e})$. By Proposition 2, however, the group $\pi(H)$ must be *equal* to the canonical copy of $M(p^{2e})$ in G . In particular, all subgroups of G isomorphic to $M(p^{2e})$ are conjugate in G_0 . $\square$

We conclude with a sufficient condition for a subgroup of G to be twisted; this result will be of key importance later. In order to state it, we will say that a subgroup H of G *stabilises a point* if, in the natural action of G on the set $F \cup \{\infty\}$ via linear fractional mappings from section 2, there exists a point in $F \cup \{\infty\}$ fixed by all linear fractional mappings corresponding to elements of H . Also, for any positive divisor g of $2f$ let G_g be the canonical copy of $\text{PSL}(2, p^g)$ in the group $G = M(p^{2f})$. Moreover, if g is even, we let G_g^* denote the copy of $\text{PGL}(2, p^{g/2})$ in G_g formed by (equivalence classes of) non-singular 2×2 matrices over $\text{GF}(p^{g/2})$.

Proposition 4 *Let H be a subgroup of $G = M(p^{2f})$ not contained in the subgroup $G_0 = \text{PSL}(2, p^{2f})$ and let $H_0 = H \cap G_0$. If H_0 does not stabilise a point and is neither dihedral nor isomorphic to A_4 , S_4 or A_5 , then H is conjugate in $\overline{G}$ to a subgroup isomorphic to $M(p^{2e})$ for some positive divisor e of f such that f/e is odd.*

Proof. By a handy summary [20] of Dickson's classification of subgroups of projective special linear groups over finite fields, subgroups of G_0 comprise point stabilisers, dihedral groups, A_4 , S_4 , A_5 , and $\text{PSL}(2, p^g)$ for divisors g of $2f$ together with $\text{PGL}(2, p^{g/2})$ for even divisors g of $2f$. Our assumptions imply that H_0 must be isomorphic to one of the last two types of subgroups.

Let e be the smallest positive integer such that f/e is odd and g divides $2e$. Invoking the classification of subgroups of G_0 again [20], and specifically the fact that any two copies of $\text{PSL}(2, p^{2e})$ in G are conjugate (by an element of G), we may assume that H_0 is a subgroup of the canonical copy G_{2e} of $\text{PSL}(2, p^{2e})$ in G . Now, by [20], the number of conjugacy classes of copies of $\text{PSL}(2, p^g)$ in G_{2e} is 1 or 2 according to whether $2e/g$ is odd or even. The two classes if g divides e are, however, by [2], fused under conjugacy in $\text{PGL}(2, p^{2e})$ and hence also under conjugacy in $\overline{G}$; the same holds for copies of $\text{PGL}(2, p^{g/2})$ if g is even. We therefore may assume that H_0 is *equal* either to the canonical copy G_g of $\text{PSL}(2, p^g)$ in G , or to the canonical copy G_g^* of $\text{PGL}(2, p^{g/2})$ in G if g is even.

Let $[A, 1] \in H \setminus H_0$ and $[C, 0] \in H_0$ be the elements from the proof of Proposition 2. Considering the conjugation $[A, 1][C, 0][A, 1]^{-1} = [ACA^{-1}, 0] \in H_0$ and verbatim repeating the arguments from the proof of Proposition 2, except with F_g in place of F_{2e} , we conclude that $\delta \in F_g$. However, a non-square element $\delta \in F = F_{2f}$ can live in a subfield $F_g < F_{2f}$ only if $2f/g$ is odd. As $g \mid 2e$ with f/e odd and e was the smallest positive integer with this property, oddness of $2f/g$ implies that $g = 2e$.

We have arrived at the conclusion that H_0 is *equal* either to the canonical copy G_{2e} of $\text{PSL}(2, p^{2e})$ or to the canonical copy G_{2e}^* of $\text{PGL}(2, p^e)$. But in the second case, by the argument in the preceding paragraph, we would have $\delta \in F_e$ while $2f/e$ is not odd, a contradiction. Therefore H_0 must be equal to G_{2e} . The conclusion that $H \cong M(p^{2e})$ now follows from Proposition 2. $\square$

4 Representatives of twisted elements

Recalling the notation introduced in section 2, we begin by identifying elements in conjugacy classes of $K \setminus K_0$ that have a particularly simple form. To facilitate the description here and also in the sections that follow, we let $\text{dia}(\alpha, \beta)$ and $\text{off}(\alpha, \beta)$, respectively, denote the 2×2 matrix with diagonal entries α, β (from the top left corner) and zero off-diagonal entries, and the 2×2 matrix with off-diagonal entries α, β (from the top right corner) and zero diagonal entries.

For every element $(A, 1) \in K \setminus K_0$ we have $(A, 1)^2 = (AA^\sigma, 0)$. In the study of conjugacy in $K \setminus K_0$ it turns out to be important to understand the behaviour of the products AA^σ . Observe that if $\delta = \det(A) \in N(F)$, then $\det(AA^\sigma) = \delta\delta^\sigma \in N(F_0)$.

Let $(A, 1) \in K \setminus K_0$ and let $\{\lambda_1, \lambda_2\}$ be the spectrum of AA^σ in a smallest extension F' of F of degree at most two in which σ may still be assumed to be given by $x \mapsto x^q$. Since $A^\sigma A$ is both a conjugate and also a σ -image of AA^σ , we have $\{\lambda_1, \lambda_2\}^\sigma = \{\lambda_1, \lambda_2\}$. This

means that either (1) $\lambda_i^\sigma = \lambda_i$ for $i = 1, 2$, or (2) $\lambda_1^\sigma = \lambda_2$ and $\lambda_2^\sigma = \lambda_1$. Note that (2) implies $\lambda_1^{q^2} = (\lambda_1^q)^q = \lambda_2^q = \lambda_1$ and, similarly, $\lambda_2^{q^2} = \lambda_2$. We conclude that $F' = F$ in both the situations (1) and (2) and so both λ_1, λ_2 are in F . Observe that $\lambda_1 \neq \lambda_2$, as otherwise we would have $\lambda_1 = \lambda_1^\sigma \in F_0$ and $\det(AA^\sigma) = \lambda_1^2 \in S(F_0)$, a contradiction. Moreover, it follows that in the case (1) we have $\lambda_i \in F_0$ for $i = 1, 2$ with $\lambda_1\lambda_2 \in N(F_0)$, and in the case (2) $\lambda_i \in N(F) \subset F \setminus F_0$ since $\det(AA^\sigma) = \lambda_1\lambda_1^\sigma \in N(F_0)$.

We will now refine our considerations of AA^σ . As before, let $q = p^f$ for some odd prime p and let e be the *smallest* positive divisor of f with f/e odd such that $AA^\sigma = \varepsilon C$ for some $C \in \text{SL}(2, p^{2e})$ and for some $\varepsilon \in F^*$. In other words, we look for the smallest subfield F_{2e} of F , with f/e odd, such that all entries of C lie in F_{2e} ; note that we may assume C to have determinant 1 since the determinant of AA^σ is a non-zero square of F . If $\{\mu, \mu^{-1}\}$ is the spectrum of C , we have, without loss of generality, $\lambda_1 = \varepsilon\mu$ and $\lambda_2 = \varepsilon\mu^{-1}$. Observe that since $\lambda_1, \lambda_2, \varepsilon \in F$, we have $\mu, \mu^{-1} \in F$. Now, μ, μ^{-1} are roots of a quadratic polynomial over F_{2e} and therefore both belong to F_{2e} or to a quadratic extension of F_{2e} . But as f/e is odd, the field F does not contain a quadratic extension of F_{2e} . We conclude that $\mu, \mu^{-1} \in F_{2e}$.

The facts in the previous paragraphs imply that if $(A, 1) \in K \setminus K_0$, then the matrix AA^σ is diagonalisable over F and C is diagonalisable over F_{2e} . In particular, there exists a $P \in \text{GL}(2, p^{2e})$ such that $P^{-1}CP = D'$ for $D' = \text{dia}(\mu, \mu^{-1})$; multiplying by ε then gives $P^{-1}AA^\sigma P = D$ for $D = \text{dia}(\lambda_1, \lambda_2)$. Here, either $\lambda_1, \lambda_2 \in F_0$ with $\lambda_1\lambda_2 \in N(F_0)$, or $\lambda_1, \lambda_2 \in F \setminus F_0$ and $\lambda_1^\sigma = \lambda_2$. With A, P, D and D' as above, in K we let $(B, 1) = (P, 0)^{-1}(A, 1)(P, 0) = (P^{-1}AP^\sigma, 1)$. Then,

$$(BB^\sigma, 0) = (P, 0)^{-1}(A, 1)(A, 1)(P, 0) = (P^{-1}AA^\sigma P, 0) = (D, 0) = (\varepsilon D', 0)$$

and it follows that $BB^\sigma = D = \varepsilon D'$.

We now derive more details about the matrix $B = P^{-1}AP^\sigma$; recall that $P \in \text{GL}(2, p^{2e})$. Let u_1, u_2 be linearly independent (column) eigenvectors of C and AA^σ for the eigenvalues μ, μ^{-1} and λ_1, λ_2 , respectively; we have $Cu_1 = \mu u_1$, $Cu_2 = \mu^{-1}u_2$, and $AA^\sigma u_i = \lambda_i u_i$ for $i \in \{1, 2\}$. Taking the σ -image of the last equation and then multiplying by A from the left we obtain $AA^\sigma(Au_i^\sigma) = \lambda_i^\sigma(Au_i^\sigma)$ for $i = 1, 2$. This means that the column vectors Au_i^σ are also eigenvectors of AA^σ for the eigenvalues λ_i^σ , $i = 1, 2$. It follows that if $\lambda_i = \lambda_i^\sigma$ for $i = 1, 2$, then we must have $Au_i^\sigma = \varepsilon_i u_i$, and if $\lambda_i = \lambda_{3-i}^\sigma$, then $Au_i^\sigma = \varepsilon_{3-i} u_{3-i}$, in both cases for some $\varepsilon_1, \varepsilon_2 \in F$. The last bit we need is the fact that for the matrix P we may take $P = (u_1, u_2)$, i.e., the matrix formed by the columns u_1, u_2 , with entries in F_{2e} . Now, for $i = 1, 2$, in the case $\lambda_i = \lambda_i^\sigma$ we have $AP^\sigma = (Au_1^\sigma, Au_2^\sigma) = (\varepsilon_1 u_1, \varepsilon_2 u_2) = P \text{dia}(\varepsilon_1, \varepsilon_2)$, and in the case $\lambda_i = \lambda_{3-i}^\sigma$ a similar calculation gives $AP^\sigma = (Au_1^\sigma, Au_2^\sigma) = (\varepsilon_2 u_2, \varepsilon_1 u_1) = P \text{off}(\varepsilon_1, \varepsilon_2)$. This shows that our matrix $B = P^{-1}AP^\sigma$ is equal to $\text{dia}(\varepsilon_1, \varepsilon_2)$ or to $\text{off}(\varepsilon_1, \varepsilon_2)$ for suitable $\varepsilon_1, \varepsilon_2 \in F$, depending on whether λ_1^σ is equal to λ_1 or λ_2 . In both cases, of course, $\varepsilon_1 \varepsilon_2 \in N(F)$.

Recalling our notation $[A, i]$ for the cosets $(A, i)L = \{(\delta A, i); \delta \in F^*\}$, the above calculations lead to the following result.

Proposition 5 *Let $G = M(p^{2f})$ for some odd prime p . Then, every element of the form $[A, 1] \in G$ is conjugate in $\overline{G}$ to $[B, 1]$ with $B = \text{dia}(\lambda, 1)$ or $B = \text{off}(\lambda, 1)$ for some $\lambda \in N(F)$. If, in addition, $[AA^\sigma, 0] = [C, 0]$ for some $C \in \text{SL}(2, p^{2e})$ with f/e odd, then $[B, 1] = [P, 0]^{-1}[A, 1][P, 0]$ for some $P \in \text{GL}(2, p^{2e})$, and $\lambda\lambda^\sigma \in F_{2e}$ or $\lambda/\lambda^\sigma \in F_{2e}$, depending on whether B is equal to $\text{dia}(\lambda, 1)$ or to $\text{off}(\lambda, 1)$.*

Proof. We have proven everything except for the last assertion. We have seen that if $[AA^\sigma, 0] = [C, 0]$ for some $C \in \text{SL}(2, p^{2e})$ with f/e odd, then $BB^\sigma = P^{-1}(AA^\sigma)P = \varepsilon \text{dia}(\mu, \mu^{-1})$ for some $\varepsilon \in F^*$, $\mu \in F_{2e}$ and some $P \in \text{GL}(2, p^{2e})$. If $B = \text{dia}(\lambda, 1)$, then we have $\text{dia}(\lambda\lambda^\sigma, 1) = BB^\sigma = \varepsilon C = \varepsilon \text{dia}(\mu, \mu^{-1})$, which implies that $\varepsilon = \mu$ and $\lambda\lambda^\sigma = \mu^2 \in F_{2e}$. In the case when $B = \text{off}(\lambda, 1)$ we have $\text{off}(\lambda, \lambda^\sigma) = BB^\sigma = \varepsilon C = \varepsilon \text{dia}(\mu, \mu^{-1})$, from which we obtain $\lambda/\lambda^\sigma = \mu^2 \in F_{2e}$. $\square$

Let us have another look at conjugation in the group $\overline{G} = J/L$. Observe that if $(P, i) \in J$, then $(P, i)^{-1} = ((P^{\sigma^i})^{-1}, i)$. Conjugates of $(B, 1) \in K$ by (P, i) have the form $(P, 0)^{-1}(B, 1)(P, 0) = (P^{-1}BP^\sigma, 1)$ if $i = 0$, and $(P, 1)^{-1}(B, 1)(P, 1) = ((P^\sigma)^{-1}B^\sigma P, 1)$ if $i = 1$. It follows that two elements $(B, 1)$ and $(B', 1)$ of K are conjugate in J if and only if $B' = P^{-1}BP^\sigma$ or $B' = (P^\sigma)^{-1}B^\sigma P$ for some $P \in \text{GL}(2, F)$. Taking the σ -image in the second case and passing onto $G = K/L$ we have:

Proposition 6 *Two elements $[B, 1]$ and $[B', 1]$ of G are conjugate in $\overline{G}$ if and only if $P^{-1}BP^\sigma = \varepsilon B'$ or $P^{-1}BP^\sigma = \varepsilon B'^\sigma$ for some $\varepsilon \in F^*$ and some $P \in \text{GL}(2, F)$.* $\square$

We will write the two conditions of Proposition 6 in the unified form $P^{-1}BP^\sigma = \varepsilon B'^{(\sigma)}$, or, equivalently, $BP^\sigma = \varepsilon PB'^{(\sigma)}$ where $B'^{(\sigma)}$ is equal to B' or B'^σ , depending on whether $i = 0$ or $i = 1$ when using the element $[P, i]$ for conjugation.

5 Conjugacy of representatives of twisted elements

We continue with identification of elements of $\overline{G}$ that conjugate a diagonal (or an off-diagonal) element from Proposition 5 to another such element. As a by-product we will be able to identify $\overline{G}$ -stabilisers of our representatives of twisted elements in G .

We begin with the case when $B = \text{dia}(\lambda, 1)$ and $B' = \text{dia}(\lambda', 1)$; by Proposition 6 it is sufficient to find the nonsingular matrices $P \in \text{GL}(2, F)$ and $\varepsilon \in F^*$ for which $BP^\sigma = \varepsilon PB'^{(\sigma)}$ in the sense of the notation introduced at the end of section 4. Throughout the computation we will use the symbols $\lambda^{(\sigma)}$ and $\lambda'^{(\sigma)}$ in an analogous way as explained for $B'^{(\sigma)}$. Assuming that P has entries $\alpha, \beta, \gamma, \delta$, the above condition says that

$$\begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha^\sigma & \beta^\sigma \\ \gamma^\sigma & \delta^\sigma \end{pmatrix} = \varepsilon \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} \lambda'^{(\sigma)} & 0 \\ 0 & 1 \end{pmatrix}.$$

Evaluating the products we obtain:

$$\lambda\alpha^\sigma = \varepsilon\lambda'^{(\sigma)}\alpha, \quad \lambda\beta^\sigma = \varepsilon\beta, \quad \gamma^\sigma = \varepsilon\lambda'^{(\sigma)}\gamma, \quad \delta^\sigma = \varepsilon\delta.$$

By straightforward manipulation this gives the following system of four equations:

$$\alpha(\lambda\lambda^\sigma - \varepsilon\varepsilon^\sigma\lambda'\lambda'^\sigma) = 0, \quad \beta(\lambda\lambda^\sigma - \varepsilon\varepsilon^\sigma) = 0, \quad \gamma(\varepsilon\varepsilon^\sigma\lambda'\lambda'^\sigma - 1) = 0, \quad \delta(\varepsilon\varepsilon^\sigma - 1) = 0.$$

From non-singularity of P it follows that $\delta \neq 0$ or $\beta \neq 0$, that is, $\varepsilon\varepsilon^\sigma = 1$ or $\varepsilon\varepsilon^\sigma = \lambda\lambda^\sigma$.

Consider first the case $\varepsilon\varepsilon^\sigma = 1$, that is, $\varepsilon^{q+1} = 1$ for $q = p^f$. Since $\lambda, \lambda' \in N(F)$, we have $\lambda\lambda^\sigma \neq 1 \neq \lambda'\lambda'^\sigma$. Our equations together with $\varepsilon\varepsilon^\sigma = 1$ then imply that $\beta = \gamma = 0$. Hence $\alpha, \delta \neq 0$, by non-singularity of P ; in particular, $\lambda\lambda^\sigma = \lambda'\lambda'^\sigma$, or, equivalently, $(\lambda'/\lambda)^{q+1} = 1$. We are interested in conjugation in the group $\overline{G} = J/L$ and so we may assume that $\delta = 1$, which reduces the relations below our matrix equation to $\varepsilon = 1$ and $\alpha^{q-1} = \lambda'^{(\sigma)}/\lambda$. Since $(\lambda'/\lambda)^{q+1} = 1$, the equation $\eta^{q-1} = \lambda'/\lambda$ has $q-1$ solutions $\eta \in F^*$ (note that $|F^*| = q^2 - 1$). If $\lambda'^{(\sigma)} = \lambda'$, then all solutions of the equation $\alpha^{q-1} = \lambda'^{(\sigma)}/\lambda$ have the form $\alpha = \eta$, and if $\lambda'^{(\sigma)} = \lambda'^\sigma = \lambda'\lambda'^{q-1}$, then all solutions of this equation are $\alpha = \eta\lambda'$.

The second case to consider is $\varepsilon\varepsilon^\sigma = \lambda\lambda^\sigma (\neq 1)$, which implies that $\alpha = \delta = 0$, and also $\lambda\lambda^\sigma\lambda'\lambda'^\sigma = 1$ since γ, β now must be non-zero. By the same token as above we may let $\gamma = 1$ without loss of generality. Then, our equations for γ and β in this case reduce to $\varepsilon\lambda'^{(\sigma)} = 1$ and $\lambda\beta^\sigma = \varepsilon\beta$, the latter now being equivalent to $\beta^{q-1} = 1/(\lambda\lambda'^{(\sigma)})$. Since now $(\lambda\lambda')^{q+1} = 1$, there are $q-1$ solutions ζ of the equation $\zeta^{q-1} = 1/(\lambda\lambda')$ in F^* . If $\lambda'^{(\sigma)} = \lambda'$, then we have $\beta = \zeta$, and if $\lambda'^{(\sigma)} = \lambda'^\sigma = \lambda'\lambda'^{q-1}$, we have $\beta = \zeta/\lambda'$. Summing up, we arrive at the following:

Proposition 7 *Let $B = \text{dia}(\lambda, 1)$ and $B' = \text{dia}(\lambda', 1)$ for $\lambda, \lambda' \in N(F)$. If an element $[P, i] \in \overline{G}$ conjugates $[B, 1]$ to $[B', 1]$, then, without loss of generality, $P = \text{dia}(\omega, 1)$ or $P = \text{off}(\omega, 1)$ for suitable $\omega \in F^*$. Moreover:*

1. *If $P = \text{dia}(\omega, 1)$, then $\lambda\lambda^\sigma = \lambda'\lambda'^\sigma$, and if this condition is satisfied, then $[B, 1]$ conjugates to $[B', 1]$ in $\overline{G}$ exactly by the $q-1$ elements $[P, 0]$ such that $\omega = \eta$ and the $q-1$ elements $[P, 1]$ with $\omega = \eta\lambda'$, where $\eta \in F^*$ is one of the $q-1$ solutions of the equation $\eta^{q-1} = \lambda'/\lambda$.*
2. *If $P = \text{off}(\omega, 1)$, then $\lambda\lambda^\sigma\lambda'\lambda'^\sigma = 1$, and if this holds, then $[B, 1]$ conjugates to $[B', 1]$ in $\overline{G}$ exactly by the $q-1$ elements $[P, 0]$ with $\omega = \zeta$ and the $q-1$ elements $[P, 1]$ such that $\omega = \zeta/\lambda'$, where $\zeta \in F^*$ is one of the $q-1$ solutions of the equation $\zeta^{q-1} = 1/(\lambda\lambda')$. $\square$*

We now repeat this process but now with matrices $B = \text{off}(\lambda, 1)$ and $B' = \text{off}(\lambda', 1)$. Conjugating by $[P, i]$ and assuming that P has entries $\alpha, \beta, \gamma, \delta$, the unified form $BP^\sigma = \varepsilon PB'^{(\sigma)}$ of the condition of Proposition 6 now translates into the matrix equation

$$\begin{pmatrix} 0 & \lambda \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \alpha^\sigma & \beta^\sigma \\ \gamma^\sigma & \delta^\sigma \end{pmatrix} = \varepsilon \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} 0 & \lambda'^{\sigma^i} \\ 1 & 0 \end{pmatrix}.$$

It follows that

$$\alpha^\sigma = \varepsilon\delta, \quad \beta^\sigma = \varepsilon\lambda'^{\sigma^i}\gamma, \quad \lambda\gamma^\sigma = \varepsilon\beta, \quad \lambda\delta^\sigma = \varepsilon\lambda'^{\sigma^i}\alpha,$$

which, after some manipulation, yield the following two equations:

$$\gamma(\lambda^\sigma - \varepsilon\varepsilon^\sigma\lambda'^{\sigma^i}) = 0 \quad \text{and} \quad \delta^\sigma(\lambda - \varepsilon\varepsilon^\sigma\lambda'^{\sigma^i}) = 0.$$

This all means that either (a) $\lambda = \varepsilon\varepsilon^\sigma\lambda'^{\sigma^i}$, and then $\beta = \gamma = 0$ and we may assume $\delta = 1$, or else (b) $\lambda^\sigma = \varepsilon\varepsilon^\sigma\lambda'^{\sigma^i}$, and then we have $\alpha = \delta = 0$ and, without loss of generality, $\gamma = 1$. Since $\varepsilon\varepsilon^\sigma, \lambda\lambda^\sigma \in F_0^*$, these conditions are equivalent to $\lambda/\lambda' \in F_0^*$ or $\lambda\lambda' \in F_0^*$, independently of the value of i , but in our analysis below it is still useful to refer to i .

In the case (a), when $\lambda/\lambda'^{\sigma^i} = \varepsilon\varepsilon^\sigma \in F_0^*$, for every $i \in \{0, 1\}$ there are $q+1$ $(q+1)^{\text{th}}$ roots $\eta_{(i)}$ of $\lambda/\lambda'^{\sigma^i}$ in F^* . From $\delta = 1$ we have $\alpha^\sigma = \varepsilon$ and $\lambda = \varepsilon\lambda'^{\sigma^i}\alpha$, that is, $\alpha^{q+1} = \lambda/\lambda'^{\sigma^i}$. This implies that $\alpha = \eta_{(i)}$ is one of the $(q+1)^{\text{th}}$ roots of $\lambda/\lambda'^{\sigma^i}$, giving $q+1$ conjugation elements $[P, i]$ such that $P = \text{dia}(\eta_{(i)}, 1)$. In the case (b), $\lambda^\sigma/\lambda'^{\sigma^i} = \varepsilon\varepsilon^\sigma \in F_0^*$ and since also $\lambda\lambda^\sigma \in F_0^*$, we have $\lambda\lambda'^{\sigma^i} \in F_0^*$. It follows that for every $i \in \{0, 1\}$ there are $q+1$ $(q+1)^{\text{th}}$ roots $\zeta_{(i)}$ of $\lambda\lambda'^{\sigma^i}$ in F^* . From $\gamma = 1$ we obtain $\lambda = \varepsilon\beta$ and $\beta^\sigma = \varepsilon\lambda'^{\sigma^i}$, which means that $\beta^{q+1} = \lambda\lambda'^{\sigma^i}$. Consequently, $\beta = \zeta_{(i)}$ and we have in this second case $q+1$ conjugation elements $[P, i]$ such that $P = \text{off}(\zeta_{(i)}, 1)$. Realising that the condition (a) for $i = 0$ is equivalent to (b) for $i = 1$ (and equivalent to $\lambda/\lambda' \in F_0^*$) and, similarly, the condition (a) for $i = 1$ is equivalent to (b) for $i = 0$ (and equivalent to $\lambda\lambda' \in F_0^*$), we conclude that:

Proposition 8 *Let $B = \text{off}(\lambda, 1)$ and $B' = \text{off}(\lambda', 1)$ for $\lambda, \lambda' \in N(F)$. Further, for $i \in \{0, 1\}$, let $\eta_{(i)}, \zeta_{(i)} \in F^*$ be any of the $q+1$ roots of the equation $\eta_{(i)}^{q+1} = \lambda/\lambda'^{\sigma^i}$ and $\zeta_{(i)}^{q+1} = \lambda\lambda'^{\sigma^i}$, respectively. Then, an element $[P, i] \in \overline{G}$ conjugates $[B, 1]$ to $[B', 1]$ if and only if $\lambda/\lambda'^{\sigma^i}$ or $\lambda\lambda'^{\sigma^i}$ are elements of F_0^* . In an equivalent form, $[B, 1]$ is conjugate to $[B', 1]$ if and only if either*

1. $\lambda/\lambda' \in F_0^*$, in which case the conjugation is realised by exactly $q+1$ elements $[P, 0]$ with $P = \text{dia}(\eta_{(0)}, 1)$ and exactly $q+1$ elements $[P, 1]$ with $P = \text{off}(\zeta_{(1)}, 1)$, or
2. $\lambda\lambda' \in F_0^*$, by the conjugation realised by exactly $q+1$ elements $[P, 0]$ with $P = \text{dia}(\eta_{(1)}, 1)$ and exactly $q+1$ elements $[P, 1]$ with $P = \text{off}(\zeta_{(0)}, 1)$. $\square$

6 Conjugacy classes of twisted elements

With the help of the calculations in the previous section we can now prove a useful result about identification of suitable representatives of conjugacy classes (in the group $\overline{G}$) of elements of $G \setminus G_0$.

Theorem 1 *Let ξ be a primitive element of F and let $[A, 1]$ be an element of G . Then, exactly one of the following two cases occur:*

1. *There exists exactly one odd $i \in \{1, 2, \dots, (q-1)/2\}$ such that $[A, 1]$ is conjugate in $\overline{G}$ to $[B, 1]$ with $B = \text{dia}(\xi^i, 1)$; the order of $[A, 1]$ in G is then $2(q-1)/\gcd\{q-1, i\}$.*

2. *There exists exactly one odd $i \in \{1, 2, \dots, (q+1)/2\}$ such that $[A, 1]$ is conjugate in $\overline{G}$ to $[B, 1]$ with $B = \text{off}(\xi^i, 1)$, and the order of $[A, 1]$ in G is $2(q+1)/\gcd\{q+1, i\}$.*

Furthermore, we have:

3. *The stabiliser of $[B, 1]$ for $B = \text{dia}(\lambda, 1)$, $\lambda \in N(F)$, in the group $\overline{G}$ is isomorphic to the cyclic group $Z_{2(q-1)}$ generated by (conjugation by) $[P, 1]$ for $P = \text{dia}(\mu\lambda, 1)$ with a suitable $(q-1)^{\text{th}}$ root of unity μ , except when λ is a $(q+1)^{\text{th}}$ root of -1 and $q \equiv -1 \pmod{4}$, in which case the stabiliser is isomorphic to $Z_{2(q-1)} \cdot Z_2$.*
4. *The stabiliser of $[B, 1]$ for $B = \text{off}(\lambda, 1)$, $\lambda \in N(F)$, in the group $\overline{G}$ is isomorphic to the cyclic group $Z_{2(q+1)}$ generated by (conjugation by) $[P, 1]$ for $P = \text{off}(\mu\lambda, 1)$, where μ is a suitable $(q+1)^{\text{th}}$ root of unity, except when λ is a $(q-1)^{\text{th}}$ root of -1 and $q \equiv 1 \pmod{4}$, when the stabiliser is isomorphic to $Z_{2(q+1)} \cdot Z_2$.*

Proof. By Proposition 5, each element $[A, 1] \in G$ is conjugate in G to $[B, 1]$ with $B = \text{dia}(\lambda, 1)$ or $B = \text{off}(\lambda, 1)$ for some $\lambda \in N(F)$.

1. Suppose that $[A, 1]$ is conjugate in G to $[B, 1]$, $B = \text{dia}(\lambda, 1)$, $\lambda \in N(F)$. Letting $B' = \text{dia}(\lambda', 1)$, Proposition 7 implies that the elements $[B, 1]$ and $[B', 1]$ are conjugate in $\overline{G}$ if and only if $\lambda\lambda^\sigma = \lambda'\lambda'^\sigma$ or $\lambda\lambda^\sigma\lambda'\lambda'^\sigma = 1$. If $\lambda = \xi^i$ and $\lambda' = \xi^{i'}$ for some odd i and i' , the condition translates into $\xi^{(q+1)i} = \xi^{\pm(q+1)i'}$, which is equivalent to $(q+1)i \equiv \pm(q+1)i' \pmod{(q^2-1)}$ and which simplifies to $i \equiv \pm i' \pmod{(q-1)}$. This proves Part 1 except for the order assertion. But, in G , for $B = \text{dia}(\xi^i, 1)$, the order of $[B, 1]$ is twice the order of $[B, 1]^2 = [BB^\sigma, 0]$, which is equal to the order of $BB^\sigma = \text{dia}(\xi^{(q+1)i}, 1)$ in $PSL(2, q^2)$, which is known to be equal to $(q^2-1)/\gcd\{q^2-1, (q+1)i\} = (q-1)/\gcd\{q-1, i\}$.

2. Now, let $[A, 1]$ be conjugate in G to $[B, 1]$ for $B = \text{off}(\lambda, 1)$ and $\lambda \in N(F)$. If $B' = \text{off}(\lambda', 1)$, by Proposition 8 we see that $[B, 1]$ is conjugate to $[B', 1]$ in $\overline{G}$ if and only if λ/λ' or $\lambda\lambda'$ are elements of F_0^* , or, equivalently, are powers of ξ^{q+1} . Letting $\lambda = \xi^i$ and $\lambda' = \xi^{i'}$ for odd i, i' , this condition translates into the congruence $i \equiv \pm i' \pmod{(q+1)}$, proving Part 2 apart from the claim about the order. To fill in this last bit, the order of $[B, 1]$ in G for $B = \text{off}(\xi^i, 1)$ is two times the order of $[B, 1]^2 = [BB^\sigma, 0]$, which is equal to the order of $BB^\sigma = \text{dia}(\xi^i, \xi^{qi})$ in $PSL(2, q^2)$. The latter is equal to the order of $\text{dia}(1, \xi^{(q-1)i})$ in $PSL(2, q^2)$, i.e., to $(q^2-1)/\gcd\{q^2-1, (q-1)i\} = (q+1)/\gcd\{q+1, i\}$.

3. The values $\lambda = \lambda'$ automatically satisfy the first condition in Proposition 7, by which the stabiliser of $[B, 1]$ for $B = \text{dia}(\lambda, 1)$ includes a subgroup H formed by (conjugation by) the $(q-1)$ elements $[P, 0]$ for $P = \text{dia}(\eta, 1)$ and the $q-1$ elements $[P, 1]$ with $P = \text{dia}(\eta\lambda, 1)$, where η ranges over the set of all $(q-1)^{\text{th}}$ roots of unity in F (and this set is simply equal to F_0^*). We claim that H is (isomorphic to) a cyclic group of order $2(q-1)$. Indeed, let $\lambda = \xi^i$ for odd i , $1 \leq i \leq (q-1)/2$. Then, letting $\eta = \xi^{(q+1)(1-i)/2}$, the group H is generated by $[P, 1]$, where $P = \text{dia}(\eta\lambda)$. To see this, observe that $[P, 1]^2 = [PP^\sigma, 0] = [Q, 0]$, where $Q = \text{dia}(\eta^2\lambda\lambda^\sigma, 1)$. With λ and η as above we have $Q = \text{dia}(\xi^{q+1}, 1)$, which means that the order of $[Q, 0]$ is $q-1$ and so $[P, 1]$ generates H , as claimed.

Observe that the values $\lambda = \lambda'$ may also satisfy our second condition $\lambda\lambda^\sigma\lambda'\lambda'^\sigma = 1$ from Proposition 7. This happens if and only if λ is a $(q+1)^{\text{th}}$ root of -1 in F and $q \equiv -1 \pmod{4}$

4, which is what we now assume; note that now $\lambda^\sigma = -\lambda^{-1}$. Then, if ζ and ζ' are $(q-1)^{\text{th}}$ roots of $(\lambda\lambda')^{-1} = \lambda^{-2}$ as in the calculations immediately preceding Proposition 7, we have $\zeta'/\zeta \in F_0^*$, $\lambda^{q-1} = -\zeta^{q-1}$, and $[\text{off}(\zeta'/\lambda, 1), i][\text{off}(\zeta/\lambda, 1), i] = [\text{dia}((-1)^i \zeta'/\zeta, 1), 0]$ for $i \in \{0, 1\}$. In particular, letting $Q = \text{off}(\zeta/\lambda, 1)$, the element $[Q, 1]^2 = [\text{dia}(-1, 1), 0]$ lies in the cyclic group H ; hence $[Q, 1]$ has order 4. Moreover, one may check that for every $\eta \in F_0^*$ we have $[Q, 1]^{-1}[P, 0][Q, 1] = [P, 0]^{-1}$ if $P = \text{dia}(\eta, 1)$ and $[Q, 1]^{-1}[P, 1][Q, 1] = [P, 1]^{-1}[Q, 1]^2$ if $P = \text{dia}(\eta\lambda, 1)$.

Summing up, the stabiliser of an element $[B, 1]$ with $B = \text{dia}(\lambda, 1)$ in $\overline{G}$ for $\lambda \in N(F)$ is isomorphic to $H \cong Z_{2(q-1)}$ except for the case when $q \equiv -1 \pmod{4}$ and λ is a $(q+1)^{\text{th}}$ root of -1 and then the stabiliser of $[B, 1]$ is isomorphic to an extension H^* of H by $[Q, 1]$ with $[Q, 1]^2 \in H$, so that $H^* \cong Z_{2(q-1)} \cdot Z_2$, as claimed.

4. The values $\lambda = \lambda'$ automatically satisfy the condition (1) from Proposition 8. By the calculations preceding this Proposition, the stabiliser of $[B, 1]$ for $B = \text{off}(\lambda, 1)$ contains a subgroup formed by the $(q+1)$ elements $[P, 0]$ for $P = \text{dia}(\eta, 1)$ and the $q-1$ elements $[P, 1]$ with $P = \text{off}(\eta\lambda, 1)$, where this time η ranges over the set of all $(q+1)^{\text{th}}$ roots of unity in F . The collection of these $2(q+1)$ elements is closed under multiplication and hence forms a group H of order $2(q+1)$; we claim that H is cyclic. To demonstrate this, let $\lambda = \xi^i$ for some odd i such that $1 \leq i \leq (q+1)/2$. Then, using $\eta = \xi^{(q-1)(1+i)/2}$, the element $[P, 1]$ with $P = \text{off}(\eta\lambda, 1)$ is a generator of H . Indeed, a calculation shows that $[P, 1]^2 = [PP^\sigma, 0] = [Q, 0]$, where $Q = \text{dia}(\eta^2\lambda^{1-q}, 1)$, and with the given λ and our choice of η we have $Q = \text{dia}(\xi^{q-1}, 1)$. It follows that the order of $[Q, 0]$ is $q+1$ and so $[P, 1]$ generates H .

Here, the values $\lambda = \lambda'$ may also fulfil the condition (2) from Proposition 8, that is, $\lambda^2 \in F_0^*$. Since we know that $\lambda \in N(F)$, it can be checked that $\lambda^2 \in F_0^*$ is equivalent to $\lambda^{q-1} = -1$ and implies $q \equiv 1 \pmod{4}$. Again, the calculations made immediately before Proposition 8 imply the following: Letting ζ be any of the $q+1$ $(q+1)^{\text{st}}$ roots of λ^2 in F , in the stabiliser of $[B, 1]$ we have additional $q+1$ elements $[P, 1]$ such that $P = \text{dia}(\zeta/\lambda, 1)$, as well as $q+1$ elements $[P, 0]$ such that $P = \text{off}(\zeta, 1)$. Further calculations show that if ζ and ζ' are $(q+1)^{\text{th}}$ roots of λ^2 , $P = \text{off}(\zeta, 1)$ and $P' = \text{off}(\zeta', 1)$, then we have $[P', 0][P, 0] = [\text{dia}(\zeta'/\zeta, 1), 0] \in H$ since it can be checked that $\zeta'/\zeta \in F_0^*$. Similarly, if $Q = \text{dia}(\zeta/\lambda, 1)$ and $Q' = \text{dia}(\zeta'/\lambda, 1)$, then $[Q', 1][Q, 1] = [Q'Q^\sigma, 0] = [\text{dia}((\zeta'/\zeta)(\zeta/\lambda)^{q+1}, 1), 0] \in H$. In particular letting $\zeta' = \zeta$ we have $(\zeta/\lambda)^{q+1} = \lambda^2/(\lambda^{q-1}\lambda^2) = -1$ and so $[Q, 1]^2 = [\text{dia}(-1, 1), 0] \in H$. We thus have a situation analogous to the Case 3 of this proof: $[Q, 1]$ has order 4, conjugation of an element $[P, 0] \in H$ with $P = \text{dia}(\eta, 1)$ by $[Q, 1]$ inverts $[P, 0]$, while conjugation of a $[P, 1] \in H$ such that $P = \text{off}(\eta\lambda, 1)$ by $[Q, 1]$ gives $[P, 1]^{-1}[Q, 1]^2$.

In summary, the stabiliser of an element $[B, 1]$ for $B = \text{off}(\lambda, 1)$ is isomorphic to $H \cong Z_{2(q+1)}$ if λ is not a $(q-1)^{\text{th}}$ root of -1 , and isomorphic to an extension H^* of H by $[Q, 1]$ (where $[Q, 1]^2 \in H$), implying that $H^* \cong Z_{2(q-1)} \cdot Z_2$. This completes the proof. $\square$

Let us remark that the exceptional cases in the items 3 and 4 above correspond precisely to elements $[B, 1]$ of order 4 in G . By inspecting possible orders of $[B, 1]$ we also have:

Corollary 1 *Every element of $G \setminus G_0$ has order divisible by 4.* □

7 Non-singular pairs and twisted subgroups

Our aim in this and the following two sections is to determine representatives of selected conjugacy classes $\{(x, y)^g; g \in \overline{G} = G\langle\sigma\rangle\}$ of elements $x, y \in G$ satisfying $(xy)^2 = 1$, and make important conclusions about subgroups the corresponding pairs (x, y) generate. We note that the action of σ need not be considered separately, because $[I, 1][A, i][I, 1] = [A^\sigma, i]$ for $i \in \{0, 1\}$, which means that the action of σ is equivalent to conjugation by the element $[I, 1] \in \overline{G}$.

Since we want xy to have order 2, both x and y as above must lie in $G \setminus G_0$ because, by Corollary 1, there are no involutions in $G \setminus G_0$. By Theorem 1 we may assume that $y = [B, 1]$ for $B = \text{dia}(\lambda, 1)$ or $B = \text{off}(\lambda, 1)$ for a suitable $\lambda \in N(F)$. Letting $x = [A, 1]$, the pair x, y may in general generate a proper subgroup of $G = M(p^{2f})$; such cases will still be of interest for our intended classification of orientably-regular maps as long as the subgroup $\langle x, y \rangle$ is twisted, that is, isomorphic to $M(p^{2e})$ for a suitable divisor e of f .

We now identify conditions on A implied by the requirement that $([A, 1][B, 1])^2$ be the identity in G and begin with the case when $B = \text{dia}(\lambda, 1)$. Let $A \in \text{GL}(2, F)$ have rows (a, b) and (c, d) , with determinant $ad - bc \in N(F)$. Then, $[A, 1][B, 1] = [AB^\sigma, 0]$, where

$$AB^\sigma = \begin{pmatrix} a\lambda^\sigma & b \\ c\lambda^\sigma & d \end{pmatrix}.$$

Since AB^σ lies in $\text{PSL}(2, F)$, it has order 2 if and only if its trace $a\lambda^\sigma + d$ is equal to zero. If one of a, d was equal to zero, both would have to be zero and then $[A, 1]$ and $[B, 1]$ would clearly not generate a twisted subgroup of G . Therefore both a, d are non-zero and we may assume without loss of generality that $a = -1$ and $d = \lambda^\sigma$. We will thus consider only elements $[A, 1] \in G$ of the form

$$A = \begin{pmatrix} -1 & b \\ c & \lambda^\sigma \end{pmatrix}, \quad u = bc + \lambda^\sigma \in N(F). \quad (1)$$

Next, consider the case when $B = \text{off}(\lambda, 1)$. For a matrix $A \in \text{GL}(2, F)$ with rows (a, b) and (c, d) such that $ad - bc \in N(F)$ we now have $[A, 1][B, 1] = [AB^\sigma, 0]$, where

$$AB^\sigma = \begin{pmatrix} b & a\lambda^\sigma \\ d & c\lambda^\sigma \end{pmatrix}.$$

Again, $AB^\sigma \in \text{PSL}(2, F)$ has order 2 if and only if its trace $b + c\lambda^\sigma$ is equal to zero. If one of b, c was equal to zero, we would have $b = c = 0$, but then $[A, 1]$ and $[B, 1]$ would again not generate a twisted subgroup of G . Therefore both b and c are non-zero and we may

assume that $c = -1$ and $b = \lambda^\sigma$. It follows that, without loss of generality, we only need to consider elements $[A, 1] \in G$ such that

$$A = \begin{pmatrix} a & \lambda^\sigma \\ -1 & d \end{pmatrix}, \quad u = ad + \lambda^\sigma \in N(F). \quad (2)$$

With A and B as above we can still identify obvious instances when $[A, 1]$ and $[B, 1]$ do not generate a twisted subgroup of G . This is certainly the case if

- (i) both $[A, 1]$ and $[B, 1]$ have order 4, as then the two elements generate a solvable group, cf. [8], or
- (ii) $B = \text{dia}(\lambda, 1)$ and A is an upper- or a lower-triangular matrix, as then $[A, 1]$ and $[B, 1]$ generate a triangular subgroup of G , or else
- (iii) $B = \text{off}(\lambda, 1)$ and A is an off-diagonal matrix, as then $[A, 1]$ and $[B, 1]$ clearly do not generate a twisted subgroup of G .

For $B = \text{dia}(\lambda, 1)$ and A given by (1) and for $B = \text{off}(\lambda, 1)$ and A given by (2), an ordered pair $([A, 1], [B, 1])$ not satisfying any of (i), (ii) and (iii) will be called *non-singular*.

We are now in position to classify the subgroups of $G = M(p^{2f})$ generated by non-singular pairs. To do so we will again use knowledge of the situation in the subgroup $G_0 \simeq \text{PSL}(2, p^{2f})$ of G . Recall that a subgroup H of $G = M(p^{2f})$ was said to stabilise a point if there exists an element in $F \cup \{\infty\}$ fixed by all linear fractional mappings corresponding to elements of H ; also, G_0 denotes the (unique) copy of $\text{PSL}(2, p^{2f})$ in G .

Proposition 9 *Let H be a subgroup of G generated by a non-singular pair $([A, 1], [B, 1])$. Then H is isomorphic to $M(p^{2e})$ for some positive divisor e of f such that f/e is odd.*

Proof. Let $H_0 = H \cap G_0$. The classification of [20] tells us that subgroups of G_0 fall into the following categories: point stabilisers, dihedral groups, A_4 , S_4 , A_5 , and $\text{PSL}(2, p^g)$ for divisors g of $2f$ together with $\text{PGL}(2, p^{g/2})$ for even divisors g of $2f$. For our subgroup H_0 we subsequently rule out all but the penultimate case.

We first show that H_0 neither stabilises a point nor is dihedral. The first coordinate of every element of the group H_0 can be written as a product of the form $X_1 Y_1^\sigma \dots X_m Y_m^\sigma$, where $X_i, Y_i \in \{A, B\}$, $1 \leq i \leq m$. It follows that H_0 is generated by the three elements $[BB^\sigma, 0]$, $[BA^\sigma, 0]$ and $[AB^\sigma, 0]$; note that $AA^\sigma = AB^\sigma(BB^\sigma)^{-1}BA^\sigma$. A straightforward calculation (details of which we leave to the reader) shows that for our generators $[A, 1]$ and $[B, 1]$ of H , there is no point in $F \cup \{\infty\}$ fixed by all three of the linear fractional mappings corresponding to BB^σ , BA^σ and AB^σ . It follows that the subgroup $H_0 = \langle [BB^\sigma, 0], [BA^\sigma, 0], [AB^\sigma, 0] \rangle$ cannot stabilise a point.

Suppose that H_0 is dihedral. An easy calculation shows that we always have $AB^\sigma \neq BA^\sigma$. It follows that one of AB^σ , BA^σ is an involution outside the cyclic part of H_0 and hence inverts one of AA^σ and BB^σ . If conjugation by AB^σ is considered, the conditions $(B^\sigma)^{-1}A^{-1}XX^\sigma AB^\sigma = (X^\sigma)^{-1}X^{-1}$ for $X \in \{A, B\}$ both reduce to $(B^{-1}A)^2 = I$

in $\text{PSL}(2, p^{2f})$. A similar calculation shows that the condition of XX^σ being inverted by conjugation by BA^σ for $X \in \{A, B\}$ both reduce to $(B^{-1}A)^2 = I$ again. The condition $(B^{-1}A)^2 = I$ in $\text{PSL}(2, p^{2f})$ is equivalent to $(B^{-1}A)^2 = \varepsilon I$ in $\text{SL}(2, p^{2f})$ for some $\varepsilon \in \text{GF}(p^{2f})$. If $B = \text{dia}(\lambda, 1)$ and $b = c = 0$ in A , then $\lambda\lambda^\sigma = -1$ which, by part 3 of Theorem 1, means that the orders of both $[A, 1]$ and $[B, 1]$ would be equal to 4. If one of $b, c \neq 0$, then $\varepsilon = -1$ and $\lambda\lambda^\sigma = 1$, a contradiction as $\lambda \in N(F)$. An entirely similar conclusion is obtained if $B = \text{off}(\lambda, 1)$: if $a = d = 0$ in A , then $\lambda^\sigma/\lambda = -1$ and $[A, 1], [B, 1]$ have order 4 by part 4 of Theorem 1, and if one of $a, d \neq 0$, then $\varepsilon = -1$ and $\lambda^\sigma = \lambda$, a contradiction again. It follows that for all non-singular pairs $([A, 1], [B, 1])$ we have $(B^{-1}A)^2 \neq I$ in $\text{PSL}(2, p^{2f})$, and so H_0 cannot be dihedral.

By Corollary 1, the order of every element in $G \setminus G_0$ is divisible by 4. Since the order of one of $[A, 1]$ and $[B, 1]$ is assumed to be greater than 4, one of the elements $[AA^\sigma, 0], [BB^\sigma, 0]$ of H_0 has to have even order greater than 2. But as A_4 and A_5 do not contain elements of even order greater than 2, H_0 cannot be isomorphic to A_4 or A_5 . The case $H_0 \cong S_4$ is excluded as follows. If there was an orientably-regular map with automorphism group H such that $H_0 \cong S_4$, then the orders of the elements $[A, 1], [B, 1]$ would have to be in the set $\{4, 8\}$ since the largest even order of an element in S_4 is 4. According to the list of [3], however, there is no orientably-regular map of type $\{4, 8\}$ or $\{8, 8\}$ with (orientation-preserving) automorphism group of order $2|S_4| = 48$.

We may now apply Proposition 4 to the subgroup H_0 to conclude that H is conjugate in $\overline{G}$ to a subgroup isomorphic to $M(p^{2e})$ for some positive divisor e of f such that f/e is odd, completing the proof. $\square$

It follows that a pair $([A, 1], [B, 1])$ of elements of G as above generates a twisted subgroup of G if and only if the pair is non-singular.

8 Orbits of non-singular pairs: The diagonal case

We will identify representatives of $\overline{G}$ -orbits of non-singular pairs $([A, 1], [B, 1])$, dealing with $B = \text{dia}(\lambda, 1)$ and A as in (1) here and deferring the case $B = \text{off}(\lambda, 1)$ to the next section.

Instead of working with matrices, the form of A in (1) suggests to look at the corresponding quadruples (λ, b, c, u) , also called *non-singular*, under the induced action of the stabiliser of $[B, 1]$ in $\overline{G}$. We recall that the values of λ and identification of the stabiliser are in items 1 and 3 of Theorem 1. To simplify the notation in what follows, for any $\omega \in F$ we will use the symbol $\sqrt[r]{\omega}$ to denote the set of all r^{th} roots of ω in $F = \text{GF}(q^2)$, $q = p^f$. The analysis in the third part of the proof of Theorem 1 tells us that the stabiliser of $[B, 1]$ in $\overline{G}$ consists exactly of the following elements of $\overline{G}$:

$$\begin{aligned}
& [P_1(\eta), 0], \quad \text{where } P_1 = \text{dia}(\eta, 1) \text{ and } \eta \in F_0^*; \\
& [P_2(\eta), 1], \quad \text{where } P_2 = \text{dia}(\eta\lambda, 1) \text{ and } \eta \in F_0^*; \\
& [P_3(\zeta), 0], \quad \text{where } P_3 = \text{off}(\zeta, 1) \text{ if } \lambda \in {}^{q+1}\sqrt{-1} \text{ and } \zeta \in {}^{q-1}\sqrt{\lambda^{-2}}; \\
& [P_4(\zeta), 1], \quad \text{where } P_4 = \text{off}(\zeta/\lambda, 1) \text{ if } \lambda \in {}^{q+1}\sqrt{-1} \text{ and } \zeta \in {}^{q-1}\sqrt{\lambda^{-2}}.
\end{aligned}$$

To find the corresponding orbit of $[A, 1]$ we first evaluate the products $[P_j(\eta), 0]^{-1}[A, 1][P_j(\eta), 0]$ for A as in (1) and $j \in \{1, 2, 3, 4\}$:

$$\begin{aligned}
& [P_1(\eta), 0]^{-1}[A, 1][P_1(\eta), 0] = [C_1, 1], \quad \text{where } C_1 = \begin{pmatrix} -1 & b\eta^{-1} \\ c\eta & \lambda^\sigma \end{pmatrix}; \\
& [P_2(\eta), 1]^{-1}[A, 1][P_2(\eta), 1] = [C_2, 1], \quad \text{where } C_2 = \begin{pmatrix} -1, & b^\sigma(\eta\lambda)^{-1} \\ c^\sigma\eta\lambda^\sigma & \lambda^\sigma \end{pmatrix}; \\
& [P_3(\zeta), 0]^{-1}[A, 1][P_3(\zeta), 0] = [C_3, 1], \quad \text{where } C_3 = \begin{pmatrix} -1 & c\zeta/\lambda \\ b\lambda/\zeta & \lambda^\sigma \end{pmatrix}; \quad \text{and} \\
& [P_4(\zeta), 1]^{-1}[A, 1][P_4(\zeta), 1] = [C_4, 1], \quad \text{where } C_4 = \begin{pmatrix} -1 & -c^\sigma\zeta/\lambda^2 \\ b^\sigma/\zeta & \lambda^\sigma \end{pmatrix}.
\end{aligned}$$

Let $\lambda = \xi^i$ for a fixed primitive element $\xi \in F$ and some odd i such that $1 \leq i \leq (q-1)/2$; note that here $\lambda \in {}^{q+1}\sqrt{-1}$ if and only if $i = (q-1)/2$. It follows that we have either $(q-1)/4$ such odd values of i if $q \equiv 1 \pmod{4}$ and all are smaller than $(q-1)/2$, or else $(q-3)/4$ such odd $i < (q-1)/2$ together with $i = (q-1)/2$ if $q \equiv -1 \pmod{4}$.

In each of the above cases, if $i < (q-1)/2$, the stabiliser H of $[B, 1]$ for $B = \text{dia}(\lambda, 1)$ has order $2(q-1)$. By the above calculations leading to the matrices C_1 and C_2 , the orbit O of the induced action of H on a quadruple (λ, b, c, u) satisfying $bc + \lambda^\sigma = u \in N(F)$ is formed by the quadruples $(\lambda, b\eta^{-1}, c\eta, u)$ for $\eta \in F_0^*$ and $(\lambda, b^\sigma(\eta\lambda)^{-1}, c^\sigma\eta\lambda^\sigma, u^\sigma\lambda^\sigma/\lambda)$ for $\eta \in F_0^*$. It is easy to check that all these $2(q-1)$ quadruples are mutually distinct and hence each such orbit O has size $2(q-1)$. If $u = \lambda^\sigma$, then one of b, c would have to be equal to zero, contrary to non-singularity. For each of our $\lfloor (q-1)/4 \rfloor$ choices of i we therefore have a total of $(q^2-3)/2$ choices for $u \in N(F)$, $u \neq \lambda^\sigma$; for each of these we can freely choose a non-zero $b \in F$ in q^2-1 ways and then c is uniquely determined. This gives a total of $\lfloor (q-1)/4 \rfloor (q^2-1)(q^2-3)/2$ quadruples (λ, b, c, u) and the number of H -orbits on these is given by dividing by $|H| = 2(q-1)$. Denoting by n_1 the number of $\overline{G}$ -equivalence classes of non-singular quadruples (λ, b, c, u) for $i < (q-1)/2$ for diagonal B , we obtain

$$n_1 = (q+1) \left\lfloor \frac{q-1}{4} \right\rfloor \frac{q^2-3}{4}. \quad (3)$$

The enumeration is trickier for $i = (q-1)/2$, that is, for $\lambda = \xi^{(q-1)/2}$ and $q \equiv -1 \pmod{4}$. This is the case when the stabiliser H^* of $[B, 1]$ for $B = \text{dia}(\lambda, 1)$ has order $4(q-1)$ and the orbit O^* of the induced action of H^* on a quadruple (λ, b, c, u) satisfying $bc + \lambda^\sigma = u \in N(F)$ includes: (i) the $2(q-1)$ quadruples in O listed in the previous paragraph, and (ii) by the form of the matrices C_3 and C_4 also the $(q-1)$ quadruples (λ, b', c', u) for $b' = c\zeta/\lambda$, $c' = b\lambda/\zeta$ as well as the $(q-1)$ quadruples $(\lambda, b'', c'', u/\lambda^2)$ for $b'' = -c^\sigma\zeta/\lambda^2$, $c'' = b^\sigma/\zeta$, where ζ can be equal to any of the $q-1$ values of ${}^{q-1}\sqrt{\lambda^{-2}}$. Clearly, either $|O^*| = 4(q-1)$ or $O^* = O$, and we will investigate the second possibility.

We have $O^* = O$ if and only if b is equal to one of b' or b'' . For our $\lambda = \xi^{(q-1)/2}$ we find that ${}^{q-1}\sqrt{\lambda^{-2}} = \{\xi^{-1}\omega; \omega \in F_0^*\}$. It can be checked that b cannot be equal to any of the values of b'' but it may happen that $b = b' = c\zeta/\lambda = c\xi^{-1}\xi^{(1-q)/2}\omega'$ for some $\omega' \in F_0^*$. Taking into account the fact that we are only interested in non-zero b, c it follows that $O^* = O$ if and only if $c/b = \xi^{(q+1)/2}\omega$ for some $\omega \in F_0^*$. But this is equivalent to $(c/b)^{q-1} = -1$, which can be rewritten in the form $bc^\sigma + b^\sigma c = 0$. Since $bc^\sigma + b^\sigma c$ is the trace of AA^σ for A as in (1), the fact that the trace is zero is equivalent to $[A, 1]^2$ having order 2 in G . This all means that $O^* = O$ if and only if the element $[A, 1]$ has order 4 in G . But for $\lambda = \xi^{(q-1)/2}$ the order of $[B, 1]$ in G is then equal to 4 as well, a contradiction with non-singularity. It follows that $|O^*| = 4(q-1)$.

Observe that if $c/b = \xi^{(q+1)/2}\omega$ for some $\omega \in F_0^*$, then the determinant equation gives $u = bc + \lambda^\sigma = b^2\xi^{(q+1)/2}\omega + \lambda^\sigma$, and since $q \equiv -1 \pmod{4}$ this means that $u - \lambda^\sigma \in S(F)$. Conversely, if this holds, then for every $\omega \in F_0^*$ the equation $b^2\xi^{(q+1)/2}\omega = u - \lambda^\sigma$ has two values of b as a solution. For a fixed $v \in N(F)$ let $N(v)$ denote the set of all $u \in N(F) \setminus \{v\}$ for which $u - v \in S(F)$. It is easy to see that the value of $|N(v)|$ does not depend on the choice of $v \in N(F)$. Let n_F denote this common value; it was shown in [17] that $n_F = (q^2 - 1)/4$.

It follows that for every $u \in N(\lambda^\sigma)$ we have $2(q-1)$ values of b satisfying $b^2\xi^{(q+1)/2}\omega = u - \lambda^\sigma$ for some $\omega \in F_0^*$, and these yield the singular pairs identified in the previous paragraph. Thus, for any of the n_F values of $u \in N(\lambda^\sigma)$ we have $(q^2 - 1) - 2(q-1) = (q-1)^2$ values of $b \in F^*$ that furnish a non-singular pair $[A, 1], [B, 1]$ for G , giving a total of $n_F(q-1)^2$ quadruples of the form $(\xi^{(q-1)/2}, b, c, u)$. Any of the remaining $(q^2 - 3)/2 - n_F$ non-square values of u then simply give $((q^2 - 3)/2 - n_F)(q^2 - 1)$ non-singular quadruples $(\xi^{(q-1)/2}, b, c, u)$ that we have to consider. As established earlier, the induced action of the group H^* on these quadruples is semi-regular. Hence, letting n_2 be the number of $\overline{G}$ -equivalence classes of non-singular quadruples (λ, b, c, u) for $\lambda = \xi^{(q-1)/2}$ and $q \equiv -1 \pmod{4}$ and for diagonal B , we have

$$n_2 = \frac{n_F(q-1)^2 + ((q^2 - 3)/2 - n_F)(q^2 - 1)}{4(q-1)} = \frac{1}{8}((q+1)(q^2 - 3) - 4n_F) . \quad (4)$$

9 Orbits of non-singular pairs: The off-diagonal case

We continue looking for representatives of $\overline{G}$ -orbits of non-singular pairs $([A, 1], [B, 1])$ as in the previous section but this time for $B = \text{off}(\lambda, 1)$ and A as in (2). We again reduce the task to finding representatives of the corresponding quadruples (λ, a, d, u) , called *non-singular* as well, under the induced action of the stabiliser of $[B, 1]$ in $\overline{G}$, with λ and the stabiliser being determined by items 2 and 4 of Theorem 1. The orbit of $[A, 1]$ given by (2) under conjugation by elements stabilising $[B, 1]$ is found by calculating the conjugates by the following elements of $\overline{G}$ identified in the fourth part of the proof of Theorem 1:

$$\begin{aligned}
& [P_1(\eta), 0], \quad \text{where } P_1 = \text{dia}(\eta, 1) \text{ and } \eta \in {}^{q+1}\sqrt{1}; \\
& [P_2(\eta), 1], \quad \text{where } P_2 = \text{off}(\eta\lambda, 1) \text{ and } \eta \in {}^{q+1}\sqrt{1}; \\
& [P_3(\zeta), 0], \quad \text{where } P_3 = \text{off}(\zeta, 1) \text{ if } \lambda \in {}^{q-1}\sqrt{-1} \text{ and } \zeta \in {}^{q+1}\sqrt{\lambda^2}; \\
& [P_4(\zeta), 1], \quad \text{where } P_4 = \text{dia}(\zeta/\lambda, 1) \text{ if } \lambda \in {}^{q-1}\sqrt{-1} \text{ and } \zeta \in {}^{q+1}\sqrt{\lambda^2}.
\end{aligned}$$

Evaluation of the products $[P_j(\eta), 0]^{-1}[A, 1][P_j(\eta), 0]$ for A as in (2) and $j \in \{1, 2, 3, 4\}$ gives:

$$\begin{aligned}
& [P_1(\eta), 0]^{-1}[A, 1][P_1(\eta), 0] = [D_1, 1], \quad \text{where } D_1 = \begin{pmatrix} a\eta^{-1} & \lambda^\sigma \\ -1 & d\eta \end{pmatrix}; \\
& [P_2(\eta), 1]^{-1}[A, 1][P_2(\eta), 1] = [D_2, 1], \quad \text{where } D_2 = \begin{pmatrix} -\eta^{-1}d^\sigma\lambda^\sigma/\lambda & \lambda^\sigma \\ -1 & -\eta a^\sigma \end{pmatrix}; \\
& [P_3(\zeta), 0]^{-1}[A, 1][P_3(\zeta), 0] = [D_3, 1], \quad \text{where } D_3 = \begin{pmatrix} d\zeta/\lambda & \lambda^\sigma \\ -1 & a\lambda/\zeta \end{pmatrix}; \text{ and} \\
& [P_4(\zeta), 1]^{-1}[A, 1][P_4(\zeta), 1] = [D_4, 1], \quad \text{where } D_4 = \begin{pmatrix} -a^\sigma\zeta/\lambda & \lambda^\sigma \\ -1 & d^\sigma\lambda/\zeta \end{pmatrix}.
\end{aligned}$$

Let $\lambda = \xi^i$ for a fixed primitive element $\xi \in F$ and some odd i such that $1 \leq i \leq (q+1)/2$; here we have $\lambda \in {}^{q-1}\sqrt{-1}$ if and only if $i = (q+1)/2$. This gives either $(q+1)/4$ such odd values of i if $q \equiv -1 \pmod{4}$, all smaller than $(q+1)/2$, or $(q-1)/4$ such odd $i < (q+1)/2$ together with $i = (q+1)/2$ if $q \equiv 1 \pmod{4}$.

In each of these instances, if $i < (q+1)/2$, the stabiliser H of $[B, 1]$ for $B = \text{off}(\lambda, 1)$ has order $2(q+1)$. By the above calculations the orbit O of the induced action of H on a quadruple (λ, a, d, u) satisfying $ad + \lambda^\sigma = u \in N(F)$ consists of the quadruples $(\lambda, a\eta^{-1}, d\eta, u)$ and $(\lambda, -d^\sigma\lambda^\sigma(\eta\lambda)^{-1}, -a^\sigma\eta, u^\sigma\lambda^\sigma/\lambda)$ for $\eta \in F_0^*$. By non-singularity, at least one of a, d is non-zero. This implies, by an easy computation, that all the above $2(q+1)$ quadruples are mutually distinct and hence each such orbit O on our quadruples under the action of H has size $2(q+1)$. If $u = \lambda^\sigma$, then exactly one of a, d is equal to zero while the other one can assume any of the $q^2 - 1$ values in F^* . For each of the $\lfloor (q+1)/4 \rfloor$ choices of i considered here, we have:

(α) $(q^2 - 3)/2$ choices of $u \in N(F) \setminus \{\lambda^\sigma\}$, each admitting a choice of $a \in F^*$ in $q^2 - 1$ ways, uniquely determining d and making $\lfloor (q+1)/4 \rfloor (q^2 - 1)(q^2 - 3)/2$ quadruples; and

(β) for $u = \lambda^\sigma$ we have $2(q^2 - 1)$ choices of non-zero a (for $d = 0$) and non-zero d (for $a = 0$), giving $\lfloor (q+1)/4 \rfloor \cdot 2(q^2 - 1)$ quadruples.

Since the number of H -orbits on quadruples in (α) and (β) is given by dividing by $|H| = 2(q+1)$, we arrive at a total of n_3 $\overline{G}$ -equivalence classes of non-singular quadruples (λ, a, d, u) for $i < (q+1)/2$ and off-diagonal B , where

$$n_3 = (q-1) \left\lfloor \frac{q+1}{4} \right\rfloor \frac{q^2 + 1}{4}. \quad (5)$$

The enumeration is again more tricky for $i = (q+1)/2$, that is, for $\lambda = \xi^{(q+1)/2}$ and $q \equiv 1 \pmod{4}$. The stabiliser H^* of $[B, 1]$ for $B = \text{off}(\lambda, 1)$ has in this case order $4(q+1)$ and the orbit O^* of the induced action of H^* on a quadruple (λ, a, d, u) with values fulfilling the equation $ad + \lambda^\sigma = u \in N(F)$ contains the $2(q+1)$ quadruples in O listed in the

previous paragraph together with the $2(q+1)$ distinct quadruples $(\lambda, d\zeta/\lambda, a\lambda/\zeta, u)$ and $(\lambda, -a^\sigma\zeta/\lambda, d^\sigma\lambda/\zeta, -u^\sigma)$, where ζ can be any of the $q+1$ values of $\sqrt[q+1]{\lambda^2}$. As before, either $|O^*| = 4(q+1)$ or $O^* = O$, and we will study the second eventuality in detail.

By inspection of the orbits O and O^* in this case, the orbit O^* coincides with O if and only if either

$$(\gamma) \quad a\eta^{-1} = d\zeta/\lambda, \text{ or}$$

$$(\delta) \quad a\eta^{-1} = -a^\sigma\zeta/\lambda \text{ and } d\eta = d^\sigma\lambda/\zeta,$$

all for some $\eta \in \sqrt[q+1]{1}$. For our calculations we may without loss of generality assume that $\zeta = \xi$, as this is one of the $(q+1)^{\text{th}}$ roots of $\lambda^2 = \xi^{q+1}$. Note that we now have $\lambda^\sigma = -\lambda$, $\eta^\sigma = \eta^{-1}$, $\zeta^\sigma = \lambda^2/\zeta$, and therefore also $(\zeta\eta/\lambda)^\sigma = -(\zeta\eta/\lambda)^{-1}$.

The case (γ) : From $a = d(\zeta\eta/\lambda)$ and its σ -image $a^\sigma = d^\sigma(\zeta\eta/\lambda)^\sigma = -d^\sigma(\zeta\eta/\lambda)^{-1}$ we obtain $aa^\sigma + dd^\sigma = 0$. But $aa^\sigma + dd^\sigma$ is the trace of AA^σ , which means that the order of both $[A, 1]$ and $[B, 1]$ is 4. It follows that (γ) cannot occur for a non-singular pair.

The case (δ) : From $a = -a^\sigma(\zeta\eta/\lambda)$ and its σ -image $a^\sigma = -a(\zeta\eta/\lambda)^\sigma = a(\zeta\eta/\lambda)^{-1}$ we have $a = -a$, that is, $a = 0$. Similarly, from $d = d^\sigma\lambda/(\zeta\eta)$ one concludes that $d = 0$, contrary to non-singularity.

We can now complete the enumeration of non-singular quadruples (λ, a, d, u) in the case $B = \text{off}(\lambda, 1)$ for $\lambda = \xi^{(q+1)/2}$. As we saw above, the case (δ) is excluded. The case (γ) does not yield non-singular pairs but we still need more details about the corresponding quadruples to be excluded. We saw that (γ) occurs if and only if $d = a\lambda/(\zeta\eta)$, and this gives $u = ad + \lambda^\sigma = a^2\xi^{(q-1)/2}\xi^{i(q-1)} + \lambda^\sigma$, $i \in \{0, 1, \dots, q\}$. Since now $q \equiv 1 \pmod{4}$, it follows that $u - \lambda^\sigma \in S(F)$, that is, $u \in N(\lambda^\sigma)$. Conversely, if $u \in N(\lambda^\sigma)$, then for each of the $q+1$ values of $i \in \{0, 1, \dots, q\}$ we obtain two solutions for a as above (and for each such a a unique d), not giving a non-singular pair. It follows that for each of the n_F elements $u \in N(\lambda^\sigma)$ we can choose $(q^2 - 1) - 2(q+1) = (q+1)(q-3)$ values $a \in F^*$ (and for each such a a unique d) such that $ad + \lambda^\sigma = u$, so that the order of $[A, 1]$ is not equal to 4. This gives a total of $n_F(q+1)(q-3)$ quadruples $(\xi^{(q+1)/2}, a, d, u)$ collected so far. Any of the remaining $(q^2 - 3)/2 - n_F$ non-square values of u distinct from λ^σ then give $((q^2 - 3)/2 - n_F)(q^2 - 1)$ quadruples $(\xi^{(q+1)/2}, a, d, u)$ that we have to consider for non-singular pairs. Finally, if $u = \lambda^\sigma$, then either $a = 0$ with $q^2 - 1$ choices for d , or $d = 0$ with $q^2 - 1$ choices for a , giving further $2(q^2 - 1)$ quadruples. Since we have shown earlier that the induced action of the group H^* on these quadruples is semi-regular with orbits of length $4(q+1)$, the number n_4 of $\overline{G}$ -equivalence classes of non-singular quadruples (λ, a, d, u) for off-diagonal B and $\lambda = \xi^{(q+1)/2}$, $q \equiv 1 \pmod{4}$, is

$$n_4 = \frac{n_F(q+1)(q-3) + ((q^2 - 3)/2 - n_F)(q^2 - 1) + 2(q^2 - 1)}{4(q+1)},$$

which simplifies to

$$n_4 = \frac{1}{8} ((q-1)(q^2 + 1) - 4n_F). \quad (6)$$

10 Enumeration of orientably-regular maps on $M(q^2)$

We have seen in section 7 that a pair $([A, 1], [B, 1])$ of elements of G , with diagonal B and A given by (1) or with off-diagonal B and A given by (2), and with product of order two, generates a twisted subgroup of G if and only if the pair is non-singular. In the previous two sections we have counted orbits of non-singular pairs in G under conjugation in $\overline{G}$, with no regard to subgroups the pairs generate. The number of these orbits turns out to be $n_1 + n_3 + n_4$ if $q \equiv 1 \pmod{4}$ and $n_1 + n_2 + n_3$ if $q \equiv -1 \pmod{4}$; in both cases the sum is equal to $(q^2 - 1)(q^2 - 2)/8$. We state this as a separate result.

Proposition 10 *The number of $\overline{G}$ -orbits of non-singular pairs in $G = M(q^2)$ is equal to $(q^2 - 1)(q^2 - 2)/8$. $\square$*

We will now refine our considerations and take into account subgroups generated by non-singular pairs. For our group $G = G_{2f} = M(p^{2f})$ and for any positive divisor e of f such that f/e is odd we let G_{2e} denote the canonical copy of $M(p^{2e})$ in G . In Lemma 2 we saw that the automorphism σ of $F = F_{2f} = \text{GF}(p^{2f})$ of order two restricts to an automorphism σ_{2e} of order two of the subfield $F_{2e} = \text{GF}(p^{2e})$ of F . We recall that $\overline{G} = \overline{G}_{2f} = G_{2f}\langle\sigma\rangle$ and we similarly introduce $\overline{G}_{2e}$ for every e as above by letting $\overline{G}_{2e} = G_{2e}\langle\sigma_{2e}\rangle$.

Let $\text{orb}_f(e)$ denote the number of $\overline{G}_{2f}$ -orbits of non-singular pairs $([A, 1], [B, 1])$ of G that generate a subgroup of G isomorphic to $M(p^{2e})$. At the same time, let $\text{orb}(e)$ be the number of orbits of non-singular pairs of G_{2e} which generate G_{2e} . The two quantities, are, in fact, equal, which is fundamental for our enumeration.

Proposition 11 *For each positive divisor e of f with f/e odd, we have $\text{orb}_f(e) = \text{orb}(e)$.*

Proof. It is clear that every $\overline{G}_{2e}$ -orbit of a non-singular pair in the canonical copy $G_{2e} \cong M(p^{2e})$ in G_{2f} is contained in a $\overline{G}_{2f}$ -orbit of the same pair. In the reverse direction, let a non-singular pair in G generate a subgroup isomorphic to $M(p^{2e})$. Since, by the important Proposition 3, all such subgroups are $\overline{G}_{2f}$ -conjugate in G_{2f} , we may assume that the non-singular pair is contained in G_{2e} . But then the $\overline{G}_{2f}$ -orbit of this pair obviously contains a $\overline{G}_{2e}$ -orbit of the same pair. The result will now be a consequence of the following claim:

Let $([A, 1], [B, 1])$ and $([A', 1], [B', 1])$ be two non-singular pairs of G_{2e} both generating G_{2e} and lying in the same $\overline{G}_{2f}$ -orbit of G_{2f} . Then the two pairs are contained in the same $\overline{G}_{2e}$ -orbit of G_{2e} .

We have $B = \text{dia}(\lambda, 1)$ and $B' = \text{dia}(\lambda', 1)$, or $B = \text{off}(\lambda, 1)$ and $B' = \text{off}(\lambda', 1)$, in both cases for some $\lambda, \lambda' \in F_{2e}$. The assumption that the two pairs are contained in the same $\overline{G}_{2f}$ -orbit means, by Proposition 6, that there exists an element $P \in \text{GL}(2, p^{2f})$ such that $AP^\sigma = \varepsilon PA'^{(\sigma)}$ and $BP^\sigma = \delta PB'^{(\sigma)}$ for some $\varepsilon, \delta \in F^*$. Propositions 7 and 8 applied to the equation $BP^\sigma = \delta PB'^{(\sigma)}$ tell us that $P = \text{dia}(\omega, 1)$ or $P = \text{off}(\omega, 1)$ for some $\omega \in F^*$. We will show that for all choices of A, A' and P the equation $AP^\sigma = \varepsilon PA'^{(\sigma)}$ implies that $\omega \in F_{2e}$, which will prove that our two non-singular pairs are in one G_{2e} -orbit.

There are two possibilities for A and A' , given in (1) and (2), and two possibilities for P , the diagonal and the off-diagonal case, to substitute in the equation $AP^\sigma = \varepsilon PA'^{(\sigma)}$; in all the four cases we need to conclude that $\omega \in F_{2e}$. We demonstrate it on the case when $P = \text{off}(\omega, 1)$ and when A, A' are given by (2), that is, when A has rows (a, λ^σ) , $(-1, d)$ and A' has rows (a', λ'^σ) , $(-1, d')$ for some $a, d, \lambda, a', d', \lambda' \in F_{2e}$. The above equation then reads

$$\begin{pmatrix} a & \lambda^\sigma \\ -1 & d \end{pmatrix} \begin{pmatrix} 0 & \omega^\sigma \\ 1 & 0 \end{pmatrix} = \varepsilon \begin{pmatrix} 0 & \omega \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a' & \lambda'^\sigma \\ -1 & d' \end{pmatrix}^{(\sigma)},$$

giving the equations

$$\lambda^\sigma = -\varepsilon\omega, \quad \omega^\sigma a = \varepsilon\omega d'^{(\sigma)}, \quad d = \varepsilon a'^{(\sigma)}, \quad -\omega = \varepsilon^\sigma \lambda'^{(\sigma)}.$$

If $a', d \neq 0$, then the third equation in combination with the first gives $\omega = -\lambda^\sigma/\varepsilon = -\lambda^\sigma a'^{(\sigma)}/d$, showing that $\omega \in F_{2e}$. If $a' = d = 0$, we must have $a, d' \neq 0$ and then a combination of the first two equations gives $\omega^\sigma = \varepsilon\omega d'^{(\sigma)}/a = -\lambda^\sigma d'^{(\sigma)}/a$, showing again that $\omega \in F_{2e}$. The remaining three cases are similar (and easier) to check. This completes the proof. $\square$

For positive integers x let us define a function h by $h(x) = (p^{2x} - 1)(p^{2x} - 2)/8$. In terms of h and the numbers $\text{orb}_f(e)$, Proposition 10 simply says that $\sum_e \text{orb}_f(e) = h(f)$, where summation is taken over all positive divisors e of f such that f/e is odd. By Proposition 11 we may replace $\text{orb}_f(e)$ with $\text{orb}(e)$ and obtain $\sum_e \text{orb}(e) = h(f)$, with the same summation convention. This miniature but important detail enables us to make a substantial advance in the enumeration.

Let $f = 2^\alpha o$ where o is an odd integer and let e be a divisor of f such that f/e is odd; equivalently, $e = 2^\alpha d$ where d is a positive (and necessarily odd) divisor of o . Taking the above notes into account, Proposition 10 may then be restated as follows:

$$\sum_{d|o} \text{orb}(2^\alpha d) = h(2^\alpha o). \quad (7)$$

Using the Möbius inversion we obtain $\text{orb}(f) = \text{orb}(2^\alpha o) = \sum_{d|o} \mu(o/d)h(2^\alpha d)$, where μ is the classical number-theoretic Möbius function μ on positive integers. We thus arrive at our first main result.

Theorem 2 *Let $q = p^f$ for an odd prime p , let $G = M(q^2)$, and let $f = 2^\alpha o$ with o odd. The number of $\overline{G}$ -orbits of non-singular generating pairs of G is equal to*

$$\sum_{d|o} \mu(o/d)h(2^\alpha d), \quad \text{where } h(x) = (p^{2x} - 1)(p^{2x} - 2)/8.$$

The last step is to study conjugacy of non-singular pairs of $M(q^2)$ under the action of the group $\text{Aut}(M(q^2))$ which, as we know by Proposition 1, is isomorphic to $\text{P}\Gamma\text{L}(2, q^2)$. Since

for $q = p^f$ we have $\text{PTL}(2, q^2) \cong \text{PGL}(2, q^2) \rtimes Z_{2f} \cong \overline{G} \rtimes Z_f$, it is sufficient to investigate the induced action of the Galois automorphisms $\sigma_j : z \mapsto z^{p^j}$ for $z \in F = \text{GF}(p^{2f})$ and $1 \leq j \leq f-1$ on the $\overline{G}$ -orbits of our non-singular pairs $([A, 1], [B, 1])$. We will use the natural notation O^{σ_j} for the σ_j -image of a $\overline{G}$ -orbit O of a pair $([A, 1], [B, 1])$ of elements of G . Note that $\sigma_f = \sigma$, and we also have $O^{\sigma_f} = O$, by the remark made at the beginning of section 7. Clearly, if $O^{\sigma_j} \cap O \neq \emptyset$, then $O^{\sigma_j} = O$.

Proposition 12 *Let O be the orbit of a non-singular pair $([A, 1], [B, 1])$ of elements of G under conjugation in $\overline{G}$ and let j be the smallest positive integer for which $O^{\sigma_j} = O$. If $[A, 1]$ and $[B, 1]$ generate G , then $j = f$.*

Proof. We may assume that $f \geq 2$, otherwise the result is trivial. Suppose that the pair $([A, 1], [B, 1])^{\sigma_j} = ([A, 1]^{\sigma_j}, [B, 1]^{\sigma_j})$ is $\overline{G}$ -conjugate to the pair $([A, 1], [B, 1])$, that is, there exists some $C \in \text{GL}(2, q^2)$ and $i \in Z_2$ such that $[A, 1]^{\sigma_j} = [C, i]^{-1}[A, 1][C, i]$ and $[B, 1]^{\sigma_j} = [C, i]^{-1}[B, 1][C, i]$. It follows that for every $[X, 1] \in \langle [A, 1], [B, 1] \rangle$ we have $[X, 1]^{\sigma_j} = [C, i]^{-1}[X, 1][C, i]$. Using our assumption that $\langle [A, 1], [B, 1] \rangle = G$, we conclude that the above is valid also for $X = \text{dia}(\xi, 1)$, where ξ is a primitive element of $F = \text{GF}(p^{2f})$. Letting C have elements $\alpha, \beta, \gamma, \delta$ in the usual order, the equivalent form $[C, i][\text{dia}(\xi^{p^j}, 1), 1] = [\text{dia}(\xi, 1), 1][C, i]$ of the above equation yields

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} (\xi^{p^j})^{(\sigma)} & 0 \\ 0 & 1 \end{pmatrix} = \varepsilon \begin{pmatrix} \xi & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha^\sigma & \beta^\sigma \\ \gamma^\sigma & \delta^\sigma \end{pmatrix}$$

for some $\varepsilon \in F^*$; here we used the (σ) -convention introduced at the end of section 4. This gives the system of equations

$$\alpha(\xi^{p^j})^{(\sigma)} = \varepsilon \xi \alpha^\sigma, \quad \beta = \varepsilon \xi \beta^\sigma, \quad \gamma(\xi^{p^j})^{(\sigma)} = \varepsilon \gamma^\sigma, \quad \delta = \varepsilon \delta^\sigma.$$

Consider first the case $\delta \neq 0$; without loss of generality we then may assume $\delta = 1$. Then $\varepsilon = 1$, and the equation for α gives $(\xi^{p^j})^{(\sigma)} \xi^{-1} = \alpha^\sigma \alpha^{-1}$, or, equivalently, $\xi^{p^{j+if}-1} = \alpha^{p^f-1}$. It follows that $p^f - 1$ is a divisor of $p^{j+if} - 1$, which implies that f divides $j + if$ and hence f divides j , which, since $j \leq f$, shows that $j = f$. If $\delta = 0$ then, without loss of generality, $\beta = 1$ and so $\varepsilon = \xi^{-1}$. The equation for γ now implies $\xi^{p^{j+if}+1} = \gamma^{p^f-1}$. It follows that $p^f - 1$ divides $p^{j+if} + 1$ and hence also $p^{2(j+if)} - 1$ and so f must divide $2(j + if)$. Thus, f is a divisor of $2j$ and as $j \leq f$, we have either $j = f$ or $j = f/2$ (assuming f is even). But the last case is easily seen to be impossible since $p^f - 1$ is not a divisor of $p^{f/2} + 1$ or $p^{3f/2} + 1$. This completes the proof. $\square$

Proposition 12 tells us that if a non-singular pair $([A, 1], [B, 1])$ of elements of G actually generates G and gives rise to an orbit O under conjugation in $\overline{G}$, then the action of the group $\text{Aut}(M(q^2))$ fuses the f orbits O^{σ_j} for $j \in \{0, 1, \dots, f-1\}$ into a single orbit. Recalling the one-to-one correspondence between isomorphism classes of orientably-regular maps supported by the group $G = M(q^2)$ and orbits of (necessarily non-singular) generating pairs of G under conjugation by $\text{Aut}(G)$, Theorem 2 then immediately implies our second main result.

Theorem 3 *Let $q = p^f$ for an odd prime p and let $f = 2^\alpha o$ with o odd. The number of orbits of non-singular generating pairs of $M(q^2)$ under the action of the group $\text{Aut}(M(q^2))$, and hence the number of isomorphism classes of orientably-regular maps $\mathcal{M}$ with $\text{Aut}^+(\mathcal{M}) \cong M(q^2)$, is equal to*

$$\frac{1}{f} \sum_{d|o} \mu(o/d) h(2^\alpha d), \quad \text{where } h(x) = (p^{2x} - 1)(p^{2x} - 2)/8.$$

11 Enumeration of reflexible maps

A map is called *reflexible* if it admits an automorphism reversing the orientation of the surface. For orientably-regular maps represented by triples (G, x, y) as indicated in the Introduction, reflexivity is equivalent with the existence of an automorphism θ of the group G such that $\theta(x) = x^{-1}$ and $\theta(y) = y^{-1}$. Note that if such a θ exists, then $\theta^2 = \text{id}$.

In the specific situation considered in this paper, namely, when $G = M(q^2)$ for $q = p^f$, we established in Proposition 1 that $\text{Aut}(G) \cong \text{P}\Gamma\text{L}(2, q^2)$. Moreover, it is well known that every automorphism in $\text{P}\Gamma\text{L}(2, q^2) \cong \text{PGL}(2, q^2) \rtimes Z_{2f}$ is a composition of a conjugation by some element of $\text{PGL}(2, q^2)$ and a power of the Frobenius automorphism $z \mapsto z^p$ of the Galois field $F = GF(p^{2f})$. It follows that an *involution* automorphism θ of $G = M(q^2)$ is a composition of a conjugation as above with σ^i for $i \in \{0, 1\}$, where σ is the automorphism of F sending z to z^q . By the remark at the beginning of Section 7, however, the action of σ is equivalent to conjugation in $\overline{G} = G\langle\sigma\rangle$ by the element $[I, 1]$. Consequently, an orientably-regular map on the group $G = M(q^2)$ generated by a pair of elements $x = [A, 1]$ and $y = [B, 1]$ is reflexible if and only if the ordered pairs (x, y) and (x^{-1}, y^{-1}) are conjugate by an *involution* element of $\overline{G}$.

In this section we will count the number of reflexible orientably-regular maps on $M(q^2)$. In particular, we will see that not all orientably-regular maps with automorphism group $M(q^2)$ are reflexible, in contrast with the position for $\text{PGL}(2, q^2)$, see e.g. [5]. We will use techniques similar to the main enumeration in previous sections and structure our explanations accordingly.

11.1 Conjugating involutions

In view of the above findings, an orientably-regular map on $M(q^2)$ given by the generating pair $([A, 1], [B, 1])$ is reflexible if and only if there is an involution $[C, i] \in \overline{G}$ for some $i \in \{0, 1\}$ such that $[C, i][A, 1][C, i] = [A, 1]^{-1}$ and $[C, i][B, 1][C, i] = [B, 1]^{-1}$. As before, we will be dealing separately with the diagonal and off-diagonal cases for the matrix B .

11.1.1 The diagonal case $B = \text{dia}(\lambda, 1)$

We can write $(B^\sigma)^{-1} = \begin{pmatrix} \varepsilon \lambda^{-q} & 0 \\ 0 & \varepsilon \end{pmatrix}$ for some $\varepsilon \neq 0$. We seek a conjugating involution $[C, i]$ as above and begin with $i = 0$. For $[C, 0]$ to be an involution we must have $\text{tr}(C) = 0$ and

we can without loss of generality write $C = \begin{pmatrix} 1 & \beta \\ \gamma & -1 \end{pmatrix}$ or $C = \begin{pmatrix} 0 & \beta \\ 1 & 0 \end{pmatrix}$.

For the first form of C the conjugation equation becomes:

$$\begin{pmatrix} 1 & \beta \\ \gamma & -1 \end{pmatrix} \begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \beta^q \\ \gamma^q & -1 \end{pmatrix} = \begin{pmatrix} \varepsilon \lambda^{-q} & 0 \\ 0 & \varepsilon \end{pmatrix}$$

This leads to the system of equations $\lambda + \beta\gamma^q = \varepsilon\lambda^{-q}$, $\lambda\beta^q - \beta = 0$, $\gamma\lambda - \gamma^q = 0$, and $\beta^q\gamma\lambda + 1 = \varepsilon$. If $\beta = 0$ then $\varepsilon = 1$ and so $\lambda = \lambda^{-q}$ or $\lambda \in {}^{q+1}\sqrt{1}$. If $\beta \neq 0$ then $\lambda = \beta^{1-q}$. In either case, $\lambda \in S(F)$ contrary to the construction of B . Hence there are no conjugating matrices C of this form.

For the second possibility we have:

$$\begin{pmatrix} 0 & \beta \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & \beta^q \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \varepsilon \lambda^{-q} & 0 \\ 0 & \varepsilon \end{pmatrix}$$

This leads to $\beta = \varepsilon\lambda^{-q}$ and $\lambda\beta^q = \varepsilon$, giving potential solutions of this form with β satisfying $\beta^{q-1} = \lambda^{q-1}$.

Now consider the case $i = 1$. Since $[C, 1]$ is an involution, it cannot be in G ; it follows that $CC^\sigma = \varepsilon I$ for some $\varepsilon \neq 0$ and $\det(C) \in S(F)$. Letting $C = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ we may either have $\gamma = 0$ in which case we can take $\alpha = 1$, or $\gamma = 1$, furnishing again two possible forms for C .

The first form of C for $i = 1$ we will consider is $C = \begin{pmatrix} 1 & \beta \\ 0 & \delta \end{pmatrix}$ for some $\delta \in S(F)$. Since $CC^\sigma = \varepsilon I$ we must have $\varepsilon = 1$ and $\delta \in {}^{q+1}\sqrt{1}$. In this case the matrix equations are:

$$\begin{pmatrix} 1 & \beta \\ 0 & \delta \end{pmatrix} \begin{pmatrix} \lambda^q & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \beta \\ 0 & \delta \end{pmatrix} = \begin{pmatrix} \varepsilon \lambda^{-q} & 0 \\ 0 & \varepsilon \end{pmatrix}$$

It follows that $\lambda^q = \varepsilon\lambda^{-q}$ and $\delta^2 = \varepsilon$, implying $\delta^2 = \lambda^{2q}$, but as $\delta \in S(F)$ and $\lambda \in N(F)$ this is impossible. We conclude that the first form cannot occur.

The second form of C for $i = 1$ to consider is $C = \begin{pmatrix} \alpha & \beta \\ 1 & \delta \end{pmatrix}$ with $\alpha\delta - \beta \in S(F)$. The requirement for $[C, 1]$ to have order two gives:

$$\begin{pmatrix} \alpha & \beta \\ 1 & \delta \end{pmatrix} \begin{pmatrix} \alpha^q & \beta^q \\ 1 & \delta^q \end{pmatrix} = \begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon \end{pmatrix}$$

Thus, $\delta = -\alpha^q$ and C has the form $\begin{pmatrix} \alpha & \beta \\ 1 & -\alpha^q \end{pmatrix}$. The conjugation equation then becomes:

$$\begin{pmatrix} \alpha & \beta \\ 1 & -\alpha^q \end{pmatrix} \begin{pmatrix} \lambda^q & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ 1 & -\alpha^q \end{pmatrix} = \begin{pmatrix} \varepsilon \lambda^{-q} & 0 \\ 0 & \varepsilon \end{pmatrix}$$

Now, $\alpha\lambda^q - \alpha^q = 0$. If $\alpha \neq 0$, then $\lambda^q = \alpha^{q-1} \in S(F)$, which is impossible. So $\alpha = 0$ and C must have the form $\begin{pmatrix} 0 & \beta \\ 1 & 0 \end{pmatrix}$ for some β . Since $CC^\sigma = \varepsilon I$ we must have $\beta = \beta^q$ and hence $\beta \in F'^*$.

11.1.2 The off-diagonal case $B = \text{off}(\lambda, 1)$

We can write $(B^\sigma)^{-1} = \begin{pmatrix} 0 & \varepsilon \\ \varepsilon\lambda^{-q} & 0 \end{pmatrix}$ for some $\varepsilon \neq 0$. We begin with the case $i = 0$, and as before we can without loss of generality set $C = \begin{pmatrix} 1 & \beta \\ \gamma & -1 \end{pmatrix}$ or $C = \begin{pmatrix} 0 & \beta \\ 1 & 0 \end{pmatrix}$. For the first form of C we have:

$$\begin{pmatrix} 1 & \beta \\ \gamma & -1 \end{pmatrix} \begin{pmatrix} 0 & \lambda \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & \beta^q \\ \gamma^q & -1 \end{pmatrix} = \begin{pmatrix} 0 & \varepsilon \\ \varepsilon\lambda^{-q} & 0 \end{pmatrix}$$

It follows that $\beta + \gamma^q\lambda = 0$, $\beta^{q+1} - \lambda = \varepsilon$, $\gamma^{q+1}\lambda - 1 = \varepsilon\lambda^{-q}$ and $-\beta^q - \gamma\lambda = 0$. If $\beta = 0$ then $\gamma = 0$ and so $\varepsilon = -\lambda$ and $\lambda^{1-q} = 1$. If $\beta, \gamma \neq 0$, then $\beta + \gamma^q\lambda^q = 0$, giving $\lambda = \lambda^q$. In either case we conclude that $\lambda \in S(F)$, contrary to the construction of B . Consequently, there are no matrices C of this form.

For the possibility $C = \begin{pmatrix} 0 & \beta \\ 1 & 0 \end{pmatrix}$ we have:

$$\begin{pmatrix} 0 & \beta \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & \lambda \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & \beta^q \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \varepsilon \\ \varepsilon\lambda^{-q} & 0 \end{pmatrix}$$

This leads to $\beta^{q+1} = \varepsilon$, $\lambda = \varepsilon\lambda^{-q}$ and we have potential solutions of this form with β satisfying $\beta^{q+1} = \lambda^{q+1}$.

Now consider the second case $i = 1$. As before the first possible form for C is $\begin{pmatrix} 1 & \beta \\ 0 & \delta \end{pmatrix}$ for some $\delta \in {}^{q+1}\sqrt{1}$. In this case the matrix equations are:

$$\begin{pmatrix} 1 & \beta \\ 0 & \delta \end{pmatrix} \begin{pmatrix} 0 & \lambda^q \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & \beta \\ 0 & \delta \end{pmatrix} = \begin{pmatrix} 0 & \varepsilon \\ \varepsilon\lambda^{-q} & 0 \end{pmatrix}$$

We must have $\beta = 0$ and $\delta = \varepsilon\lambda^{-q}$, giving potential solutions of this form for any $\delta \in {}^{q+1}\sqrt{1}$.

The second possible form of C for $i = 1$ is $C = \begin{pmatrix} \alpha & \beta \\ 1 & -\alpha^q \end{pmatrix}$ for which the equations are:

$$\begin{pmatrix} \alpha & \beta \\ 1 & -\alpha^q \end{pmatrix} \begin{pmatrix} 0 & \lambda^q \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ 1 & -\alpha^q \end{pmatrix} = \begin{pmatrix} 0 & \varepsilon \\ \varepsilon\lambda^{-q} & 0 \end{pmatrix}$$

Here we have $\alpha\beta + \alpha\lambda^q = 0$, $\beta^2 - \alpha^{q+1}\lambda^q = \varepsilon$, $-\alpha^{q+1} + \lambda^q = \varepsilon\lambda^{-q}$, and $-\alpha^q\beta - \alpha^q\lambda^q = 0$. If $\alpha \neq 0$, then $\lambda^q = -\beta$ but $\beta \in F'^*$, so $-\beta \in S(F)$ and this is impossible. It follows that $\alpha = 0$ but then $\varepsilon = \beta^2 = \lambda^{2q}$, while at the same time $\beta \in S(F)$ and $\lambda^q \in N(F)$, a contradiction. Hence there are no matrices C of this form.

11.1.3 Summary of possible conjugating elements

In the case $B = \text{dia}(\lambda, 1)$ the possible conjugating elements can have the following forms.

$$[C, 0]; \quad C = \begin{pmatrix} 0 & \beta \\ 1 & 0 \end{pmatrix}; \quad \beta^{q-1} = \lambda^{q-1} \quad (8)$$

$$[C, 1]; \quad C = \begin{pmatrix} 0 & \beta \\ 1 & 0 \end{pmatrix}; \quad \beta \in F'^* \quad (9)$$

In the case $B = \text{off}(\lambda, 1)$ the possible conjugating elements can have the following forms.

$$[C, 0]; \quad C = \begin{pmatrix} 0 & \beta \\ 1 & 0 \end{pmatrix}; \quad \beta^{q+1} = \lambda^{q+1} \quad (10)$$

$$[C, 1]; \quad C = \begin{pmatrix} 1 & 0 \\ 0 & \delta \end{pmatrix}; \quad \delta \in {}^{q+1}\sqrt{1} \quad (11)$$

11.2 Enumeration

We now proceed to the actual enumeration and follow the same strategy as we used for general maps, namely, counting orbits in the diagonal and off-diagonal cases for B as in Sections 8 and 9 and then deriving the final enumeration result using the Möbius inversion formula as in Section 10.

11.2.1 Counting orbits: The case $B = \text{dia}(\lambda, 1)$

In this case we have $A = \begin{pmatrix} -1 & b \\ c & \lambda^q \end{pmatrix}$. We can write $[A, 1]^{-1}$ as $[(A^\sigma)^{-1}, 1]$ where $(A^\sigma)^{-1}$ has the form $\varepsilon \begin{pmatrix} -\lambda & b^q \\ c^q & 1 \end{pmatrix}$ for some $\varepsilon \neq 0$.

The first possibility is that the conjugating element has the form $[C, 0]$ with $C = \begin{pmatrix} 0 & \beta \\ 1 & 0 \end{pmatrix}$ for some β such that $\beta^{q-1} = \lambda^{q-1}$. In this case the equations are:

$$\begin{pmatrix} 0 & \beta \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -1 & b \\ c & \lambda^q \end{pmatrix} \begin{pmatrix} 0 & \beta^q \\ 1 & 0 \end{pmatrix} = \varepsilon \begin{pmatrix} -\lambda & b^q \\ c^q & 1 \end{pmatrix}$$

After some manipulation the resulting equations lead to $\beta = -b^q/c$; $\beta^{q-1} = \lambda^{q-1}$. Another way to express the second equation is $(bc)^{1-q} = \lambda^{q-1}$ or $(bc\lambda)^{q-1} = 1$ so that $bc\lambda \in F'^*$.

Since $u = bc + \lambda^q \in N(F) \setminus \{\lambda^q\}$ we can count the possible values of u by noting that $u\lambda = bc\lambda + \lambda^{q+1}$. So $u\lambda \in F'^*$ but we must have $u \neq \lambda^q$ and therefore there are exactly $q-2$ choices for u . For each choice of u we then set $bc = u - \lambda^q$ and then there are $q^2 - 1$ choices for b and c is determined.

For given $\lambda = \xi^i$, the number of matrices A satisfying the equations is then $(q-2)(q^2-1)$. The $\overline{G}$ -orbits have length $2(q-1)$ if $i < (q-1)/2$ and $4(q-1)$ if $i = (q-1)/2$. So the number of orbits of generating pairs of this form for each i is:

$$r_{1,i} = \begin{cases} (q-2)\frac{q+1}{2} & \text{for } \lambda = \xi^i, i < (q-1)/2 \\ (q-2)\frac{q+1}{4} & \text{for } \lambda = \xi^i, i = (q-1)/2 \end{cases} \quad (12)$$

The total number of orbits is therefore

$$r_1 = \sum_{i \leq (q-1)/2} r_{1,i} = \frac{(q^2-1)(q-2)}{8} \quad (13)$$

The second possibility is that the conjugating element has the form $[C, 1]$ with $C = \begin{pmatrix} 0 & \beta \\ 1 & 0 \end{pmatrix}$ for some $\beta \in F'^*$. In this case the equations are:

$$\begin{pmatrix} 0 & \beta \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -1 & b^q \\ c^q & \lambda \end{pmatrix} \begin{pmatrix} 0 & \beta \\ 1 & 0 \end{pmatrix} = \varepsilon \begin{pmatrix} -\lambda & b^q \\ c^q & 1 \end{pmatrix}$$

The resulting equations become $\beta = -b^q/c^q$; $\beta \in F'^*$; in particular, $b/c \in F'^*$. Now note that $bc = c^2b/c \in S(F)$ and so the number of possible values for $u = bc + \lambda^q$ is just $n_F = (q^2-1)/4$ from before. It follows that there are $(q^2-1)/4$ possible values for the product bc and we can count the number of ways we can choose b .

Since $b/c \in F'^*$ we can write $x = b/c$, $y = bc$ and there are $q-1$ choices for x . For each choice we need b to satisfy the equations $x = b/c$, $y = bc$ so $xy = b^2$. Knowing that $xy \in S(F)$, there are exactly two choices for b and so the total number of matrices A satisfying the equations for a given λ is $(q-1)(q^2-1)/2$.

Then, given the orbit lengths as before, the number of orbits of pairs of this form for each i is:

$$r_{2,i} = \begin{cases} \frac{q^2-1}{4} & \text{for } \lambda = \xi^i, i < (q-1)/2 \\ \frac{q^2-1}{8} & \text{for } \lambda = \xi^i, i = (q-1)/2 \end{cases} \quad (14)$$

$$r_2 = \sum_{i \leq (q-1)/2} r_{2,i} = \frac{(q^2-1)(q-1)}{16} \quad (15)$$

Note also that in the set of matrices counted by r_1 above, we have $bc \in N(F)$ and for those counted by r_2 we have $bc \in S(F)$, so that the sets are disjoint.

11.2.2 Counting orbits: The case $B = \text{off}(\lambda, 1)$

In this case we have $A = \begin{pmatrix} a & \lambda^q \\ -1 & d \end{pmatrix}$. We can write $[A, 1]^{-1}$ as $[(A^\sigma)^{-1}, 1]$ where $(A^\sigma)^{-1}$ has the form $\varepsilon \begin{pmatrix} d^q & -\lambda \\ 1 & a^q \end{pmatrix}$ for some $\varepsilon \neq 0$.

The first possibility is that the conjugating element has the form $[C, 0]$ with $C = \begin{pmatrix} 0 & \beta \\ 1 & 0 \end{pmatrix}$ for some β such that $\beta^{q+1} = \lambda^{q+1}$. In this case the equations are:

$$\begin{pmatrix} 0 & \beta \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a & \lambda^q \\ -1 & d \end{pmatrix} \begin{pmatrix} 0 & \beta^q \\ 1 & 0 \end{pmatrix} = \varepsilon \begin{pmatrix} d^q & -\lambda \\ 1 & a^q \end{pmatrix}$$

It follows that $\beta d = \varepsilon d^q$, $-\beta^{q+1} = -\varepsilon \lambda$, $\lambda^q = \varepsilon$, and $a\beta^q = \varepsilon a^q$. So either $ad = 0$ or $(\lambda ad)^{q-1} = 1$. If $ad \neq 0$ then as before there are exactly $q-2$ choices for $u = ad + \lambda^q$, each with $q^2 - 1$ choices for a and then d is determined. If $ad = 0$ there are $q^2 - 1$ choices for a if $d = 0$ and $q^2 - 1$ choices for d if $a = 0$.

Thus for a given $\lambda = \xi^i$ there are $q(q^2 - 1)$ matrices A satisfying the equations. The orbit lengths are $2(q+1)$ if $i < (q+1)/2$ and $4(q+1)$ if $i = (q+1)/2$. So the number of orbits of this form for each i is:

$$r_{3,i} = \begin{cases} \frac{q(q-1)}{2} & \text{for } \lambda = \xi^i, i < (q+1)/2 \\ \frac{q(q-1)}{4} & \text{for } \lambda = \xi^i, i = (q+1)/2 \end{cases} \quad (16)$$

$$r_3 = \sum_{i \leq (q+1)/2} r_{3,i} = \frac{q(q^2 - 1)}{8} \quad (17)$$

The remaining possibility is that the conjugating element has the form $[C, 1]$ with $C = \begin{pmatrix} 1 & 0 \\ 0 & \delta \end{pmatrix}$ for some δ such that $\delta^{q+1} = 1$. In this case the equations are:

$$\begin{pmatrix} 1 & 0 \\ 0 & \delta \end{pmatrix} \begin{pmatrix} a^q & \lambda \\ -1 & d^q \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \delta \end{pmatrix} = \varepsilon \begin{pmatrix} d^q & -\lambda \\ 1 & a^q \end{pmatrix}$$

This leads to $a^q = \varepsilon d^q$, $\lambda \delta = -\lambda \varepsilon$, $-\delta = \varepsilon$, and $\delta^2 d^q = \varepsilon a^q$. So $\delta = -a^q/d^q$ and $\delta \in {}^{q+1}\sqrt{1}$. Now $ad \in S(F)$ since $a^q/d^q = -\delta \in S(F)$, which implies that the number of possible $u = ad + \lambda^q$ is, as before, $n_F = (q^2 - 1)/4$. To proceed, write $x = a/d$, $y = ad$ and then $x^{q(q+1)} = 1$. This gives $x \in {}^{q+1}\sqrt{1}$ and so there are $q+1$ choices for x and for each x there are $(q^2 - 1)/4$ choices for y , with still having to choose d to satisfy $d^2 = y/x$. Since we know $y/x \in S(F)$ there are 2 choices for d .

Thus, for a given $\lambda = \xi^i$, there are $(q+1)(q^2 - 1)/2$ matrices A satisfying the above equations. The orbit lengths are as before, so the number of orbits of this form for each i

is:

$$r_{4,i} = \begin{cases} \frac{q^2 - 1}{4} & \text{for } \lambda = \xi^i, i < (q+1)/2 \\ \frac{q^2 - 1}{8} & \text{for } \lambda = \xi^i, i = (q+1)/2 \end{cases} \quad (18)$$

$$r_4 = \sum_{i \leq (q+1)/2} r_{4,i} = \frac{(q+1)(q^2 - 1)}{16} \quad (19)$$

Again, the sets of matrices counted by r_3 and r_4 are disjoint since in the first case $ad \in N(F)$ and in the second $ad \in S(F)$.

11.2.3 Summary of counting orbits

For $B = \text{dia}(\lambda, 1)$ the total number of $\overline{G}$ -orbits of generating pairs for reflexible maps is

$$R_1 = r_1 + r_2 = \frac{(q^2 - 1)(3q - 5)}{16} \quad (20)$$

For $B = \text{off}(\lambda, 1)$ the total number of such orbits is

$$R_2 = r_3 + r_4 = \frac{(q^2 - 1)(3q + 1)}{16} \quad (21)$$

The total number of orbits is therefore:

$$R = R_1 + R_2 = \frac{(q^2 - 1)(3q - 2)}{8} \quad (22)$$

11.3 Counting reflexible maps

We may enumerate the orientably-regular reflexible maps on $M(q^2)$ by using the above calculations in conjunction with the logic of Section 10. Since details of this process are exactly as in Section 10 except for using the input on counting orbits from Subsection 11.2.3 we present just the final result.

Theorem 4 *Let $q = p^f$ be an odd prime power, with $f = 2^\alpha o$ where o is odd. The number of orientably-regular reflexible maps $\mathcal{M}$ with $\text{Aut}^+(\mathcal{M}) \cong M(q^2)$ is, up to isomorphism, equal to*

$$\frac{1}{f} \sum_{d|o} \mu(o/d) \tilde{h}(2^\alpha d) ,$$

where $\tilde{h}(x) = (p^{2x} - 1)(3p^x - 2)/8$ and μ is the Möbius function.

12 Remarks

As stated in the Introduction, orientably-regular maps have been enumerated for very sparse classes of non-trivial groups, and in terms of *all* such maps (not just maps of restricted types) only for almost-Sylow-cyclic groups [4] and the linear fractional groups $\text{PSL}(2, q)$ and $\text{PGL}(2, q)$ [5, 22]. It should be noted, however, that the available results for $\text{PSL}(2, q)$ and $\text{PGL}(2, q)$ are more detailed by giving ‘closed formulae’ for the number of orientably-regular maps of every given type, whereas our main results in Theorems 3 and 4 contain formulae for the total number of such maps.

In order to obtain a refined version of our enumeration of orientably-regular maps with automorphism group isomorphic to a twisted linear fractional group $G = M(q^2)$ one could follow [12], which requires setting up both a character table for G and the Möbius function for the lattice of subgroups of G . The number of orientably-regular maps on the group G is then obtained as a combination of a character-theoretic formula for counting solutions of the equation $xyz = 1$ for x, y, z in given conjugacy classes of G (a special case of a general formula of Frobenius [9]) combined with Möbius inversion, which is a forthcoming project of the authors. Whether the project will return a ‘nice’ formula, however, is not clear due to another significant difference between the family of orientably-regular maps on $M(q^2)$ compared to those on $\text{PGL}(2, q)$. Namely, in the case of $\text{PGL}(2, q)$, for any even $k, \ell \geq 4$ not both equal to 4 there is an orientably-regular map for infinitely many values of q , cf. [5]. Our next result shows that this fails to hold in the case of $M(q^2)$.

Proposition 13 *If $k, \ell \equiv 0 \pmod{8}$ and $k \not\equiv \ell \pmod{16}$ then there is no orientably-regular map of type (k, ℓ) on $M(q^2)$ for any q .*

Proof. By Theorem 1 orders of elements in $G \setminus G_0$ are $o_i = 2(q-1)/\gcd\{q-1, i\}$ and $o'_i = 2(q+1)/\gcd\{q+1, i\}$ for odd i such that $1 \leq i \leq (q-1)/2$ and $1 \leq i \leq (q+1)/2$, respectively. Note that if $o_i \equiv 0 \pmod{8}$ then $o'_i \equiv 4 \pmod{8}$ and vice versa. Further, if $o_i \equiv 8 \pmod{16}$ then $q-1 \equiv 4 \pmod{8}$ since i is odd, and if $o_i \equiv 0 \pmod{16}$ then $q-1 \equiv 0 \pmod{8}$. It follows that for a given q we cannot have a non-singular generating pair of orders $o_i \equiv 0 \pmod{16}$ and $o_j \equiv 8 \pmod{16}$. The argument for orders of the form o'_i is similar. $\square$

Besides reflexivity, another frequently studied property of orientably-regular maps is self-duality. In general, an oriented map is *positively self-dual* if it is isomorphic to its dual with the same orientation, and *negatively self-dual* if it is isomorphic to its oppositely oriented dual map. In terms of orientably-regular maps represented by triples (G, x, y) , positive and negative self-duality is equivalent to the existence of an (involutory) automorphism of G sending the ordered pair (x, y) onto (y, x) and (y^{-1}, x^{-1}) , respectively. By the same arguments as in the second paragraph of Section 11 one concludes that for our group $G = M(q^2)$, an orientably-regular map defined by a generating pair $[A, 1], [B, 1]$ will be positively self-dual if and only if there exists an involution $[C, i] \in \overline{G}$ conjugating the two generators, and the map will be negatively self-dual if there is such an involution conjugating $[A, 1]$ to $[B, 1]^{-1}$.

Setting up the corresponding matrix equations for such conjugations, however, lead to enormously complicated formulae from which we were not able to extract ‘nice’ closed formulae. Clearly in a self-dual map we have $k = \ell$ so that the orders of $[A, 1]$ and $[B, 1]$ must be equal. We used GAP[10] to construct all the regular maps on $M(q^2)$, for small values of q , with the generators of equal order, and then tested for self-duality by determining if a conjugating element $[C, i]$ as above exists. The results of this computation are given in Table 1, showing the numbers $\#(k=\ell)$ of maps that have generators of equal orders, those which are positively or negatively self-dual, and those which are both.

q	$B = \text{dia}(\lambda, 1)$				$B = \text{off}(\lambda, 1)$			
	$\#(k=\ell)$	+ self-dual	– self-dual	both	$\#(k=\ell)$	+ self-dual	– self-dual	both
3	0	0	0	0	3	3	3	3
5	15	15	5	5	10	10	6	6
7	28	28	8	8	78	42	14	14
9	95	45	9	9	68	36	10	10
11	276	132	24	24	265	165	33	33
13	469	273	39	39	666	234	42	42
17	2556	612	68	68	1312	544	72	72
19	1960	760	80	80	2799	855	95	95

Table 1: Numbers of self-dual maps on $M(q^2)$

Note that the computational evidence suggests that a negatively self-dual map on $M(q^2)$ is also positively self-dual.

Acknowledgement. The authors thank Marston Conder for independently checking our computational results displayed in Table 1 using the MAGMA package.

The last two authors gratefully acknowledge support of this work by the research grants APVV 0136/12, APVV-15-0220, VEGA 1/0007/14, VEGA 1/0026/16, and VEGA 1/0142/17.

References

- [1] Y. F. Ban, S. F. Du, Y. Liu, A. Malnič, R. Nedela and M. Škoviera, Regular maps with nilpotent automorphism groups, Preprint, 2012.
- [2] P. J. Cameron, G. R. Omidi, B. Tayfeh-Rezaie, 3-designs from $\text{PGL}(2, q)$, Electronic J. Combin. 13 (2006), Art. R50.
- [3] M. Conder, List of orientably-regular chiral and reflexible maps up to genus 301, available at <https://www.math.auckland.ac.nz/~conder/> (2012).

- [4] M. Conder, P. Potočnik and J. Širáň, Regular maps with almost Sylow-cyclic automorphism groups, and classification of regular maps with Euler characteristic $-p^2$, *J. Algebra* 324 (2010), 2620-2635.
- [5] M. Conder, P. Potočnik and J. Širáň, Regular hypermaps over projective linear groups, *J. Australian Math. Soc.* 85 (2008), 155-175.
- [6] M. Conder, J. Širáň and T. Tucker, The genera, reflexivity and simplicity of regular maps, *J. Europ. Math. Soc.* 12 (2010), 343-364.
- [7] M. Conder and T. Tucker, Regular Cayley maps for cyclic groups, *Trans. Amer. Math. Soc.* 366 (2014) 7, 3585–3609.
- [8] H. S. M. Coxeter and W. O. J. Moser, ‘Generators and Relations for Discrete Groups’, Springer, 1980.
- [9] F. G. Frobenius, Über Gruppencharaktere, *Sitzber. Königlich Preuss. Akad. Wiss. Berlin* (1896), 985–1021.
- [10] The GAP Group, GAP – Groups, Algorithms, and Programming, Version 4.7.8; 2015. (<http://www.gap-system.org>)
- [11] M. Giudici, Maximal subgroups of almost simple groups with socle $\text{PSL}(2, q)$, *arXiv:math/0703685* (2007).
- [12] G. A. Jones, Combinatorial categories and permutation groups, *Ars Math. Contemp.* 10 (2016) 2, 237–254.
- [13] G. A. Jones, Ree groups and Riemann surfaces. *J. Algebra* **165** (1994), 41-62.
- [14] G. A. Jones and S. A. Silver, Suzuki groups and surfaces, *J. London Math. Soc.* (2) 48 (1993), 117-125.
- [15] G. A. Jones and D. Singerman, Theory of maps on orientable surfaces, *Proc. London Math. Soc.* (3) 37 (1978) 2, 273-307.
- [16] G. A. Jones and D. Singerman, Belyi functions, hypermaps and Galois groups, *Bull. London Math. Soc.* 28 (1996) 6, 561-590.
- [17] J. B. Kelly, A characteristic property of quadratic residues, *Proc. Amer. Math. Soc.* 5 (1954), 38-46.
- [18] A. M. Macbeath, Generators of the linear fractional groups, in: 1969 Number Theory (Proc. Sympos. Pure Math., Vol. XII, Houston, Tex.), Amer. Math. Soc., Providence, R.I., 1967, 14–32.
- [19] R. Nedela, Regular maps - combinatorial objects relating different fields of mathematics, *J. Korean Math. Soc.* 38 (2001) 5, 1069-1105.

- [20] D. Leemans and J. de Saadeleer, On the rank two geometries of the groups $\mathrm{PSL}(2, q)$: part I, *Ars Math. Contemp.* 3 (2010), 177–192.
- [21] J. Širáň, How symmetric can maps on surfaces be? In: *Surveys in Combinatorics* (London Mathematical Society Lecture Note Series 409), Cambridge University Press, 2013, 161–238.
- [22] C. H. Sah, Groups related to compact Riemann surfaces, *Acta Math.* 123 (1969), 13–42.
- [23] H. Zassenhaus, Über endliche Fastkörper, *Abh. Math. Sem. Univ. Hamburg* 11 (1936), 187–220.