Colourings of planar quasicrystals

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Colourings of Planar Quasicrystals

Michael Baake[(1)], Uwe Grimm[(2)], and Max Scheffer[(3)]

(1) Institut für Mathematik, Universität Greifswald,
Jahnstr. 15a, 17487 Greifswald, Germany
(2) Applied Mathematics Department, The Open University,
Walton Hall, Milton Keynes MK7 6AA, UK
(3) Institut für Physik, Technische Universität,
09107 Chemnitz, Germany

Abstract

The investigation of colour symmetries for periodic and aperiodic systems consists of two steps. The first concerns the computation of the possible numbers of colours and is mainly combinatorial in nature. The second is algebraic and determines the actual colour symmetry groups. Continuing previous work, we present the results of the combinatorial part for planar patterns with n-fold symmetry, where $n = 7, 9, 15, 16, 20, 24$. This completes the cases with $\phi(n) \leq 8$, where ϕ is Euler's totient function.

Colour symmetries of crystals and, more recently, of quasicrystals continue to attract a lot of attention, simply because so little is known about their classification, see [4] for a recent review. A first step in this analysis consists in answering the question how many different colourings of an infinite point set exist which are compatible with its underlying symmetry. More precisely, one has to determine the possible numbers of colours, and to count the corresponding possibilities to distribute the colours over the point set (up to permutations).

In this generality, the problem has not even been solved for simple lattices. One common restriction is to demand that one colour occupies a subset which is of the same Bravais type as the original set, while the other colours encode the cosets. In this situation, to which we will also restrict ourselves, several results are known and can be given in closed form [1, 2, 3, 4, 5]. Of particular interest are planar cases because, on the one hand, they show up in quasicrystalline T-phases, and, on the other hand, they are linked to the rather interesting classification of planar Bravais classes with n-fold symmetry [6]. They are unique for the following 29 choices of n:

$$\begin{aligned} n \;&=\; 3,4,5,7,8,9,11,12,13,15,16,17,19,20,21,24,\\ &\quad\;\; 25,27,28,32,33,35,36,40,44,45,48,60,84. \end{aligned}$$

The canonical representatives are the sets of cyclotomic integers $\mathcal{M}_n = \mathbb{Z}[\xi_n]$ where $\xi_n = \exp(2\pi i/n)$ with n from our list. Except $n = 1$ (where $\mathcal{M}_1 = \mathbb{Z}$ is one-dimensional), these are all cases where $\mathbb{Z}[\xi_n]$ is a principal ideal domain and thus has class number one, see [7, 3] for details. If n is odd, we have $\mathcal{M}_n = \mathcal{M}_{2n}$ and $\mathcal{M}_n$ thus has $2n$-fold symmetry. To avoid duplication of results, values $n \equiv 2 \pmod 4$ do not appear in the above list. There are systematic mathematical reasons to prefer this traditional convention to the notation of [4, 6].

It is this very connection to algebraic number theory which gives the Bravais classification [6], and also allows for a solution of the combinatorial part of the colouring problem by means of Dirichlet series generating functions. The latter is explained in [1], where the solutions for all n with $\phi(n) \leq 4$ and for $n = 7$ are given explicitly. Here, $\phi(n)$ is Euler's totient function which is the number of integers k, $1 \leq k \leq n$, which are coprime with n. Note that $\phi(n) = 2$ are the crystallographic cases $n = 3$ (triangular lattice) and $n = 4$ (square lattice), while $\phi(n) = 4$ means $n \in \{5, 8, 12\}$ which are the standard symmetries of planar quasicrystals. Again, $n = 10$ is covered implicitly, as explained above.

The methods emerging from this connection allow the full treatment of all 29 cases listed above, and this will be spelled out in detail in a forthcoming publication. Here, we present the results for $\phi(n) = 6$ (i.e., $n \in \{7, 9\}$) and for $\phi(n) = 8$ (i.e., $n \in \{15, 16, 20, 24\}$), thus completing the cases with $\phi(n) \leq 8$, where partial results had been given in [1, 4].

Let us consider $\mathcal{M}_n$ with a fixed n from our list. Then, $\mathcal{M}_n$ is an Abelian group, and also a $\mathbb{Z}$-module of rank $\phi(n)$. We view it as a subset of the Euclidean plane, and hence as a geometric object. Our combinatorial problem is now to determine the values of the multiplicative arithmetic function $a_n(k)$ which counts the possibilities to colour an n-fold symmetric submodule of $\mathcal{M}_n$ and its cosets with k different colours, see [1, 4] for details. These submodules are then necessarily principal ideals of $\mathcal{M}_n = \mathbb{Z}[\xi_n]$, i.e., sets of the form $g\mathcal{M}_n$ with $g \in \mathcal{M}_n$.

On the other hand, for the n from the above list, *all* ideals of $\mathcal{M}_n$ are principal, so our combinatorial problem amounts to counting all (non-zero) ideals, which is achieved by the Dedekind zeta-function of the cyclotomic field $\mathbb{Q}(\xi_n)$. The number of colours then corresponds to the norm of the ideal [3], and we obtain the following Dirichlet series generating function

$$\zeta_{\mathcal{M}_n}(s) \;:=\; \sum_{k=1}^{\infty} \frac{a_n(k)}{k^s} \;=\; \zeta_{\mathbb{Q}(\xi_n)}(s) \;=\; \prod_{p \text{ prime}} E(p^{-s}) \,. \tag{1}$$

The last expression is called the Euler product expansion of the Dirichlet series. Each Euler factor is of the form

$$E(p^{-s}) \;=\; \frac{1}{(1 - p^{-\ell s})^m} \;=\; \sum_{k=0}^{\infty} \binom{k+m-1}{m-1} \frac{1}{(p^s)^{k\ell}} \tag{2}$$

from which one deduces the value of $a_n(p^r)$ for $r \geq 0$. The indices ℓ, m depend on the prime p, and on the choice of n.

$\phi(n)$	n	special p	general p
2	3	3	p_1, p_2^2
	4	2	p_1, p_3^2
4	5	5	p_1, p_2^4, p_3^4, p_4^2
	8	2	p_1, p_3^2, p_5^2, p_7^2
	12	2^2, 3^2	p_1, p_5^2, p_7^2, p_{11}^2
6	7	7	p_1, p_2^3, p_3^6, p_4^3, p_5^6, p_6^2
	9	3	p_1, p_2^6, p_4^3, p_5^6, p_7^3, p_8^2
8	15	3^4, 5^2	p_1, p_2^4, p_4^2, p_7^4, p_8^4, p_{11}^2, p_{13}^4, p_{14}^2
	16	2	p_1, p_3^4, p_5^4, p_7^2, p_9^2, p_{11}^4, p_{13}^4, p_{15}^2
	20	2^4, $5\,[m=2]$	p_1, p_3^4, p_7^4, p_9^2, p_{11}^2, p_{13}^4, p_{17}^4, p_{19}^2
	24	2^2, $3^2\,[m=2]$	p_1, p_5^2, p_7^2, p_{11}^2, p_{13}^2, p_{17}^2, p_{19}^2, p_{23}^2

Table 1: Basic indices for the numbers of colours. Composite numbers (i.e., possible numbers of colours) are all products of basic indices.

If $p \equiv k \pmod{n}$ with k and n coprime, then $\ell \cdot m = \phi(n)$. Such primes are listed as p_k^ℓ in Table 1. In addition, there are finitely many primes p which divide n, the so-called ramified primes. They are listed as p^ℓ in Table 1, and here we have $m = 1$ except for two cases (where $m = 2$). With this information, one can easily calculate the possible numbers of colours and the generating functions by inserting (2) into (1) and expanding the Euler product.

The result for $n = 7$ was already given in equation (11) of reference [1]. The Euler product is correct, but the explicit first few terms contain one misprint, which we correct here

$$\zeta_{\mathcal{M}_7}(s) \;=\; 1 + \tfrac{1}{7^s} + \tfrac{2}{8^s} + \tfrac{6}{29^s} + \tfrac{6}{43^s} + \tfrac{1}{49^s} + \tfrac{2}{56^s} + \tfrac{3}{64^s} + \tfrac{6}{71^s} + \tfrac{6}{113^s} + \tfrac{6}{127^s} + \tfrac{3}{169^s} + \dots$$

The case $n = 9$, in full detail, reads

$$\begin{aligned}
\zeta_{\mathcal{M}_9}(s) \;&=\; \frac{1}{1-3^{-s}} \prod_{p\equiv 1 \bmod 9} \frac{1}{(1-p^{-s})^6} \prod_{p\equiv 2,5 \bmod 9} \frac{1}{1-p^{-6s}} \\
&\quad\cdot \prod_{p\equiv 4,7 \bmod 9} \frac{1}{(1-p^{-3s})^2} \prod_{p\equiv 8 \bmod 9} \frac{1}{(1-p^{-2s})^3} \\
&=\; 1 + \tfrac{1}{3^s} + \tfrac{1}{9^s} + \tfrac{6}{19^s} + \tfrac{1}{27^s} + \tfrac{6}{37^s} + \tfrac{6}{57^s} + \tfrac{1}{64^s} + \tfrac{6}{73^s} + \tfrac{1}{81^s} + \tfrac{6}{109^s} + \tfrac{6}{111^s} + \dots
\end{aligned}$$

The remaining cases can be calculated in the same way, using the data of Table 1. Here, we just spell out the first few terms of the Dirichlet series for the four solutions of $\phi(n) = 8$:

$$
\begin{aligned}
\zeta_{\mathcal{M}_{15}}(s) &= 1 + \tfrac{2}{16^s} + \tfrac{1}{25^s} + \tfrac{8}{31^s} + \tfrac{8}{61^s} + \tfrac{1}{81^s} + \tfrac{4}{121^s} + \tfrac{8}{151^s} + \tfrac{8}{181^s} + \tfrac{8}{211^s} + \tfrac{8}{241^s} + \dots \\
\zeta_{\mathcal{M}_{16}}(s) &= 1 + \tfrac{1}{2^s} + \tfrac{1}{4^s} + \tfrac{1}{8^s} + \tfrac{1}{16^s} + \tfrac{8}{17^s} + \tfrac{1}{32^s} + \tfrac{8}{34^s} + \tfrac{4}{49^s} + \tfrac{1}{64^s} + \tfrac{8}{68^s} + \tfrac{2}{81^s} + \dots \\
\zeta_{\mathcal{M}_{20}}(s) &= 1 + \tfrac{2}{5^s} + \tfrac{1}{16^s} + \tfrac{3}{25^s} + \tfrac{8}{41^s} + \tfrac{8}{61^s} + \tfrac{2}{80^s} + \tfrac{2}{81^s} + \tfrac{8}{101^s} + \tfrac{4}{121^s} + \tfrac{4}{125^s} + \dots \\
\zeta_{\mathcal{M}_{24}}(s) &= 1 + \tfrac{1}{4^s} + \tfrac{2}{9^s} + \tfrac{1}{16^s} + \tfrac{4}{25^s} + \tfrac{2}{36^s} + \tfrac{4}{49^s} + \tfrac{1}{64^s} + \tfrac{8}{73^s} + \tfrac{3}{81^s} + \tfrac{8}{97^s} + \tfrac{4}{100^s} + \dots
\end{aligned}
$$

To illustrate our findings, we present a colouring of a sevenfold rhombus tiling with eight colours. In Figure 1, the vertices of the tiling are coloured, which are a subset of the module $\mathcal{M}_7$. This case corresponds to one of the two possible colouring with eight colours (up to permutations), the other inequivalent colouring being obtained by a reflection. Alternatively, one can colour the tiles (which is some sort of complementary problem), by locally assigning a point of $\mathcal{M}_7$ to each tile and transferring the corresponding colour. An example is shown in Figure 2.

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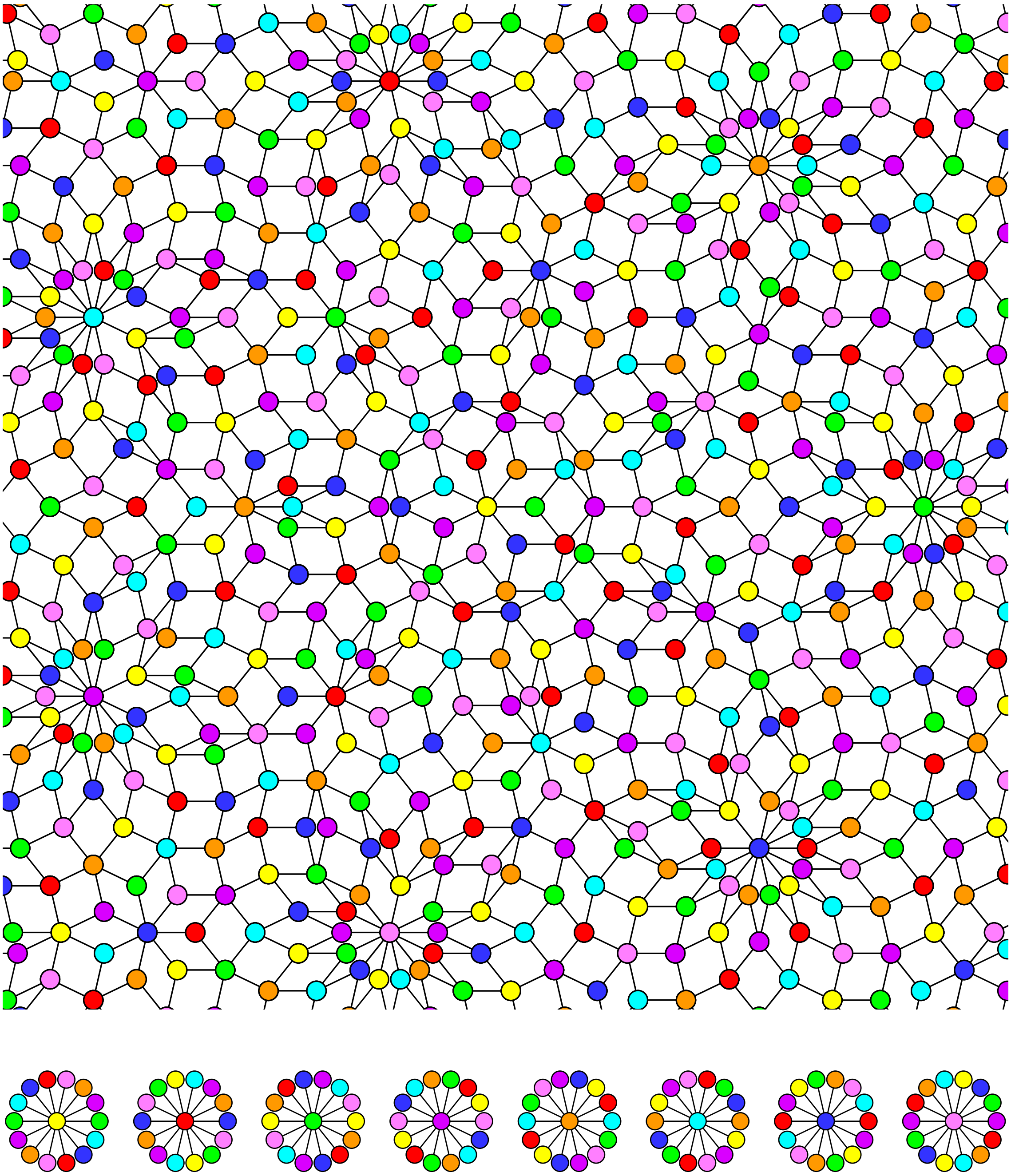

Figure 1: Vertex colouring of a sevenfold tiling with eight colours. The eight colour stars encode the relations between the colours of neighbouring vertices.

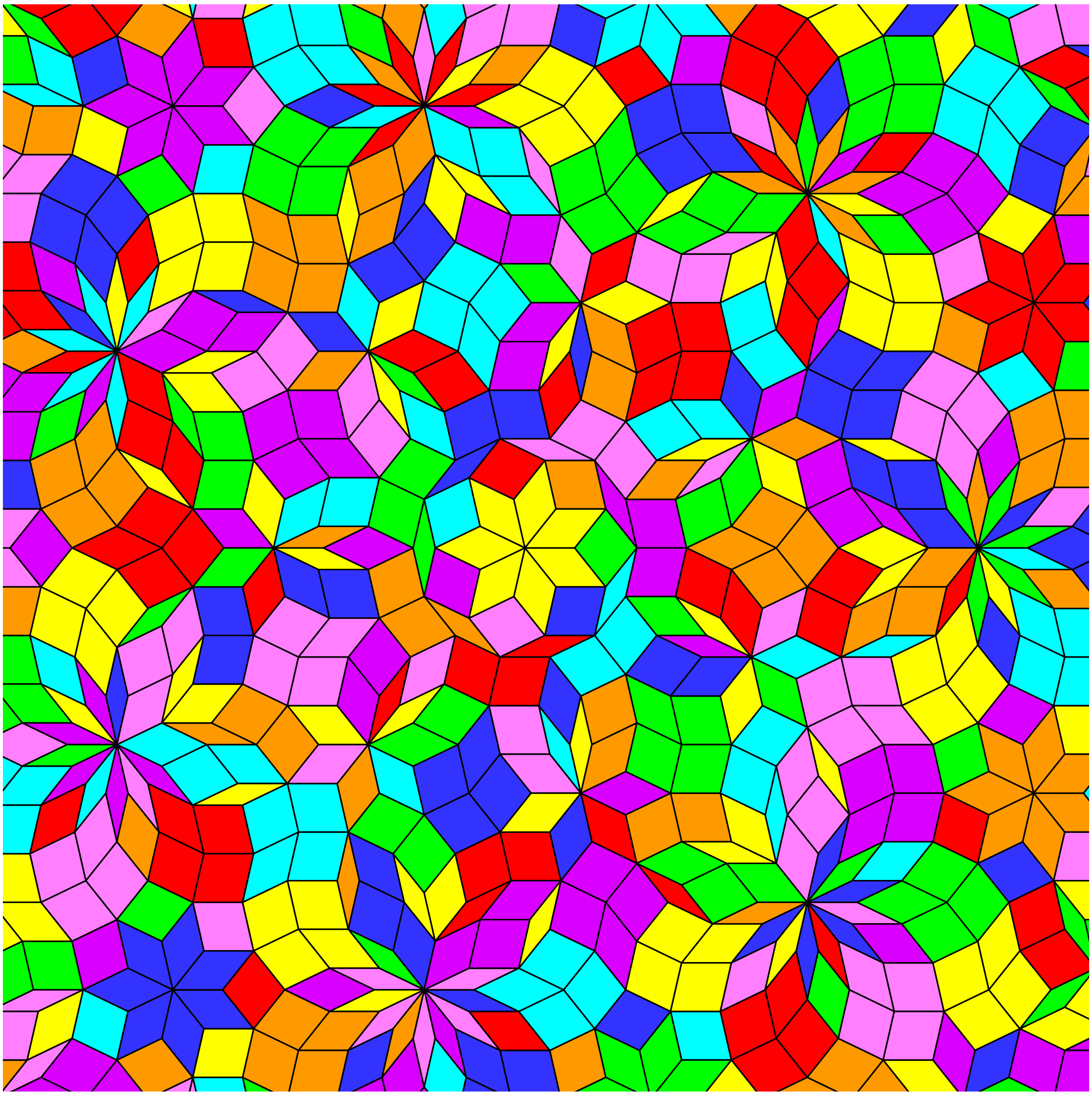

Figure 2: Colouring of a sevenfold tiling with eight colours.