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ORDERS THAT YIELD HOMEOMORPHISMS ON BRATTELI DIAGRAMS

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Abstract. We call an order ω on a Bratteli diagram B *perfect* if its Vershik map is a homeomorphism. In this paper we study the set of orders on a diagram and find necessary and sufficient conditions for an order to be perfect, in particular when the order has several extremal paths. This work generalizes previous results, obtained for finite rank Bratteli diagrams. We describe an explicit procedure to create perfect orderings on Bratteli diagrams based on the study of certain relations between the entries of the diagram's incidence matrices and properties of the associated graphs, with the latter relations characterizing diagrams which support perfect orderings. Also, we apply our theory to give a new combinatorial proof of the fact that the dimension group of a diagram supporting perfect orderings with k maximal paths has a copy of $\mathbb{Z}^{k-1}$ contained in their infinitesimal subgroup. Under certain conditions, we show that a similar result holds if the diagram supports countably many maximal paths. Our results are illustrated by numerous examples.

1. Introduction

A *Bratteli diagram* B (Definition 2.1) is an infinite graph which encodes how a space X_B is to be cut up into arbitrarily small pieces, in order that a dynamics be defined on it, by shifting these pieces. The space X_B is represented as the set of infinite paths on B starting at a distinguished vertex, and the dynamics is given by an *order* ω (Section 2.2) which describes how those pieces are to be shifted, or *stacked*. The dynamics φ_ω, called a *Vershik map,* consists of moving up the stack, and φ_ω is defined and continuous everywhere except at points which are always at the top of a tower. In this article we are concerned with characterising *perfect* orders, which are those where φ_ω extends to a homeomorphism. We recall the following crucial fact that emphasizes the importance of perfect orderings: the class of *aperiodic* (Definition 2.2) Bratteli diagrams with perfect orderings is in a one-to-one correspondence with the set of aperiodic homeomorphisms of a Cantor set [Med06].

This paper is a natural extension of our previous work [BKY13] that was devoted to the study of perfect orderings on Bratteli diagrams of *finite* rank. We summarize in Section 3.1 the relevant results from [BKY13], and refer to the article for a more detailed analysis. We restrict our attention to *regular* Bratteli diagrams (Definition 2.3). For these diagrams, if ω is any order on B that admits a Vershik map on X_B, then this map is unique. For any order ω on a regular diagram B, the sets $X_{\max}(\omega)$ and $X_{\min}(\omega)$ are closed and nowhere dense in the path set X_B.

Suppose that the Vershik map φ_ω can be extended to a homeomorphism. Then the order's extremal edge structure, called its *skeleton* (Definition 3.9), constrains its *languages* (Definition 2.5): words in this language must generate paths in certain directed graphs $\mathcal{H}_n$ (Definition

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3.19) that are defined by the skeleton. The skeleton also defines the extension of φ_ω to the set of maximal paths of ω. This extension must satisfy the properties described in Definition 3.13; we call orders which satisfy these properties a *correspondence*. Conversely, given a Bratteli diagram, skeleton and a correspondence, our main result is that we characterise, in Theorem 4.6, which orders extend to homeomorphisms, using the fact that their language must correspond to valid Eulerian paths in $\mathcal{H}_n$.

The last section contains a new proof of known results stated in [GPS95] which, in turn, are based on the earlier paper [Put89]: namely that if G is the dimension group of a simple Bratteli diagram B, such that B supports a perfect order with exactly j maximal paths and j minimal paths, then the infinitesimal subgroup of G contains a subgroup isomorphic to $\mathbb{Z}^{j-1}$. Our combinatorial proof uses extensively the machinery of skeletons and associated graphs that we have developed, as well as our characterization of diagrams that support perfect orderings in Theorem 4.6. We also show that our proof can be extended to a work for a class of Bratteli diagrams that have countably many extremal paths, and believe that an appropriate version of these results holds for dimension groups of aperiodic diagrams. Here we mention recent results of [Han13], where given a dimension group whose infinitesimal subgroup contains $\mathbb{Z}^k$, concrete (equal row and column sum) Bratteli diagram representations of these dimension groups are found. Some of the examples in [Han13] can be shown to satisfy the conditions of Theorem 4.6. It would be interesting to characterize the dimension groups of diagrams that support perfect orderings.

We end with a few remarks. We find it useful to include a number of examples in the text that will help the reader to understand the concepts and statements. Our examples are mainly of simple diagrams, but the constructs extend to aperiodic diagrams in a similar fashion. The words “order” and “ordering” are mostly used as synonyms, although we often use the former for a specific order, and the latter for an arbitrary order chosen from $\mathcal{O}_B$.

2. Definitions and notation

2.1. **Bratteli diagrams.**

Definition 2.1. A *Bratteli diagram* is an infinite graph $B = (V, E)$ such that the vertex set $V = \bigcup_{i\geq 0} V_i$ and the edge set $E = \bigcup_{i\geq 1} E_i$ are partitioned into disjoint subsets V_i and E_i such that

(i) $V_0 = \{v_0\}$ is a single point;

(ii) V_i and E_i are finite sets;

(iii) there exist a range map r and a source map s from E to V such that $r(E_i) = V_i$, $s(E_i) = V_{i-1}$, and $s^{-1}(v) \neq \emptyset$, $r^{-1}(v') \neq \emptyset$ for all $v \in V$ and $v' \in V \setminus V_0$.

The pair (V_i, E_i) or just V_i is called the i-th level of the diagram B. A finite or infinite sequence of edges $(e_i : e_i \in E_i)$ such that $r(e_i) = s(e_{i+1})$ is called a *finite* or *infinite path* respectively. We write $e(v, v')$ to denote a finite path $e = (e_1, ..., e_k)$ such that $s(e) = v(= s(e_1))$ and $r(e) = v'(= r(e_k))$, and let $E(v, v')$ denote the set of all such paths. For a Bratteli diagram B, let X_B denote the set of infinite paths starting at the top vertex v_0. We endow X_B with the topology generated by cylinder sets $U(e_k, \ldots, e_n) := \{x \in X_B : x_i = e_i,\ i = k, \ldots, n\}$, where$(e_k, \ldots, e_n)$ is a finite path in B from level k to level n. Then X_B is a 0-dimensional compact metric space with respect to this topology. *We assume throughout the paper that X_B has no isolated points for all considered Bratteli diagrams B.*

Given a Bratteli diagram B, the n-th *incidence matrix* $F_n = (f^{(n)}_{v,w}),\ n \geq 0$, is a $|V_{n+1}| \times |V_n|$ matrix whose entries $f^{(n)}_{v,w}$ are equal to the number of edges between the vertices $v \in V_{n+1}$ and $w \in V_n$, i.e.

$$f^{(n)}_{v,w} = |\{e \in E_{n+1} : r(e) = v, s(e) = w\}|.$$

Observe that every vertex $v \in V$ is connected to v_0 by a finite path and the set $E(v_0, v)$ of all such paths is finite. Set $h^{(n)}_v = |E(v_0, v)|$ for $v \in V_n$. Then

$$h^{(n+1)}_v = \sum_{w \in V_n} f^{(n)}_{v,w} h^{(n)}_w \ \text{ or } \ h^{(n+1)} = F_n h^{(n)}$$

where $h^{(n)} = (h^{(n)}_w)_{w \in V_n}$.

Next we define some popular families of Bratteli diagrams. We say that B is *simple* if for any level n there is $m > n$ such that $E(v, w) \neq \emptyset$ for all $v \in V_n$ and $w \in V_m$. We say B is *stationary* if $F_n = F_1$ for all $n \geq 2$. We say B has *finite rank* if for some k, $|V_n| \leq k$ for all $n \geq 1$. Let B have finite rank. We say B has *rank d* if d is the smallest integer such that $|V_n| = d$ infinitely often. We say that B has *strict rank d* if $|V_n| = d$ for each $n \geq 1$, and if $|r^{-1}(v)| \geq 2$ for each $v \in V \backslash V_0$. If V has strict rank d, we will assume that for any n the vertex set V_n does not depend on n and equals the set V that has exactly d vertices. In this article we will consider general Bratteli diagrams, which are not necessarily of finite rank.

For a Bratteli diagram B, the *tail (cofinal) equivalence* relation $\mathcal{E}$ on the path space X_B is defined as $x\mathcal{E}y$ if $x_n = y_n$ for all n sufficiently large, where $x = (x_n)$, $y = (y_n)$. Let $X_{per} = \{x \in X_B : |[x]_{\mathcal{E}}| < \infty\}$, with $[x]_{\mathcal{E}}$ denoting the $\mathcal{E}$ equivalence class of x. By definition, we have $X_{per} = \{x \in X_B : \exists n > 0 \text{ such that } (|r^{-1}(r(x_i))| = 1\ \forall i \geq n)\}$. A Bratteli diagram B is called *aperiodic* if $X_{per} = \emptyset$, i.e., every $\mathcal{E}$-orbit is countably infinite. *Throughout the paper, we only consider aperiodic Bratteli diagrams B. For these diagrams X_B is a Cantor set and $\mathcal{E}$ is a Borel equivalence relation on X_B with uncountably infinitely many equivalence classes.*

Let B be a Bratteli diagram, and $n_0 = 0 < n_1 < n_2 < \ldots$ be a strictly increasing sequence of integers. The *telescoping of B to (n_k)* is the Bratteli diagram B', whose k-level vertex set $V'_k = V_{n_k}$ and whose incidence matrices (F'_k) are defined by

$$F'_k = F_{n_{k+1}-1} \circ \ldots \circ F_{n_k},$$

where (F_n) are the incidence matrices for B. Thus when one telescopes B one takes a subsequence of levels $\{n_k\}$ and considers the set $E(n_k, n_{k+1})$ of all finite paths between the levels $\{n_k\}$ and $\{n_{k+1}\}$ as edges of the new diagram. In particular, a Bratteli diagram B has rank d if and only if there is a telescoping B' of B such that B' has exactly d vertices at each level.

Next we define a family of aperiodic diagrams that have an incidence matrix structure that is useful for our purposes.

Definition 2.2. We define the family $\mathcal{A}$ of Bratteli diagrams, all of whose incidence matrices are of the form

$$F_n := \begin{pmatrix} A_n^{(1)} & 0 & \dots & 0 & 0 \\ 0 & A_n^{(2)} & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & A_n^{(k)} & 0 \\ B_n^{(1)} & B_n^{(2)} & \dots & B_n^{(k)} & C_n \end{pmatrix} \tag{2.1}$$

where for some sequences $(d_1^{(n)})_n, \dots, (d_k^{(n)})_n$ and $(d^{(n)})_n$,

(1) $A_n^{(i)}$ is a $d_i^{(n+1)} \times d_i^{(n)}$ matrix,
(2) all matrices $A_n^{(i)}$, $B_n^{(i)}$ and C_n are strictly positive, and
(3) C_n is a $d^{(n+1)} \times d^{(n)}$ matrix.

2.2. **Orders on a Bratteli diagram.** A Bratteli diagram $B = (V, E)$ is called *ordered* if a linear order '>' is defined on every set $r^{-1}(v)$, $v \in \bigcup_{n\geq 1} V_n$. We use ω to denote the corresponding partial order on E and write (B, ω) when we consider B with the ordering ω. Denote by $\mathcal{O}_B$ the set of all orderings on B.

Every $\omega \in \mathcal{O}_B$ defines the *lexicographic* ordering on the set $E(k, l)$ of finite paths between vertices of levels V_k and V_l: $(e_{k+1}, ..., e_l) > (f_{k+1}, ..., f_l)$ if and only if there is i with $k+1 \leq i \leq l$, $e_j = f_j$ for $i < j \leq l$ and $e_i > f_i$. It follows that, given $\omega \in \mathcal{O}_B$, any two paths from $E(v_0, v)$ are comparable with respect to the lexicographic ordering generated by ω. If two infinite paths are tail equivalent, and agree from the vertex v onwards, then we can compare them by comparing their initial segments in $E(v_0, v)$. Thus ω defines a partial order on X_B, where two infinite paths are comparable if and only if they are tail equivalent. We call a finite or infinite path $e = (e_i)$ *maximal (minimal)* if every e_i is maximal (minimal) amongst the edges from $r^{-1}(r(e_i))$.

Notice that, for $v \in V_i$, $i \geq 1$, the minimal and maximal (finite) paths in $E(v_0, v)$ are unique. Denote by $X_{\max}(\omega)$ and $X_{\min}(\omega)$ the sets of all maximal and minimal infinite paths in X_B, respectively. It is not hard to show that $X_{\max}(\omega)$ and $X_{\min}(\omega)$ are *non-empty closed subsets* of X_B; in general, $X_{\max}(\omega)$ and $X_{\min}(\omega)$ may have interior points. For a finite rank Bratteli diagram B, the sets $X_{\max}(\omega)$ and $X_{\min}(\omega)$ are always finite for any ω, and if B has rank d, then each of them have at most d elements ([BKMS10]). An ordered Bratteli diagram (B, ω) is called *properly ordered* if the sets $X_{\max}(\omega)$ and $X_{\min}(\omega)$ are singletons. We denote by $\mathcal{O}_B(j)$ the set of all orders on B which have j maximal and j minimal paths. Thus, in our notation, $\mathcal{O}_B(1)$ is the set of proper orders on B.

Let (B, ω) be an ordered Bratteli diagram, and suppose that $B' = (V', E')$ is the telescoping of B to levels (n_k). Let $v' \in V'$ and suppose that the two edges e'_1, e'_2, both with range v', correspond to the finite paths e_1, e_2 in B, both with range v. Define the order ω' on B' by $e'_1 < e'_2$ if and only if $e_1 < e_2$. Then ω' is called the *lexicographic order generated by* ω and is denoted by $\omega' = L(\omega)$. It is not hard to see that if $\omega' = L(\omega)$, then

$$|X_{\max}(\omega)| = |X_{\max}(\omega')|, \quad |X_{\min}(\omega)| = |X_{\min}(\omega')|.$$

If B is an aperiodic Bratteli diagram and $\omega \in \mathcal{O}_B$, then $X_{\max}(\omega) \cap X_{\min}(\omega) = \emptyset$.

Definition 2.3. A Bratteli diagram B is called *regular* if for any ordering $\omega \in \mathcal{O}_B$ the sets $X_{\max}(\omega)$ and $X_{\min}(\omega)$ have empty interior.

In particular, finite rank Bratteli diagrams are regular, and if all incidence matrix entries of B are at least 2, then B is regular. *In this article we assume that all diagrams are regular.*

Definition 2.4. Let (B,ω) be an ordered, regular Bratteli diagram. We say that $\varphi = \varphi_\omega : X_B \to X_B$ is a *Vershik map* if it satisfies the following conditions:
(i) φ is a homeomorphism of the Cantor set X_B;
(ii) $\varphi(X_{\max}(\omega)) = X_{\min}(\omega)$;
(iii) if an infinite path $x = (x_1, x_2, \ldots)$ is not in $X_{\max}(\omega)$, then $\varphi(x_1, x_2, \ldots) = (x_1^0, \ldots, x_{k-1}^0, \overline{x_k}, x_{k+1}, x_{k+2}, \ldots)$, where $k = \min\{n \geq 1 : x_n \text{ is not maximal}\}$, $\overline{x_k}$ is the successor of x_k in $r^{-1}(r(x_k))$, and $(x_1^0, \ldots, x_{k-1}^0)$ is the minimal path in $E(v_0, s(\overline{x_k}))$.

If ω is an ordering on B, then one can always define the map φ_0 that maps $X_B \setminus X_{\max}(\omega)$ onto $X_B \setminus X_{\min}(\omega)$ according to (iii) of Definition 2.4. The question about the existence of the Vershik map is equivalent to that of an extension of $\varphi_0 : X_B \setminus X_{\max}(\omega) \to X_B \setminus X_{\min}(\omega)$ to a homeomorphism of the entire set X_B. If ω is a proper ordering, then φ_ω is a homeomorphism. For a finite rank Bratteli diagram B, the situation is simpler than for a general Bratteli diagram because the sets $X_{\max}(\omega)$ and $X_{\min}(\omega)$ are finite. Note that Vershik did not assume that his maps where homeomorphisms, as he was working in the measurable context. We say that an ordering $\omega \in \mathcal{O}_B$ is *perfect* if ω admits a Vershik map φ_ω on X_B. Denote by $\mathcal{P}_B$ the set of all perfect orderings on B. We observe that for a regular Bratteli diagram with an ordering ω, the Vershik map φ_ω, if it exists, is defined in a unique way. Also, a necessary condition for $\omega \in \mathcal{P}_B$ is that $|X_{\max}(\omega)| = |X_{\min}(\omega)|$. Given (B,ω) with $\omega \in \mathcal{P}_B$, the uniquely defined system (X_B, φ_ω) is called a *Bratteli-Vershik* system.

2.3. **The languages of an ordered Bratteli diagram.** If V is a finite alphabet, let V^+ denote the set of nonempty words over V. We use the notation $W' \subseteq W$ to indicate that W' is a subword of W. If $W_1, W_2, \ldots, W_n$, are words, then we let $\prod_{i=1}^n W_i$ refer to their concatenation.

Let ω be an order on a Bratteli diagram B. Fix a vertex $v \in V_n$ and some level $m < n$, consider the set $E(V_m, v) = \bigcup_{v' \in V_m} E(v', v)$ of all finite paths between vertices of level m and v. This set can be ordered by ω: $E(V_m, v) = \{e_1, \ldots e_p\}$ where $e_i < e_{i+1}$ for $1 \leq i \leq p-1$. Define the word $w(v, m, n) := s(e_1)s(e_2)\ldots s(e_p)$ over the alphabet V_m. If $W = v_1 \ldots v_r \in V_n^+$, let $w(W, n-1, n) := \prod_{i=1}^r w(v_i, n-1, n)$.

Definition 2.5. The *level-n language* $\mathcal{L}(B,\omega,n)$ *of* (B,ω) is

$$\mathcal{L}(B,\omega,n) := \{W : \; W \subset w(v,n,N), \quad \text{for some } v \in V_N, N > n\}.$$

If B has strict rank d, then each of the level-n languages can be defined on a common alphabet V, and in this case we recover our definition of *the* language $\mathcal{L}(B,\omega)$ that we introduced in [BKY13]:

$$\mathcal{L}(B,\omega) := \limsup_n \mathcal{L}(B,\omega,n).$$

3. Skeletons on Bratteli diagrams

3.1. **Skeletons and associated graphs on finite rank Bratteli diagrams.** In this section we review definitions and results from [BKY13] on finite rank diagrams. We do this mainly to set the stage for generalizing these notions to nonfinite rank diagrams, in Section 3.2. Thus our discussion of finite rank ordered diagrams will be concise. For more details on definitions and results, we refer the reader to Section 3 in [BKY13].

Suppose that B has strict rank d. If a maximal (minimal) path M (m) goes through the same vertex v_M (v_m) at each level of B, we will call this path *vertical*. The following proposition characterizes when ω is a perfect order on a finite rank Bratteli diagram, and was proved in ([BKY13, Proposition 3.2, Lemma 3.3]) for finite rank diagrams.

Proposition 3.1. *Let (B,ω) be an ordered Bratteli diagram.*

(1) Suppose that B has strict rank d and that the ω-maximal and ω-minimal paths $M_1,...,M_k$ and $m_1,...,m_{k'}$ are vertical passing through the vertices $v_{M_1},\dots,v_{M_k}$ and $v_{m_1},\dots,v_{m_{k'}}$ respectively. Then ω is perfect if and only if
  (a) $k=k'$,
  (b) there is a permutation σ of $\{1,\dots k\}$ such that for each $i\in\{1,...,k\}$, $v_{M_i}v_{m_j}\in\mathcal{L}(B,\omega)$ if and only if $j=\sigma(i)$.

(2) Let B' be a telescoping of B. Then $\omega\in\mathcal{P}_B$ if and only if $\omega'=L(\omega)\in\mathcal{P}_{B'}$.

We give an example of how one would apply Proposition 3.1. This example also answers negatively the following question, that is related to Statement 2 of Proposition 3.1. *Let B be a Bratteli diagram and B' a telescoping of B. Is it true that any perfect order on B' is obtained by telescoping of a perfect order on B?*

Example 3.2. We define a stationary Bratteli diagram B such that for a telescoped diagram B' there is a perfect order $\omega'\in\mathcal{P}_{B'}$ satisfying the condition $\omega'\neq L(\omega)$ for any perfect order ω on B.

Let B be a stationary Bratteli diagram defined on the set of four vertices $\{a,b,c,d\}$ by the incidence matrix

$$F=\begin{pmatrix}2&1&1&1\\1&2&1&1\\1&1&2&1\\1&1&1&2\end{pmatrix}.$$

Let B' be the diagram obtained by telescoping B to all odd levels; it has incidence matrix

$$F'=F^2=\begin{pmatrix}7&6&6&6\\6&7&6&6\\6&6&7&6\\6&6&6&7\end{pmatrix}.$$

In order to define a perfect order ω', we let, for each n, $w(a,n,n+1)=(adbc)^6a$, $w(b,n,n+1)=(bcad)^6b$, $w(c,n,n+1)=(cadb)^6c$, and, finally, for $r^{-1}(d)$ we set $w(d,n,n+1)=bcad^7(bca)^5$. This appearance of d^7 prevents ω' from being a lexicographical order on B' generated by any choice of ω on B. On the other hand, using Proposition 3.1, we can verify that $\omega'\in\mathcal{P}_{B'}$. For, the set of words of length 2 that belong to $\mathcal{L}(B',\omega')$ are $\{ab,ad,bc,ca,db,dd\}$ and σ :

$\{a,b,c\} \to \{a,b,c\}$ defined by $\sigma(a)=b$, $\sigma(b)=c$ and $\sigma(c)=a$ satisfies part (1) of Proposition 3.1.

Let ω be an order on a Bratteli diagram B. If $v \in V\backslash V_0$, we denote the minimal edge with range v by $\overline{e}_v$, and we denote the maximal edge with range v by $\widetilde{e}_v$.

Definition 3.3. Let (B,ω) be an ordered rank d diagram. We say that (B,ω) is *well telescoped* if

(1) B has strict rank d,
(2) all ω-extremal paths are vertical, with $\widetilde{V}$, $\overline{V}$ denoting the sets of vertices through which maximal and minimal paths run respectively, and
(3) $s(\widetilde{e}_v) \in \widetilde{V}$ and $s(\overline{e}_v) \in \overline{V}$ for each $v \in V\backslash(V_0 \cup V_1)$, and this is independent of n.

If (B,ω) is perfectly ordered, for it to be considered well telescoped, it will also have to satisfy

(4) if $\widetilde{v}\overline{v}$ appears as a subword of some $w(v,m,n)$ with $m \geq 1$, then , then $\sigma(\widetilde{v}) = \overline{v}$ defines a one-to-one correspondence between the sets $\widetilde{V}$ and $\overline{V}$.

Given an ordered finite rank (B,ω), it can always be telescoped so that it is well telescoped. For details of how this can be done, see Lemma 3.11 in [BKY13]. Thus, when we talk about a (finite rank) ordered diagram, we assume without loss of generality that it is well telescoped. For well telescoped ordered diagrams (B,ω), we have $s(\widetilde{e}_v) \in \widetilde{V}_n$ and $s(\overline{e}_v) \in \overline{V}_n$ for any $v \in V_{n+1},\ n \geq 1$. Given a well telescoped (B,ω), we call the set $\mathcal{F}_\omega = (\widetilde{V}, \overline{V}, \{\widetilde{e}_v, \overline{e}_v : v \in V_n,\ n \geq 2\})$ the *skeleton associated to* ω. If ω is a perfect order on B, it follows that $|\overline{V}| = |\widetilde{V}|$, and if $\sigma : \widetilde{V} \to \overline{V}$ is the permutation given by Proposition 3.1, we call σ the *accompanying permutation.*

The notion of a skeleton of an ordered diagram can be extended to an unordered diagram. Namely, given a strict rank d diagram B, we select, two subsets $\widetilde{V}$ and $\overline{V}$ of V, of the same cardinality, and, for each $v \in V\backslash V_0 \cup V_1$, we select two edges $\widetilde{e}_v$ and $\overline{e}_v$, both with range v, and such that $s(\widetilde{e}_v) \in \widetilde{V}$, $s(\overline{e}_v) \in \overline{V}$. In this way we can extend the definition of a skeleton $\mathcal{F} = (\widetilde{V}, \overline{V}, \{\widetilde{e}_v, \overline{e}_v : v \in V, \})$ to an unordered strict rank d Bratteli diagram, with the objective of creating well-telescoped orders. A more detailed discussion can be found in [BKY13]). Arbitrarily choosing a bijection $\sigma : \widetilde{V} \to \overline{V}$, we can consider the set of orders on B which have $\mathcal{F}$ as skeleton and σ as accompanying permutation.

Given a skeleton $\mathcal{F}$ on a finite rank diagram B, for any vertices $\widetilde{v} \in \widetilde{V}$ and $\overline{v} \in \overline{V}$, we set

$$W_{\widetilde{v}} = \{w \in V : s(\widetilde{e}_w) = \widetilde{v}\} \text{ and } W'_{\overline{v}} = \{w \in V : s(\overline{e}_w) = \overline{v}\}. \tag{3.1}$$

Then $W = \{W_{\widetilde{v}} : \widetilde{v} \in \widetilde{V}\}$ and $W' = \{W'_{\overline{v}} : \overline{v} \in \overline{V}\}$ are both partitions of V. We call W and W' the *partitions generated by* $\mathcal{F}$. Let $[\overline{v}, \widetilde{v}] := W'_{\overline{v}} \cap W_{\widetilde{v}}$, and define the partition

$$W \cap W' := \{[\overline{v}, \widetilde{v}] : \overline{v} \in \overline{V}, \widetilde{v} \in \widetilde{V}, [\overline{v}, \widetilde{v}] \neq \emptyset\}.$$

Let $\mathcal{F}$ be a skeleton on the strict finite rank B with accompanying permutation σ. Let $\mathcal{H} = (T,P)$ be the directed graph where the set T of vertices of $\mathcal{H}$ consists of partition elements $[\overline{v}, \widetilde{v}]$ of $W' \cap W$, and where there is an edge in P from $[\overline{v}, \widetilde{v}]$ to $[\overline{v}', \widetilde{v}']$ if and only if $\overline{v}' = \sigma(\widetilde{v})$. We call $\mathcal{H}$ the *directed graph associated to* $(B, \mathcal{F}, \sigma)$.

Example 3.4. Suppose that $\widetilde{V} = \overline{V} = \{a,b\}$, $\sigma(a)=a$, $\sigma(b)=b$, and the set of vertices of $\mathcal{H}$ is $[a,a],[a,b],[b,a],[b,b]$. Then $\mathcal{H}$ is illustrated in Figure 1. Note that we do not specify a skeleton here. In general, it is possible that for some skeletons one of the vertices $[b,a]$ or $[a,b]$ be degenerate, for example, if $W'_a \cap W_b = \emptyset$, then the vertex $[a,b]$ is not present in $\mathcal{H}$. If the permutation is $\sigma(a) = b$, $\sigma(b) = a$, then $\mathcal{H}$ is identical to that of Figure 1, except that the vertices $\{[a,a], [b,b], [a,b], [b,a]\}$ of the new graph are relabelled $\{[b,a], [a,b], [b,b], [a,a]\}$ respectively.

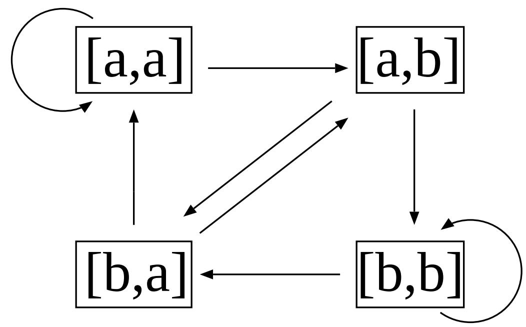

FIGURE 1. The associated graph $\mathcal{H}$ for $\omega \in \mathcal{P}_B(2)$, $\sigma(a) = a$, $\sigma(b) = b$.

Suppose that the strict finite rank B has skeleton $\mathcal{F}$ and accompanying permutation σ. Then any path in $\mathcal{H}$ corresponds to a family of words in V^+: for if $p_1 p_2 \dots p_k$ is a path in $\mathcal{H}$ where p_i has source $[\overline{v}_i, \widetilde{v}_i]$, and for each i, v_i is any vertex in $[\overline{v}_i, \widetilde{v}_i]$, then the path $p_1 p_2 \dots p_k$ corresponds to the word $v_1 v_2 \dots v_k$ (and many such words can exist). Conversely, if the word $v_1 v_2 \dots v_k$ is such that $v_i \in [\overline{v}_i, \widetilde{v}_i]$ for each i, and $[\overline{v}_1, \widetilde{v}_1]\,[\overline{v}_2, \widetilde{v}_2] \dots [\overline{v}_k, \widetilde{v}_k]$ is a path in $\mathcal{H}$, then we say that $v_1 v_2 \dots v_k$ *corresponds to a valid path in* $\mathcal{H}$. The relevance of $\mathcal{H}$ for perfect orders is described by the following lemma, which was proved in [BKY13, Remark 3.16, Lemma 3.17].

Lemma 3.5. *Let B be a strict finite rank Bratteli diagram, $\mathcal{F}$ be a skeleton on B and $\sigma : \widetilde{V} \to \overline{V}$ be a bijection. Let $\mathcal{H}$ be the associated directed graph. Suppose that the ordering ω on B has skeleton and accompanying permutation $(\mathcal{F}, \sigma)$, and such that each word in $\mathcal{L}(B, \omega)$ corresponds to a valid path in $\mathcal{H}$. Then ω is perfect. Conversely, if (B, ω) is a well telescoped, perfectly ordered diagram with skeleton and accompanying permutation $(\mathcal{F}, \sigma)$, then every word in $\mathcal{L}(B, \omega)$ corresponds to a valid path in $\mathcal{H}$.*

3.2. **Skeletons, associated graphs, and correspondences on general Bratteli diagrams.** If B is not of finite rank, the notion of a skeleton can be generalized, although the notation is more technical.

Definition 3.6. Suppose that (B, ω) is an ordered Bratteli diagram. To each maximal path M and minimal path m we associate the sequences $(v_n(M))$ and $(v_n(m))$ of vertices that M and m pass through. For each n, let $\widetilde{V}_n := \{v \in V_n : v = v_n(M) \text{ for some maximal path } M\}$; we call vertices in $\widetilde{V}_n$ *maximal vertices*. Similarly we can define $\overline{V}_n$, the set of *minimal vertices* in V_n.

In other words, for each $\widetilde{v} \in \widetilde{V}_n$, there is at least one infinite maximal path passing through $\widetilde{v}$, and for each $\overline{v} \in \overline{V}_n$, there is at least one infinite minimal path passing through $\overline{v}$. The following lemma tells us that the notion of a well telescoped ordered Bratteli diagram can be extended to general Bratteli diagrams:

Proposition 3.7. *Let (B,ω) be an ordered aperiodic Bratteli diagram. Then there exists a telescoping $(B',\omega') = ((V',E'),\omega')$ of (B,ω) to a sequence of levels (n_k), such that*

- *for every vertex $v \in V'$, any maximal edge $\widetilde{e}_v \in E'_k$ has source in $\widetilde{V}'_{k-1}$ and any minimal edge $\overline{e}_v$ has source in $\overline{V}'_{k-1}$, and*
- *all of B's incidence matrix entries that are nonzero are at least two.*

Proof. We use the idea of the proof of Proposition 2.8 from [HPS92]. Take $n_1 = 1$, write $V_1 = \widetilde{V}_1 \cup (V_1 \backslash \widetilde{V}_1)$. Take a vertex $v \in V_1 \backslash \widetilde{V}_1$ and consider all maximal edges e such that $s(e) = v$. We say that a maximal edge e is *extendable* if there is a maximal edge e' such that $r(e) = s(e')$. Let E_v be the set of all finite maximal paths consisting of extendable edges starting from v. The set E_v is finite, and this is true for any $v \in V_1 \backslash \widetilde{V}_1$. Therefore we can find n_2 such that if f is a maximal finite path with source in $V_1 \backslash \widetilde{V}_1$, then $r(f) \in V_n$ with $n < n_2$. Thus, all maximal paths with source in V_1 and range in V_{n_2} must have source in $\widetilde{V}_1$. We describe only the next step, the rest following by induction. Write $V_{n_2} = \widetilde{V}_{n_2} \cup (V_{n_2} \backslash \widetilde{V}_{n_2})$. Find an n_3 such that if f is a maximal finite path with source in $V_{n_2} \backslash \widetilde{V}_{n_2}$, then $r(f) \in V_n$ with $n < n_3$. Therefore all maximal paths with source in V_{n_2} and range in V_{n_3} must have source in $\widetilde{V}_{n_2}$. Continue, and telescope (B,ω) via levels (n_k). Since maximal edges in E'_k in the telescoped diagram (B',ω') correspond to maximal paths between $V_{n_{k-1}}$ and V_{n_k} in B, the result follows. An identical argument yields the result for minimal edges. □

Definition 3.8. Let (B,ω) be an ordered Bratteli diagram, where the sequences $(\widetilde{V}_n)$ and $(\overline{V}_n)$ consist of the maximal and minimal vertices, respectively. Suppose that for each $n \geq 2$ and each $v \in V_{n+1}$, $s(\widetilde{e}_v) \in \widetilde{V}_n$ and $s(\overline{e}_v) \in \overline{V}_n$. Suppose also that all of B's nonzero incidence matrix entries are at least two. Then we say that (B,ω) is *well telescoped*.

Proposition 3.7 tells us that we can assume that (B,ω) is well telescoped. Next we generalize the notion of a skeleton to an unordered, non-finite rank Bratteli diagram. Let B be a Bratteli diagram, where we assume that all nonzero entries of its incidence matrices are at least two.

Definition 3.9. Let $\{M_\alpha : \alpha \in I\}$, and $\{m_\beta : \beta \in J\}, |I| = |J|$, be closed, nowhere dense sets of infinite paths. For $n \in \mathbb{N}$, let

$$\widetilde{V}_n := \{\widetilde{v} : \widetilde{v} = v_n(M_\alpha) \text{ for some } \alpha \in I\} \text{ and } \overline{V}_n := \{\overline{v} : \overline{v} = v_n(m_\beta) \text{ for some } \beta \in J\}.$$

Let $\{\widetilde{e}_v : v \in V_{n+1}\}$ and $\{\overline{e}_v : v \in V_{n+1}\}$ be sets of edges such that $r(\widetilde{e}_v) = r(\overline{e}_v) = v$, $s(\widetilde{e}_v) \in \widetilde{V}_n$ and $s(\overline{e}_v) \in \overline{V}_n$. Suppose that for any n and any $N > n$,

(1) if $v_N(M_\alpha) = v_N(M'_\alpha)$, then $v_n(M_\alpha) = v_n(M'_\alpha)$, with the analogous condition holding for paths m_β and $m_{\beta'}$,
(2) if $v \in \widetilde{V}_n \cap \overline{V}_n$, then $\widetilde{e}_v \neq \overline{e}_v$, and
(3) if $n \in \mathbb{N}$, $\alpha \in I$ and $\beta \in J$, then $M_\alpha \in U(\widetilde{e}_v)$ whenever $v = v_n(M_\alpha)$ and $m_\beta \in U(\overline{e}_v)$ whenever $v = v_n(m_\beta)$.

Then we call $\mathcal{F} = (\widetilde{V}_{n-1}, \overline{V}_{n-1}, \{\widetilde{e}_v, \overline{e}_v : v \in V_n\} : n \geq 2)$ the *skeleton associated to* $\{M_\alpha : \alpha \in I\}$ *and* $\{m_\beta : \beta \in J\}$. Vertices in the sets $\widetilde{V}_n$ and $\overline{V}_n$ are called *maximal* and *minimal* vertices respectively. Paths M_α, $\alpha \in I$, are called *maximal* and form the set $X_{\max}(\mathcal{F})$, and paths m_β, $\beta \in J$, are called *minimal* and form the set $X_{\min}(\mathcal{F})$.

Remark 3.10. We note the following:

- If we are given a well telescoped ordered diagram (B,ω) and $|X_{\max}(\omega)| = |X_{\min}(\omega)|$, then we can define the skeleton generated by ω, by letting $\{M_\alpha : \alpha \in I\} = X_{\max}(\omega)$, $\{m_\beta : \beta \in J\} = X_{\min}(\omega)$, and choosing for each $v \in V \backslash V_0 \cup V_1$, $\widetilde{e}_v, \overline{e}_v$ to be the maximal and minimal edges with range v respectively. It is clear that if $\mathcal{F}$ is defined by such an order ω, then $X_{\max}(\mathcal{F}) = X_{\max}(\omega)$ and $X_{\min}(\mathcal{F}) = X_{\min}(\omega)$.
- We will not be concerned with orders ω such that $|X_{\max}(\omega)| \neq |X_{\min}(\omega)|$, as our aim is to characterize perfect orders and such a condition would prevent an order from being perfect.
- The requirement that $X_{\max}(\mathcal{F})$ and $X_{\min}(\mathcal{F})$ be closed is natural: after all, we are introducing skeletons to build orders, and in that case the latter sets must be closed.
- The regularity of B leads to the requirement that the sets $X_{\max}(\omega)$ and $X_{\min}(\omega)$ are nowhere dense.

Example 3.11. Suppose that $V_0 = \{v_0\}$, and for $n \geq 1$, $V_n = \widetilde{V}_n = \overline{V}_n = \{v_1, \dots v_n\}$, and all nonzero incidence matrix entries are at least two. Suppose also that for $v_i \in V_{n+1}$, we choose $\widetilde{e}_i \neq \overline{e}_i$ and define

$$s(\widetilde{e}_{v_i}) = \begin{cases} v_i & \text{if } i \neq n+1 \\ v_n & \text{if } i = n+1 \end{cases}, \text{ and } \ s(\overline{e}_{v_i}) = \begin{cases} v_i & \text{if } i \neq n+1 \\ v_1 & \text{if } i = n+1 \end{cases}.$$

In Figure 2 we have drawn (only) the extremal edges in B, with dashed lines represented maximal edges and solid lines representing minimal edges. Consider the sets of infinite paths $\{M_\alpha : \alpha \in \mathbb{N} \cup \{\infty\}\}$ whose edges consist of the identified maximal edges, so that M_1 passes vertically through the vertex v_1 at all levels, and for $i > 1$, M_i passes through vertices v_1, $v_2, \dots v_{i-1}, v_i$, and then goes down vertically through v_i. Finally, M_∞ passes through vertices $v_1, v_2, v_3, \dots$. Similarly consider the set of infinite paths $\{m_\beta : \beta \in \mathbb{N}\}$ whose edges consist of the identified minimal edges, so that m_1 passes vertically through v_1, and for $i > 1$, m_i passes through v_1 exactly $i-1$ times, then jumps to v_i and goes down vertically through v_i. It is straightforward to verify that the sets $\{M_\alpha : \alpha \in \mathbb{N} \cup \{\infty\}\}$ and $\{m_\beta : \beta \in \mathbb{N}\}$ are both countable, closed, and nowhere dense, and that $\mathcal{F} = (\widetilde{V}_{n-1}, \overline{V}_{n-1}, \{\widetilde{e}_v, \overline{e}_v : v \in V_n\} : n \geq 2)$ is the skeleton associated to $\{M_\alpha : \alpha \in \mathbb{N} \cup \{\infty\}\}$ and $\{m_\beta : \beta \in \mathbb{N}\}$.

Example 3.12. Suppose that for $n \geq 1$, $V_n = \widetilde{V}_n = \overline{V}_n = \{v_1, \dots, v_{2^n}\}$, and all incidence matrix entries are at least two. We label each vertex $v_i \in V_n$ using a binary string $(x_1, \dots x_n)$ of length n which denotes i's binary expansion, starting with the least significant digit. For example, we label vertex $v_5 \in V_4$ with the string (1010). Suppose that for $v_i \in V_{n+1}$,

$$s(\widetilde{e}_{(x_1,\dots,x_{n+1})}) = s(\overline{e}_{(x_1,\dots,x_{n+1})}) = (x_1, \dots, x_n).$$

In this case the sets $\{M_\alpha : \alpha \in \{0,1\}^{\mathbb{N}}\}$ and $\{m_\alpha : \alpha \in \{0,1\}^{\mathbb{N}}\}$ are uncountable.

Note that the previous examples illustrate the fact that when defining a skeleton, we do not need complete information about the Bratteli diagram B. In particular, at this point, we need to know very little about the incidence matrices (F_n) of B. The skeleton $\mathcal{F}$ is simply a constrained set of choices for all extremal edges when building an order.

Next we discuss a way of building a homeomorphism $\sigma : X_{\max}(\mathcal{F}) \to X_{\min}(\mathcal{F})$ which is amenable to being extended to a Vershik map. This will be the analogue of the permutation associated to a skeleton in the finite rank case.

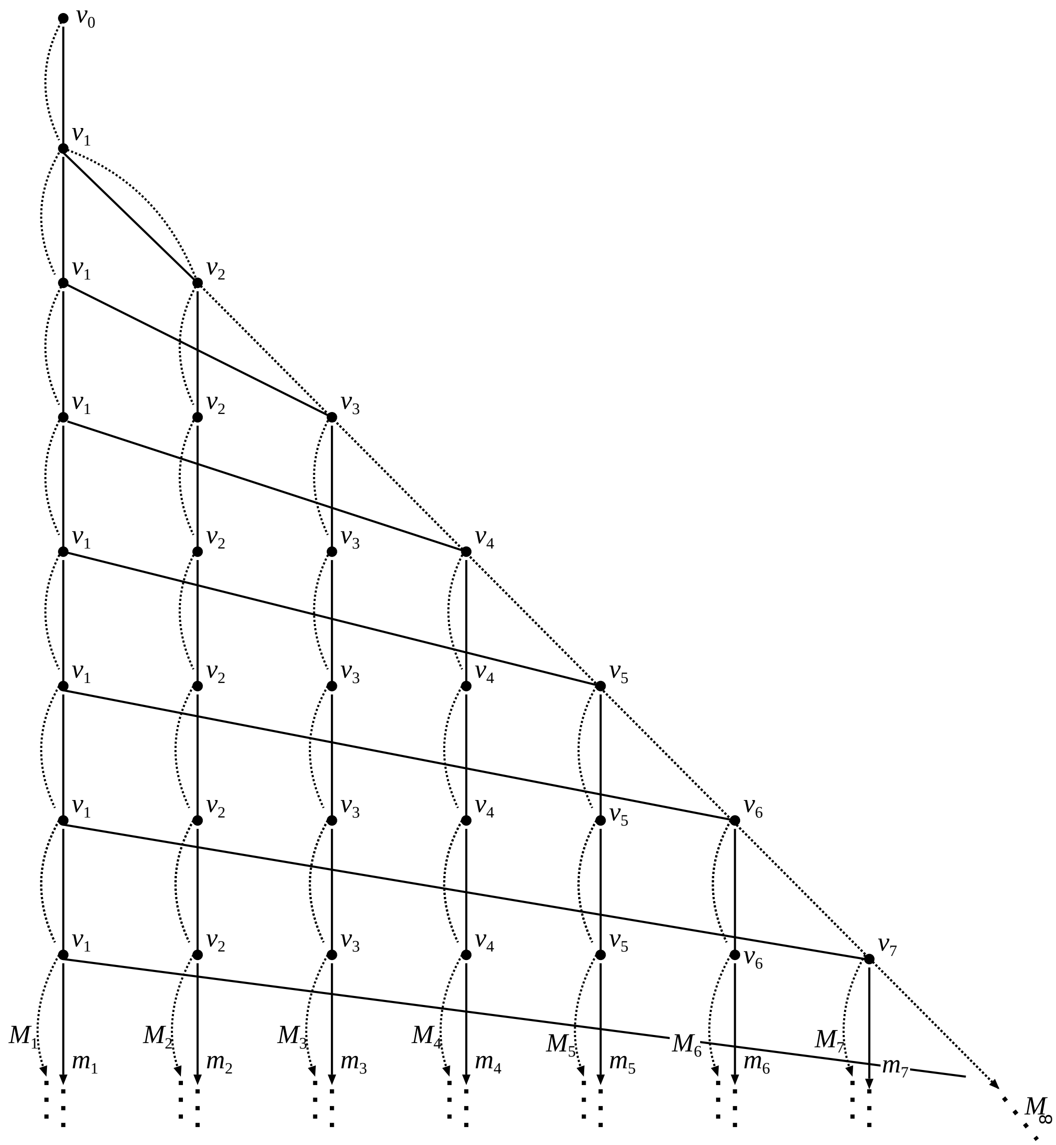

FIGURE 2. The minimal and maximal edge structure for Example 3.11: solid lines are minimal edges, dashed lines are maximal.

Suppose that ω is an order on B. Let $\sigma_n : \widetilde{V}_n \to 2^{\overline{V}_n}$ be defined by $\overline{v} \in \sigma_n(\widetilde{v})$ if and only if $\widetilde{v}\overline{v} \in \mathcal{L}(B, \omega, n)$. If ω is perfect, then for any sequence $(\widetilde{v}_n) = (v_n(M_\alpha))$, there is a unique sequence $(\overline{v}_n) = (v_n(m_\beta))$ with $\overline{v}_n \in \sigma_n(\widetilde{v}_n)$ for each n. If a perfect order ω has a finite number of extremal paths, we can say more. In that case, for all large enough n, and all maximal $\widetilde{v} \in \widetilde{V}_n$, $\sigma_n(\widetilde{v})$ is an element, not a subset, of $\overline{V}_n$ - i.e. each maximal vertex can only be followed by a unique minimal vertex in $\mathcal{L}(B, \omega, n)$. One can then say that for all large n, $|\widetilde{V}_n| = |\overline{V}_n|$ and $\sigma_n : \widetilde{V}_n \to \overline{V}_n$ is a bijection. In the case of finite rank, the sets $\widetilde{V}_n$ and $\overline{V}_n$, and the maps $\sigma_n : \widetilde{V}_n \to \overline{V}_n$ could be taken to be equal for all n, and in that case we called $\sigma := (\sigma_n)$ a *permutation* in [BKY13].

In the general case of a perfect order with infinitely many extremal paths, though, this fact - that for any sequence $(\widetilde{v}_n) = (v_n(M_\alpha))$, there is a unique sequence $(\overline{v}_n) = (v_n(m_\beta))$ with $\overline{v}_n \in \sigma_n(\widetilde{v}_n)$ for each n - does not generally imply that $(\widetilde{v}_n) = (v_n(M_\alpha))$ is eventually a sequence of singletons. The main obstacle is that one can have pairs of distinct maximal paths that agree on an arbitrarily large initial segment. For, suppose that $\varphi_\omega(M) = m$, and assume that M' is another maximal path that coincides with M for the first n segments till the vertex $\widetilde{v}_n$. By continuity of φ_ω, the minimal path $m' = \varphi_\omega(M')$ must be close to m, but it can be that $\overline{v}_n = v_n(m) \neq v_n(m') = \overline{w}_n$. So, we see that not only $\widetilde{v}_n\overline{v}_n \in L(B, \omega, n)$ but also $\widetilde{v}_n\overline{w}_n \in L(B, \omega, n)$, that is $\{\overline{v}_n, \overline{w}_n\} \in \sigma_n(\widetilde{v}_n)$. One can build such orders on any Bratteli diagram: see Example 3.

We use these observations to make the following definition, which generalizes the concept of a permutation for finite rank diagrams. Some notation: if $\sigma : \widetilde{V} \to 2^{\overline{V}}$ and $\overline{v} \in \overline{V}$, we define $\sigma^{-1}(\overline{v}) := \{\widetilde{v} : \overline{v} \in \sigma(\widetilde{v})\}$.

Definition 3.13. Let $\mathcal{F}$ be a skeleton for an unordered Bratteli diagram B. Suppose that $\sigma = (\sigma_n)_n$ is a sequence of maps $\sigma_n : \widetilde{V}_n \to 2^{\overline{V}_n}$, such that for each n, $\bigcup_{\widetilde{v} \in \widetilde{V}_n} \sigma_n(\widetilde{v}) = \overline{V}_n$, and

(1) σ is *composition consistent:* let $M(n, N, \widetilde{v})$ and $m(n, N, \overline{v})$ denote the maximal and minimal paths from level n to level $N > n$ with range $\widetilde{v}$ and $\overline{v}$ respectively. If $\overline{v} \in \sigma_N(\widetilde{v})$, then $s(m(n, N, \overline{v})) \in \sigma_n(s(M(n, N, \widetilde{v})))$,
(2) for any $M \in X_{\max}(\mathcal{F})$, there is a unique $m \in X_{\min}(\mathcal{F})$ with $(v_n(m))_n \in \prod_n \sigma_n(v_n(M))$,
(3) for any $m \in X_{\min}(\mathcal{F})$, there is a unique $M \in X_{\max}(\mathcal{F})$ with $(v_n(M))_n \in \prod_n \sigma_n^{-1}(v_n(m))$, and
(4) the bijection $\sigma : X_{\max}(\mathcal{F}) \to X_{\min}(\mathcal{F})$ defined using properties 2 and 3 is a homeomorphism.

Then we say that $\sigma = (\sigma_n)$ is a *correspondence associated to* $\mathcal{F}$.

Example 3.14. We continue with the skeleton defined in Example 3.11 and Figure 2. Let σ be defined by

$$\sigma_n(v_i) = \begin{cases} \{v_{i+1}\} & \text{if } 1 \leq i \leq n-1 \\ \{v_1\} & \text{if } i = n \end{cases}$$

for each $n \geq 1$; then one can verify that σ is composition consistent. Since each $\sigma_n : \widetilde{V}_n \to \overline{V}_n$ is in fact a point map, this means that items (2) and (3) of Definition 3.13 are satisfied. The homeomorphism $\sigma : X_{\max}(\mathcal{F}) \to X_{\min}(\mathcal{F})$ satisfies $\sigma(M_\infty) = m_1$, and for $i \geq 1$, $\sigma(M_i) = m_{i+1}$.

Example 3.15. We continue with the skeleton defined in Example 3.12. Let '+1' denote addition with carry, so that for example, $(1010) + 1 = (0110)$. Let $\sigma = (\sigma_n)$ be defined by

$$\sigma_n((x_1, \ldots, x_n)) = \begin{cases} \{(x_1, \ldots, x_n) + 1\} & \text{if } (x_1, \ldots, x_n) \neq (1, \ldots 1) \\ \{(0, \ldots, 0)\} & \text{if } (x_1, \ldots, x_n) = (1, \ldots 1) \end{cases}$$

for each $n \geq 1$; then one can verify that σ is composition consistent. Since each $\sigma_n : \widetilde{V}_n \to \overline{V}_n$ is in fact a point map, this means that items (2) and (3) of Definition 3.13 are satisfied. Note that the bijection $\sigma : X_{\max}(\mathcal{F}) \to X_{\min}(\mathcal{F})$ satisfies $\sigma(M_{111\ldots}) = m_{000\ldots}$, and $\sigma(M_{x_1x_2\ldots}) = m_{y_1y_2\ldots}$, where $(y_1y_2\ldots) = (x_1x_2\ldots) + (100\ldots)$; in other words, σ is the binary odometer map.

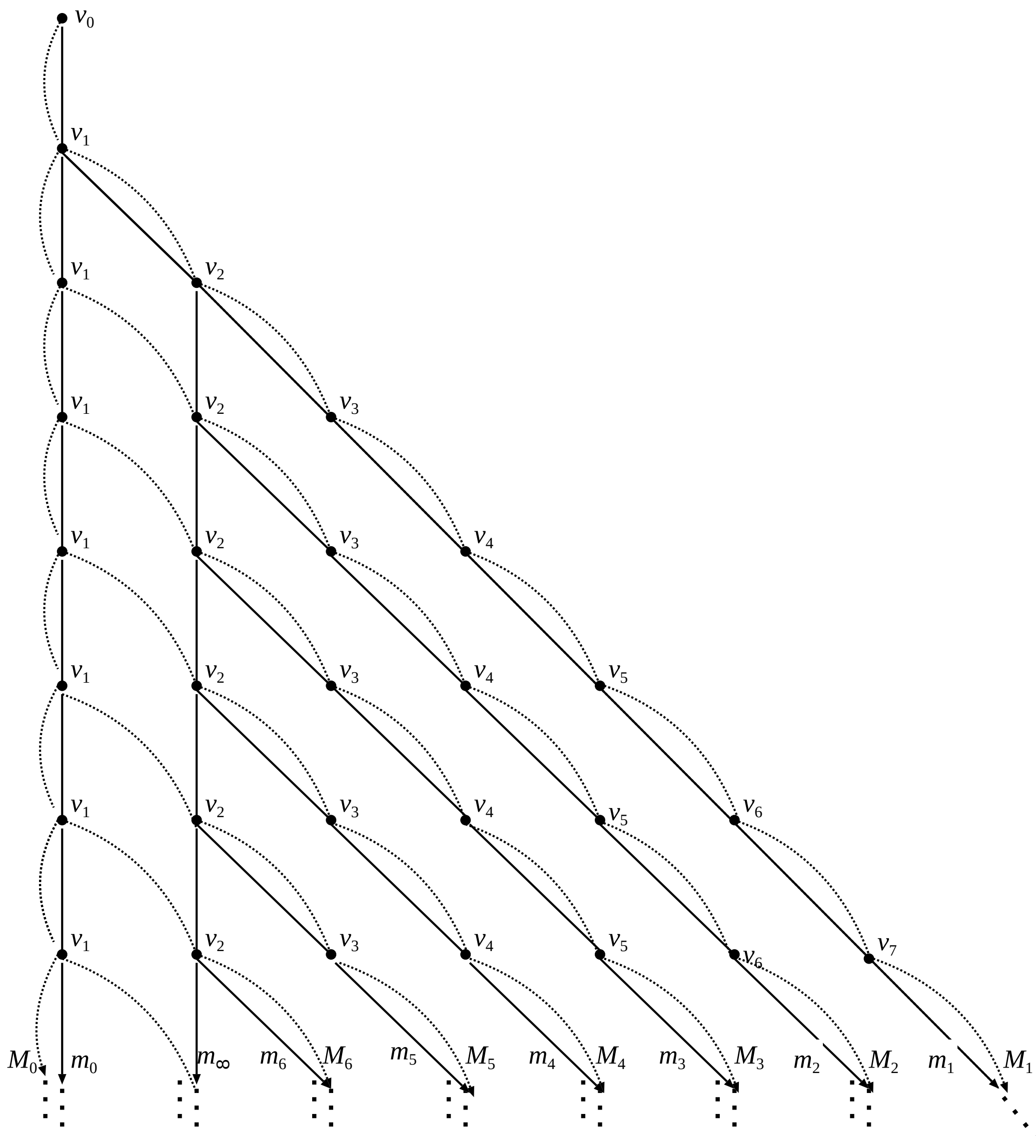

FIGURE 3. The minimal and maximal edge structure for Example 3.16

Example 3.16. It seems to be difficult to find examples of skeletons and accompanying correspondences where the maps σ_n are not eventually point maps. If both $v_N(m)$ and $v_N(m')$ both belong to $\sigma_N(v_N(M))$, the composition consistency condition forces $v_n(m)$ and $v_n(m')$ to belong to $\sigma_n(v_n(M))$ for $n < N$, making it hard for points (2) and (3) of the definition of a correspondence to be satisfied. Here is one example, illustrated in Figure 3. The vertex structure of this diagram is as in Example 3.11: $V_n = \widetilde{V}_n = \overline{V}_n = \{v_1, \dots, v_n\}$. To define the skeleton, we let $s(\widetilde{e}_{v_i}) = s(\overline{e}_{v_i}) = v_{i-1}$ for $i > 2$, $s(\widetilde{e}_{v_2}) = v_1$, $s(\overline{e}_{v_2}) = v_2$, and $s(\widetilde{e}_{v_1}) = s(\overline{e}_{v_1}) = v_1$. This skeleton is illustrated in Figure 3. We define, for each n, $\sigma_n(v_1) = \{v_2, v_3\}$, $\sigma_n(v_i) = v_{i+1}$ for $i = 2, \dots, n-1$, and $\sigma_n(v_n) = v_1$. Then $(\sigma_n)_n$ defines a correspondence, with $\sigma(M_n) = m_{n-1}$ for $n \geq 1$, and $\sigma(M_0) = m_\infty$.

In the case where $\mathcal{F}$ is the skeleton associated to a well telescoped ordered diagram (B, ω), we will *always* take σ_n to be that defined by $\mathcal{L}(B, \omega, n)$, as discussed in the paragraph preceding Definition 3.13. Namely, given an order ω with skeleton $\mathcal{F}$, we define $\overline{v} \in \sigma_n(\widetilde{v})$ if and only if $\widetilde{v}\overline{v} \in \mathcal{L}(B, \omega, n)$. Whether or not $\sigma = (\sigma_n)$ is a correspondence depends on whether ω is perfect, as seen in the following proposition:

Theorem 3.17. *Let (B, ω) be a well telescoped ordered Bratteli diagram with skeleton $\mathcal{F}$ and accompanying maps $\sigma = (\sigma_n)$. Then ω is perfect if and only if σ is a correspondence.*

Proof. Suppose that ω is perfect. The fact that (σ_n) is composition consistent follows from the definition of the level n languages $\mathcal{L}(B, \omega, n)$. If it is the case that for distinct minimal paths m, and m', and a maximal path M, the two sequences $(v_n(m))$ and $(v_n(m'))$ belong to $(\sigma_n(v_n(M)))$ then we can build two sequences of paths (x_n) and (y_n), both converging to M, where $\varphi_\omega(x_n) \to m$ and $\varphi_\omega(y_n) \to m'$, contradicting continuity of φ_ω. Thus for each maximal path M, there is a unique $m \in X_{\min}(\omega)$ with $(v_n(m))_n \in \prod_n \sigma_n(v_n(M))$, and the continuity of φ_ω implies that in fact $m = \varphi_\omega(M)$. Similarly for any $m \in X_{\min}(\omega)$, there is a unique $M = \varphi_\omega^{-1}(m) \in X_{\max}(\omega)$ with $(v_n(M))_n \in \prod_n \sigma_n^{-1}(v_n(m))$. Thus $\sigma : X_{\max}(\omega) \to X_{\min}(\omega)$ coincides with $\varphi_\omega : X_{\max}(\omega) \to X_{\min}(\omega)$, and the fact that φ_ω is a homeomorphism and $X_{\min}(\omega)$, $X_{\max}(\omega)$ are both closed implies that $\sigma : X_{\max}(\omega) \to X_{\min}(\omega)$ is a homeomorphism.

Conversely, suppose that σ is a correspondence. The Vershik map φ_ω is well defined everywhere, and continuous, outside the sets of extreme paths. We use σ to define φ_ω on $X_{\max}(\omega)$, so that σ equals the restriction of φ_ω to $X_{\max}(\omega)$, a similar statement holding for σ^{-1}. As σ is a correspondence, φ_ω is continuous on $X_{\max}(\omega)$; thus to check continuity of φ_ω, it is sufficient to consider a convergent sequence (x_n) of non-maximal paths, where $x_n \to M$ with M maximal. We claim that $(\varphi_\omega(x_n))$ converges to some minimal sequence m (and in fact this m does not depend on the choice of (x_n)). Suppose not. Then for two subsequences (y_n) and (y'_n) of (x_n), we have $\varphi_\omega(y_n) \to m$ and $\varphi_\omega(y'_n) \to m'$ for two paths $m \neq m'$, which are necessarily minimal paths.

Since each y_n is not maximal, this implies that for some subsequence (n_k), $v_{n_k}(m) \in \sigma_{n_k}(v_{n_k}(M))$, which implies, by composition consistency, that $v_n(m) \in \sigma_n(v_n(M))$ for each $n \geq 1$. Similarly, as each y'_n is not maximal for some subsequence (n'_k), $v_{n'_k}(m') \in \sigma_{n'_k}(v_{n'_k}(M))$, which implies that $v_n(m') \in \sigma_n(v_n(M))$ for each n. Since we have assumed that σ is a correspondence, this contradicts the fact that there is a unique minimal element $m = \sigma(M)$ such that $(v_n(m))_n \in \prod_n(\sigma_n(v_n(M)))$. Compactness ensures the continuity of φ_ω^{-1}.

□

Theorem 3.17 tells us that behind every perfect order ω on a diagram B, there is an underlying skeleton $\mathcal{F}$ and correspondence σ. More generally, a correspondence accompanying a skeleton $\mathcal{F}$ will contain the information that allows us to extend the partial definition of orders using $\mathcal{F}$ to construct perfect orders. As in the finite rank case, the notions of accompanying partitions and associated directed graphs will be useful.

Definition 3.18. Suppose that B is a Bratteli diagram with skeleton $\mathcal{F} = (\widetilde{V}_{n-1}, \overline{V}_{n-1}, \{\widetilde{e}_v, \overline{e}_v : v \in V_n\} : n \geq 2)$ associated to the set $\{M_\alpha : \alpha \in I\}$ of maximal paths and the set $\{m_\beta : \beta \in J\}$ of minimal paths. For any vertices $\widetilde{v} \in \widetilde{V}_{n-1}$ and $\overline{v} \in \overline{V}_{n-1}$, we set

$$W_{\widetilde{v}}(n) = \{w \in V_n : s(\widetilde{e}_w) = \widetilde{v}\}, \quad W'_{\overline{v}}(n) = \{w \in V_n : s(\overline{e}_w) = \overline{v}\} \tag{3.2}$$

where $n \geq 2$. It is obvious that $W(n) = \{W_{\widetilde{v}}(n) : \widetilde{v} \in \widetilde{V}_{n-1}\}$ and $W'(n) = \{W'_{\overline{v}}(n) : \overline{v} \in \overline{V}_{n-1}\}$ form two partitions of V_n. We call the sequence of partitions $W = (W(n))_n$ and $W' = (W'(n))_n$ the *partitions generated by* $\mathcal{F}$.

The intersection of $W(n)$ and $W'(n)$ is the partition $W'(n) \cap W(n)$ whose elements are non-empty sets $W'_{\overline{v}}(n) \cap W_{\widetilde{v}}(n)$ where $(\overline{v}, \widetilde{v}) \in \overline{V}_{n-1} \times \widetilde{V}_{n-1}$. We shall use the notation $[\overline{v}, \widetilde{v}, n] := W'_{\overline{v}}(n) \cap W_{\widetilde{v}}(n)$ for shorthand.

Definition 3.19. Let $\mathcal{F}$ be a skeleton on B with an associated correspondence σ. Let $\mathcal{H}_n = (T_n, P_n)$ be the directed graph where the set T_n of vertices of $\mathcal{H}_n$ will consist of partition elements $[\overline{v}, \widetilde{v}, n]$ of $W(n) \cap W'(n)$, and where there is an edge in P_n from $[\overline{v}, \widetilde{v}, n]$ to $[\overline{v}', \widetilde{v}', n]$ if and only if $\overline{v}' \in \sigma_{n-1}(\widetilde{v})$. We call $(\mathcal{H}_n)$ the *sequence of directed graphs associated to* $(\mathcal{F}, \sigma)$.

Remark 3.20. The vertices of $\mathcal{H}_n$ are labeled by $[\overline{v}, \widetilde{v}, n]$ where $(\overline{v}, \widetilde{v}) \in \overline{V}_{n-1} \times \widetilde{V}_{n-1}$. On the other hand, the set $[\overline{v}, \widetilde{v}, n]$ is a set of vertices in V_n. When we speak about a path in $\mathcal{H}_n$, we mean a concatenated sequence of directed edges between vertices of $\mathcal{H}_n$; these paths will correspond to families of words in V_n^+. The next proposition tells us that if ω is perfect, then words in $\mathcal{L}(B, \omega, n)$ must come from a valid path in $\mathcal{H}_n$; it is the appropriate generalization of Lemma 3.5.

Proposition 3.21. *Suppose that $\mathcal{F}$ is a skeleton on B, with a correspondence σ. Let $(\mathcal{H}_n)_n$ be the sequence of directed graphs associated to $(\mathcal{F}, \sigma)$.*

1. *If the perfect order ω has associated skeleton and correspondence $(\mathcal{F}, \sigma)$, then words in $\mathcal{L}(B, \omega, n)$ correspond to paths in $\mathcal{H}_n$.*
2. *Let ω be defined to have $\mathcal{F}$ and correspondence σ, and where for each n, all words in $\mathcal{L}(B, \omega, n)$ correspond to paths in $\mathcal{H}_n$. Then ω is perfect.*

Proof. For a given perfect order ω, the map σ_n is defined using the language $\mathcal{L}(B, \omega, n)$. If $vw \in \mathcal{L}(B, \omega, n)$, where $v \in [\overline{v}, \widetilde{v}, n]$ and $w \in [\overline{v}', \widetilde{v}', n]$, then $\widetilde{v}\overline{v}' \in \mathcal{L}(B, \omega, n-1)$, so that $\overline{v}' \in \sigma_{n-1}(\widetilde{v})$. Thus vw corresponds to a path in $\mathcal{H}_n$. The argument for longer words in $\mathcal{L}(B, \omega, n)$ is similar.

To prove the second statement, take non-maximal paths (x_n) converging to a maximal M. We shall show that in fact $\varphi_\omega(x_n) \to \sigma(M)$. This implies that φ_ω is continuous, and also that $\varphi_\omega : X_{\max}(\omega) \to X_{\min}(\omega)$ can be defined coinciding with $\sigma : X_{\max}(\mathcal{F}) \to X_{\min}(\mathcal{F})$. Suppose that x_n agrees with M to level k_n. Then, since σ is composition consistent, $v_j(\varphi_\omega(x_n)) \in \sigma_j(v_j(M))$ for each $j \leq k_n$. For some subsequence n_l, $\varphi_\omega(x_{n_l}) \to m$ where m is a minimal path. This implies that $(v_j(m)) \in \prod_{j=1}^{\infty} \sigma(v_j(M))$, and by conditions (2) and (4) of Definition 3.13, $m = \sigma(M)$. Since any subsequence of $(\varphi_\omega(x_n))$ has a subsequence that converges to m, it follows that $\varphi_\omega(x_n) \to m$. □

Example 3.22. We continue Examples 3.11 and 3.14: illustrated in Figure 4 is the graph $\mathcal{H}_n$ associated to the skeleton and correspondence considered in those examples. Let $w^{(n)} = v_1 \dots v_n$; then $w^{(n)}$ is generated from a path in $\mathcal{H}_n$. Suppose that all words $w(v, n, n+1)$ are defined using $w^{(n)}$, subject to the constraints of the skeleton $\mathcal{F}$ defined in Example 3.11, for example, $w(v_1, n, n+1)$ must both start and end with v_1. Then, *provided* that the incidence matrices $(\mathcal{F}_n)$ of B allow us, we can define a perfect order ω with skeleton $\mathcal{F}$ as in Example 3.11 and accompanying correspondence σ as in Example 3.14. For example, the v_1-indexed row of $\mathcal{F}_n$ must be of the form $(\alpha_n + 1, \alpha_n, \ldots, \alpha_n, \beta_n)$ where α_n and β_n are positive integers. This will be further elucidated in Theorem 4.6.

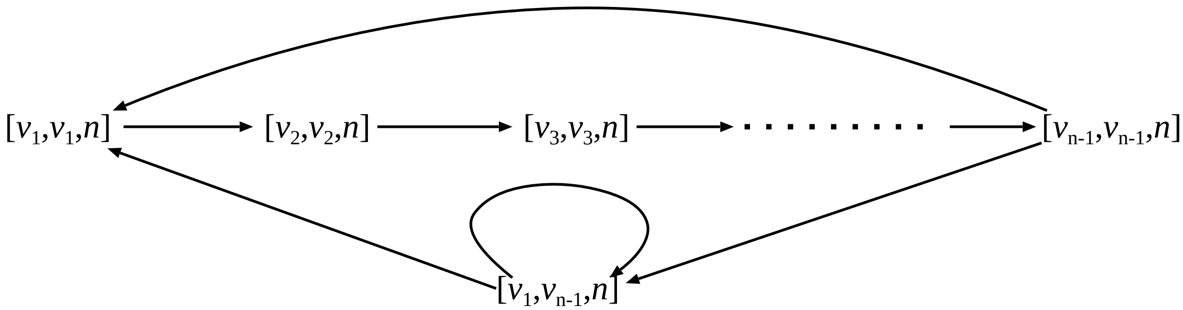

FIGURE 4. The graph $\mathcal{H}_n$ in Example 3.22

Finally we state and prove the analogue of Proposition 3.19 in [BKY13]. Recall that a directed graph is *strongly connected* if for any two vertices v, v', there are paths from v to v', and also from v' to v. If at least one of these paths exist, then G is *weakly connected*. Recall also the definition of the family $\mathcal{A}$ of Bratteli diagrams (Definition 2.2).

Proposition 3.23. *Let (B, ω) be a well telescoped, perfectly ordered Bratteli diagram with skeleton $\mathcal{F}_\omega$, and correspondence σ. Let $(\mathcal{H}_n)_n$ be the sequence of associated directed graphs.*

(1) If B is simple, then $\mathcal{H}_n$ is strongly connected for any n.
(2) If $B \in \mathcal{A}$, $\mathcal{H}_n$ is weakly connected for any n.

Proof. We prove (1) - the proof of (2) is similar (if we focus on $w(v, n-1, n)$ where v is the vertex which indexes the strictly positive row in F_n). In the case of simple diagrams, we can assume that all entries of F_n are positive for each n.

Take two vertices $[\overline{v}_1, \widetilde{v}_1, n]$ and $[\overline{v}_3, \widetilde{v}_3, n]$ in $\mathcal{H}_n$. If $\overline{v}_2 \in \sigma_{n-1}(\widetilde{v}_1)$, then there is some vertex $\overline{\overline{v}} \in \overline{V}_n$ such that $s(\overline{e}_{\overline{\overline{v}}}) = \overline{v}_2$. Let $\overline{\overline{v}} \in [\overline{v}_2, \widetilde{v}_2, n]$. Clearly there is an edge from $[\overline{v}_1, \widetilde{v}_1, n]$ to $[\overline{v}_2, \widetilde{v}_2, n]$ in $\mathcal{H}_n$.

Let $v \in V_{n+1}$ be such that $s(\overline{e}_v) = \overline{\overline{v}}$. Let $v_3 \in [\overline{v}_3, \widetilde{v}_3, n]$. Since B is simple, $f^{(n)}_{v,v_3} > 0$; this means that $\overline{\overline{v}} \ldots v_3$ is a prefix of $w(v, n, n+1)$. This implies that there is path in $\mathcal{H}_n$ from $[\overline{v}_2, \widetilde{v}_2, n]$ to $[\overline{v}_3, \widetilde{v}_3, n]$. $\square$

Remark 3.24. It is not hard to see that the converse statement to Proposition 3.23 is not true. There are examples of non-simple diagrams of finite rank whose associated graphs are strongly connected.

Note also that the assumption that ω is perfect is crucial. Moreover, there are examples of *simple* finite rank Bratteli diagrams and skeletons none of whose associated graphs are strongly connected. Indeed, let B be a stationary diagram with $V = \{a, b, c\}$ with the skeleton $\mathcal{F} = \{M_a, M_b, m_a, m_b; \widetilde{e}_c, \overline{e}_c\}$ where $s(\widetilde{e}_c) = b, s(\overline{e}_c) = a$. Let $\sigma(a) = a, \sigma(b) = b$. Constructing the associated graph $\mathcal{H}$, we see that there is no path from $[b, b]$ to $[a, a]$. It can be also shown that there is no perfect ordering ω such that $\mathcal{F} = \mathcal{F}_\omega$. This observation complements Proposition 3.23 by stressing the importance of the strong connectedness of $\mathcal{H}_n$ for the existence of perfect orderings.

4. Characterizing Bratteli diagrams that support perfect orders

In this section we characterize Bratteli diagrams that support perfect, non-proper orders via their incidence matrices. Our main result is Theorem 4.6, which extends a similar result proved in [BKY13, Theorem 4.6] for finite rank diagrams. We define the class $\mathcal{P}_B^*$, a set of perfect orders whose language properties are similar to those of orders in $\mathcal{P}_B(j)$, j finite, and for whom a refined version of Theorem 4.6 holds, namely Corollary 4.9.

The intuition behind the proof of Theorem 4.6 is the following idea. If a diagram B is to support a perfect, well-telescoped order ω, then ω would define a skeleton $\mathcal{F}$ and correspondence σ. The correspondence intrinsically contains the information about the languages defined by ω, and this is further expressed with the sequence $(\mathcal{H}_n)$ of directed graphs, in that words in $\mathcal{L}(B, \omega, n)$ must correspond to paths in $\mathcal{H}_n$. Words in $\mathcal{L}(B, \omega, n)$ are generated by the orders placed on the edges in $r^{-1}(u)$, where $u \in V_m$ and $m > n$. To define $w(u, n, n+1)$ for $u \in V_{n+1}$, then, we use $\mathcal{H}_n$. The edge structure of $\mathcal{H}_n$ implies that if a word vw lies in $w(u, n, n+1)$ with v belonging to $W_{\widetilde{v}}(n)$, then w must belong to $W_{\overline{v}}(n)$ for some $\overline{v} \in \sigma_{n-1}(\widetilde{v})$ - this is the statement of Lemma 4.2. This means that every time we leave a vertex in $\mathcal{H}_n$ of the form $[*, \widetilde{v}, n]$, we must go to a vertex of the form $[\overline{v}, *, n]$ for some $\overline{v} \in \sigma_{n-1}(\widetilde{v})$. Thus the u-th row of the n-th incidence matrix F_n must have a 'balance' between entries $f_{u,v}^{(n)}$, where $v \in W_{\widetilde{v}}(n)$, and entries $f_{u,v'}^{(n)}$, where $v' \in \bigcup_{\overline{v} \in \sigma_{n-1}(\widetilde{v})} W_{\overline{v}}(n)$. This is more precisely stated in Corollaries 4.3 and 4.4. It turns out that these 'balance' requirements, the system of relations (4.8), along with the system (4.7), are also sufficient for the existence of a perfect order on B.

First, we define the class $\mathcal{P}_B^*$, a class of perfect orders that naturally generalizes the class of perfect orders with finitely many extremal paths, and also introduce/re-introduce notation we shall need.

Definition 4.1. Let (B, ω) be a well telescoped, perfectly ordered diagram with skeleton $\mathcal{F}$ and correspondence $\sigma = (\sigma_n)$. We say that ω belongs to $\mathcal{P}_B^*$ if ω satisfies the following conditions: for each maximal path M with $\widetilde{v}_n = v_n(M)$, $\sigma_n(\widetilde{v}_n) \in \overline{V}_n$ for all n sufficiently large, and for each minimal path m with $\overline{v}_n = v_n(m)$, $\sigma_n(\overline{v}_n)^{-1} \in \widetilde{V}_n$ for all n sufficiently large.

In fact, it seems (see the comment in Example 3.16) unnatural for a perfect order to not belong to $\mathcal{P}_B^*$. All well telescoped perfect orders with finitely many maximal paths belong to $\mathcal{P}_B^*$, i.e. $\mathcal{P}_B \cap \mathcal{O}_B(j) \subset \mathcal{P}_B^*$ for each finite j. Also, suppose the perfect well telescoped order ω is such that for each maximal path M_α, and each minimal path m_β, there exist neighborhoods $U(M_\alpha)$ and $U(m_\beta)$ such that no other maximal paths belong to $U(M_\alpha)$, and no other minimal paths are in $U(m_\beta)$; then $\omega \in \mathcal{P}_B^*$. Note that such an order can have at most countably many extremal paths, since the correspondence $M_\alpha \to U(M_\alpha)$ is injective, and the set of clopen sets is countable.

Let ω be a perfect order on B. Recall that ω generates the skeleton $\mathcal{F}_\omega = (\widetilde{V}_{n-1}, \overline{V}_{n-1}, \{\widetilde{e}_v, \overline{e}_v : v \in V_n\}, n > 1)$ and two partitions $W(n) = \{W_{\widetilde{v}}(n) : \widetilde{v} \in \widetilde{V}_{n-1}\}$ and $W'(n) = \{W'_{\overline{v}}(n) : \overline{v} \in \overline{V}_{n-1}\}$ of V_n. Moreover, we have also a sequence of correspondences $\sigma_n : \widetilde{V}_n \to 2^{\overline{V}_n}, n \geq 1$, defined by ω. We recall also the notation used for maximal (minimal) paths: if M is a maximal path then it determines uniquely a sequence of maximal vertices $(\widetilde{v}_n = v_n(M))$.

Let $E(V_n, u)$ be the set of all finite paths between vertices of level n and a vertex $u \in V_m$ where $m > n$. The symbols $\widetilde{e}(V_n, u)$ and $\overline{e}(V_n, u)$ are used to denote the maximal and minimal finite paths in $E(V_n, u)$, respectively; if $m = n+1$ so that $u \in V_{n+1}$, then we revert to the shorter notation $\widetilde{e}_u$ and $\overline{e}_u$. Fix a maximal and minimal vertex $\widetilde{v} \in \widetilde{V}_{n-1}$ and $\overline{v} \in \overline{V}_{n-1}$ respectively. Denote $E(W_{\widetilde{v}}(n), u) = \{e \in E(V_n, u) : s(e) \in W_{\widetilde{v}}(n), r(e) = u\}$ and $E(W'_{\overline{v}}(n), u) = \{e \in E(V_n, u) : s(e) \in W'_{\overline{v}}(n), r(e) = u\}$. Clearly, the sets $\{E(W_{\widetilde{v}}(n), u) : \widetilde{v} \in \widetilde{V}\}$ and $\{E(W_{\overline{v}}(n), u) : \overline{v} \in \overline{V}\}$ form two partitions of $E(V_n, u)$. It may happen that the maximal finite path $\widetilde{e}(V_n, u)$ has its source in $W_{\widetilde{v}}(n)$. In this case, we define $\widetilde{E}(W_{\widetilde{v}}(n), u) = E(W_{\widetilde{v}}(n), u) \setminus \{\widetilde{e}(V_n, u)\}$. Otherwise, $\widetilde{E}(W_{\widetilde{v}}(n), u) = E(W_{\widetilde{v}}(n), u)$. Similarly we define the set $\overline{E}(W'_{\overline{v}}(n), u)$ using the minimal finite path $\overline{e}(V_n, u)$.

In the next few results we assume that a perfect, well telescoped order ω has attached its skeleton $\mathcal{F}$, correspondence σ and partitions $(W(n))$ and $(W'(n))$. Also, for brevity we shall abuse notation: if e is an edge, (or a set of edges), we will write $\varphi_\omega(e)$ instead of the more correct $\varphi_\omega(U(e))$.

Lemma 4.2. *Suppose (B, ω) is a well telescoped, perfectly ordered Bratteli diagram. Let $\widetilde{v} \in \widetilde{V}_{n-1}$. If $u \in V_m$, $m > n$, and $e \in \widetilde{E}(W_{\widetilde{v}}(n), u)$, then $\varphi_\omega(e) \in \bigcup_{\overline{v} \in \sigma_{n-1}(\widetilde{v})} \overline{E}(W_{\overline{v}}(n), u)$.*

Proof. Extend e to a path $e*$ in $\widetilde{E}(\widetilde{v}, u)$ by concatenating the maximal edge in $E(\widetilde{v}, s(e))$ to e. Similarly, extend $\varphi_\omega(e)$ to a path e^{**} in $\overline{E}(\overline{v}, u)$ by concatenating the minimal edge in $E(\overline{v}, \varphi_\omega(e))$ to $\varphi_\omega(e)$. Since $\varphi_\omega(e^*) = (e^{**})$, this means that $\widetilde{v}\overline{v} \subset w(u, n-1, m)$, so that $\widetilde{v}\overline{v} \in \mathcal{L}(B, \omega, n-1)$. By definition of σ_{n-1}, $\overline{v} \in \sigma_{n-1}(\widetilde{v})$.

$\square$

The following corollary can be easily deduced from Lemma 4.2.

Corollary 4.3. *Let (B, ω) be a well telescoped, perfectly ordered Bratteli diagram. Then for any $n \geq 2$, $\widetilde{v} \in \widetilde{V}_{n-1}$ and $u \in V_m$, $m > n$, we have*

$$\sum_{\overline{v} \in \sigma_{n-1}(\widetilde{v})} |\widetilde{E}(W_{\widetilde{v}}(n), u) \cap \varphi_\omega^{-1}(\overline{E}(W'_{\overline{v}}(n), u))| = |\widetilde{E}(W_{\widetilde{v}}(n), u)|. \tag{4.1}$$

Also if $\overline{v} \in \overline{V}_{n-1}$, then

$$\sum_{\widetilde{v} : \overline{v} \in \sigma_{n-1}(\widetilde{v})} |\widetilde{E}(W_{\widetilde{v}}(n), u) \cap \varphi_\omega^{-1}(\overline{E}(W'_{\overline{v}}(n), u))| = |\overline{E}(W'_{\overline{v}}(n), u)|. \tag{4.2}$$

We can refine the statement of Corollary 4.3 in some special cases:

Corollary 4.4. *Let (B, ω) be a well telescoped, perfectly ordered Bratteli diagram.*

(1) Suppose that $\omega \in \mathcal{P}_B \cap \mathcal{O}_B(j)$. Then there exists an n_0 such that for any $n \geq n_0$, any vertex $\widetilde{v} \in \widetilde{V}_{n-1}$, any $m > n$, and any $u \in V_m$, $\sigma_{n-1}(\widetilde{v})$ is a singleton and one has

$$|\widetilde{E}(W_{\widetilde{v}}(n), u)| = |\overline{E}(W'_{\sigma_{n-1}(\widetilde{v})}(n), u)|. \tag{4.3}$$

(2) Suppose that $\omega \in \mathcal{P}_B^*$. Then for any maximal path M (and hence any sequence $(\widetilde{v}_n = v_n(M))$), there exists an n_0 such that for any $n \geq n_0$, $\sigma_n(\widetilde{v}_n)$ is a singleton, and for

any $m > n > n_0$ and $u \in V_m$, one has

$$|\widetilde{E}(W_{\widetilde{v}_{n-1}}(n), u)| = |\overline{E}(W'_{\sigma_{n-1}(\widetilde{v}_{n-1})}(n), u)|. \tag{4.4}$$

Given the incidence matrices (F_n) for B, where $F_n = \{(f^{(n)}_{u,w}) : u \in V_{n+1}, w \in V_n\}$, we define the sequences of modified incidence matrices $(\widetilde{F}_n)$ and $(\overline{F}_n)$ as in Section 4 of [BKY13]. Namely, define $\widetilde{F}_n = (\widetilde{f}^{(n)}_{u,w})$ and $\overline{F}_n = (\overline{f}^{(n)}_{u,w})$ by the following rule (here $w \in V_n,\ u \in V_{n+1}$ and $n \geq 1$):

$$\widetilde{f}^{(n)}_{u,w} = \begin{cases} f^{(n)}_{u,w} - 1 & \text{if } \widetilde{e}_u \in E(w,u) \text{ and} \\ f^{(n)}_{u,w} & \text{otherwise,} \end{cases} \tag{4.5}$$

and

$$\overline{f}^{(n)}_{u,w} = \begin{cases} f^{(n)}_{u,w} - 1 & \text{if } \overline{e}_u \in E(w,u) \text{ and} \\ f^{(n)}_{u,w}, & \text{otherwise.} \end{cases} \tag{4.6}$$

Relation (4.1) implies that for each $u \in V_{n+1}$ and each $\widetilde{v} \in \widetilde{V}_{n-1}$, if $w \in W_{\widetilde{v}}(n)$, then each $\widetilde{f}^{(n)}_{u,w}$ can be written as

$$\widetilde{f}^{(n)}_{u,w} = \sum_{\overline{v} \in \sigma_{n-1}(\widetilde{v})} \widetilde{f}^{(n)}_{u,w,\overline{v}} \tag{4.7}$$

and relation (4.2) says that for each $u \in V_{n+1}$ and each $\overline{v} \in \overline{V}_{n-1}$,

$$\sum_{\widetilde{v} : \overline{v} \in \sigma_{n-1}(\widetilde{v})} \sum_{w \in W_{\widetilde{v}}(n)} \widetilde{f}^{(n)}_{u,w,\overline{v}} = \sum_{w' \in W'_{\overline{v}}(n)} \overline{f}^{(n)}_{u,w'}\,. \tag{4.8}$$

We will refer to the relations in (4.8) above as the *balance relations*. If ω is perfect and has finitely many extremal paths, then the balance relations have the following form: for $n > n_0$, $\widetilde{v} \in \widetilde{V}_{n-1}$ and $u \in V_{n+1}$:

$$\sum_{w \in W_{\widetilde{v}}(n)} \widetilde{f}^{(n)}_{u,w} = \sum_{w' \in W'_{\sigma_{n-1}(\widetilde{v})}(n)} \overline{f}^{(n)}_{u,w'}, \quad u \in V_{n+1}. \tag{4.9}$$

If $\omega \in \mathcal{P}^*_B$ and the maximal path M is given, then there exists n_0 such that for each $n > n_0$, if $v_{n-1}(M) = \widetilde{v}$ and $u \in V_{n+1}$, then relation (4.9) is satisfied.

The content of Theorem 4.6 is that given a skeleton and correspondence on B, relations (4.7) and (4.8) are sufficient conditions on the incidence matrices of a Bratteli diagram, in order that it supports a perfect order ω. Our proof is constructive in that given a diagram, skeleton and correspondence,we use an algorithm to define, for each $u \in V_{n+1}$ and $n \in \mathbb{N}$, the word $w(u, n, n+1)$ - i.e. we order the set $r^{-1}(u)$. We do this by constructing a path in $\mathcal{H}_n$ that starts in $[\overline{v}_0, \widetilde{v}_0, n]$, where $s(\overline{e}_u) \in [\overline{v}_0, \widetilde{v}_0, n]$, terminates at $[\overline{v}_t, \widetilde{v}_t, n]$, where $s(\widetilde{e}_u) \in [\overline{v}_t, \widetilde{v}_t, n]$, and passes through each vertex in $\mathcal{H}_n$ a prescribed number of times that we now make precise.

Proposition 3.23 tells us that we have to assume that the directed graphs $\mathcal{H}_n$ are strongly connected. We make clear what we mean by this as follows. Fix $n \in \mathbb{N}$ and $u \in V_{n+1}$. If $[\overline{v}, \widetilde{v}, n] \in \mathcal{H}_n$, we associate a number $P_u([\overline{v}, \widetilde{v}, n]) := \sum_{w \in [\overline{v}, \widetilde{v}, n]} \widetilde{f}^{(n)}_{u,w}$ to the vertex $[\overline{v}, \widetilde{v}, n]$. This *crossing number* represents the number of times that we will have to pass through the vertex $[\overline{v}, \widetilde{v}, n]$ when we define an order on $r^{-1}(u)$, and here we emphasize that if we terminate at $[\overline{v}, \widetilde{v}, n]$, we do not consider this final visit as contributing to the crossing number - this is why we use the terms $\widetilde{f}^{(n)}_{u,w}$, and not $f^{(n)}_{u,w}$. We say that $\mathcal{H}_n$ is *positively strongly connected* if for each $u \in V_{n+1}$, the set of vertices $\{[\overline{v}, \widetilde{v}, n] : P_u([\overline{v}, \widetilde{v}, n]) > 0\}$, along with all the relevant edges of $\mathcal{H}_n$, form a strongly connected subgraph of $\mathcal{H}_n$. If $s(\widetilde{e}_u) \in [\overline{v}, \widetilde{v}, n]$ we shall call this vertex in $\mathcal{H}_n$ the *terminal vertex*, as when defining the order on $r^{-1}(u)$, we need a path that ends at this vertex (although it can obviously go through this vertex several times - in fact precisely $P_u([\overline{v}, \widetilde{v}, n])$ times).

Example 4.5. In this example we drop the dependence on n and consider the stationary diagram $B = (V, E)$ that was used above in Example 3.2. Suppose that $V = \{a, b, c, d\}$, $\overline{V} = \widetilde{V} = \{a, b, c\}$, with $a \in [a, a]$, $b \in [b, b]$, $c \in [c, c]$ and $d \in [b, a]$. Let $\sigma(a) = b$, $\sigma(b) = c$ and $\sigma(c) = a$. Suppose that the incidence matrix F of B is

$$F := \begin{pmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{pmatrix}$$

Then if $u = d$, $P_d([a, a]) = 0$, and the remaining three vertices $[b, b]$, $[c, c]$ and $[b, a]$ do not form a strongly connected subgraph of $\mathcal{H}$, - for example there is no path from $[c, c]$ to $[b, a]$. Hence for us $\mathcal{H}$ is not positively strongly connected.

Note also that although the rows of this incidence matrix satisfy the balance relations, there is no way to define an order on $r^{-1}(d)$ so that the resulting global order is perfect. The lack of positive strong connectivity of the graph $\mathcal{H}$ is precisely the impediment.

The following theorem is our main result, extending a similar result (Theorem 4.6) in [BKY13].

Theorem 4.6. *Let B be a Bratteli diagram with incidence matrices (F_n). Let $\mathcal{F}$ be a skeleton on B and σ an associated correspondence, such that the graphs $\mathcal{H}_n$ are all positively strongly connected. Suppose that the collection of natural numbers*

$$\{\widetilde{f}^{(n)}_{u,w,\overline{v}} : u \in V_{n+1}, w \in W_{\widetilde{v}}(n), \overline{v} \in \sigma_{n-1}(\widetilde{v})\}, \quad \widetilde{v} \in \widetilde{V}_{n-1}, \tag{4.10}$$

is such that relations (4.7) and (4.8) are satisfied.

Then there exists a perfect order ω on B having $\mathcal{F}$ and σ as associated skeleton and correspondence, respectively. Conversely, suppose that a perfect ω has accompanying skeleton and correspondence $(\mathcal{F}, \sigma)$. Then there exists a set of natural numbers as in (4.10) such that relations (4.7) and (4.8) hold.

Proof. As the preceding discussion deals with the necessity a perfect order having to satisfy relations (4.7) and (4.8), we prove here only the sufficiency of these relations.

Our goal is to define a linear order on $r^{-1}(u)$ for each $u \in V_{n+1}$ and $n > 1$ - in other words to define $w(u, n, n+1)$ - so that the corresponding partial ordering ω on B is perfect. Recall that each set $r^{-1}(u)$ contains two pre-selected edges $\widetilde{e}_u, \overline{e}_u$ and they should be the maximal and minimal edges in $r^{-1}(u)$ after defining $w(u, n, n+1)$.

Our proof is based on an inductive procedure that is applied to each row of the incidence matrices. We first describe in details the first step of the procedure that will be applied repeatedly. It will be seen from our proof that for given B, $\mathcal{F}$ and σ, neither is the word $w(u, n, n+1)$ that we define unique, nor will our procedure give all possible valid words.

We will first consider the particular case when the associated graphs $\mathcal{H} = (\mathcal{H}_n)$ defined by $\mathcal{F}$ do not have loops. After that, we will modify the construction to include possible loops.

Case I: There are no loops in the graphs $\mathcal{H}_n$. To begin with, we take some $u \in V_{n+1}$ and consider the u-th rows of matrices $\overline{F}_n$ and $\widetilde{F}_n$. They coincide with the row $(f^{(n)}_{u,v_1}, ..., f^{(n)}_{u,v_d})$ of the matrix F_n except only one entry either corresponding to $|E(s(\overline{e}_u), u)|$ and $|E(s(\widetilde{e}_u), u)|$ in $\overline{F}_n$ and $\widetilde{F}_n$, respectively. Take $\overline{e}_u$ and assign the number 0 to it, i.e. $\overline{e}_u$ is the minimal edge in $r^{-1}(u)$. Let $[\overline{v}_0, \widetilde{v}_0, n]$ be the vertex[1] of $\mathcal{H}_n$ such that $s(\overline{e}_u) \in [\overline{v}_0, \widetilde{v}_0, n]$. Consider the set

$$\{\widetilde{f}^{(n)}_{u,w} : w \in [\overline{v}, \widetilde{v}, n] : \overline{v} \in \sigma_{n-1}(\widetilde{v}_0)\}\,,$$

and let $\widetilde{f}^{(n)}_{u,w'}$ be the maximum of this set, where $w' \in [\overline{v}_1, \widetilde{v}_1, n]$. If there are several entries that are the maximal value, we chose one arbitrarily amongst them. Take any edge $e_1 \in E(w', u)$. In the case where $\widetilde{e}_u \in E(w', u)$, we choose $e_1 \neq \widetilde{e}_u$. Assign the number 1 to e_1 so that e_1 becomes the successor of $e_0 = \overline{e}_u$.

Two edges were labeled in the above procedure, e_0 and e_1. We may think of this step as if these edges were 'removed' from the set of all edges in $r^{-1}(u)$. In the collection of relations (4.8) we have worked with the relation defined by u and $\overline{v}_1$. On the left hand side, the entry $\widetilde{f}^{(n)}_{u,s(\overline{e}_u),\overline{v}_1}$ was reduced by 1, and on the right hand side, $\overline{f}^{(n)}_{u,w'}$ was reduced by 1. We need to verify that neither side was reduced by more than 1, i.e. we claim that the remaining non-enumerated edges satisfy the relation

$$\sum_{\widetilde{v}:\overline{v}_1 \in \sigma_{n-1}(\widetilde{v})} \sum_{w \in W_{\widetilde{v}}(n)} \widetilde{f}^{(n)}_{u,w,\overline{v}} - 1 = \sum_{w' \in W'_{\overline{v}_1}(n)} \overline{f}^{(n)}_{u,w',} - 1\,. \tag{4.11}$$

The choice of $w' \in [\overline{v}_1, \widetilde{v}_1, n]$ actually means that we take the edge from $[\overline{v}_0, \widetilde{v}_0, n]$ to $[\overline{v}_1, \widetilde{v}_1, n]$ in the associated graph $\mathcal{H}_n$. Note that $\overline{v}_1 \notin \widetilde{\sigma}_{n-1}(\widetilde{v}_1)$, otherwise there would be a loop at $[\overline{v}_1, \widetilde{v}_1, n]$ in $\mathcal{H}_n$, a contradiction to our assumption. This is why there is exactly one edge removed in each side of (4.11) so that our resulting row still satisfies (4.8). This completes the first step of the construction.

We are now at the vertex $[\overline{v}_1, \widetilde{v}_1, n]$ in $\mathcal{H}_n$. To repeat the above procedure, note that we now have a 'new', reduced u-th row of F_n - namely, the entry $f_{\overline{v}_0}(n)$ has been reduced by one. Thus the crossing numbers of the vertices of $\mathcal{H}_n$ change (one crossing number is reduced by one). Also note that in this new reduced row, $\overline{f}^{(n)}_{u,w'} = f^{(n)}_{u,w'} - 1$; in other words, with each step of this algorithm the row we are working with changes, and the vertex w such that $\overline{f}^{(n)}_{u,w} = f^{(n)}_{u,w} - 1$ changes. For, the vertex such that $\overline{f}^{(n)}_{u,w} = f^{(n)}_{u,w} - 1$ belongs to the vertex in $\mathcal{H}_n$ where we currently are, and this changes at every step of the algorithm. Let us assume that all crossing numbers ares still positive for the time being to describe the second step of the algorithm.

We apply the above described procedure again, this time to $w' = s(e_1)$, to show how we should proceed to complete the next step. Consider the set

$$\{\widetilde{f}^{(n)}_{u,w} : w \in [\overline{v}, \widetilde{v}, n] : \overline{v} \in \sigma_{n-1}(\widetilde{v}_1)\}\,,$$

and let $\widetilde{f}^{(n)}_{u,w''}$ be the maximum of this set, where $w'' \in [\overline{v}_3, \widetilde{v}_3, n]$. Once again, if there are several entries that are the maximal value, we chose one arbitrarily amongst them. Take any

[1]The same word 'vertex' is used in two meanings: for elements of the set T_n of the graph $\mathcal{H}_n$ and for elements of the set V_n of the Bratteli diagram B. To avoid any possible confusion, we point out explicitly what vertex is meant in the context.

edge $e_2 \in E(w'', u)$. In the case where $\widetilde{e}_u \in E(w'', u)$, we choose $e_2 \neq \widetilde{e}_u$. Assign the number 2 to e_2 so that e_2 becomes the successor of e_1.

We note that in the collection of relations (4.7), indexed by the vertices u, $\widetilde{v}_1$ and $w' = s(e_1)$, one entry was 'removed' from each side of the relation: on the right hand side, the entry $\widetilde{f}^{(n)}_{u,w',\overline{v}_2}$ was reduced by 1.

In the collection of relations (4.8) we have worked with the relation defined by u and $\overline{v}_2$. On the left hand side, the entry $\widetilde{f}^{(n)}_{u,w',\overline{v}_2}$ was reduced by 1, and on the right hand side, $\overline{f}^{(n)}_{u,w''}$ was reduced by 1. As we saw in (4.11), the relevant relation in (4.8) becomes

$$\sum_{\widetilde{v}:\overline{v}_2\in\sigma_{n-1}(\widetilde{v})}\ \sum_{w\in W_{\widetilde{v}}(n)} \widetilde{f}^{(n)}_{u,w,\overline{v}} - 1 = \sum_{w'\in W'_{\overline{v}_2}(n)} \overline{f}^{(n)}_{u,w'} - 1\,. \tag{4.12}$$

We remark also that the choice that we made of w'' (or e_2) allows us to continue the existing path (in fact, the edge) in $\mathcal{H}_n$ from $[\overline{v}_0, \widetilde{v}_0, n]$ to $[\overline{v}_1, \widetilde{v}_1, n]$ with the edge from $[\overline{v}_1, \widetilde{v}_1, n]$ to $[\overline{v}_2, \widetilde{v}_2, n]$, where $\widetilde{v}_2$ is defined by the property that $s(e_2) \in [\overline{v}_2, \widetilde{v}_2, n]$.

This process can be continued. At each step we apply the following rules:

(1) the edge e_i, that must be chosen next after e_{i-1}, is taken from the set $E(w^*, u)$ where w^* is such that $f^{(n)}_{u,w^*}$ is maximal amongst $f^{(n)}_{u,w}$, as w runs over $[\overline{v}, \widetilde{v}, n]$ where $\overline{v} \in \sigma_n(\widetilde{v}_{i-1})$, and

(2) the edge e_i is always taken not equal to $\widetilde{e}_u$ unless no more edges except $\widetilde{e}_u$ are left.

After every step of the construction, we see that the following statements hold.

(i) Relations (4.7), (with $\widetilde{v} = \widetilde{v}_i$) and (4.8) (with $\overline{v} = \overline{v}_i$) remain true when we treat them as the number of non-enumerated edges left in $r^{-1}(u)$. In other words, when a pair of vertices $\widetilde{v}_i$ and $\overline{v}_i$ is considered, we reduce by 1 each side of the relevant relations.

(ii) The used procedure allows us to build a path p from the starting vertex $[\overline{v}_0, \widetilde{v}_0, n]$ going through other vertices of the graph $\mathcal{H}_n$ according to the choice we make at each step. We need to guarantee that at each step, we are able to move to a vertex in $\mathcal{H}_n$ whose crossing number is still positive (unless we are at the terminal stage). As long as the crossing numbers of vertices in $\mathcal{H}_n$ are positive, there is no concern. Suppose thought that we land at a (non-terminal) vertex $[\overline{v}, \widetilde{v}, n]$ in $\mathcal{H}_n$ whose crossing number is one (and this is the first time this happens). When we leave this vertex, to go to $[\overline{v}', \widetilde{v}', n]$, the crossing number for $[\overline{v}, \widetilde{v}, n]$ will become 0 and therefore it will no longer be a vertex of $\mathcal{H}_n$. *Thus at this point, with each step, the graph $\mathcal{H}_n$ is also changing (being reduced).* We need to ensure that there is a way to continue the path out of $[\overline{v}', \widetilde{v}', n]$. Since

$$\sum_{w\in W_{\widetilde{v}'}(n)} \widetilde{f}^{(n)}_{u,w} \geq P_u[\overline{v}', \widetilde{v}', n] \geq 1,$$

then for some $\overline{v} \in \sigma_{n-1}(\widetilde{v}')$,

$$\sum_{\widetilde{v}:\overline{v}\in\sigma_{n-1}(\widetilde{v})}\ \sum_{w\in W_{\widetilde{v}}(n)} \widetilde{f}^{(n)}_{u,w,\overline{v}} \geq 1,$$

so that by the balance relations, $\sum_{w'\in W'_{\overline{v}}(n)} \overline{f}^{(n)}_{u,w'} \geq 1$. If the crossing number of all the vertices $[\overline{v}, *, n]$ have been reduced to 0, then this means that $\sum_{w'\in W_{\overline{v}}(n)} \overline{f}^{(n)}_{u,w'} = 1$, this tells us that we have to move into the terminal vertex for the last time. Then the balance relations, which continue to be respected, ensure we are done. Otherwise, the balance relations guarantee that $\sum_{w'\in W_{\overline{v}}(n)} \overline{f}^{(n)}_{u,w'} > 1$, which means there is a valid continuation of our path out of $[\overline{v}', \widetilde{v}', n]$

and to a new vertex in $\mathcal{H}_n$, and we are not at the end of the path. It is these balance relations which always ensure that the path can be continued until it reaches its terminal vertex.

(iii) In accordance with (i), the u-th row of F_n is transformed by a sequence of steps in such a way that entries of the obtained rows form decreasing sequences. These entries show the number of non-enumerated edges remaining after the completed steps. It is clear that, by the rule used above, we decrease the largest entries first. It follows from the simplicity of the diagram that, for sufficiently many steps, the set $\{s(e_i)\}$ will contain all vertices $v_1, ..., v_d$ from V_n. This means that the transformed u-th row consists of entries which are strictly less than those of F_n. After a number of steps the u-th row will have a form where the difference between any two entries is ± 1. After that, this property will remain true.

(iv) It follows from (iii) that we finally obtain that all entries of the resulting u-th row are zeros or ones. We apply the same procedure to enumerate the remaining edges from $r^{-1}(u)$ such that the number $|r^{-1}(u)| - 1$ is assigned to the edge $\widetilde{e}_u$. This means that we have constructed the word $W_u = s(\overline{e}_u)s(e_1) \cdots s(\widetilde{e}_u)$, ie we have ordered $r^{-1}(u)$.

Looking at the path p that is simultaneously built in $\mathcal{H}_n$, we see that the number of times this path comes into and leaves a vertex $[\overline{v}, \widetilde{v}, n]$ is precisely that vertex's crossing number $P_u([\overline{v}, \widetilde{v}, n])$. The path p is an Eulerian path of $\mathcal{H}_n$ that finally arrives to the vertex of $\mathcal{H}_n$ defined by $s(\widetilde{e}_u)$.

Case II: there is a loop in $\mathcal{H}_n$. To deal with this case, we have to refine the described procedure to avoid a possible situation when the algorithm cannot be finished properly. Suppose that the graph $\mathcal{H}_n$ has some loops.

We start as in Case I, and continue until we have arrived to a vertex $[\overline{v}_1, \widetilde{v}_1, n]$, where, for the first time, $[\overline{v}_1, \widetilde{v}_1, n]$ has a successor $[\overline{v}_2, \widetilde{v}_2, n]$ with a loop, ie $\overline{v}_2 \in \sigma_n(\widetilde{v}_2)$. If $[\overline{v}_2, \widetilde{v}_2, n]$ has crossing number zero, - ie it is the terminal vertex - and we are not at the terminal stage of defining the order, we ignore this vertex and continue as in Case I. If $[\overline{v}_2, \widetilde{v}_2, n]$ has a positive crossing number, i.e. $P_u([\overline{v}_2, \widetilde{v}_2, n]) > 0$, then at this point, we continue the path to $[\overline{v}_2, \widetilde{v}_2, n]$, and then traverse this loop $(\sum_{w \in [\overline{v}_2, \widetilde{v}_2, n]} \widetilde{f}^{(n)}_{u,w,\overline{v}_2}) - 1$ times. This means we are traversing this loop enough times that it is effectively no longer part of the resulting $\mathcal{H}_n$ that we have at the end of this step - we will no longer need, or even be able, to traverse the loop.

Looking at the relation

$$\sum_{\widetilde{v}: \overline{v}_2 \in \sigma_{n-1}(\widetilde{v})} \sum_{w \in W_{\widetilde{v}}(n)} \widetilde{f}^{(n)}_{u,w,\overline{v}_2} = \sum_{w' \in W'_{\overline{v}_2}(n)} \overline{f}^{(n)}_{u,w'}, \tag{4.13}$$

we see that by the time we have arrived at the vertex $[\overline{v}_2, \widetilde{v}_2, n]$, traversed it exactly $(\sum_{w \in [\overline{v}_2, \widetilde{v}_2, n]} \widetilde{f}^{(n)}_{u,w,\overline{v}_2}) - 1$ times, and left it, we see that we have removed $\sum_{w \in [\overline{v}_2, \widetilde{v}_2, n]} \widetilde{f}^{(n)}_{u,w,\overline{v}_2}$ from each side of (4.13). We consequently enumerate all edges whose source lies in $[\overline{v}_2, \widetilde{v}_2, n]$ in *any* arbitrary order.

We also need to ensure that once we have 'removed' the loop at $[\overline{v}_2, \widetilde{v}_2, n]$ from the graph $\mathcal{H}_n$, we do not disrupt future movement of our path, i.e. we do not disconnect $\mathcal{H}_n$ in a damaging way. To see this, suppose we have a loop at $[\overline{v}_2, \widetilde{v}_2, n]$ whose crossing number is positive. If $[\overline{v}_1, \widetilde{v}_1, n]$ is a (non-looped) vertex with a positive crossing number which has $[\overline{v}_2, \widetilde{v}_2, n]$ as a successor, then for some $[\overline{v}_3, \widetilde{v}_3, n] \neq [\overline{v}_2, \widetilde{v}_2, n]$ with $\overline{v}_3 \in \sigma_{n-1}(\widetilde{v}_1)$, the vertex $[\overline{v}_3, \widetilde{v}_3, n]$ will (if we are not at the terminal stage) have a positive crossing number. This is because of our discussion above concerning (4.13): the crossing number at the looped vertex appears on both

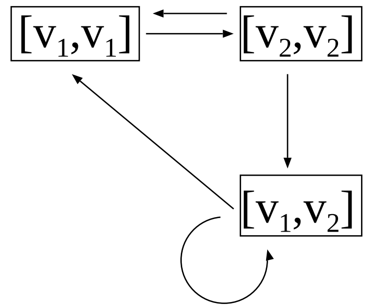

Figure 5. The graph H_3 for Example 4.7

sides, and cancels. So if $[\overline{v}_1, \widetilde{v}, n]$ has a positive crossing number, this contributes positive values to the left hand side of (4.13); and so there is some vertex $[\overline{v}_3, \widetilde{v}_3, n]$ with a positive value on the right hand side. All this means that we are able to continue our path out of the looped vertex $[\sigma_n(\widetilde{v}), \widetilde{v}, n]$ when we arrive there at some future stage.

We now revert to the old procedure. We are at the vertex $[\overline{v}_2, \widetilde{v}_2, n]$ in $\mathcal{H}_n$. If none of its successors have a loop, we revert to the algorithm in Case I, and continue with that algorithm until we reach a vertex in $\mathcal{H}_n$ one of whose successors has a loop, and then repeat the procedure described in Case II. We continue until we have defined the order on $r^{-1}(u)$. To summarize the general procedure, we notice that, constructing the Eulerian path p, the following rule is used: as soon as p arrives before a loop around a vertex in $\mathcal{H}_n$, then p makes as many loops around that vertex as needed so that this loop never needs or can be used again.Then p leaves the looped vertex and proceeds to a vertex according to the procedure in Case I, or, if there is as a follower a vertex with a loop, according to the procedure in Case II.

As noticed above, the fact that all edges e from $r^{-1}(u)$ are enumerated is equivalent to defining a word formed by the sources of e. In our construction, we obtain the word $w(u, n, n+1) = s(\overline{e}_u)s(e_1)\cdots s(e_j)\cdots\cdots s(\widetilde{e}_u)$.

Applying these arguments to every vertex u of the diagram, we define an ordering ω on B. That ω is perfect follows from Lemma 3.21: we chose ω to have skeleton $\mathcal{F}$, and for each n, constructed all words $w(v, n, n+1)$ to correspond to paths in $\mathcal{H}_n$. The result follows. □

Example 4.7. We continue with Examples 3.11 and 3.14, defining an order on $r^{-1}(v_2)$ where $v_2 \in V_3$, if $(f^{(3)}_{v_2,v_1}, f^{(3)}_{v_2,v_2}, f^{(3)}_{v_2,v_3}) = (1, 2, 1)$. In what follows we drop the superscript (3). This simple example illustrates why loop in the graphs $\mathcal{H}_n$ can cause a problem. The graph $\mathcal{H} = \mathcal{H}_3$ is shown in Figure 5; recall that $v_1 \in [v_1, v_1]$, $v_2 \in [v_2, v_2]$ and $v_3 \in [v_1, v_2]$. Since all the maps σ_n are point maps, the system of relations (4.7) becomes trivial. The balance relations (4.8) become

$$\widetilde{f}_{v_2,v_1} = \overline{f}_{v_2,v_2}, \text{ and } \widetilde{f}_{v_2,v_2} + \widetilde{f}_{v_2,v_3} = \overline{f}_{v_2,v_1} + \overline{f}_{v_2,v_3}$$

respectively, and our vector $(1, 2, 1)$ satisfies these constants. The only valid choice of an ordering of $r^{-1}(v_2)$ obtained using our algorithm is $w(v_2, 2, 3) = v_2v_3v_1v_2$, and in fact this is the only valid ordering possible.

Example 4.8. We continue with Example 3.16, and illustrate how to define an order on $r^{-1}(v_1)$ where $v_1 \in V_4$, if $(f^{(4)}_{v_1,v_1}, f^{(4)}_{v_1,v_2}, f^{(4)}_{v_1,v_3}, f^{(4)}_{v_1,v_4}) = (4, 2, 2, 3)$. In what follows we drop the

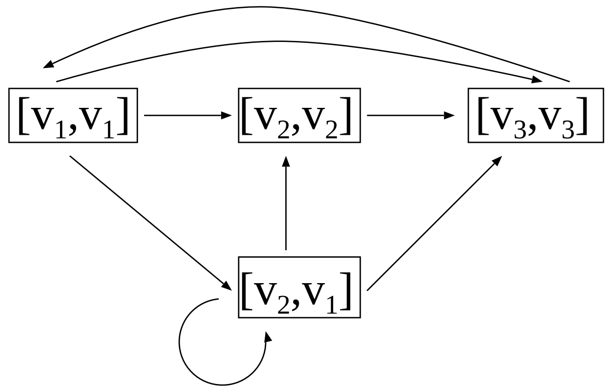

FIGURE 6. The graph H_4 for Example 4.8

superscript (4). The graph $\mathcal{H} = \mathcal{H}_4$ is shown in Figure 6; recall that $v_1 \in [v_1, v_1]$, $v_2 \in [v_2, v_1]$, $v_3 \in [v_2, v_2]$ and $v_4 \in [v_3, v_3]$. The (nontrivial part of the) system of relations (4.7) become

$$\widetilde{f}_{v_1,v_1} = \widetilde{f}_{v_1,v_1,v_2} + \widetilde{f}_{v_1,v_1,v_3} \text{ and } \widetilde{f}_{v_1,v_2} = \widetilde{f}_{v_1,v_2,v_2} + \widetilde{f}_{v_1,v_2,v_3},$$

and the balance relations (4.8) become, with $\overline{v} = v_1,\ v_2$ and v_3

$$\widetilde{f}_{v_1,v_4} = \overline{f}_{v_1,v_1},\ \widetilde{f}_{v_1,v_1,v_2} + \widetilde{f}_{v_1,v_2,v_2} = \overline{f}_{v_1,v_3} + \overline{f}_{v_1,v_2} \text{ and } \widetilde{f}_{v_1,v_1,v_3} + \widetilde{f}_{v_1,v_3} + \widetilde{f}_{v_1,v_2,v_3} = \overline{f}_{v_1,v_4}$$

respectively. If we let

$$\widetilde{f}_{v_1,v_1} = \widetilde{f}_{v_1,v_1,v_2} + \widetilde{f}_{v_1,v_1,v_3} = 2 + 1 \text{ and } \widetilde{f}_{v_1,v_2} = \widetilde{f}_{v_1,v_2,v_2} + \widetilde{f}_{v_1,v_2,v_3} = 2 + 0,$$

then the balance relations are satisfied. A valid choice of an ordering on $r^{-1}(v_1)$ obtained using our algorithm is $w(v_1, 3, 4) = v_1 v_2^2 v_3 v_4 v_1 v_3 v_4 v_1 v_4 v_1$; the other is $w(v_1, 3, 4) = v_1 v_2^2 v_3 v_4 v_1 v_4 v_1 v_3 v_4 v_1$. Note that there are other valid choices of orderings on $r^{-1}(v_1)$, but they are not achieved with this algorithm.

As a corollary we identify the conditions needed so that a perfect order in $\mathcal{P}_B^*$ is supported by B. Note that one can talk of a skeleton and correspondence $\mathcal{F}$ and σ of being capable of generating orders that belong to $\mathcal{P}_B^*$: either all perfect orders ω having skeleton $(\mathcal{F}, \sigma)$ belong to $\mathcal{P}_B^*$, or none of them do.

Corollary 4.9. *Let B be a Bratteli diagram with incidence matrices (F_n). Let $\mathcal{F}$ be a skeleton on B and σ an associated correspondence that can generate orders in $\mathcal{P}_B^*$, and suppose that all associated graphs $\mathcal{H}_n$ are positively strongly connected. Suppose also that for each M (and hence each sequence $(\widetilde{v}_n = v_n(M))$), there exists an n_0 such that for any $n \geq n_0$, any $m > n$, and any $u \in V_m$, the entries of incidence matrices (F_n) satisfy condition (4.9). Then there is a perfect ordering ω on B such that $\mathcal{F} = \mathcal{F}_\omega$ and the Vershik map φ_ω satisfies the relation $\varphi_\omega = \sigma$ on $X_{\max}(\mathcal{F})$.*

In the remaining part of this section, we will consider the class of Bratteli diagrams $\mathcal{A}$ that is close, by its structure, to diagrams of finite rank. We refer to the notation in Definition 2.2.

Take a Bratteli diagram B and let the columns of $A_n^{(i)}$ be indexed by vertices $V_n^{(i)}$. Define $X_B^{(i)} = \{x = (x_n) \in X_B : s(x_n) \in V_n^{(i)} \text{ for each } n\}$, and let $X_B^{(a)} = X_B \setminus \bigcup_i X_B^{(i)}$. We will call $X_B^{(i)}$ the i-th minimal component of X_B (here minimality is considered with respect to the tail equivalence relation $\mathcal{E}$). If $B \in \mathcal{A}$, and has incidence matrices of the form as in (2.1) we will say that B has k *minimal components*.

Given a skeleton $\mathcal{F}$ on B, let $X_{\max}^{(i)}(\mathcal{F}) = X_{\max}(\mathcal{F}) \cap X_B^{(i)}$; define $X_{\min}^{(i)}(\mathcal{F})$ similarly. Also, if ω is an order on B, define $X_{\max}^{(i)}(\omega)$ and $X_{\min}^{(i)}(\omega)$ analogously. Note that for any statement that we make about a skeleton, we can make an analogous statement about an order - simply consider the skeleton associated with the well-telescoped order.

The following lemma is straightforward.

Lemma 4.10. *Let $B \in \mathcal{A}$. Then*

(1) If $\mathcal{F}$ is a skeleton on B, then, for each i, the sets $X_{\max}^{(i)}(\mathcal{F})$ and $X_{\min}^{(i)}(\mathcal{F})$ are closed.
(2) If ω is a perfect order on B, then, for each i, $\varphi_\omega : X_{\max}^{(i)}(\omega) \to X_{\min}^{(i)}(\omega)$ is a homeomorphism.

We use Proposition 3.23 to prove the following generalization of Proposition 3.26 in [BKY13].

Proposition 4.11. *Let $B \in \mathcal{A}$ have k minimal components. Suppose that C_n is a $d \times d$ matrix where $1 \leq d \leq k-1$. If $k = 2$, then there are perfect orderings on B only if $C_n = (1)$ for all but finitely many n. If $k > 2$, then there is no perfect ordering on B.*

Proof. We first claim that in $\mathcal{H}_n$, there are k connected components of vertices $T_n^{(1)}, \dots T_n^{(k)}$, such that there are no edges from vertices in $T_n^{(i)}$ to vertices in $T_n^{(j)}$ if $i \neq j$. To see this, if $1 \leq i \leq k$, let $T_n^{(i)} = \{[\overline{v}, \widetilde{v}, n] : \overline{v} \in V_n^{(i)}, \widetilde{v} \in V_n^{(i)}\}$. By Lemma 4.10, for large n, if $\widetilde{v} \in \widetilde{V}_n^{(i)}$ it is not possible that $\overline{v} \in \sigma_n(\widetilde{v})$ if $\overline{v} \notin \overline{V}_n^{(i)}$.

If ω is a perfect order on B, we assume that (B, ω) is well telescoped and has skeleton $\mathcal{F}_\omega$. (Otherwise we work with the diagram B' on which $L(\omega)$ is well telescoped: Note that if B has incidence matrices of the given form, then so does any telescoping.) By Proposition 3.23, the graphs $\mathcal{H}_n$ are weakly connected. The only way that this can happen is if there are $k-1$ remaining vertices in $\mathcal{H}_n$ (so that d must equal $k-1$), each have one incoming edge from one of the components $T_n^{(i)}$, and one outgoing edge into one of the components $T_n^{(j)}$. Each of these remaining vertices in $\mathcal{H}_n$ corresponds to exactly one vertex in $V_n \setminus \bigcup_{i=1}^k V_n^{(i)}$. Thus at least one of the components, say $T_n^{(1)}$, has no incoming edges. Take now a vertex t, not belonging to any of the $T_n^{(j)}$'s, such that if $v \in t$, then $s(\overline{e}_v) \in T_n^{(p)}$, where $p \neq 1$. Since the row in F_n corresponding to v is strictly positive, we have a contradiction as there is no path from $T_n^{(p)}$ to $T_n^{(1)}$. □

5. The infinitesimal subgroup of diagrams that support perfect orders

We will use our results from Section 4 to give an alternative proof of the following result that was proved in [GPS95] (Corollary 2). We recall that one can associate the so called *dimension group* with each simple Bratteli diagram $B = (V, E)$. Let (F_n) be the sequence of incidence matrices of B, then the dimension group G is defined as the inductive limit:

$$G := \varinjlim_{n\to\infty} \mathbb{Z}^{|V_n|} \xrightarrow{F_n} \mathbb{Z}^{|V_{n+1}|}.$$

For all information on dimension groups, we refer the reader to [Eff81].

If G is a simple dimension group, the elements g of G of the infinitesimal subgroup $Inf(G)$ are defined by the relation: $\tau(g) = 0$ for every normalized trace τ on G. Any normalized trace is defined by a probability measure invariant with respect to the tail equivalence relation. If B' is a telescoping of B, then their dimension groups G, G' are group order isomorphic, and this isomorphism maps $Inf(G)$ onto $Inf(G')$.

Theorem 5.1. *Let B be a simple Bratteli diagram and G its dimension group. Suppose $\omega \in \mathcal{P}_B \cap \mathcal{O}_B(j)$, that is there is a perfect order ω on B with exactly j maximal paths and j minimal paths where $j \geq 2$. Then $Inf(G)$, the infinitesimal subgroup of G, contains a subgroup isomorphic to $\mathbb{Z}^{j-1}$.*

Proof. Given a simple Bratteli diagram B and $\omega \in \mathcal{P}_B \cap \mathcal{O}_B(j)$, we assume that (B, ω) is well telescoped. For if not, we will telescope (B, ω) to (B', ω'), and working with (B', ω'), we shall show that if G' is the dimension group defined by B', then $Inf(G')$ contains a subgroup isomorphic to $\mathbb{Z}^{j-1}$. Since G, G' are order isomorphic groups with the isomorphism mapping $Inf(G)$ to $Inf(G')$, then $Inf(G)$ will also contain a subgroup isomorphic to $\mathbb{Z}^{j-1}$.

(1) In the proof, we will use the notation defined in Sections 3 and 4. This means that we will freely operate with such objects as the skeleton $\mathcal{F}_\omega$, correspondence $\sigma = (\sigma_n)$, sets of maximal and minimal vertices $\widetilde{V}_n, \overline{V}_n$, partitions $W(n), W'(n)$, sets $\widetilde{E}(W_{\widetilde{v}}(n), u), \overline{E}(W'_{\overline{v}}(n), u)$, maximal and minimal finite paths $\widetilde{e}_v, \overline{e}_v, \widetilde{e}(V_n, u), \overline{e}(V_n, u)$ (defined just before Lemma 4.2), etc.

Fix a level n such that $n > n_0$ where n_0 is as in the first statement of Corollary 4.4, so that $|\widetilde{V}_i| = |\overline{V}_i| = j$ for all $i \geq n-1$, and take a maximal vertex $\widetilde{v}^* \in \widetilde{V}_{n-1}$. We construct a sequence of vectors $(\varepsilon_{\widetilde{v}^*}^{(n+k)})_{k \geq 1}$ with $\varepsilon_{\widetilde{v}^*}^{(n+k)} \in \mathbb{Z}^{|V_{n+k}|}$ as follows. Take first a vertex $v \in V_{n+1}$ and set

$$\varepsilon_{\widetilde{v}*}^{(n+1)}(v) := \begin{cases} -1 & \text{if } s(\overline{e}_v) \in W'_{\sigma_{n-1}(\widetilde{v}^*)}(n), s(\widetilde{e}_v) \notin W_{\widetilde{v}^*}(n), \\ 1 & \text{if } s(\overline{e}_v) \notin W'_{\sigma_{n-1}(\widetilde{v}^*)}(n), s(\widetilde{e}_v) \in W_{\widetilde{v}^*}(n), \\ 0 & \text{otherwise} \end{cases} \tag{5.1}$$

to obtain the v-th entry of $\varepsilon_{\widetilde{v}^*}^{(n+1)}$. In general, let v be any vertex from V_{n+k}. Then we define $\varepsilon_{\widetilde{v}^*}^{(n+k)}$ as follows:

$$\varepsilon_{\widetilde{v}*}^{(n+k)}(v) := \begin{cases} -1 & \text{if } s(\overline{e}(V_n, v)) \in W'_{\sigma_{n-1}(\widetilde{v}^*)}(n), s(\widetilde{e}(V_n, v)) \notin W_{\widetilde{v}^*}(n), \\ 1 & \text{if } s(\overline{e}(V_n, v)) \notin W'_{\sigma_{n-1}(\widetilde{v}^*)}(n), s(\widetilde{e}(V_n, v)) \in W_{\widetilde{v}^*}(n), \\ 0 & \text{otherwise.} \end{cases} \tag{5.2}$$

(2) We will show that for any $k \geq 1$

$$F_{n+k} \varepsilon_{\widetilde{v}*}^{(n+k)} = \varepsilon_{\widetilde{v}*}^{(n+k+1)}. \tag{5.3}$$

To prove (5.3), we use another representation of entries of the vector $\varepsilon_{\widetilde{v}*}^{(n+k)}(v)$. Indeed, since ω is perfect we have that relation (4.9) in Section 4 holds. Also, if $F(n, n+k) = F_{n+k-1} \circ \cdots \circ F_n, k \geq 1$, and $u \in V_{n+k}$, then relation (4.9) becomes

$$\sum_{w \in W_{\widetilde{v}}(n)} \widetilde{f}_{u,w}^{(n,n+k)} = \sum_{w' \in W'_{\sigma_{n-1}(\widetilde{v})}(n)} \overline{f}_{u,w'}^{(n,n+k)}, \qquad u \in V_{n+k}\,.$$

It is straightforward to check that

$$\varepsilon_{\widetilde{v}*}^{(n+k)}(v) = \sum_{w \in W_{\widetilde{v}^*}(n)} f_{v,w}^{(n,n+k)} - \sum_{w' \in W'_{\sigma_{n-1}(\widetilde{v}^*)}(n)} f_{v,w'}^{(n,n+k)}, \quad v \in V_{n+k}. \tag{5.4}$$

Then (5.3) can be proved by induction. Indeed, compute for $k = 1$

$$\begin{aligned} F_{n+1}\varepsilon_{\widetilde{v}*}^{(n+1)}(v) &= \sum_{u\in V_{n+1}} f_{v,u}^{(n+1)} \left(\sum_{w\in W_{\widetilde{v}^*}(n)} f_{u,w}^{(n)} - \sum_{w'\in W'_{\sigma_{n-1}(\widetilde{v}^*)}(n)} f_{u,w'}^{(n)} \right) \\ &= \sum_{w\in W_{\widetilde{v}^*}(n)} f_{v,w}^{(n,n+2)} - \sum_{w'\in W'_{\sigma_{n-1}(\widetilde{v}^*)}(n)} f_{v,w'}^{(n,n+2)} \\ &= \varepsilon_{\widetilde{v}*}^{(n+2)}(v), \quad v \in V_{n+2}. \end{aligned}$$

The induction step can be computed in a similar way. We omit the details.

(3) It follows from relation (5.3) that every vector $\varepsilon_{\widetilde{v}^*}^{(n+1)}, \widetilde{v}^* \in \widetilde{V}_{n-1}$, generates the element $g_{\widetilde{v}^*} = (\varepsilon_{\widetilde{v}^*}^{(n+1)}, \varepsilon_{\widetilde{v}^*}^{(n+2)}, \varepsilon_{\widetilde{v}^*}^{(n+3)}, \dots)$ of the dimension group G of B.

Next, since $W(n)$ and $W'(n)$ constitute partitions of V_n, we have

$$\begin{aligned} \sum_{\widetilde{v}\in\widetilde{V}_{n-1}} \varepsilon_{\widetilde{v}}^{(n+k)}(v) &= \sum_{\widetilde{v}\in\widetilde{V}_{n-1}} \left(\sum_{w\in W_{\widetilde{v}^*}(n)} f_{v,w}^{(n,n+k)} - \sum_{w'\in W'_{\sigma_{n-1}(\widetilde{v}^*)}(n)} f_{v,w'}^{(n,n+k)} \right) \\ &= \sum_{w\in V_n} f_{v,w}^{(n,n+k)} - \sum_{w'\in V_n} f_{v,w'}^{(n,n+k)} \\ &= 0 \end{aligned}$$

for any k and v. That is the vectors $\{\varepsilon_{\widetilde{v}}^{(n+k)} : \widetilde{v} \in \widetilde{V}_{n-1}\}$ are linearly dependent.

On the other hand, *we claim that any subset of this set containing $j-1$ vectors is linearly independent over* $\mathbb{Z}$. To simplify our notation we consider the set of vectors $\{\varepsilon_{\widetilde{v}}^{(n+1)} : \widetilde{v} \in \widetilde{V}_{n-1}\}$ only, since the case for the vectors $\{\varepsilon_{\widetilde{v}}^{(n+k)} : \widetilde{v} \in \widetilde{V}_{n-1}\}$ where $k > 1$ is considered similarly. Suppose that $\widetilde{V}_{n-1} = \{\widetilde{v}_1, \dots, \widetilde{v}_j\}$. Consider the $|V_{n+1}| \times j$-matrix whose i-th column is the vector $\varepsilon_{\widetilde{v}_i}^{(n+1)}$. We fix a vertex $u \in V_{n+1}$ and look at the u-th row R_u of this matrix; this is the row formed by the u-th entries of the vectors $\{\varepsilon_{\widetilde{v}_i}^{(n+1)}\}, 1 \le i \le j$. The definition of the entries of the row R_u (see (5.1)) shows that either they are all 0 or each value 1 and -1 is taken exactly once, and all remaining entries in this row are zero. Note that since this property holds for any vertex u, we have another proof of the fact that the sum of all vectors $\{\varepsilon_{\widetilde{v}}^{(n+1)}\}$ is zero. Also, for every $\widetilde{v}$, the strong connectivity of the graph $\mathcal{H}_n$ implies that we have $|\{u \in V_{n+1} : \varepsilon_{u,\widetilde{v}}^{(n+1)} = 1\}| \ge 1$ and $|\{u \in V_{n+1} : \varepsilon_{u,\widetilde{v}}^{(n+1)} = -1\}| \ge 1$ (but these sets may be of different cardinalities).

We pick any vector $\varepsilon_{\widetilde{v}}^{(n+1)}$ in the set $\{\varepsilon_{\widetilde{v}_i}^{(n+1)}\}$ and show that the set of remaining vectors $\{\varepsilon_{\widetilde{v}_i}^{(n+1)}\} \setminus \varepsilon_{\widetilde{v}}^{(n+1)}$ is linearly independent. Indeed, assume that

$$\sum_{\widetilde{v}_i\neq\widetilde{v}} m_{\widetilde{v}_i}\varepsilon_{\widetilde{v}_i}^{(n+1)} = 0; \tag{5.5}$$

we shall show that all m_{v_i}'s in (5.5) must be equal to m. If one assumes that $m \neq 0$ we obtain two contradictory equalities $\sum_{\widetilde{v}\neq\widetilde{v}_0} \varepsilon_{\widetilde{v}}^{(n+1)} = 0$ and $\sum_{\widetilde{v}\neq\widetilde{v}_0} \varepsilon_{\widetilde{v}}^{(n+1)} = -\varepsilon_{\widetilde{v}_0}^{(n+1)}$. Pick any two maximal vertices $\widetilde{v}_p$ and $\widetilde{v}_t$ that are not equal to $\widetilde{v}$. Firstly, by the strong connectivity of $\mathcal{H}_n$, there exist vertices $[\overline{v}_p, \widetilde{v}_p]$ and $[\overline{v}_t, \widetilde{v}_t]$ in $\mathcal{H}_n$ that do not have loops. Secondly, also by the strong connectivity of $\mathcal{H}_n$, we can find a path from $[\overline{v}_p, \widetilde{v}_p]$ to $[\overline{v}_t, \widetilde{v}_t]$, and we can also assume that for any vertex along this path, there are no loops (otherwise these vertices can be removed from the path, and we still have a valid path). Finally, note that if there is an edge from $[\overline{v}', \widetilde{v}']$ to $[\overline{v}^*, \widetilde{v}^*]$ and there is no loop at $[\overline{v}^*, \widetilde{v}^*]$, then $m_{\widetilde{v}^*} = m_{\widetilde{v}'}$. The result follows.

(4) It remains to show that the elements $g_{\widetilde{v}}, \widetilde{v} \in \widetilde{V}_{n-1}$, belong to the infinitesimal subgroup $Inf(G)$. Since G is a simple dimension group, it suffices to check that $\tau(g_{\widetilde{v}}) = 0$ for any trace τ on G or, equivalently, for any invariant measure μ.

Fix a probability φ_ω-invariant measure μ on the path space of the diagram B. Consider the vector $p^{(n)} = (p_v^{(n)}) \in \mathbb{R}^{|V_n|}$ whose entries are μ-measures of a finite path (cylinder set) with source v_0 and range v. This means that, in particular, $\mu(\widetilde{e}(v_0, w)) = p_w^{(n)}, w \in W_{\widetilde{v}}(n)$ and $\mu(\overline{e}(v_0, w')) = p_{w'}^{(n)}, w' \in W_{\sigma_{n-1}(\widetilde{v})}(n)$. As noticed in [BKMS10], we have $F_n^T p^{(n+1)} = p^{(n)}, \forall n \in \mathbb{N}$, or

$$\sum_{v \in V_{n+1}} f_{v,w}^{(n)} p_v^{(n+1)} = p_w^{(n)}, \quad w \in V_n.$$

Compute $< p^{(n+1)}, \varepsilon_{\widetilde{v}}^{(n+1)} >$ where $< \cdot, \cdot >$ is the inner product:

$$\begin{aligned}
< p^{(n+1)}, \varepsilon_{\widetilde{v}}^{(n+1)} > &= < F_{n+1}^T p^{(n+2)}, \varepsilon_{\widetilde{v}}^{(n+1)} > \\
&= < p^{(n+2)}, F_{n+1} \varepsilon_{\widetilde{v}}^{(n+1)} > \\
&= < p^{(n+2)}, \varepsilon_{\widetilde{v}}^{(n+2)} > \\
&\cdots \\
&= < p^{(n+j)}, \varepsilon_{\widetilde{v}}^{(n+j)} >
\end{aligned}$$

for any j. On the other hand,

$$\begin{aligned}
< p^{(n+1)}, \varepsilon_{\widetilde{v}}^{(n+1)} > &= \sum_{v \in V_{n+1}} p_v^{(n+1)} \left(\sum_{w \in W_{\widetilde{v}}(n)} f_{v,w}^{(n)} - \sum_{w' \in W'_{\sigma_{n-1}(\widetilde{v})}(n)} f_{v,w'}^{(n)} \right) \\
&= \sum_{w \in W_{\widetilde{v}}(n)} \sum_{v \in V_{n+1}} f_{v,w}^{(n)} p_v^{(n+1)} - \sum_{w' \in W'_{\sigma_{n-1}(\widetilde{v})}(n)} \sum_{v \in V_{n+1}} f_{v,w'}^{(n)} p_v^{(n+1)} \\
&= \sum_{w \in W_{\widetilde{v}}(n)} p_w^{(n)} - \sum_{w' \in W'_{\sigma_{n-1}(\widetilde{v})}(n)} p_{w'}^{(n)} \\
&= 0
\end{aligned}$$

because

$$\begin{aligned}
\sum_{w \in W_{\widetilde{v}}(n)} p_w^{(n)} &= \mu(\bigcup_{w \in W_{\widetilde{v}}(n)} \widetilde{e}(v_0, w)) \\
&= \mu(\varphi_\omega(\bigcup_{w \in W_{\widetilde{v}}(n)} \widetilde{e}(v_0, w))) \\
&= \mu(\bigcup_{w' \in W'_{\sigma_{n-1}(\widetilde{v})}(n)} \overline{e}(v_0, w')) \\
&= \sum_{w' \in W'_{\sigma_{n-1}(\widetilde{v})}(n)} p_{w'}^{(n)}.
\end{aligned}$$

This proves that $g_{\widetilde{v}} \in Inf(G)$. □

Remark 5.2. In the simpler case when B is of finite rank, then for each n, $\varepsilon_{\widetilde{v}*}^{(n)} = \varepsilon_{\widetilde{v}*}$, each of the latter j vectors correspond to an infinitesimal, and $\varepsilon_{\widetilde{v}*} = -\sum_{\widetilde{v} \neq \widetilde{v}*} \varepsilon_{\widetilde{v}}$, while $\{\varepsilon_{\widetilde{v}} : \widetilde{v} \neq \widetilde{v}*\}$

is a linearly independent set, so that there are $j-1$ identified copies of $\mathbb{Z}$ in the infinitesimal subgroup of $dim(B)$.

Example 5.3. Let

$$F_n = \begin{pmatrix} f_{aa}^{(n)} & f_{ab}^{(n)} & \alpha^{(n)} & \alpha^{(n)} \\ f_{ba}^{(n)} & f_{bb}^{(n)} & \beta^{(n)} & \beta^{(n)} \\ f_{ca}^{(n)} & f_{cb}^{(n)} & \gamma^{(n)}+1 & \gamma^{(n)} \\ f_{da}^{(n)} & f_{db}^{(n)} & \delta^{(n)} & \delta^{(n)}+1 \end{pmatrix}; \tag{5.6}$$

then there exist orders ω on B that belong to $\mathcal{P}_B \cap \mathcal{O}_B(2)$, and such that the associated graph and correspondence is as in Figure 1, where $a \in [a,a]$, $b \in [b,b]$, $c \in [a,b]$ and $d \in [b,a]$. In this case,

$$\varepsilon_a = \begin{pmatrix} 0 \\ 0 \\ -1 \\ 1 \end{pmatrix} \text{ and } \varepsilon_b = \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix}; \tag{5.7}$$

so that $F_n \varepsilon_a = \varepsilon_a$ as claimed and ε_a corresponds to an element of $Inf(G)$.

Example 5.4. One can vary the given skeleton and correspondence in Theorem 5.1 to maximize the number of copies of $\mathbb{Z}$ that one can find in $Inf(G)$. For example, if a finite rank simple diagram B supports orders in $\omega \in \mathcal{P}_B(2)$ that can have either of the two possible graphs described in Example 3.4, this means that there is a copy of $\mathbb{Z} \times \mathbb{Z}$ in $Inf(G)$. The incidence matrices of this diagram must have a very restrictive structure. For example, if B has rank 4, then the incidence matrices must be of the form

$$F_n = \begin{pmatrix} a_n^{(1)}+1 & a_n^{(1)} & a_n^{(2)} & a_n^{(2)} \\ b_n^{(1)} & b_n^{(1)}+1 & b_n^{(2)} & b_n^{(2)} \\ c_n^{(1)} & c_n^{(1)} & c_n^{(2)}+1 & c_n^{(2)} \\ d_n^{(1)} & d_n^{(1)} & d_n^{(2)} & d_n^{(2)}+1 \end{pmatrix} \tag{5.8}$$

Bratteli diagrams with these incidence matrices have 2 orders in $\mathcal{P}_B \cap \mathcal{O}_B(2)$, each with a different associated graph $\mathcal{H}$. This implies that they have (at least) two independent infinitesimals, corresponding to the elements.

$$\varepsilon = \begin{pmatrix} 0 \\ 0 \\ -1 \\ 1 \end{pmatrix} \text{ and } \varepsilon' = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}. \tag{5.9}$$

Next we extend Theorem 5.1 to diagrams supporting perfect orders that belong to $\mathcal{P}_B^*$. This result overlaps Corollary 3 in [GPS95].

Theorem 5.5. *Let B be a simple Bratteli diagram and G its dimension group. Suppose ω belongs to $\mathcal{P}_B^* \setminus \bigcup_{j=1}^{\infty} \mathcal{O}_B(j)$. Then $Inf(G)$, the infinitesimal subgroup of G, contains as a subgroup the free abelian group on countably many generators.*

Proof. The proof is similar to that of Theorem 5.1. The second statement of Corollary 4.4 tells us that for each maximal path M, with $\widetilde{v}_n = v_n(M)$, there exists some level n_0 such that if $n \geq n_0 - 1$, $\sigma_n(\widetilde{v}_n)$ is a singleton. We construct a sequence of vectors $(\varepsilon_M^{(n_0+k)})_{k\geq 1}$ with $\varepsilon_M^{(n+k)} \in \mathbb{Z}^{|V_{n+k}|}$ as follows. Take first a vertex $v \in V_{n_0+1}$ and set

$$\varepsilon_M^{(n_0+1)}(v) := \begin{cases} -1 & \text{if } s(\overline{e}_v) \in W'_{\sigma_{n_0-1}(\widetilde{v}_{n_0-1})}(n_0), s(\widetilde{e}_v) \notin W_{\widetilde{v}_{n_0-1}}(n_0), \\ 1 & \text{if } s(\overline{e}_v) \notin W'_{\sigma_{n_0-1}(\widetilde{v}_{n_0-1})}(n_0), s(\widetilde{e}_v) \in W_{\widetilde{v}_{n_0-1}}(n_0), \\ 0 & \text{otherwise} \end{cases}$$

to obtain the v-th entry of $\varepsilon_M^{(n_0+1)}$. In general, let v be any vertex from V_{n_0+k}. Then we define $\varepsilon_M^{(n_0+k)}$ as follows:

$$\varepsilon_{\widetilde{v}*}^{(n_0+k)}(v) := \begin{cases} -1 & \text{if } s(\overline{e}(V_{n_0}, v)) \in W'_{\sigma_{n_0-1}(\widetilde{v}_{n_0-1})}(n_0), s(\widetilde{e}(V_{n_0}, v)) \notin W_{\widetilde{v}_{n_0-1}}(n_0), \\ 1 & \text{if } s(\overline{e}(V_{n_0}, v)) \notin W'_{\sigma_{n_0-1}(\widetilde{v}_{n_0-1})}(n_0), s(\widetilde{e}(V_{n_0}, v)) \in W_{\widetilde{v}_{n_0-1}}(n_0), \\ 0 & \text{otherwise.} \end{cases}$$

As in (2) of Theorem 5.1, we can show that for any $k \geq 1$

$$F_{n_0+k}\varepsilon_M^{(n_0+k)} = \varepsilon_M^{(n_0+k+1)}. \tag{5.10}$$

This means that we can define, from relation (5.10), the element $g_M = (\varepsilon_M^{(n_0+1)}, \varepsilon_M^{(n_0+2)}, \varepsilon_M^{(n_0+3)}, \ldots)$ of the dimension group G of B. In this way, we get a countably infinite collection of elements $\{g_M : M \text{ maximal}\}$.

The argument that the collection $\{g_M : M \text{ maximal}\}$ generates a free abelian group, as the case of (3) of Theorem 5.1, depends on the strong connectivity of the graphs $\mathcal{H}_n$. Take a finite set of maximal paths $(M_1, ..., M_k)$ and suppose that there is a linear relation

$$\sum_{i=1}^{k} m_i g_{M_i} = 0,$$

where the m_i,s are nonzero. Let $\widetilde{v}_n^i = \widetilde{v}_n(M_i)$ and $\overline{v}_n^i = \overline{v}_n(M_i)$, and choose an N large enough so that $\sigma_n(\widetilde{v}_n^i)$ is a singleton for each $n \geq N-1$ and $1 \leq i \leq k$. Consider the $|V_{N+1}| \times k$-matrix whose i-th column is the vector $\varepsilon_{M_i}^{(N+1)}$. We fix a vertex $u \in V_{N+1}$ and look at the u-th row R_u of this matrix; this is the row formed by the u-th entries of the vectors $\{\varepsilon_{M_i}^{(n+1)}\}, 1 \leq i \leq k$. The definition of the entries of the row R_u shows that apart from at most one occurrence of 1 and of -1, they are all 0. Note that unlike the case in the proof of (3) of Theorem 5.1, it is possible that only one of the values 1, -1 appear in any row R_u. Also, for every $\widetilde{v}$, the strong connectivity of the graph $\mathcal{H}_n$ implies that we have $|\{u \in V_{n+1} : \varepsilon_{u,\widetilde{v}}^{(n+1)} = 1\}| \geq 1$ and $|\{u \in V_{n+1} : \varepsilon_{u,\widetilde{v}}^{(n+1)} = -1\}| \geq 1$. The linear relation implies that $\sum_{i=1}^{k} m_i \varepsilon_{M_i}^{(N+1)} = 0$. It follows that if a 1 occurs in the row R_u a -1 must also occur; i.e. if 1 occurs in R_u, then $u \in [\overline{v}_N^i, \widetilde{v}_N^j]$ for some $1 \leq i,\, j \leq k$. Otherwise - if R_u consists only of zeros - $u \in [\overline{v}, \widetilde{v}]$ with $\overline{v} \notin \{\overline{v}_N^1, \ldots, \overline{v}_N^k\}$ and $\widetilde{v} \notin \{\widetilde{v}_N^1, \ldots, \widetilde{v}_N^k\}$. Thus, we have partitioned $\mathcal{H}_{N+1}$ into two disconnected sets of vertices, contradicting its strong connectivity.

The proof that each g_M is an infinitesimal is now very similar to part (4) of the proof of Theorem 5.1.

$\square$

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