Embedding Bratteli-Vershik systems in cellular automata

Journal Item

Embedding Bratteli-Vershik systems in cellular automata*

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Abstract

Many dynamical systems can be naturally represented as *Bratteli-Vershik* (or *adic*) systems, which provide an appealing combinatorial description of their dynamics. If an adic system X satisfies two technical conditions (*focus* and *bounded width*) then we show how to represent X using a two-dimensional subshift of finite type Y; each 'row' in a Y-admissible configuration corresponds to an infinite path in the Bratteli diagram of X, and the vertical shift on Y corresponds to the 'successor' map of X. Any Y-admissible configuration can then be recoded as the spacetime diagram of a one-dimensional cellular automaton Φ; in this way X is *embedded* in Φ (i.e. X is conjugate to a subsystem of Φ). With this technique, we can embed many odometers, Toeplitz systems, and constant-length substitution systems in one-dimensional cellular automata.

1 Introduction

The Bratteli-Vershik ('adic') transformations are a large and diverse class of symbolic dynamical systems which are of interest partly because many other dynamical systems can be naturally represented as adic transformations, thereby yielding an appealing combinatorial description of their dynamics. Cellular automata (CA) are another class of symbolic dynamical system with versatile representation capabilities. Recent work shows that certain simple adic systems arise naturally as subsystems of CA. For example, if $\Phi : \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$ is a non-injective, right-sided, left-permutative CA, then many orbit-closures of Φ are conjugate to odometers [CPY07]; a more precise characterization of some of these odometers appears in [CY07]. Similarly, in the *Game of Life* CA, the orbit closure of a certain configuration can be represented as a substitution system [GM05]. The main result of this article is the following:

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Theorem 1 *Let $(X_{\mathcal{B}}, V_{\mathcal{B}})$ be an adic system, where $\mathcal{B}$ is a properly ordered Bratteli diagram which is focused, and has bounded width. Then there exist cellular automata which embed $(X_{\mathcal{B}}, V_{\mathcal{B}})$.*

For definitions of terms in Theorem 1, and notation, see Section 2. Theorem 1 follows directly from Theorem 5 (in Section 3). In Section 4 we give examples of families of dynamical systems whose adic representations satisfy the conditions of Theorem 1. This includes all primitive substitutions, and many Toeplitz systems (Corollaries 8 and 12). This technique also reproves the embedding of certain odometers in a CA, a result in [CPY07]; however the family of CA that we identify here are disjoint from those found in [CPY07].

2 Preliminaries

2.1 Notation

Let $\mathcal{A}$ be a finite alphabet; elements in $\mathcal{A}$ will be denoted by a, b, c etc. A *word* $\mathbf{a}$ from $\mathcal{A}$ is a finite concatenation of elements from $\mathcal{A}$. Let $\mathcal{A}^k$, $\mathcal{A}^{\leq k}$, and $\mathcal{A}^+$ denote the set of all words of length k, length at most k, and any finite length respectively. We include the empty word, and use boldface to denote words. If $\mathbf{a} = a_0 a_1 \ldots a_j$ and $\mathbf{a}^* = a_0^* a_1^* \ldots a_k^*$, then $\mathbf{a}\mathbf{a}^* := a_0 a_1 \ldots a_j a_0^* a_1^* \ldots a_k^*$. If $\mathbf{a} = a_0 a_1 \ldots a_j$, define $|\mathbf{a}| := j$. The space of all bi-infinite sequences from $\mathcal{A}$ is written as $\mathcal{A}^{\mathbb{Z}}$. Elements of the latter are written $\mathbf{a} = \ldots a_{-1} \cdot a_0 a_1 \ldots$. If $\mathbb{N} := \{0, 1, 2, \ldots\}$, and $\mathbb{N}^+ := \{1, 2, \ldots\}$, elements of $\mathcal{A}^{\mathbb{N}}$ and $\mathcal{A}^{\mathbb{N}^+}$ are defined analagously. The notation $\{x_n\}_{n=\infty}^{k}$ is used to denote the left-infinite sequence $\ldots x_{k+2}\, x_{k+1}\, x_k$. If $\mathbf{x} = \{x_n\}_{n=\infty}^{1}$ and $\mathbf{y} = \{y_n\}_{n=0}^{\infty}$, then by $\mathbf{x} \cdot \mathbf{y}$ we mean the bi-infinite sequence $\ldots x_2\, x_1 \cdot y_0\, y_1 \ldots$. If $\mathbb{L} = \mathbb{Z}$, $\mathbb{N}$, or $\mathbb{N}^+$, then $\mathcal{A}^{\mathbb{L}}$ is a *Cantor space*: a zero-dimensional compact metric space, when $\mathcal{A}$ is endowed with the discrete topology and $\mathcal{A}^{\mathbb{L}}$ with the product topology. If $b \in \mathcal{A}$, let $[b]_j := \{\mathbf{a} : \ a_j = b\}$; these (clopen) sets form a countable basis for the topology on $\mathcal{A}^{\mathbb{L}}$. This topology is also generated by the Hamming metric. The (left) *shift map* $\sigma : \mathcal{A}^{\mathbb{L}} \to \mathcal{A}^{\mathbb{L}}$ is the map defined as $(\sigma(x))_n = x_{n+1}$. X is a *subshift of* $(\mathcal{A}^{\mathbb{L}}, \sigma)$ if it is a closed σ-invariant subset of $\mathcal{A}^{\mathbb{L}}$. A (one dimensional) *cellular automaton* is a continuous, σ-commuting map $\Phi : \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$. The Curtis-Hedlund-Lyndon theorem [Hed69] states that every CA is given by a *local rule* $\phi : \mathcal{A}^{l+r+1} \to \mathcal{A}$ for some $l \geq 0$ (the *left radius of* ϕ) and $r \geq 0$ (the *right radius of* ϕ), where for all $x \in \mathcal{A}^{\mathbb{Z}}$, and all $i \in \mathbb{Z}$,

$$[\Phi(x)]_i = \phi(x_{i-l}, x_{i-l+1}, \ldots, x_{i+r}) \, .$$

If $X \subset \mathcal{A}^{\mathbb{Z}}$ is closed and $\Phi(X) \subseteq X$, then (X, Φ) is a *subsystem* of $(\mathcal{A}^{\mathbb{Z}}, \Phi)$. If the dynamical system (Y, T) is topologically conjugate to the subsystem (X, Φ) of $(\mathcal{A}^{\mathbb{Z}}, \Phi)$ (i.e. if there exists a homeomorphism $f : Y \to X$ with $f \circ T = \Phi \circ f$), we say that $(\mathcal{A}^{\mathbb{Z}}, \Phi)$ *embeds* (Y, T).

2.2 Bratteli Diagrams

A *Bratteli diagram* $\mathcal{B} = (\mathcal{V}, \mathcal{E})$ is an infinite directed graph with *vertex set* $\mathcal{V} = \bigsqcup_{n=0}^{\infty} \mathcal{V}_n$ and *edge set* $\mathcal{E} = \bigsqcup_{n=1}^{\infty} \mathcal{E}_n$, where all $\mathcal{V}_n$'s and $\mathcal{E}_n$'s are finite, $\mathcal{V}_0 = \{⧗\}$, and, if x is an edge in $\mathcal{E}_n$, the source $s(x)$ of x lies in $\mathcal{V}_n$ and the range $r(x)$ of x lies in $\mathcal{V}_{n-1}$. We remark that in all references to Bratteli diagrams that we have consulted, edges move in the opposite direction: they have source in $\mathcal{V}_n$ and range $\mathcal{V}_{n+1}$; however for the purposes of our results in Section 3, we find our representation more visually intuitive. We will use $x, y \ldots$ when referring to edges, and $a, b, \ldots$ when referring to vertices, although edges in $\mathcal{E}_1$ will be given special labels ⧗_i, where i varies. A finite set of edges $\{x_{n+k}\}_{k=1}^{K}$, with $s(x_{n+k}) = r(x_{n+k+1})$ for $1 \leq k \leq K-1$, is called a *path* from $s(x_{n+K})$ to $r(x_{n+1})$. Similarly an *infinite path* in $\mathcal{B}$ is a sequence $\mathbf{x} = \{x_n\}_{n=\infty}^{1}$, with $x_n \in \mathcal{E}_n$ for $n \geq 1$, and $s(x_n) = r(x_{n+1})$ for $n \geq 1$. We will often write

$$\ldots a_{n+1} \overset{x_{n+1}}{\rightarrow} a_n \overset{x_n}{\rightarrow} \ldots \overset{x_2}{\rightarrow} a_1 \overset{x_1}{\rightarrow} \text{⧗} \tag{1}$$

where $s(x_n) = a_n$. The set of all infinite paths in $\mathcal{B}$ will be denoted $X_{\mathcal{B}}$ (a subset of $\Pi_{n\geq 1}\mathcal{E}_n$), and $X_{\mathcal{B}}$ is endowed with the topology induced from the product topology on $\Pi_{n\geq 1}\mathcal{E}_n$. Thus $X_{\mathcal{B}}$ is a compact metric space.

If $\mathbf{x} = \{x_n\}_{n=\infty}^{1}$ and $\mathbf{x}' = \{x'_n\}_{n=\infty}^{1}$ are two elements in $X_{\mathcal{B}}$, we write $\mathbf{x} \sim \mathbf{x}'$ if the tails of $\mathbf{x}$ and $\mathbf{x}'$ are equal. It follows that $\sim$ is an equivalence relation, and primitivity of τ implies that each equivalence class of $\sim$ is dense. We will mostly write $(\mathbf{a}, \mathbf{x}) = \{(a_n, x_n)\}_{n\geq 1}$, where $s(x_n) = a_n$, when referring to an element $\mathbf{x}$ in $X_{\mathcal{B}}$.

Two Bratteli diagrams $\mathcal{B} = (\mathcal{V}, \mathcal{E})$ and $\mathcal{B}' = (\mathcal{V}', \mathcal{E}')$ are *isomorphic* if there exists a pair of bijections $f_{\mathcal{V}} : \mathcal{V} \to \mathcal{V}'$ and $f_{\mathcal{E}} : \mathcal{E} \to \mathcal{E}'$ satisfying $f_{\mathcal{V}}(a) \in \mathcal{V}'_n$ if $a \in \mathcal{V}_n$, and $s(f_{\mathcal{E}}(x)) = f_{\mathcal{V}}(s(x))$, $r(f_{\mathcal{E}}(x)) = f_{\mathcal{V}}(r(x))$ whenever $x \in \mathcal{E}$.

Let $\{n_k\}_{k=0}^{\infty}$ be a sequence of increasing integers with $n_0 = 0$. Then $\mathcal{B}' = (\mathcal{V}', \mathcal{E}')$ is a *telescoping* of $\mathcal{B} = (\mathcal{V}, \mathcal{E})$ if $\mathcal{V}'_k = V_{n_k}$ (with the vertex $v \in \mathcal{V}_{n_k}$ labelled as $v' \in \mathcal{V}'_k$), and the number of edges from $v'_{k+1} \in \mathcal{V}'_{k+1}$ to $v'_k \in \mathcal{V}'_k$ is the number of paths from $v_{k+1} \in \mathcal{V}_{n_{k+1}}$ to $v_k \in \mathcal{V}_{n_k}$. We consider two Bratteli diagrams $\mathcal{B}$ and $\mathcal{B}'$ *equivalent* if $\mathcal{B}'$ can be obtained from $\mathcal{B}$ by isomorphism and telescoping. Thus when we talk about a Bratteli diagram we are talking about an equivalence class of diagrams. We say that $\mathcal{B}$ is *simple* if there exists a telescoping $\mathcal{B}' = (\mathcal{V}', \mathcal{E}')$ of $\mathcal{B}$ so that, for any $a \in \mathcal{V}'_{n+1}$ and $b \in \mathcal{V}'_n$, there is at least one edge from a to b. If $X_{\mathcal{B}}$ is simple, then $X_{\mathcal{B}}$ has no isolated points, making it a Cantor space.

2.2.1 Ordering $X_{\mathcal{B}}$

For each $a \in \mathcal{V}_n$, let $\mathcal{E}_n(a) = \{x \in \mathcal{E}_n : s(x) = a\}$. Say $\mathcal{B}$ is *ordered* if for each $n \geq 1$ and $a \in \mathcal{V}_n$, there is a linear order $\geq$ on $\mathcal{E}_n(a)$; elements of $\mathcal{E}_n$ will then be labelled $0, 1, \ldots$ according to their order. If $a \in \mathcal{V}\backslash\{\text{⧗}\}$, define $|a| := |\mathcal{E}_n(a)| - 1$, so that $\mathcal{E}_n(a) = \{0, 1, \ldots |a|\}$.

The linear order on edges in each $\mathcal{E}_n(a)$ induces a partial ordering on paths from $\mathcal{V}_n$ to $\mathcal{V}_m$: the two paths $\mathbf{x} = a_n \overset{x_n}{\rightarrow} a_{n-1} \overset{x_{n-1}}{\rightarrow} \ldots \overset{x_{m+1}}{\rightarrow} a_m$ and $\mathbf{x}' = a'_n \overset{x'_n}{\rightarrow} a'_{n-1} \overset{x'_{n-1}}{\rightarrow} \ldots \overset{x'_{m+1}}{\rightarrow} a'_m$ from $\mathcal{E}_n$ to $\mathcal{E}_m$ are comparable with $\mathbf{x} < \mathbf{x}'$ if $a_n = a'_n$ and if there is some $k \in [n, m]$ with $x_k < x'_k$ and $x_j = x'_j$ for $k+1 \leq j \leq n$.

Finally, two elements $\mathbf{x}$, $\mathbf{x}' \in X_{\mathcal{B}}$ are *comparable* with $\mathbf{x} < \mathbf{x}'$ if there is a k such that $x_n = x'_n$ for all $n > k$, and $x_k < x'_k$. Thus each equivalence class for $\sim$ is ordered. There is the obvious notion of ordered isomorphism of two ordered Bratteli diagrams $\mathcal{B}$, $\mathcal{B}'$: the isomorphism between $\mathcal{B}$ and $\mathcal{B}'$ also has to satisfy $f_{\mathcal{E}}(x) \leq f_{\mathcal{E}}(y)$ if $x \leq y$. If $\mathcal{B}'$ is a telescoping of the ordered Bratteli diagram $\mathcal{B}$, then the order induced on $\mathcal{B}'$ from the order on $\mathcal{B}$ makes $\mathcal{B}'$ an ordered Bratteli diagram. We say that the ordered Bratteli diagrams $\mathcal{B}$, $\mathcal{B}'$ are *equivalent* if $\mathcal{B}'$ is the image of $\mathcal{B}$ by telescoping and order isomorphism.

An infinite path is *maximal* (*minimal*) if all the edges making up the path are maximal (minimal). If $\mathbf{x} = \{x_n\}_{n=\infty}^{1}$ is not maximal, let k be the smallest integer such that x_k is not a maximal edge, and let y_k be the successor of x_k. Then the *successor* of x is the path $\ldots x_{k+2}\, x_{k+1}\, y_k\, 0 \ldots 0$. Similarly, every non-minimal path has a *predecessor*. Let $X_{\min}$, $(X_{\max}) \subset X_{\mathcal{B}}$ be defined as the set of minimal (maximal) elements of $X_{\mathcal{B}}$. By compactness, these sets are non empty. Let $V_{\mathcal{B}} : X_{\mathcal{B}} \backslash X_{max} \to X_{\mathcal{B}} \backslash X_{min}$ be the successor map. Simple ordered Bratteli diagrams which have a unique minimal and maximal element (called $\mathbf{x}_{\min}$ and $\mathbf{x}_{\max}$ respectively) are called *properly ordered*. If $\mathcal{B}$ is properly ordered, then $V_{\mathcal{B}}$ can be extended to a homeomorphism on $X_{\mathcal{B}}$ by setting $V_{\mathcal{B}}(\mathbf{x}_{\max}) = \mathbf{x}_{\min}$. We call $(X_{\mathcal{B}}, V_{\mathcal{B}})$ the *Bratteli-Vershik* or *adic* system associated with τ. Note that $(X_{\mathcal{B}}, V_{\mathcal{B}})$ is a minimal system, since $V_{\mathcal{B}}$ orbits are equivalence classes for $\sim$. Let us say that two Cantor systems $(X_i, T_i, \mathbf{x}_i)$, $i = 1, 2$ are *pointedly isomorphic* if there exists a homeomorphism $f : X_1 \to X_2$ with $f \circ T_1 = T_2 \circ f$ and $f(x_1) = x_2$. The Bratteli- Vershik system associated to an equivalence class of properly ordered Bratteli diagrams is well defined up to pointed isomorphism:

Theorem 2 *Let $\mathcal{B}$ and $\mathcal{B}'$ be properly ordered Bratteli diagrams. Then $\mathcal{B}$ is equivalent to $\mathcal{B}'$ if and only if $(X_{\mathcal{B}}, V_{\mathcal{B}}, \mathbf{x}_{\min})$ is pointedly isomorphic to $(X_{\mathcal{B}'}, V_{\mathcal{B}'}, \mathbf{x}_{\min})$.* [HPS92, §4]

We say that $\mathcal{B}$ has *bounded width* if there exists a constant K such that $|\mathcal{V}_n| \leq K$ and $|\mathcal{E}_n| \leq K$ for each $n \geq 1$. If $\mathcal{B}$ is ordered, we say that $\mathcal{B}$ is *focused* if, for each $n \geq 1$, all minimal edges in $\mathcal{E}_n$ have the same range. For example, any *proper substitution* (see Section 4) has a focused representation. If $\mathcal{B}$ is focused, we use $\underline{a}$ to denote the range of any minimal edge in $\mathcal{E}_n$. Note that if $\mathcal{B}$ is focused, then its unique minimal element is $(\underline{\mathbf{a}}, \mathbf{0})$.

3 The spacetime diagrams of $(X_{\mathcal{B}}, V_{\mathcal{B}})$ and its associated subshift of finite type

In this section, $\mathcal{B}$ is a properly ordered Bratteli diagram, which has bounded width K. In this case vertices in $\mathcal{V}$ can be labelled from a finite alphabet $\mathcal{A}$. For each $n \geq 1$ there is a function $\boldsymbol{\tau}^n : \mathcal{A} \to \mathcal{A}^{\leq K}$ such that $\boldsymbol{\tau}^n$ completely describes $\mathcal{E}_n$ and its ordering. In other words, if we write $\boldsymbol{\tau}^n(a) = \tau_0^n(a)\tau_2^n(a)\ldots\tau_{l_a^n}^n(a)$ where $l_a^n = |\boldsymbol{\tau}^n(a)|$, for each $a \in \mathcal{A}$ and $n \geq 1$, then there is an edge from $a \in \mathcal{V}_n$ to b in $\mathcal{V}_{n-1}$, and it is labelled k, if and only $\tau_k^n(a) = b$. If $\boldsymbol{\tau}_a^n$ is the empty word, this is taken to mean that a does not appear as a vertex in $\mathcal{V}_n$. Implicit in this description is some arbitrary but pre-assigned labelling of vertices in $\mathcal{V}$ from $\mathcal{A}$. We will also assume that if $\mathcal{B}$ is focused, then its labelling reflects this. We shall use 'stars' such as $*, \star, \ldots$ to indicate l_a^n, so that "$\tau_*^n(a)$" means the last letter of $\boldsymbol{\tau}^n(a)$, provided that there is no ambiguity.

Define $\mathcal{A}_{\mathcal{B}}$ to be the alphabet

$$\begin{aligned} \mathcal{A}_{\mathcal{B}} \;&:=\; \{(a, \boldsymbol{\tau}^n, x) : a \in \mathcal{A}, n \in \mathbb{N}, x \in [0, l_a^n]\} \\ &\cup\; \{(a, \boldsymbol{\tau}^1, \Sigma_x) : a \in \mathcal{A}, x \in [0, l_a^1]\} \cup \{\Sigma\}\,. \end{aligned}$$

An element $(\mathbf{a}, \mathbf{x}) \in X_{\mathcal{B}}$ can also be seen as an element of $\mathcal{A}_{\mathcal{B}}^{\mathbb{N}}$ by writing $(\mathbf{a}, \mathbf{x}) = (\mathbf{a}, \{\boldsymbol{\tau}^n\}_{n\geq 1}, \mathbf{x})$. Conversely, when we write "$(\mathbf{a}, \mathbf{x}) \in \mathcal{A}_{\mathcal{B}}^{\mathbb{N}}$", we mean $(\mathbf{a}, \mathbf{x}) = (\mathbf{a}, \{\boldsymbol{\tau}^n\}_{n\geq 1}, \mathbf{x})$. Let $\tilde{Y} \subset \mathcal{A}_{\mathcal{B}}^{\mathbb{Z}\times\mathbb{N}^+}$ be the set of all *space-time diagrams* for $(X_{\mathcal{B}}, V_{\mathcal{B}})$: elements $\tilde{\mathbf{y}} = \{\tilde{\mathbf{y}}_m\}_{m\in\mathbb{Z}} \in \tilde{Y}$ (with the row $\tilde{\mathbf{y}}_m = \{\tilde{y}_n^m\}_{n=\infty}^{1}$) are such that $\tilde{\mathbf{y}}_0 = (\mathbf{a}, \mathbf{x})$ for some $(\mathbf{a}, \mathbf{x}) \in X_{\mathcal{B}}$ and $\tilde{\mathbf{y}}_m = \{V_{\mathcal{B}}^m(\mathbf{a}, \mathbf{x})\}$ for each $m \in \mathbb{Z}$. Let $⧗^\infty = ⧗, ⧗, \ldots$; We extend $\tilde{Y}$ to $Y \subset \mathcal{A_B}^{\mathbb{Z}\times\mathbb{Z}}$ by letting

$$Y := \{\mathbf{y} = \{\mathbf{y}_m\}_{m\in\mathbb{Z}} : \mathbf{y} = \tilde{\mathbf{y}} \cdot ⧗^\infty \text{ for some } \tilde{\mathbf{y}} \in \tilde{Y}\}.$$

There is a natural identification of elements in Y with elements in $\tilde{Y}$. Note that we use positive integers to denote column locations in $\mathbf{y}$, unless there is a possibility of confusion.

A (two-dimensional) *subshift* is a closed, shift-invariant subset $\mathcal{S} \subset \mathcal{A}^{\mathbb{Z}^2}$. If $\mathcal{S} \subset \mathcal{A}^{\mathbb{Z}^2}$ is a subshift, then for any finite $\mathbb{K} \subset \mathbb{Z}^2$, let $\mathcal{S}_{\mathbb{K}} := \{\mathbf{s}_{\mathbb{K}} \ ; \ \mathbf{s} \in \mathcal{S}\}$ be the set of all $\mathcal{S}$-*admissible* $\mathbb{K}$*-blocks*. If $\mathsf{z} \in \mathbb{Z}^2$, then $\mathcal{S}_{\mathbb{K}} = \mathcal{S}_{\mathbb{K}+\mathsf{z}}$ (because $\mathcal{S}$ is shift-invariant). We say $\mathcal{S}$ is a *subshift of finite type* (SFT) if there is some finite *neighbourhood* $\mathbb{K} \subset \mathbb{Z}^2$ such that $\mathcal{S} = \{\mathbf{a} \in \mathcal{A}^{\mathbb{Z}^2} \ ; \ \mathbf{a}_{\mathbb{K}+\mathsf{z}} \in \mathcal{S}_{\mathbb{K}}, \ \forall \mathsf{z} \in \mathbb{Z}^2\}$. For example, if $\Phi : \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$ is a one-dimensional cellular automaton, then the set of spacetime diagrams for Φ is an SFT in $\mathcal{A}^{\mathbb{Z}^2}$.

Let $\mathbb{K} = \{0, 1\}^2$, and let $\mathcal{S}$ be the SFT defined using the $\mathbb{K}$-blocks in Figure 1. The following lemma is straightforward.

Lemma 3 *Y is the subset* $\displaystyle\bigcup_{\substack{a\in\mathcal{A}\\ 0\leq i\leq l_a^1}} \{\mathbf{y} \in \mathcal{S} \ ; \ y_0^0 = ⧗, \ y_{-1}^0 = (a, ⧗_i)\}$ *of* $\mathcal{S}$. ■

We will sometimes use the symbols $\boxed{\begin{smallmatrix}\tau & j\\ a & \end{smallmatrix}}$, $\boxed{\begin{smallmatrix}\tau' & ⧗\\ a & \end{smallmatrix}}$, ... or $\boxed{T_1}$, $\boxed{T_2}$, ... to denote elements of $\mathcal{A}_{\mathcal{B}}$. For example, if $\mathcal{B}$ is the Bratteli diagram from Figure 3, then

$$\mathcal{A}_{\mathcal{B}} = \left\{ \boxed{\begin{smallmatrix}\tau & 0\\ a & \end{smallmatrix}}, \boxed{\begin{smallmatrix}\tau & 1\\ a & \end{smallmatrix}}, \boxed{\begin{smallmatrix}\tau & 2\\ a & \end{smallmatrix}}, \boxed{\begin{smallmatrix}\tau & 0\\ b & \end{smallmatrix}}, \boxed{\begin{smallmatrix}\tau & 1\\ b & \end{smallmatrix}}, \boxed{\begin{smallmatrix}\tau & ⧗\\ a & \end{smallmatrix}}, \boxed{\begin{smallmatrix}\tau & ⧗\\ b & \end{smallmatrix}}, \boxed{⧗} \right\},$$

where the dropped index in $⧗$ indicates that there is a unique edge from each vertex in $\mathcal{V}_1$ to $⧗$. A (possibly infinite) concatenation of letters from $\mathcal{A}_{\mathcal{B}}$ is a *fragment* of a spacetime diagram if it appears in some $\mathbf{y} \in Y$.

Lemma 4 *If $\mathcal{B}$ is focused and properly ordered, and* $\begin{array}{|c|c|c|}\hline T_{j+1}^m & T_j^m & T_{j-1}^m \\ \hline \multicolumn{1}{c|}{} & T_j^{m+1} & T_{j-1}^{m+1} \\ \cline{2-3}\end{array}$ *is a fragment of* $\mathbf{y} \in Y$*, then* $\begin{array}{|c|c|c|}\hline T_{j+1}^m & T_j^m & T_{j-1}^m \\ \hline \multicolumn{2}{c|}{} & T_{j-1}^{m+1} \\ \cline{3-3}\end{array}$ *determines* $\boxed{T_j^{m+1}}$.

Proof: We prove the case where all tiles in $\begin{array}{|c|c|c|}\hline T_{j+1}^m & T_j^m & T_{j-1}^m \\ \hline \multicolumn{1}{c|}{} & T_j^{m+1} & T_{j-1}^{m+1} \\ \cline{2-3}\end{array}$ are from $\{\boxed{\begin{smallmatrix}\tau^n & x\\ a & \end{smallmatrix}} : a \in \mathcal{A}, n \geq 1, x \in [0, l_a^n]\}$; other cases are similar. If $T_{j-1}^m = T_{j-1}^{m+1}$, then $T_j^{m+1} = T_j^m$, as is the case if $T_{j-1}^m = \boxed{\begin{smallmatrix}\tau^n & x\\ a & \end{smallmatrix}}$ with $x < l_a^n$. If

$$\begin{array}{|c|c|}\hline T_j^m & T_{j-1}^m \\ \hline \multicolumn{1}{c|}{} & T_{j-1}^{m+1} \\ \cline{2-2}\end{array} \quad = \quad \begin{array}{|c|c|}\hline \begin{smallmatrix}\tau^n & x\\ b & \end{smallmatrix} & \begin{smallmatrix}\tau^{n-1} & *\\ a & \end{smallmatrix} \\ \hline \multicolumn{1}{c|}{} & \begin{smallmatrix}\tau^{n-1} & 0\\ ? & \end{smallmatrix} \\ \cline{2-2}\end{array}$$

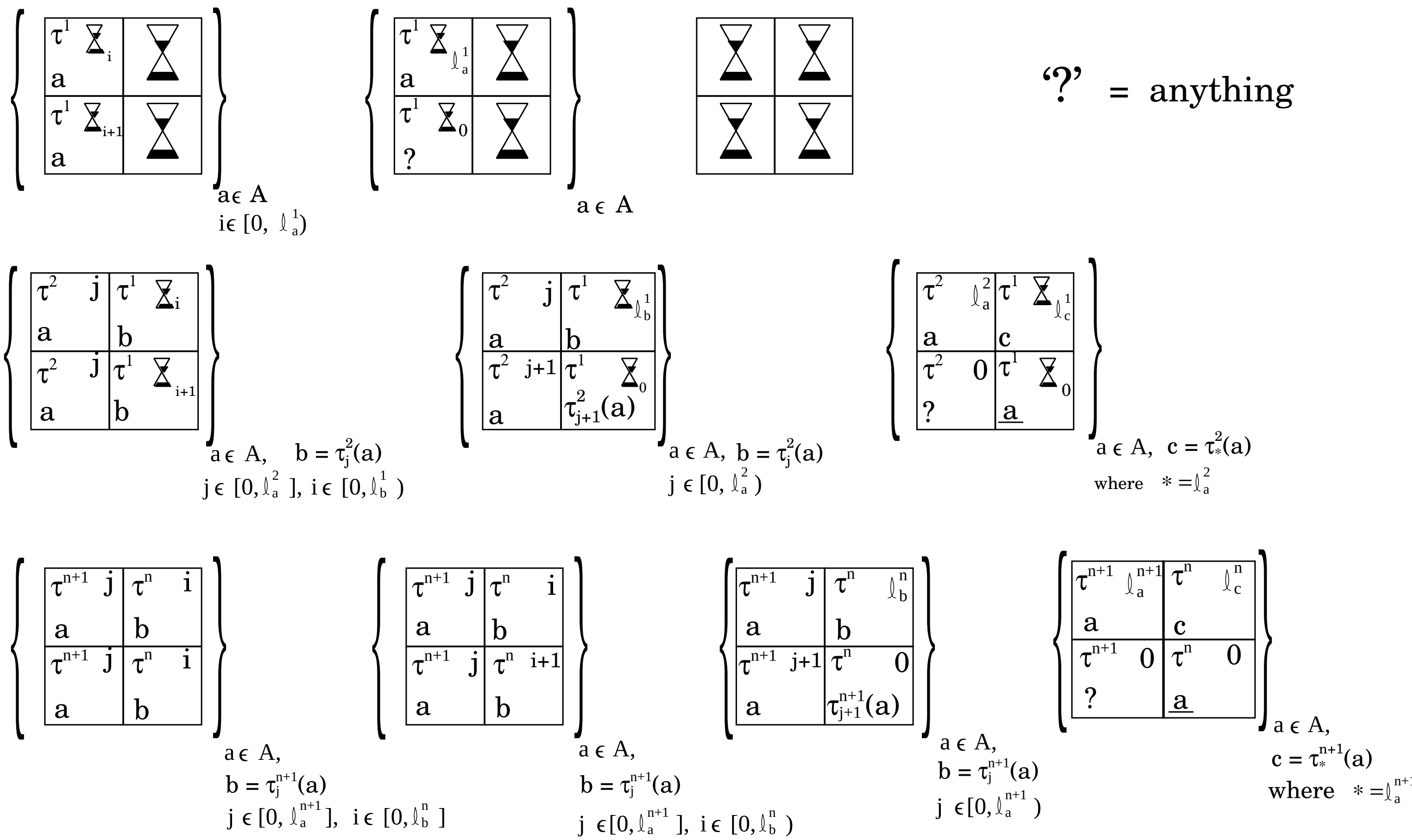

Figure 1: Admissible $\{0,1\}^2$-blocks defining the subshift of finite type $\mathcal{S}$.

with $x < l_b^n$, then $T_j^{m+1} = \boxed{\begin{smallmatrix}\tau^n & x+1\\ a & \end{smallmatrix}}$. If

$$\begin{array}{|c|c|c|}\hline T^m_{j+1} & T^m_j & T^m_{j-1}\\ \hline \multicolumn{2}{c|}{} & T^{m+1}_{j-1}\\ \cline{3-3}\end{array} \quad = \quad \begin{array}{|c|c|c|}\hline \tau^{n+1}\;\; y & \tau^n\;\; \star & \tau^{n-1}\;\; \ast \\ c & b & a\\ \hline \multicolumn{2}{c|}{} & \tau^{n-1}\;\; 0 \\ \multicolumn{2}{c|}{} & ?\\ \cline{3-3}\end{array}$$

with $y < l_c^{n+1}$, then $T_j^{m+1} = \boxed{\begin{smallmatrix}\tau^n & 0\\ \tau^{n+1}_{y+1}(c) & \end{smallmatrix}}$. Finally if

$$\begin{array}{|c|c|c|}\hline T^m_{j+1} & T^m_j & T^m_{j-1}\\ \hline \multicolumn{2}{c|}{} & T^{m+1}_{j-1}\\ \cline{3-3}\end{array} \quad = \quad \begin{array}{|c|c|c|}\hline \tau^{n+1}\;\; \bigstar & \tau^n\;\; \star & \tau^{n-1}\;\; \ast \\ c & b & a\\ \hline \multicolumn{2}{c|}{} & \tau^{n-1}\;\; 0 \\ \multicolumn{2}{c|}{} & ?\\ \cline{3-3}\end{array},$$

then since τ is focused, we must have $T_j^{m+1} = \boxed{\begin{smallmatrix}\tau^n & 0\\ \underline{a} & \end{smallmatrix}}$. ■

Let W_4 be the set of all fragments $\begin{array}{|c|c|c|}\hline T^m_{j+1} & T^m_j & T^m_{j-1}\\ \hline \multicolumn{2}{c|}{} & T^{m+1}_{j-1}\\ \cline{3-3}\end{array}$ appearing in Y, where $m,\ j,\ \in \mathbb{Z}$. Let $f : W_4 \to: \mathcal{A}_{\mathcal{B}}$ be the function such that if $\begin{array}{|c|c|c|}\hline T^m_{j+1} & T^m_j & T^m_{j-1}\\ \hline \multicolumn{1}{c|}{} & T^{m+1}_{j} & T^{m+1}_{j-1}\\ \cline{2-3}\end{array}$ is the unique extension of $\begin{array}{|c|c|c|}\hline T^m_{j+1} & T^m_j & T^m_{j-1}\\ \hline \multicolumn{2}{c|}{} & T^{m+1}_{j-1}\\ \cline{3-3}\end{array}$, then $f\left(\begin{array}{|c|c|c|}\hline T^m_{j+1} & T^m_j & T^m_{j-1}\\ \hline \multicolumn{2}{c|}{} & T^{m+1}_{j-1}\\ \cline{3-3}\end{array}\right) = \boxed{T^{m+1}_j}$.

This also means that any (diagonal) ray

$$\mathbf{R} := \{(\begin{array}{|c|c|c|}\hline T^{-k}_{2k+3} & T^{-k}_{2k+2} & T^{-k}_{2k+1}\\ \hline\end{array}) : k\ \in \mathbb{N}\}$$

which is a fragment in Y determines the horizontal $\mathbf{T}_0 := \{T_k^0 : k\ \in \mathbb{N}^+\}$. Let $\mathcal{R}$ be the collection of such rays $\mathbf{R}$, and let $F : \mathcal{R} \to \mathcal{A}_{\mathcal{B}}^{\mathbb{N}}$ be the function which maps these rays in $\mathcal{R}$ to their horizontal image $\mathbf{T}_0$. Call any 3-tuple $\begin{array}{|c|c|c|}\hline T^{-k}_{2k+3} & T^{-k}_{2k+2} & T^{-k}_{2k+1}\\ \hline\end{array}$ in a ray a *step*. Note that any ray in $\mathcal{R}$ can be extended in a unique way to a "two-sided" ray fragment from Y: simply attach infinitely many steps of the form $\begin{array}{|c|c|c|}\hline \text{⧗} & \text{⧗} & \text{⧗}\\ \hline\end{array}$.

Let $\mathcal{A}_\Phi := \mathcal{A}_{\mathcal{B}}^3$ and $W_3 \subset \mathcal{A}_\Phi$ be the set of all horizontal 3-tuple fragments from Y. We will use $S_0, S_1 \ldots$ to denote letters from $\mathcal{A}_\Phi$. Define a left radius one, right ra 3-++ dius one CA $\Phi : W_3^{\mathbb{Z}} \to W_3^{\mathbb{Z}}$ with local rule $\phi : W_3^3 \to W_3$ defined as

$$\Phi\left(\boxed{a|b|c}\,,\boxed{d|e|f}\,,\boxed{g|h|i}\right) = \begin{cases} \boxed{z|y|g} \quad \left(\begin{array}{r}\text{where } y{:=}f(d,e,f,g)\\ \text{and } z{:=}f(x,d,e,y),\\ \text{given } x{:=}f(a,b,c,d)\end{array}\right) & \text{if } \begin{array}{|c|c|c|c|c|c|c|}\hline a & b & c & \multicolumn{4}{c}{} \\ \hline \multicolumn{2}{c|}{} & d & e & f & \multicolumn{2}{c}{}\\ \cline{3-7} \multicolumn{4}{c|}{} & g & h & i\\ \cline{5-7}\end{array} \text{ is admissible;}\\ \text{arbitrary} & \text{otherwise.}\end{cases}$$

Let the maximal element of $X_{\mathcal{B}}$ be $(\{m_n\}_{n\geq 1}, \{\star_n\}_{n\geq 1})$, and define

$$\mathbf{x}_{\text{init}} :=$$

$$\ldots, \begin{array}{|c|c|c|}\hline \tau^6\ \ast_6 & \tau^5\ \ast_5 & \tau^4\ \ast_4\\ m_6 & m_5 & m_4\\ \hline\end{array}, \begin{array}{|c|c|c|}\hline \tau^4\ \ast_4 & \tau^3\ \ast_3 & \tau^2\ \ast_2\\ m_4 & m_3 & m_2\\ \hline\end{array}, \begin{array}{|c|c|c|}\hline \tau^2\ 0 & \tau^1\ \text{⧗}_0 & \text{⧗}\\ \underline{a} & \underline{a} & \\ \hline\end{array}, \begin{array}{|c|c|c|}\hline \text{⧗} & \text{⧗} & \text{⧗}\\ \hline\end{array}, \begin{array}{|c|c|c|}\hline \text{⧗} & \text{⧗} & \text{⧗}\\ \hline\end{array}, \ldots$$

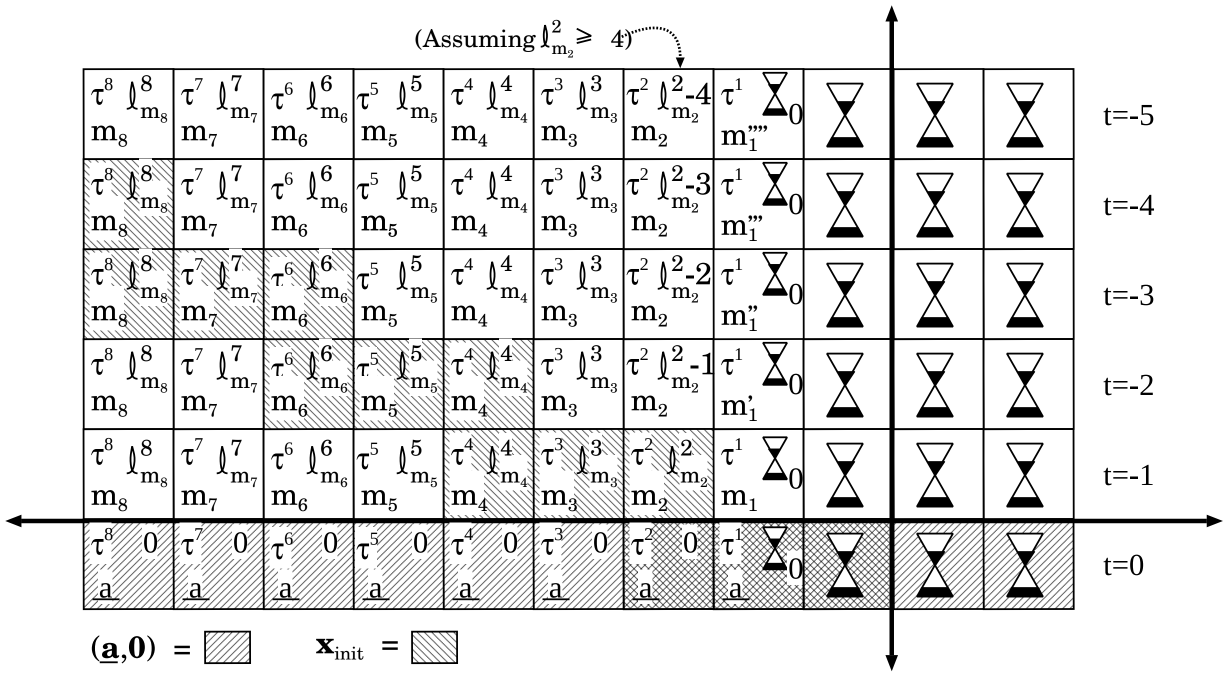

Figure 2: The spacetime diagram of $(\underline{\mathbf{a}}, \mathbf{0})$ and its five predecessors in the adic system. This $\mathbb{Z}^2$-indexed configuration is an admissible element of the subshift of finite type defined using the 2×2 tiles in Figure 1. Finally, the tinted diagonal ray is $\mathbf{x}_{\text{init}}$; if read 'diagonally' from top right to bottom left, this configuration is a the spacetime diagram of $\mathbf{x}_{\text{init}}$ under the CA Φ.

Set $\Omega = \overline{\{\Phi^n((\mathbf{x}_{\text{init}})\}_{n \geq 0}}$.

Theorem 5 (Ω, Φ) *is topologically conjugate to* $(X_{\mathcal{B}}, V_{\mathcal{B}})$.

Proof: Let $C_1 : \Omega \to \mathcal{R}$ be the map which takes a sequence $\{S_k\}_{k \in \mathbb{Z}}$ to the ray with steps S_k. Note that if $C_1(\mathbf{x})$ is a fragment from an element $\mathbf{y}$ in Y, then $C_1(\Phi(\mathbf{x}))$ is also a fragment from Y, by Lemma 4 and also by the definition of Φ. Note also (see Figure 2) that $C_1(\mathbf{x}_{\text{init}})$ is a fragment from the spacetime diagram of the minimal element $(\underline{\mathbf{a}}, \mathbf{0})$ where

$$(\underline{\mathbf{a}}, \mathbf{0}) := \dots \underline{a} \xrightarrow{0} \underline{a} \xrightarrow{0} \dots \underline{a} \xrightarrow{0} \underline{a} \xrightarrow{\text{⧗}_0} \cdot \text{⧗⧗⧗} \dots .$$

Now define $C := F \circ C_1$. By the above remarks, C maps Ω into $X_{\mathcal{B}}$. Note that $C(\mathbf{x}_{\text{init}}) = (\underline{\mathbf{a}}, \mathbf{0})$, and that by induction

$$C(\Phi^n(\mathbf{x}_{\text{init}})) = V^n_{\mathcal{B}}(C(\mathbf{x}_{\text{init}})),$$

for all $n \in \mathbb{N}$. It can be seen that $C : \{\Phi^n(\mathbf{x}_{\text{init}})\}_{n \geq 0} \to \{V^n_{\mathcal{B}}((\underline{\mathbf{a}}, \mathbf{0}))\}_{n \geq 0}$ is a uniformly continuous bijection with a uniformly continuous inverse. Thus C can be extended to a conjugacy between (Ω, Φ) and $(X_{\mathcal{B}}, V_{\mathcal{B}})$. ∎

4 Examples

4.1 Substitutions

A *substitution* is a map $\boldsymbol{\tau} : \mathcal{A} \to \mathcal{A}^+$; we will write $\boldsymbol{\tau}(a) = \mathbf{a} = a_0 a_2 \ldots a_{l_a}$. We extend $\boldsymbol{\tau}$ to a map $\boldsymbol{\tau} : \mathcal{A}^+ \to \mathcal{A}^+$ by concatenation: if $\mathbf{a} = a_0 a_1 \ldots a_k$, then $\boldsymbol{\tau}(\mathbf{a}) := \boldsymbol{\tau}(a_0)\boldsymbol{\tau}(a_1)\ldots\boldsymbol{\tau}(a_k)$. In this way the composition $\boldsymbol{\tau} \circ \boldsymbol{\tau} : \mathcal{A}^+ \to \mathcal{A}^+$ is well defined; we will write $\boldsymbol{\tau}^{\circ n}$ to refer to the n-fold composition of $\boldsymbol{\tau}$. The substitution $\boldsymbol{\tau}$ is extended to a map $\boldsymbol{\tau} : \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$ defined by $\boldsymbol{\tau}(\ldots a_{-1} \cdot a_0\, a_1 \ldots) = (\ldots \boldsymbol{\tau}(a_{-1}) \cdot \boldsymbol{\tau}(a_0)\boldsymbol{\tau}(a_1)\ldots)$. We say $\boldsymbol{\tau}$ is *proper* if there exist $\underline{a}$, $\overline{a}$ in $\mathcal{A}$ such that for each $a \in \mathcal{A}$, $\boldsymbol{\tau}(a)$ starts with $\underline{a}$ and ends with $\overline{a}$. We say $\boldsymbol{\tau}$ is *primitive* if there exists a positive integer k such that for any $a \in \mathcal{A}$, all letters of $\mathcal{A}$ appear in $\boldsymbol{\tau}^{\circ k}(a)$ (this requires that for some letter a, $l_a \geq 1$). A *language* is a collection of words from $\mathcal{A}^+$. If $\mathbf{a} \in \mathcal{A}^{\mathbb{L}}$, the word $\mathbf{W}$ is a *factor of* $\mathbf{a}$ if there is some $n \in \mathbb{L}$ such that $a_n a_{n+1} \ldots a_{n+|\mathbf{W}|} = \mathbf{W}$. If $\mathcal{L}_{\boldsymbol{\tau}}$ is the language generated by the words $\boldsymbol{\tau}^{\circ n}(a)$ with n positive and $a \in \mathcal{A}$, then let $X_{\boldsymbol{\tau}}$ be the subshift associated with $\mathcal{L}_{\boldsymbol{\tau}}$. We call $X_{\boldsymbol{\tau}}$ the *(substitution) subshift defined by* $\boldsymbol{\tau}$. Henceforth we assume that $\boldsymbol{\tau}$ is primitive. In that case, $(X_{\boldsymbol{\tau}}, \sigma)$ is *minimal* (every $x \in X_{\boldsymbol{\tau}}$ has a dense orbit). For details see [Fog02]. A *fixed point* of $\boldsymbol{\tau}$ is a sequence $\mathbf{a} \in \mathcal{A}^{Z}$ such that $\boldsymbol{\tau}(\mathbf{a}) = \mathbf{a}$. If $\mathbf{a}$ is a fixed point for $\boldsymbol{\tau}$, then $\boldsymbol{\tau}(a_0)$ starts with a_0 and $\boldsymbol{\tau}(a_{-1})$ ends with a_{-1}. Conversely, if there exist letters r, l such that $\boldsymbol{\tau}(l)$ ends with l and $\boldsymbol{\tau}(r)$ starts with r, then $\mathbf{a} := \lim_{n\to\infty} \boldsymbol{\tau}^{\circ n}(l) \cdot \boldsymbol{\tau}^{\circ n}(r)$ is the unique fixed point satisfying $a_{-1} = l$ and $a_0 = r$. The fixed point $\mathbf{a}$ is *admissible for* $\boldsymbol{\tau}$ if the word $a_{-1}a_0$ occurs in $\boldsymbol{\tau}^{\circ n}(a)$, for some $a \in \mathcal{A}$ and some positive n. If $\mathbf{a}$ is admissible, and $\boldsymbol{\tau}$ is primitive, then $\mathcal{O}_\sigma(\mathbf{a}) := \overline{\{\sigma^n(\mathbf{a})\}} = X_{\boldsymbol{\tau}}$. Using the pigeonhole principle, there exists $n \geq 1$ such that $\boldsymbol{\tau}^{\circ n}$ has at least one admissible fixed point, and since $\boldsymbol{\tau}$ and $\boldsymbol{\tau}^{\circ n}$ define the same subshift, we can assume that any primitive substitution subshift is the orbit closure of some admissible fixed point. If $\boldsymbol{\tau}$ is proper, then it has a unique (admissible) fixed point. The substitution $\boldsymbol{\tau}$ is called *aperiodic* if $\mathcal{O}_\sigma(\mathbf{a})$ is infinite, i.e. if $\mathbf{a}$ is not σ-periodic.

4.1.1 The Bratteli diagram associated with a proper substitution

Let $\boldsymbol{\tau}$ be a proper substitution on $\mathcal{A}$, with all words $\boldsymbol{\tau}(a)$ starting with $\underline{a}$. The *Bratteli diagram associated with* $\boldsymbol{\tau}$ has vertex sets $\mathcal{V}_n = \{(a,n) : a \in \mathcal{A}\}$, for each $n \geq 1$ (in the notation of Section 3, we have $\boldsymbol{\tau}^n := \boldsymbol{\tau}$ for all $n \in \mathbb{N}$). There is exactly one edge from each vertex in $\mathcal{V}_1$ to $\mathbb{X}$. If $1 \leq m < n$, then the number of paths in $\mathcal{B}$ from (a,n) to (b,m) is the number of occurences of b in $\boldsymbol{\tau}^{\circ(n-m)}(a)$. If $\boldsymbol{\tau}$ is primitive, then there is a positive k such that for any two letters a and b, there is at least one path from $(a, n+k)$ to (b,n). The Bratteli diagram $\mathcal{B}' = (\mathcal{V}', \mathcal{E}')$ for $\boldsymbol{\tau}^{\circ k}$ is the telescoping of the Bratteli diagram $\mathcal{B} = (\mathcal{V}, \mathcal{E})$ for $\boldsymbol{\tau}$, with $\mathcal{V}'_n = \mathcal{V}_{nk}$ for $n > 1$ and $\mathcal{V}'_1 = \mathcal{V}_1$. Thus, Bratteli diagrams associated with proper substitutions are simple, focused and of bounded width. An example of such a Bratteli diagram is given in Figure 3.

4.2 Odometers

Let $\mathbf{q} := (q_1, q_2, \ldots)$ be an ordered set of integers ≥ 2 (the *quotient set*). Let $\mathcal{Z}(\mathbf{q}) := \prod_{l=1}^{\infty} \mathbb{Z}_{q_l}$ be the Cartesian product set. Then $(\mathcal{Z}(\mathbf{q}), \oplus)$ is a (compact, zero-dimensional, profinite, abelian) group where "$\oplus$" is defined as addition "with carry": if $\mathbf{x} = (\ldots, x_2, x_1)$

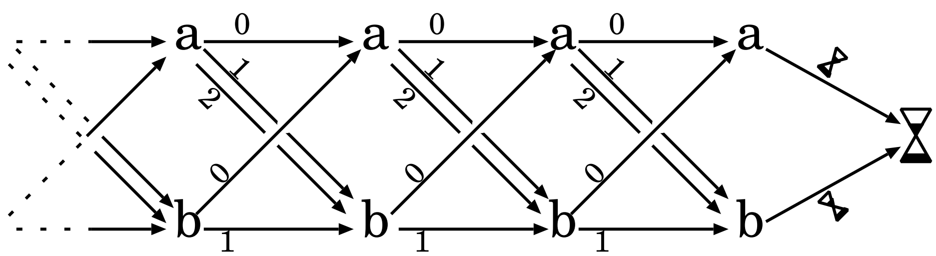

Figure 3: The Bratteli diagram associated with the substitution $\boldsymbol{\tau}(a) = abb,\ \ \boldsymbol{\tau}(b) = ab$.

and $\mathbf{y} = (\ldots, y_2, y_1)$, then $\mathbf{x} \oplus \mathbf{y} := (\ldots, r_2, r_1)$ where $x_1 + y_1 = k_1 q_1 + r_1$ for some $k_1 \geq 0$ and $0 \leq r_1 < q_1$, and for each $n \geq 2$,

$$k_{n-1} + x_n + y_n = k_n q_n + r_n,$$

with $k_n \geq 0$ and $0 \leq r_n < q_n$. If p is prime, then the *multiplicity of* p *in* $\mathbf{q}$ is the sum number of times (possibly infinite) that p occurs in the prime decomposition of all the elements of sequence $\mathbf{q}$.

Let $\mathbf{1} := (\ldots, 0, 0, 1)$. We define the *odometer* $\varsigma : \mathcal{Z}(\mathbf{q}) \to \mathcal{Z}(\mathbf{q})$ as $\varsigma(\mathbf{g}) = \mathbf{g} \oplus \mathbf{1}$. In other words, an odometer is a group rotation on $\mathcal{Z}(\mathbf{q})$.

Theorem 6 *$(\mathcal{Z}(\mathbf{q}), \varsigma)$ and $(\mathcal{Z}(\mathcal{Q}^*), \varsigma)$ are topologically conjugate if and only if every prime p has equal multiplicity in $\mathbf{q}$ and $\mathcal{Q}^*$.* [Dow05, Thm 1.2]

Thus we can assume that elements in $\mathbf{q}$ are prime. If $q_i = q$ for each i then we will write $\mathcal{Z}(q)$ for $\mathcal{Z}(\mathbf{q})$; this is know as the "q-adic" odometer, and is a model for "base q" arithmetic. Odometers have an adic representation where each $\mathcal{V}_n$ is a one-element set and for each n, there are q_{n+1} edges in $\mathcal{E}_{n+1}$. Adic systems are often seen as "generalised odometers".

The connection between adic systems, substitutions and many odometers is given by the next result:

Theorem 7 *Let $\boldsymbol{\tau}$ be a primitive substitution, with fixed point $\mathbf{x}$.*

1. *If $\boldsymbol{\tau}$ is aperiodic, then $(X_{\boldsymbol{\tau}}, \sigma, \mathbf{x})$ is pointedly isomorphic to $(X_{\mathcal{B}}, V_{\mathcal{B}}, \mathbf{x}_{\min})$, for some Bratteli diagram arising from a proper aperiodic substitution.*

2. *If $\boldsymbol{\tau}$ is periodic, then $(X_{\mathcal{B}}, V_{\mathcal{B}})$ is isomorphic to the odometer $(\mathcal{Z}(\ldots, k, k, p), \varsigma)$ where p is the periodicity of $X_{\boldsymbol{\tau}}$, and $k \in \mathbb{N}$.*

[For97, Thm 17] or [DHS99, Prop 20]

Corollary 8 1. *Aperiodic, primitive substitution systems can be embedded in a CA.*

2. *Odometers whose quotient set has finitely many primes can be embedded in a CA.*

Proof: The first statement follows from Theorem 7.

To prove the second statement: Suppose that $\mathbf{q}_f$ and $\mathbf{q}_i$ are the set of primes in $\mathbf{q}$ with finite and infinite multiplicity respectively. Let $N := (\prod_{p \in \mathbf{q}_f} p)$ and $M := (\prod_{p \in \mathbf{q}_i} p)$. Theorem 6 tells us that we can assume that $\mathbf{q} = (\dots, M, M, N)$. If $\boldsymbol{\tau}$ is the substitution on a 1-letter alphabet $\mathcal{A} = \{a\}$ defined by $\boldsymbol{\tau}(a) = a^M$, then the Bratteli system $(X_\mathcal{B}, V_\mathcal{B})$ associated to $\boldsymbol{\tau}$ is topologically isomorphic to $(\mathcal{Z}(M), \varsigma)$, by Theorem 7. If the Bratteli diagram for $\boldsymbol{\tau}$ is modified by putting N edges from the vertex a in $\mathcal{V}_1$ to ⧗, then the resulting Bratteli system is topologically isomorphic to $(\mathcal{Z}(\dots M, M, N), \varsigma)$. ∎

4.3 Toeplitz subshifts

If $\mathcal{A}$ is a finite alphabet, and $X = \mathcal{A}^\mathbb{Z}$, a *Toeplitz sequence* is a σ-aperiodic element $\mathbf{x} \in X$ such that for each $m \in \mathbb{Z}$, there is some p such that $x_{m+pn} = x_m$ for each $n \in \mathbb{Z}$. The *Toeplitz subshift* associated with $\mathbf{x}$ (also sometimes called the *Toeplitz flow*) is the subshift $(\mathcal{O}_\sigma(\mathbf{a}), \sigma)$. See [Dow05].

Given a Bratteli diagram $\mathcal{B}$ whose vertices are labelled from $\mathcal{A}$, we say that $\mathcal{B}$ has the *equal path number property* if for each $n \geq 1$, l_a^n is independent of a. A dynamical system (Y, T) with Y a metric space is *expansive* if there exists some $\epsilon > 0$ such that for each $\mathbf{x} \neq \mathbf{y}$, there is some integer n such that $d(T^n(\mathbf{x}), T^n(\mathbf{y})) \geq \epsilon$.

Theorem 9 *The family of expansive adic systems associated to properly ordered Bratteli diagrams with the equal path number property is upto conjugacy, the family of Toeplitz flows.* [GJ00, Thm8]

Lemma 10 *Given a Toeplitz system (Y, σ), the Bratteli diagram constructed in Theorem 9 is focused.*

Proof: The proof of Theorem 9 involves finding a sequence of collections of words $\{\mathcal{W}_n\}_n$, and a sequence of words $\mathbf{W}_n \in \mathcal{W}_n$ such that all words in $\mathcal{W}_{n+1}$ are concatenations of words from $\mathcal{W}_n$, and also begin with the word $\mathbf{W}_n$. In fact if $\mathbf{x}$ is the Toeplitz sequence, there will be a sequence p_n such that $\mathbf{W}_n = x_0 x_1 \dots x_{p_n - 1}$. The constructed Bratteli diagram will have vertices in $\mathcal{V}_n$ labelled by words in $\mathcal{W}_n$, and, if $\mathbf{W} \in \mathcal{W}_{n+1}$ is such that it is a concatenation $\mathbf{W} = \mathbf{W}^1 \mathbf{W}^2 \dots \mathbf{W}^k$ of words in $\mathcal{W}_n$, there will be an ordered edge with label i from the vertex representing $\mathbf{W}$ in $\mathcal{V}_{n+1}$ to the vertex in $\mathcal{V}_n$ representing $\mathbf{W}^i$. Since all words in $\mathcal{W}_{n+1}$ begin with $\mathbf{W}_n$, this means that all minimal edges in $\mathcal{E}_{n+1}$ have the same range. ∎

One can construct a Toeplitz sequence $\mathbf{x}$ in the following way, which is described in [Dow05]. Fix an alphabet $\mathcal{A}$. Pick an integer $s_1 \geq 2$, and pick a subset $F_1 \subset [0, s_1)$ and a $\mathcal{A}$-valued labelling of F_1, say $A_1 \in \mathcal{A}^{F_1}$. For each $n \in \mathbb{Z}$, let $x_{ns_1 + F_1} = A_1$. Let $\mathrm{Per}_{s_1}(\mathbf{x}) := F_1 + s_1\mathbb{Z}$. Given s_k, choose s_{k+1} such that s_{k+1} is an essential multiple of s_k. Choose $F_{k+1} \subset [0, s_{k+1} - 1) \backslash \mathrm{Per}_{s_k}(\mathbf{x})$, and choose $A_{k+1} \in \mathcal{A}^{F_{k+1}}$. Let $x_{ns_{k+1} + F_{K+1}} = A_{k+1}$ for all integers n. For any integer n, the sets $[ns_k, (n+1)s_k)$ are *k-intervals*, and the coordinates in a k-interval not filled in during the first k steps in the construction are called *k holes*. The choices above must be made so that eventually all of $\mathbf{x}$ is filled in. Also, after completing this procedure, we redo the construction, at each stage k re-defining F_k so that all essentially s_k-periodic parts of $\mathbf{x}$ are filled in.

The next lemma identifies which properly ordered Bratteli diagrams with the equal path number property have bounded width:

Lemma 11 *Let* $\mathbf{x}$ *be a Toeplitz sequence constructed in the above manner. Suppose that for each* k, $s_{k+1}/s_k \leq K$, *and also so that there are at most* $log_{|\mathcal{A}|}(K)$ k*-holes. Then the Bratteli diagram constructed in the proof of Theorem 9 is of bounded width* K.

Proof: If there are $log_{|\mathcal{A}|}(K)$ k-holes, this means that there are at most K elements in $\mathcal{W}_k$, where the sets $\mathcal{W}_k$ are those defined in the proof of Proposition 10. This bounds the number of vertices in $\mathcal{V}_k$. If $s_{k+1}/s_k \leq K$, this means that each word in $\mathcal{W}_{k+1}$ are concatenations of at most K words from $\mathcal{W}_k$. Thus there at most K edges emanating from each vertex in $\mathcal{W}_{k+1}$. ■

Corollary 12 *Toeplitz systems satisfying the conditions of Lemma 11 can be embedded in a cellular automaton.*

Remark: Some non-primitive substitutions systems also have adic representations, which are focused and of bounded width. For example, the *Chacon substitution* $\boldsymbol{\tau}(a) = aaba$, $\boldsymbol{\tau}(b) = b$ is conjugate to an *induced system* of the substitution $\boldsymbol{\tau}^*(a) = aab$, $\boldsymbol{\tau}^*(b) = abb$ ([GJ00, §4.2]). Which aperiodic substitutions have a focused bounded width representation? The same can be asked of *finite rank* transformations obtained using the "cutting and stacking" method with a bounded number of cuts and spacers: do they have a focused bounded width Bratteli representation?

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