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Monomiality principle, Sheffer-type polynomials and the normal ordering problem

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Abstract. We solve the boson normal ordering problem for $\left(q(a^\dagger)a + v(a^\dagger)\right)^n$ with arbitrary functions $q(x)$ and $v(x)$ and integer n, where a and $a^\dagger$ are boson annihilation and creation operators, satisfying $[a, a^\dagger] = 1$. This consequently provides the solution for the exponential $e^{\lambda(q(a^\dagger)a+v(a^\dagger))}$ generalizing the shift operator. In the course of these considerations we define and explore the monomiality principle and find its representations. We exploit the properties of Sheffer-type polynomials which constitute the inherent structure of this problem. In the end we give some examples illustrating the utility of the method and point out the relation to combinatorial structures.

1. Introduction

In this work we are concerned with one mode boson creation $a^\dagger$ and annihilation a operators satisfying the commutation relation

$$[a, a^\dagger] = 1. \tag{1}$$

We consider the normal ordering problem of a specific class of operator expressions. The normally ordered form of a general function $F(a, a^\dagger)$, denoted as $\mathcal{N}\left[F(a, a^\dagger)\right] \equiv F(a, a^\dagger)$, is defined by moving all the annihilation operators a to the right, using the commutation relation Eq.(1). Additionally we define the operation $:G(a, a^\dagger):$ which means normally order

$G(a,a^\dagger)$ *without* taking into account the commutation relations. Using the latter operation the normal ordering problem for $F(a,a^\dagger)$ is solved if we find an operator $G(a,a^\dagger)$ for which $F(a,a^\dagger) = :G(a,a^\dagger):$ is satisfied.

In 1974 J. Katriel [1] considered the normal ordering problem for powers of the number operator $N = a^\dagger a$ and pointed out its connection to combinatorics. It can be written as

$$\left(a^\dagger a\right)^n = \sum_{k=1}^{n} S(n,k)(a^\dagger)^k a^k, \tag{2}$$

where the integers $S(n,k)$ are the so called *Stirling numbers* of the second kind counting the number of ways of putting n different objects into k identical containers (none left empty).

In the coherent state representation $|z\rangle = e^{-|z|^2/2}\sum_{n=0}^{\infty}\frac{z^n}{\sqrt{n!}}|n\rangle$, where $a^\dagger a|n\rangle = n|n\rangle$, $\langle n|n'\rangle = \delta_{n,n'}$ and $a|z\rangle = z|z\rangle$ [2], we may write

$$\langle z|\left(a^\dagger a\right)^n|z\rangle = B(n,|z|^2), \tag{3}$$

where $B(n,x)$ are so called (exponential) *Bell polynomials* [3]

$$B(n,x) = \sum_{k=1}^{n} S(n,k)x^k. \tag{4}$$

Further development of this idea (see *e.g.* [4],[5]) provides the normally ordered expression for the exponential

$$e^{\lambda a^\dagger a} = \mathcal{N}\left[e^{\lambda a^\dagger a}\right] \equiv :e^{a^\dagger a(e^\lambda - 1)}: . \tag{5}$$

Here we shall extend these results in a very particular direction. We consider operators linear in annihilation a or creation $a^\dagger$ operators. More specifically, we consider operators which, say for linearity in a, have the form $q(a^\dagger)a + v(a^\dagger)$, where $q(x)$ and $v(x)$ are arbitrary functions. We shall find the normally ordered form of the n-th power $(q(a^\dagger)a + v(a^\dagger))^n$ and then of the exponential $e^{\lambda(q(a^\dagger)a+v(a^\dagger))}$. This is a far reaching generalization of the results of [6],[7],[8] where a special case of the operator $a^\dagger a + a^r$ was considered.

In this approach we use methods which are based on the monomiality principle [9]. First, using the methods of umbral calculus, we find a wide class of representations of the canonical commutation relation Eq.(1) in the space of polynomials. This establishes the link with Sheffer-type polynomials. Next the specific matrix elements of the above operators are derived and thereafter, with the help of coherent state representation, extended to the general form. Finally we obtain normally ordered expressions for these operators. It turns out that the exponential generating functions in the case of linear dependence on the annihilation (or creation) operator are of Sheffer-type, and that assures their convergence. In the end we give some examples with special emphasis put on their Sheffer-type origin and point out the relation to combinatorial structures.

2. Monomiality principle

Here we introduce the concept of monomiality which arises from the action of the multiplication and derivative operators on monomials. Next we provide a wide class of representations of that property in the framework of Sheffer-type polynomials. Finally we establish the correspondence to the occupation number representation.

2.1. Definition and general properties
Let us consider the Heisenberg-Weyl algebra satisfying the commutation relation

$$[P, M] = 1. \tag{6}$$

The simplest representation of Eq.(6) is by the derivative $P = D = \frac{d}{dx}$ and multiplication $M = X$ operators acting in the space of polynomials ($[D, X] = 1$). They are defined by their action on monomials

$$\begin{aligned} Xx^n &= x^{n+1}, \\ Dx^n &= nx^{n-1}. \end{aligned} \tag{7}$$

and subsequently on polynomials and formal power series.
Here we extend this framework. Suppose one wants to find the representations of Eq.(6) such that the action of M and P on certain polynomials $s_n(x)$ is analogous to the action of X and D on monomials. More specifically one searches for M and P and associated polynomials $s_n(x)$ (of degree n, $n = 0, 1, 2, ...$) which satisfy

$$\begin{aligned} &Ms_n(x) = s_{n+1}(x), \\ &Ps_n(x) = n\ s_{n-1}(x). \end{aligned} \tag{8}$$

The rule embodied in Eq.(8) is called the *monomiality principle*. The polynomials $s_n(x)$ are then called *quasi-monomials* with respect to operators M and P. These operators can be immediately recognized as raising and lowering operators acting on the $s_n(x)$'s.
The definition Eq.(8) implies some general properties besides fulfilling commutation relation of Eq.(6). First the operators M and P obviously satisfy Eq.(6). Further consequence of Eq.(8) is the eigenproperty of MP

$$MPs_n(x) = ns_n(x). \tag{9}$$

The polynomials $s_n(x)$ may be obtained through the action of M^n on s_0

$$s_n(x) = M^n s_0 \tag{10}$$

and consequently the exponential generating function of $s_n(x)$'s is

$$G(\lambda, x) \equiv \sum_{n=0}^{\infty} s_n(x) \frac{\lambda^n}{n!} = e^{\lambda M} s_0. \tag{11}$$

Also, if we write the quasimonomial $s_n(x)$ explicitly as

$$s_n(x) = \sum_{k=0}^{n} s_{n,k}\ x^k, \tag{12}$$

then

$$s_n(x) = \left(\sum_{k=0}^{n} s_{n,k}\ X^k \right) 1. \tag{13}$$

Several types of such polynomial sequences were studied recently using this monomiality principle [10],[11],[12],[13],[14].

2.2. Monomiality principle representations: Sheffer-type polynomials

Here we show that if $s_n(x)$ are of *Sheffer-type* then it is possible to give explicit representations of M and P. Conversely, if $M = M(X, D)$ and $P = P(D)$ then $s_n(x)$ of Eq.(8) are necessarily of Sheffer-type.

Properties of Sheffer-type polynomials are naturally handled within the so called *umbral calculus* [15],[16],[17]. They are usually defined through their exponential generating function

$$G(\lambda, x) = \sum_{n=0}^{\infty} s_n(x) \frac{\lambda^n}{n!} = A(\lambda) e^{xB(\lambda)} \tag{14}$$

with some (formal) power series $A(\lambda) = \sum_{n=0}^{\infty} a_n \frac{\lambda^n}{n!}$ and $B(\lambda) = \sum_{n=0}^{\infty} b_n \frac{\lambda^n}{n!}$ such that $b_0 = 0$, $b_1 \neq 0$ and $a_0 \neq 0$.

Here we focus on their ladder structure aspect and recall the following equivalent definition. Suppose we have a polynomial sequence $s_n(x)$, $n = 0, 1, 2, ...$ ($s_n(x)$ being a polynomial of degree n). It is called of a Sheffer A-type zero [18],[19] (which we shall call Sheffer-type) if there exists a function $f(x)$ such that

$$f(D) s_n(x) = n s_{n-1}(x). \tag{15}$$

Operator $f(D)$ plays the role of the lowering operator. This characterization is not unique, *i.e.* there are many Sheffer-type sequences $s_n(x)$ satisfying Eq.(15) for given $f(x)$. We can further classify them by postulating the existence of the associated raising operator. A general theorem [15],[20] states that a polynomial sequence $s_n(x)$ satisfying the monomiality principle Eq.(8) with an operator P given as a function of the derivative operator only $P = P(D)$ is *uniquely* determined by two (formal) power series $f(\lambda) = \sum_{n=0}^{\infty} f_n \frac{\lambda^n}{n!}$ and $g(\lambda) = \sum_{n=0}^{\infty} g_n \frac{\lambda^n}{n!}$ such that $f_0 = 0$, $f_1 \neq 0$ and $g_0 \neq 0$. The exponential generating function of $s_n(x)$ is then equal to

$$G(\lambda, x) = \sum_{n=0}^{\infty} s_n(x) \frac{\lambda^n}{n!} = \frac{1}{g(f^{-1}(\lambda))}\ e^{x f^{-1}(\lambda)}, \tag{16}$$

and their associated raising and lowering operators of Eq.(8) are given by

$$\begin{aligned} P &= f(D), \\ M &= \left[X - \tfrac{g'(D)}{g(D)}\right] \tfrac{1}{f'(D)}\ . \end{aligned} \tag{17}$$

Observe that X enters M only linearly and the order of X and D in $M(X, D)$ matters.

By direct calculation one may check that any pair M, P from Eq.(17) automatically satisfies Eq.(6). The detailed proof can be found in [15],[20].

Here are some examples we have obtained of representations of the monomiality principle Eq.(8) and their associated Sheffer-type polynomials:

a) $M(X, D) = 2X - D, \qquad P(D) = \frac{1}{2}D,$
$s_n(x) = H_n(x)$ - Hermite polynomials;
$G(\lambda, x) = e^{2\lambda x - \lambda^2}$.

b) $M(X, D) = -XD^2 + (2X - 1)D - X + 1, \qquad P(D) = -\sum_{n=1}^{\infty} D^n,$
$s_n(x) = n! L_n(x)$ - where $L_n(x)$ are Laguerre polynomials;
$G(\lambda, x) = \frac{1}{1-\lambda} e^{x \frac{\lambda}{\lambda - 1}}$.

c) $M(X, D) = X \frac{1}{1-D}, \qquad P(D) = -\frac{1}{2}D^2 + D,$
$s_n(x) = P_n(x)$ - Bessel polynomials [21];
$G(\lambda, x) = e^{x(1-\sqrt{1-2\lambda})}$.

d) $M(X,D) = X(1+D), \qquad P(D) = \ln(1+D),$
$s_n(x) = B_n(x)$ - (exponential) Bell polynomials;
$G(\lambda, x) = e^{x(e^\lambda - 1)}.$

e) $M(X,D) = Xe^{-D}, \qquad P(D) = e^D - 1,$
$s_n(x) = x^{\underline{n}}$ - the lower factorial polynomials [22];
$G(\lambda, x) = e^{x\ln(1+\lambda)}.$

f) $M(X,D) = (X - \tan(D))\cos^2(D), \qquad P(D) = \tan(D),$
$s_n(x) = R_n(x)$ - Hahn polynomials [23];
$G(\lambda, x) = \frac{1}{\sqrt{1+\lambda^2}} e^{x\arctan(\lambda)}.$

g) $M(X,D) = X\frac{1+W_L(D)}{W_L(D)}D, \qquad P(D) = W_L(D),$
where $W_L(x)$ is the Lambert W function [24];
$s_n(x) = I_n(x)$ - the idempotent polynomials [3];
$G(\lambda, x) = e^{x\lambda e^\lambda}.$

2.3. Monomiality vs Fock space representations

We have already called operators M and P satisfying Eq.(8) the rising and lowering operators. Indeed, their action rises and lowers the index n of the quasimonomial $s_n(x)$. This resembles the property of creation $a^\dagger$ and annihilation a operators in the Fock space given by

$$\begin{array}{rcl} a|n\rangle & = & \sqrt{n}\;|n-1\rangle, \\ a^\dagger|n\rangle & = & \sqrt{n+1}\;|n+1\rangle. \end{array} \tag{18}$$

These relations are almost the same as Eq.(8). There is a difference in coefficients. To make them analogous it is convenient to redefine the number states $|n\rangle$ as

$$\widetilde{|n\rangle} = \sqrt{n!}\;|n\rangle. \tag{19}$$

(Note that $\widetilde{|0\rangle} \equiv |0\rangle$).
Then the creation and annihilation operators act as

$$\begin{array}{rcl} a\widetilde{|n\rangle} & = & n\;\widetilde{|n-1\rangle}, \\ a^\dagger\widetilde{|n\rangle} & = & \widetilde{|n+1\rangle}. \end{array} \tag{20}$$

Now this exactly mirrors the relation of Eq.(8). So, we make the correspondence

$$\begin{array}{ccl} P & \longleftrightarrow & a \\ M & \longleftrightarrow & a^\dagger \\ s_n(x) & \longleftrightarrow & \widetilde{|n\rangle}\;, \qquad n = 0,1,2,\ldots\;. \end{array} \tag{21}$$

We note that this identification is purely algebraic, *i.e.* we are concerned here only with the commutation relation Eqs.(1) or (6). We neither impose the scalar product in the space of polynomials nor consider the conjugacy properties of the operators. These properties are irrelevant for our proceeding discussion. We note only that they may be rigorously introduced, see *e.g.* [15].

3. Normal ordering via monomiality

In this section we shall exploit the correspondence of Section 2.3 to obtain the normally ordered expression of powers and exponential of the operators $a^\dagger q(a) + v(a)$ and (by the conjugacy

property) $q(a^\dagger)a + v(a^\dagger)$. To this end we shall apply the results of Section 2.2 to calculate some specific coherent state matrix elements of operators in question and then, through the exponential mapping property, we shall extend them to general matrix elements. In conclusion we shall also comment on other forms of linear dependence on a or $a^\dagger$.
We use the correspondence of Section 2.3 cast in the simplest form for $M = X$, $P = D$ and $s_n(x) = x^n$, *i.e.*

$$\begin{array}{ccc} D & \longleftrightarrow & a, \\ X & \longleftrightarrow & a^\dagger, \\ x^n & \longleftrightarrow & \widetilde{|n\rangle}\ , \qquad n = 0, 1, 2, \ldots\ . \end{array} \tag{22}$$

Now, recall the representation Eq.(17) of operators M and P in terms of X and D. Applying the correspondence of Eq.(22) it takes the form

$$\begin{array}{l} P(a) = f(a), \\ M(a, a^\dagger) = \left[a^\dagger - \frac{g'(a)}{g(a)}\right] \frac{1}{f'(a)}\ . \end{array} \tag{23}$$

From Eqs.(10),(12) and (13) we get

$$\left[M(a, a^\dagger)\right]^n |0\rangle = \sum_{k=0}^{n} s_{n,k}\ (a^\dagger)^k |0\rangle. \tag{24}$$

In the coherent state representation it yields

$$\langle z|\ \left[M(a, a^\dagger)\right]^n |0\rangle = s_n(z^*)\langle z|0\rangle. \tag{25}$$

Exponentiating $M(a, a^\dagger)$ and using Eq.(16) we obtain

$$\langle z|e^{\lambda M(a,a^\dagger)}|0\rangle = \frac{1}{g(f^{-1}(\lambda))}\ e^{z^* f^{-1}(\lambda)}\langle z|0\rangle. \tag{26}$$

Observe that in Eq.(25) we obtain Sheffer-type polynomials (modulus coherent states overlapping factor $\langle z|0\rangle$). This connection will be explored in detail in Section 4.2.
The result of Eq.(26) can be further extended to the general matrix element $\langle z|e^{\lambda M(a,a^\dagger)}|z'\rangle$. To this end recall the property $|z\rangle = e^{-\frac{|z|^2}{2}} e^{za^\dagger}|0\rangle$ and write

$$\begin{aligned} \langle z|e^{\lambda M(a,a^\dagger)}|z'\rangle &= e^{-\frac{1}{2}|z'|^2}\langle z|e^{\lambda M(a,a^\dagger)}e^{z'a^\dagger}|0\rangle \\ &= e^{-\frac{1}{2}|z'|^2}\langle z|e^{z'a^\dagger}e^{-z'a^\dagger}e^{\lambda M(a,a^\dagger)}e^{z'a^\dagger}|0\rangle \\ &= e^{z^*z'-\frac{1}{2}|z'|^2}\langle z|e^{-z'a^\dagger}e^{\lambda M(a,a^\dagger)}e^{z'a^\dagger}|0\rangle. \end{aligned}$$

Next, using the exponential mapping formula $e^{-xa^\dagger}F(a, a^\dagger)e^{xa^\dagger} = F(a + x, a^\dagger)$ (see [2],[25]) we arrive at

$$\begin{aligned} \langle z|e^{\lambda M(a,a^\dagger)}|z'\rangle &= e^{z^*z'-\frac{1}{2}|z'|^2}\langle z|e^{\lambda M(a+z',a^\dagger)}|0\rangle \\ &= e^{z^*z'-\frac{1}{2}|z'|^2}\langle z|e^{\lambda\left(a^\dagger-\frac{g'(a+z')}{g(a+z')}\right)\frac{1}{f'(a+z')}}|0\rangle. \end{aligned}$$

Now we are almost ready to apply Eq.(26) to evaluate the matrix element on the r.h.s. of the above equation. Before doing so we have to appropriately redefine functions $f(x)$ and $g(x)$ in the following way (z' - fixed)

$$\begin{array}{lcl} f(x) & \rightarrow & \tilde{f}(x) = f(x + z') - f(z'), \\ g(x) & \rightarrow & \tilde{g}(x) = g(x + z')/g(z'). \end{array}$$

Then $\tilde{f}(0) = 0$, $\tilde{f}'(0) \neq 0$ and $\tilde{g}(0) = 1$ as required by the Sheffer property for $\tilde{f}(x)$ and $\tilde{g}(x)$. Observe that these conditions are not fulfilled by $f(x+z')$ and $g(x+z')$. This step imposes (analytical) constraints on z', *i.e.* it is valid whenever $\tilde{f}'(z') \neq 0$ (although, we note that for formal power series approach this does not present any difficulty). Now we can write

$$\begin{aligned} \langle z|e^{\lambda\left(a^\dagger - \frac{g'(a+z')}{g(a+z')}\right)\frac{1}{f'(a+z')}}|0\rangle &= \langle z|e^{\lambda\left(a^\dagger - \frac{\tilde{g}'(a)}{\tilde{g}(a)}\right)\frac{1}{\tilde{f}'(a)}}|0\rangle \\ &= \frac{1}{\tilde{g}(\tilde{f}^{-1}(\lambda))}\; e^{z^*\tilde{f}^{-1}(\lambda)}\langle z|0\rangle. \end{aligned}$$

By going back to the initial functions $f(x)$ and $g(x)$ this readily gives the final result

$$\langle z|e^{\lambda M(a,a^\dagger)}|z'\rangle = \frac{g(z')}{g(f^{-1}(\lambda+f(z')))}e^{z^*[f^{-1}(\lambda+f(z'))-z']}\langle z|z'\rangle, \tag{27}$$

where $\langle z|z'\rangle = e^{z^*z' - \frac{1}{2}|z'|^2 - \frac{1}{2}|z|^2}$ is the coherent states overlapping factor.
To obtain the normally ordered form of $e^{\lambda M(a,a^\dagger)}$ we apply the crucial property of the coherent state representation $\langle z|F(a,a^\dagger)|z'\rangle = \langle z|z'\rangle\; G(z^*,z') \Longrightarrow F(a,a^\dagger) = \,:G(a^\dagger,a):$ (see [2]). Then Eq.(27) provides the central result

$$e^{\lambda M(a,a^\dagger)} = \,:e^{a^\dagger[f^{-1}(\lambda+f(a))-a]}\frac{g(a)}{g(f^{-1}(\lambda+f(a)))}:\ . \tag{28}$$

Let us point out again that $a^\dagger$ appears linearly in $M(a,a^\dagger)$, see Eq.(23).
For simplicity we put

$$\begin{aligned} q(x) &= \frac{1}{f'(x)}, \\ v(x) &= \frac{g'(x)}{g(x)}\frac{1}{f'(x)}, \end{aligned}$$

and define

$$\begin{aligned} T(\lambda,x) &= f^{-1}(\lambda+f(x)), \\ G(\lambda,x) &= \frac{g(x)}{g(T(\lambda,x))}. \end{aligned}$$

This allows us to rewrite the main normal ordering formula of Eq.(28) as

$$e^{\lambda\left(a^\dagger q(a)+v(a)\right)} = \,:e^{a^\dagger[T(\lambda,a)-a]}\;G(\lambda,a): \tag{29}$$

where the functions $T(\lambda,x)$ and $G(\lambda,x)$ fulfill the following differential equations

$$\frac{\partial T(\lambda,x)}{\partial\lambda} = q(T(\lambda,x))\ , \qquad\qquad T(0,x) = x\ , \tag{30}$$

$$\frac{\partial G(\lambda,x)}{\partial\lambda} = v(T(\lambda,x))\cdot G(\lambda,x)\ , \qquad G(0,x) = 1\ . \tag{31}$$

Eq.(29) in the coherent state representation takes the form

$$\langle z'|e^{\lambda\left(a^\dagger q(a)+v(a)\right)}|z\rangle = \langle z'|z\rangle\; e^{z'^*[T(\lambda,z)-z]}G(\lambda,z). \tag{32}$$

We conclude by making a comment on other possible forms of linear dependence on a or $a^\dagger$. By hermitian conjugation of Eq.(29) we obtain the expression for the normal form of $e^{\lambda\left(q(a^\dagger)a+v(a)\right)}$. This amounts to the formula

$$e^{\lambda\left(q(a^\dagger)a+v(a)\right)} = :G(\lambda,a^\dagger)e^{[T(\lambda,a^\dagger)-a^\dagger]a}: \tag{33}$$

with the same differential equations Eqs.(30) and (31) for functions $T(\lambda,x)$ and $G(\lambda,x)$. In the coherent state representation it yields

$$\langle z'|e^{\lambda\left(q(a^\dagger)a+v(a)\right)}|z\rangle = \langle z'|z\rangle\; G(\lambda,z'^*)e^{[T(\lambda,z'^*)-z'^*]z} \tag{34}$$

We also note that all other operators linearly dependent on a or $a^\dagger$ may be written in just considered forms using $[a,F(a,a^\dagger)] = \frac{\partial}{\partial a^\dagger}\,F(a,a^\dagger)$ which yields $aq(a^\dagger)+v(a^\dagger) = q(a^\dagger)a + q'(a^\dagger)+v(a^\dagger)$ and $q(a)a^\dagger + v(a) = a^\dagger q(a) + q'(a) + v(a)$.
Observe that from analytical point of view certain limitations on the domains of z, z' and λ should be put in some specific cases (locally around zero all the formulas hold true). Also we point out the fact that functions $q(x)$ and $v(x)$ (or equivalently $f(x)$ and $g(x)$) may be taken as the formal power series.
In the end we note that the reverse process, *i.e.* derivation of the normally ordered form from the substitution theorem, is also possible, see [26].

4. Sheffer-type polynomials and normal ordering: Examples

We now proceed to examples putting special emphasis on their Sheffer-type origin.

4.1. Examples

We start with enumerating some examples of the evaluation of the coherent state matrix elements of Eqs.(25) and (32). We choose the $M(a,a^\dagger)$'s as in the list a) - g) in Section 2.2:

a) $\langle z|(-a+2a^\dagger)^n|0\rangle = H_n(z^*)\langle z|0\rangle$, Hermite polynomials;
$\langle z|e^{\lambda(-a+2a^\dagger)}|z'\rangle = e^{\lambda(2z^*-z')-\lambda^2}\langle z|z'\rangle$.

b) $\langle z|\left[-a^\dagger a^2 + (2a^\dagger-1)a - a^\dagger + 1\right]^n|0\rangle = n!L_n(z^*)\langle z|0\rangle$,
Laguerre polynomials;
$\langle z|e^{\lambda\left[-a^\dagger a^2+(2a^\dagger-1)a-a^\dagger+1\right]}|z'\rangle = \frac{z'^2-\lambda z'+1}{(1-z')(1-\lambda(z'-1))}e^{z^*\lambda\frac{(1-z')^2}{(\lambda(1-z')-1)}}\langle z|z'\rangle$.

c) $\langle z|\left(a^\dagger\frac{1}{1-a}\right)^n|0\rangle = P_n(z^*)\langle z|0\rangle$, Bessel polynomials;
$\langle z|e^{\lambda\left(a^\dagger\frac{1}{1-a}\right)}|z'\rangle = e^{z^*[1-\sqrt{1-2(\lambda+z'-\frac{1}{2}z'^2)}-z']}\langle z|z'\rangle$.

d) $\langle z|(a^\dagger a + a^\dagger)^n|0\rangle = B_n(z^*)\langle z|0\rangle$, Bell polynomials;
$\langle z|e^{\lambda(a^\dagger a+a^\dagger)}|z'\rangle = e^{z^*(z'+1)(e^\lambda-1)}\langle z|z'\rangle$.

e) $\langle z|(a^\dagger e^{-a})^n|0\rangle = (z^*)^{\underline{n}}\langle z|0\rangle$, the lower factorial polynomials;
$\langle z|e^{\lambda(a^\dagger e^{-a})}|z'\rangle = e^{z^*[\ln(e^{z'}+\lambda)-z']}\langle z|z'\rangle$.

f) $\langle z|\left[(a^\dagger-\tan(a))\cos^2(a)\right]^n|0\rangle = R_n(z^*)\langle z|0\rangle$, Hahn polynomials;
$\langle z|e^{\lambda(a^\dagger-\tan(a))\cos^2(a)}|z'\rangle = \frac{\cos[\arctan(\lambda+\tan(z'))]}{\cos(z')}e^{z^*[\arctan(\lambda+\tan(z'))-z']}\langle z|z'\rangle$.

g) $\langle z|\left[a^\dagger\frac{1+W_L(a)}{W_L(a)}a\right]^n|0\rangle = I_n(z^*)\langle z|0\rangle$, the idempotent polynomials;
$\langle z|e^{\lambda a^\dagger\frac{1+W_L(a)}{W_L(a)}a}|z'\rangle = e^{z^*[\lambda e^{\lambda+W_L(z')}+z'(e^\lambda-1)]}\langle z|z'\rangle$.

Note that for $z' = 0$ we obtain the exponential generating functions of appropriate polynomials multiplied by the coherent states overlapping factor $\langle z|0\rangle$, see Eq.(27).
These examples show how the Sheffer-type polynomials and their exponential generating functions arise in the coherent state representation. This generic structure is the consequence of Eqs.(25) and (27) or in general Eqs.(32) or (34) and it will be investigated in more detail now. Afterwards we shall provide more examples of combinatorial origin.

4.2. Sheffer polynomials and normal ordering
First recall the definition of the family of Sheffer-type polynomials $s_n(z)$ defined through the exponential generating function (see Eq.(14)) as

$$G(\lambda, z) = \sum_{n=0}^{\infty} s_n(z) \frac{\lambda^n}{n!} = A(\lambda)\ e^{zB(\lambda)} \tag{35}$$

where functions $A(\lambda)$ and $B(\lambda)$ satisfy: $A(0) \neq 0$ and $B(0) = 0$, $B'(0) \neq 0$.
Returning to normal ordering, recall that the coherent state expectation value of Eq.(33) is given by Eq.(34). When one fixes z' and takes λ and z as indeterminates, then the r.h.s. of Eq.(34) may be read off as an exponential generating function of Sheffer-type polynomials defined by Eq.(35). The correspondence is given by

$$A(\lambda) = g(\lambda, z'^*), \tag{36}$$

$$B(\lambda) = \left[T(\lambda, z'^*) - z'^*\right]. \tag{37}$$

This allows us to make the statement that the coherent state expectation value $\langle z'|...|z\rangle$ of the operator $\exp\left[\lambda(q(a^\dagger)a + v(a^\dagger))\right]$ for any fixed z' yields (up to the overlapping factor $\langle z'|z\rangle$) the exponential generating function of a certain sequence of Sheffer-type polynomials in the variable z given by Eqs.(36) and (37). The above construction establishes the connection between the coherent state representation of the operator $\exp\left[\lambda(q(a^\dagger)a + v(a^\dagger))\right]$ and a family of Sheffer-type polynomials $s_n^{(q,v)}(z)$ related to q and v through

$$\langle z'|e^{\lambda\left[q(a^\dagger)a+v(a^\dagger)\right]}|z\rangle = \langle z'|z\rangle \left(1 + \sum_{n=1}^{\infty} s_n^{(q,v)}(z) \frac{\lambda^n}{n!}\right), \tag{38}$$

where explicitly (again for z' fixed):

$$s_n^{(q,v)}(z) = \langle z'|z\rangle^{-1} \langle z'| \left[q(a^\dagger)a + v(a^\dagger)\right]^n |z\rangle. \tag{39}$$

We observe that Eq.(39) is an extension of the seminal formula of Katriel [1],[27] where $v(x) = 0$ and $q(x) = x$. The Sheffer-type polynomials are in this case Bell polynomials expressible through the Stirling numbers of the second kind Eq.(4).
Having established relations leading from the normal ordering problem to Sheffer-type polynomials we may consider the reverse approach. Indeed, it turns out that for any Sheffer-type sequence generated by $A(\lambda)$ and $B(\lambda)$ one can find functions $q(x)$ and $v(x)$ such that the coherent state expectation value $\langle z'|\exp\left[\lambda(q(a^\dagger)a + v(a^\dagger))\right]|z\rangle$ results in a corresponding exponential generating function of Eq.(35) in indeterminates z and λ (up to the overlapping factor $\langle z'|z\rangle$ and z' fixed). Appropriate formulas can be derived from Eqs.(36) and (37) by substitution into Eqs.(30) and (31):

$$q(x) = B'(B^{-1}(x - z'^*)), \tag{40}$$

$$v(x) = \frac{A'(B^{-1}(x - z'^*))}{A(B^{-1}(x - z'^*))}. \tag{41}$$

One can check that this choice of $q(x)$ and $v(x)$, if inserted into Eqs. (30) and (31), results in

$$\begin{aligned} T(\lambda,x) &= B(\lambda + B^{-1}(x - z'^*)) + z'^*, & (42)\\ g(\lambda,x) &= \frac{A(\lambda + B^{-1}(x - z'^*))}{A(B^{-1}(x - z'^*))}, & (43) \end{aligned}$$

which reproduce

$$\langle z'|e^{\lambda\left[q(a^\dagger)a+v(a^\dagger)\right]}|z\rangle = \langle z'|z\rangle A(\lambda)e^{zB(\lambda)}. \tag{44}$$

The result summarized in Eqs.(36) and (37) and in their 'dual' forms Eqs.(40)-(43) provide us with a considerable flexibility in conceiving and analyzing a large number of examples.

4.3. Combinatorial examples

In this section we will work out examples illustrating how the exponential generating function $G(\lambda) = \sum_{n=0}^{\infty} a_n \frac{\lambda^n}{n!}$ of certain combinatorial sequences $(a_n)_{n=0}^{\infty}$ appear naturally in the context of boson normal ordering. To this end we shall assume specific forms of $q(x)$ and $v(x)$ thus specifying the operator that we exponentiate. We then give solutions to Eqs.(30) and (31) and subsequently through Eqs.(36) and (37) we write the exponential generating function of combinatorial sequences whose interpretation will be given.

a) Choose $q(x) = x^r$, $r > 1$ (integer), $v(x) = 0$ (which implies $g(\lambda,x) = 1$). Then $T(\lambda,x) = x\left[1 - \lambda(r-1)x^{r-1}\right]^{\frac{1}{1-r}}$. This gives

$$\mathcal{N}\left[e^{\lambda(a^\dagger)^r a}\right] \equiv\, :\exp\left[\left(\frac{a^\dagger}{(1-\lambda(r-1)(a^\dagger)^{r-1})^{\frac{1}{r-1}}} - 1\right)a\right]:$$

as the normally ordered form. Now we take $z' = 1$ in Eqs.(36) and (37) and from Eq.(44) we obtain

$$\langle 1|z\rangle^{-1}\langle 1|e^{\lambda(a^\dagger)^r a}|z\rangle = \exp\left[z\left(\frac{1}{(1-\lambda(r-1))^{\frac{1}{r-1}}} - 1\right)\right] ,$$

which for $z = 1$ generates the following sequences:

$$\begin{aligned} r = 2: &\quad a_n = 1, 1, 3, 13, 73, 501, 4051, \ldots \\ r = 3: &\quad a_n = 1, 1, 4, 25, 211, 2236, 28471, \ldots \quad , \text{ etc.} \end{aligned}$$

These sequences enumerate r-ary forests [28],[29],[30].

b) For $q(x) = x\ln(ex)$ and $v(x) = 0$ (implying $g(\lambda,x) = 1$) we have $T(\lambda,x) = e^{e^\lambda - 1}x^{e^\lambda}$. This corresponds to

$$\mathcal{N}\left[e^{\lambda a^\dagger \ln(ea^\dagger)a}\right] \equiv\, :\exp\left[\left(e^{e^\lambda-1}(a^\dagger)^{e^\lambda} - 1\right)a\right]: ,$$

whose coherent state matrix element with $z' = 1$ is equal to

$$\langle 1|z\rangle^{-1}\langle 1|e^{\lambda a^\dagger \ln(ea^\dagger)a}|z\rangle = \exp\left[z\left(e^{e^\lambda - 1} - 1\right)\right] ,$$

which for $z = 1$ generates $a_n = 1, 1, 3, 12, 60, 385, 2471, \ldots$ enumerating partitions of partitions [29], [28], [30].

The following two examples refer to the reverse procedure, see Eqs.(40)-(43). We choose first a Sheffer-type exponential generating function and deduce $q(x)$ and $v(x)$ associated with it.

c) $A(\lambda) = \frac{1}{1-\lambda}$, $B(\lambda) = \lambda$, see Eq.(35). This exponential generating function for $z = 1$ counts the number of arrangements $a_n = n!\sum_{k=0}^{n}\frac{1}{k!} = 1, 2, 5, 65, 326, 1957, \ldots$ of the set of n elements [3]. The solutions of Eqs.(40) and (41) are: $q(x) = 1$ and $v(x) = \frac{1}{2-x}$. In terms of bosons it corresponds to

$$\mathcal{N}\left[e^{\lambda\left(a+\frac{1}{2-a^\dagger}\right)}\right] \equiv\ : \frac{2-a^\dagger}{2-a^\dagger-\lambda}e^{\lambda a} :\ = \frac{2-a^\dagger}{2-a^\dagger-\lambda}e^{\lambda a}.$$

d) For $A(\lambda) = 1$ and $B(\lambda) = 1 - \sqrt{1-2\lambda}$ one gets the exponential generating function of the Bessel polynomials [21]. For $z = 1$ they enumerate special paths on a lattice [31]. The corresponding sequence is $a_n = 1, 1, 7, 37, 266, 2431, \ldots$. The solutions of Eqs.(40) and (41) are: $q(x) = \frac{1}{2-x}$ and $v(x) = 0$. It corresponds to

$$\mathcal{N}\left[e^{\lambda\frac{1}{2-a^\dagger}a}\right] \equiv\ : e^{\left(1-\sqrt{(2-a^\dagger)-2\lambda}\right)a} :$$

in the boson formalism.

These examples show that any combinatorial structure which can be described by a Sheffer-type exponential generating function can be cast in boson language. This gives rise to a large number of formulas of the above type which put them in a quantum mechanical setting.

5. Conclusions

We have solved the boson normal ordering problem for the powers and exponentials of the operators linear either in the creation or in the annihilation operator, *i.e.* $(q(a^\dagger)a + v(a^\dagger))^n$ and $e^{\lambda(q(a^\dagger)a+v(a^\dagger))}$ where $q(x)$ and $v(x)$ are arbitrary functions. This was done by the use of umbral calculus methods [15] in finding representations of the monomiality principle (*i.e.* representations of the Heisenberg-Weyl algebra in the space of polynomials) and application of the coherent state methodology. Moreover, we have established one-to-one connection between this class of normal ordering problems and the family of Sheffer-type polynomials and provided a wealth of combinatorial examples.

References

[1] Katriel J 1974 *Lett. Nuovo Cimento* **10** 565

[2] Klauder J R and Skagerstam B-S 1985 *Coherent States. Application in Physics and Mathematical Physics* (Singapore: World Scientific)

[3] Comtet L 1974 *Advanced Combinatorics* (Dordrecht: Reidel)

[4] Katriel J 2002 *J. Opt. B: Quantum Semiclass. Opt.* **4** S200

[5] Blasiak P 2005 *Concepts of Physics* **2** (PhD Thesis, Institut of Nuclear Physics in Kraków and University of Paris VI, arXiv:quant-ph/0507206)

[6] Mikhailov V V 1983 *J. Phys. A : Math. Gen.* **16** 3817

[7] Mikhailov V V 1985 *J. Phys. A : Math. Gen.* **18** 231

[8] Katriel J 1983 *J. Phys. A : Math. Gen.* **16** 4171

[9] Blasiak P and Dattoli G and Horzela A and Penson K A 2005 arXiv:quant-ph/0504009

[10] Dattoli G and Ottaviani P L and Torre A and Vasquez L 1997 *Riv. Nuovo Cim.* **20** 1

[11] Dattoli G Torre A and Mazzacurati G 1997 *Nuov. Cim. B* **112** 133

[12] Dattoli G 1999 *Hermite-Bessel and Laguerre-Bessel functions: a by-product of the monomiality principle* (Advanced Special Functions and Applications) ed D Cocolicchio, G Dattoli and H M Srivastava (Rome: Aracne Rditrice)

[13] Dattoli G and Srivastava H M and Cesarano C 2001 *Appl. Math. Comput.* **124** 117

[14] Cesarano C 2000 Monomiality principle and Legendre polynomials (Advanced Special Functions and Integration Methods) ed G Dattoli, H M Srivastava and C Cesarano (Rome: Aracne Editrice)

[15] Roman S 1984 *The Umbral Calculus* (Orlando: Academic Press)

[16] Roman S and Rota G -C 1978 *Adv. Math.* **27** 95

[17] Di Bucchianico A 1998 *Introduction to Umbral Calculus* (http://www.win.tue.nl/˜sandro/)

[18] Sheffer I M 1939 *Duke Math. Journal* **5** 590

[19] Rainville E D 1965 *Special Functions* (New York: The Macmillan Company)

[20] Ben Cheikh Y 2003 *Appl. Math. Comput.* **141** 63

[21] Grosswald E 1978 *Bessel Polynomials* (Berlin: Springer)

[22] Turbiner A and Smirnov Y 1995 *Mod. Phys. Lett. A* **10** 1795

[23] Bender C M 1991 *Solution of operator equations of motion*(Rigorous Results in Quantum Dynamics) ed J Dittrich and P Exner (Singapore: World Scientific) pp 99-112

[24] Corless R M and Gonnet G H and Hare D E G and Jeffrey D J and Knuth D E 1996 *Adv. Comp. Math.* **5** 329

[25] Louisell W H 1964 *Radiation and Noise in Quantum Electronics* (New York: McGrow-Hill Co.)

[26] Blasiak P and Horzela A and Penson K A and Duchamp G H E and Solomon A I 2005 *Phys. Lett. A* **338** 108

[27] Katriel J 2000 *Phys. Lett. A* **237** 159

[28] Sloane N J A 2005 *Encyclopedia of Integer Sequences* (http://www.research.att.com/˜njas/sequences)

[29] Stanley R P 1999 Enumerative Combinatorics (Cambridge: University Press)

[30] Flajolet P and Sedgewick R 2005 *Analytic Combinatorics* (http://algo.inria.fr/flajolet/Publications/books.html)

[31] Pittman J 1999 *A lattice path model for the Bessel polynomials* (U.C. Berkeley, USA) nr 551