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THE SMALL INDEX PROPERTY FOR COUNTABLE 1-TRANSITIVE LINEAR ORDERS

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Abstract. It is shown that the countable saturated discrete linear ordering has the small index property, but that the countable 1-transitive linear orders which contain a convex subset isomorphic to $\mathbb{Z}^2$ do not. Similar results are also proved in the coloured case.

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1. Introduction. In [**9**] it was shown that the ordered set of rational numbers has the 'small index property' SIP, meaning that any subgroup of its automorphism group having index strictly less than $2^{\aleph_0}$ contains the pointwise stabilizer of a finite set. The small index property has received a great deal of attention in quite a wide variety of special cases. Its model-theoretic significance is that its truth tells us that the natural topological group associated with the structure (under the topology of pointwise convergence) can be recovered from the pure group, and from this one can deduce that the structure is interpretable in the (abstract) automorphism group [**5**].

A conjecture of Macpherson [**7**] was that the SIP should hold for every $\aleph_0$-categorical structure. This was refuted by Hrushovski, but the conjecture remains in modified form (see [**6**], bottom of page 52).

In this paper we look at a class of countable structures which are (mostly) not $\aleph_0$-categorical, namely the countable '1-transitive' linear orders classified by Morel [**8**]. (A linear order X is *1-transitive* if its automorphism group acts (singly) transitively on X.) These have arisen as building blocks for various other classes of countable structures, in particular for certain 'cycle-free' partial orders [**11**], and in [**10**] the small index property was investigated for some of these structures. Generally the SIP was established there for structures built using only the very simplest of Morel's cases, and it was left open as to whether it might hold more generally. What was wanted was to find for which of Morel's structures the SIP held.

Rather disappointingly, we are only able to establish the SIP for $\mathbb{Q}$ (already known), $\mathbb{Z}$ (trivial), and $\mathbb{Q}.\mathbb{Z}$ (new). For all Morel's other orders, $\mathbb{Z}^\alpha$ and $\mathbb{Q}.\mathbb{Z}^\alpha$ for ordinals $\alpha \geq 2$, the SIP is definitely false. Meanwhile, Duby [**3**] examined the coloured case, and was able to establish the SIP for all the $\aleph_0$-categorical coloured orders among those given in [**1**, **2**] (essentially those which only have finitely many colours, and which contain no discrete orderings in their coding trees). In view of the example of $\mathbb{Q}.\mathbb{Z}$ (which we already had shown has the SIP) he asked whether one could establish the SIP for all the saturated structures among those classified, and we answer this affirmatively here.

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A *coloured linear order* is a triple $(X, <, F)$ where $(X, <)$ is a linear order, and F maps X onto some set C (of 'colours'). Automorphisms of a coloured linear order are order-automorphisms g which also preserve the colours ($F(gx) = F(x)$ for all $x \in X$). Generalizing the notion of '1-transitivity' to this situation we say that $(X, <, F)$ is *1-transitive* if its automorphism groups acts transitively on the points of each fixed colour. For $1 \leq n \leq \aleph_0$ we denote by $\mathbb{Q}_n$ the 'n-coloured rationals'. This is the coloured linear order with colour set $n = \{0, 1, \ldots, n-1\}$ characterized uniquely up to isomorphism as being countable, dense without endpoints, and so that between any two points there are points of each possible colour. (Thus $\mathbb{Q}_1$ here just stands for $\mathbb{Q}$.) If Y_i for $i < n$ are countable coloured linear orders with pairwise disjoint colour sets, then we write $\mathbb{Q}_n(Y_0, Y_1, \ldots, Y_{n-1})$ for the result of replacing all points coloured i by Y_i for each i. We use this notation even if n is infinite (though it should then strictly speaking be $\mathbb{Q}_n(Y_0, Y_1, \ldots))$. As in [**4**], we shall write $A(X)$ throughout for the automorphism group of the chain (linearly ordered set) $(X, <)$, and also if X is a coloured chain. In this paper we write group actions on the left, and for any permutation group G acting on Ω, we write G_X and $G_{\{X\}}$ for the pointwise and setwise stabilizers of $X \subseteq \Omega$ in G respectively. The *support* of a permutation g is the set of elements moved by g, written as *supp* g.

2. Positive results.

LEMMA 2.1. *Suppose that $(X, <, F)$ is a countable coloured linear order, and Y is a convex subset of X having only countably many images under the automorphism group G of $(X, <, F)$. Then if X has the SIP, so does Y.*

Proof. If H is a subgroup of $A(Y)$ of index $< 2^{\aleph_0}$, we let $K = \pi^{-1}H$ where π is the projection of $G_{\{Y\}}$ onto $A(Y)$ (obtained by restriction). Since Y has only countably many images under G, the index of $G_{\{Y\}}$ in G is $\leq \aleph_0$, and we deduce that $|G : K| < 2^{\aleph_0}$. By the SIP for G, $K \geq G_Z$ for some finite $Z \subseteq X$. We see that $A(Y)_{Y \cap Z} \leq H$. For if $g \in A(Y)_{Y \cap Z}$, let h agree with g on Y, and fix all other points of X. Then $h \in G_Z$, so $h \in K$, and hence $g = \pi(h) \in \pi K = H$. This establishes the SIP for $A(Y)$. □

The following result is based on our original proof of the SIP for $\mathbb{Q}.\mathbb{Z}$, and is essentially the same as Theorem 4.15 from [**3**]. We concentrate on points which differ from those in [**9**] and refer the reader to that paper where appropriate. First we state the required analogue of [**9**] Lemma 3.1.

LEMMA 2.2. *For any n with $1 \leq n \leq \aleph_0$, and any countable linear orders Y_i for $i < n$, coloured by pairwise disjoint colour sets, if $G = \mathrm{Aut}\, \mathbb{Q}_n(Y_0, Y_1, \ldots, Y_{n-1})$, then the cartesian product $G^{\aleph_0}$ of countably many copies of G has no proper normal subgroup of index $<2^{\aleph_0}$.*

Proof. Let N be a given normal subgroup of $G^{\aleph_0}$ of index $<2^{\aleph_0}$. By picking an irrational (by which we mean a point of the order-completion of $\mathbb{Q}_n(Y_0, Y_1, \ldots, Y_{n-1})$ corresponding to an irrational of $\mathbb{Q}_n$) the argument of [**9**] shows that N contains an element which has a single orbital of parity $+1$ on each copy of $\mathbb{Q}_n(Y_0, Y_1, \ldots, Y_{n-1})$. By the method of [**9**] Lemma 4.2, any two such elements are conjugate, so they all lie in N. Since these elements generate G (see [**9**]), it follows that $N = G^{\aleph_0}$. □

THEOREM 2.3. *For any n with $1 \leq n \leq \aleph_0$, and any countable linear orders Y_i for $i < n$, coloured by pairwise disjoint colour sets, $\mathbb{Q}_n(Y_0, Y_1, \ldots, Y_{n-1})$ has the small index property if and only if each Y_i does.*

Proof. The fact that the SIP for $X = \mathbb{Q}_n(Y_0, Y_1, \ldots, Y_{n-1})$ implies the SIP for each Y_i follows from Lemma 2.1. This is because $\mathbb{Q}_n(Y_0, Y_1, \ldots, Y_{n-1})$ contains (many) convex copies of Y_i.

Conversely, assume the SIP holds for each Y_i, and suppose that H is a subgroup of $G = A(\mathbb{Q}_n(Y_0, Y_1, \ldots, Y_{n-1}))$ having index less than $2^{\aleph_0}$. We first consider the case in which the natural projection π from G to $A(\mathbb{Q}_n)$ maps H onto $A(\mathbb{Q}_n)$. (Note that π is well-defined, since the colour sets of the Y_i are pairwise disjoint, so that the copies of each Y_i are permuted among themselves, and any automorphism of $\mathbb{Q}_n(Y_0, Y_1, \ldots, Y_{n-1})$ must give rise to an automorphism of $\mathbb{Q}_n$.)

By a *moiety* of $\mathbb{Q}_n$ we understand a subset of the form $\bigcup_{m\in\mathbb{Z}}(a_{2m}, a_{2m+1})$ where a_m are irrationals such that $a_m < a_{m+1}$ for each m, and $a_m \to \pm\infty$ as $m \to \pm\infty$. Now let $\mathbb{Q}_n = \bigcup_{m\in\omega} M_m$ be an expression for $\mathbb{Q}_n$ as a union of $\aleph_0$ pairwise disjoint moieties M_m. For any subset Y of $\mathbb{Q}_n(Y_0, Y_1, \ldots, Y_{n-1})$, we shall write $G(Y)$ for the subgroup of G comprising those elements whose support is contained in Y. Then if K is the subgroup of G that fixes each $\pi^{-1}M_m$ setwise, and π_m is the projection from K to $G(\pi^{-1}M_m)$ (given by restriction), we have

$$\prod_{m\in\omega} |\pi_m(K) : \pi_m(K \cap H)| = \left|\prod_{m\in\omega} \pi_m(K) : \prod_{m\in\omega} \pi_m(K \cap H)\right| \leq |K : K \cap H| < 2^{\aleph_0},$$

since $K \cap H \leq \prod \pi_m(K \cap H)$. Hence $\pi_m(K) = \pi_m(K \cap H)$ for some m. (This argument will be referred to as 'the projection argument'.)

We next deduce as in the proof of **[9]** Lemma 2.2 that $H \cap G(\pi^{-1}M_m) \lhd G(\pi^{-1}M_m)$, for the above m. Now each open interval of M_m is isomorphic to $\mathbb{Q}_n$, and so each open interval of $\pi^{-1}M_m$ is isomorphic to $\mathbb{Q}_n(Y_0, Y_1, \ldots, Y_{n-1})$. It follows that $G(\pi^{-1}M_m)$ is isomorphic to $G^{\aleph_0}$. So by Lemma 2.2, $G(\pi^{-1}M_m)$ has no proper normal subgroups of index $<2^{\aleph_0}$. Therefore $G(\pi^{-1}M_m) \leq H$. Since $A(\mathbb{Q}_n)$ acts transitively on the set of moieties, and (we are assuming) H projects onto $A(\mathbb{Q}_n)$, we deduce that $G(\pi^{-1}M) \leq H$ for *every* moiety M of $\mathbb{Q}_n$. But each element of G can be written as the product of two elements, each lying in $G(\pi^{-1}M) \leq H$ for some moiety M, and therefore $H = G$.

We now pass to the general case in which π is not assumed to map H onto $A(\mathbb{Q}_n)$. It is still the case that πH has index less than $2^{\aleph_0}$ in $A(\mathbb{Q}_n)$, so by the SIP for $A(\mathbb{Q}_n)$ **[9]**, there is a finite $X \subseteq \mathbb{Q}_n$ whose pointwise stabilizer is contained in πH. In fact, the proof in **[9]** shows that we may take for X the set of all the points of $\mathbb{Q}_n$ fixed by πH. Since each open interval determined by the members of X is isomorphic to $\mathbb{Q}_n$, we may apply the above argument to each if these (finitely many) intervals separately, and deduce that H contains the setwise stabilizer of $\bigcup \pi^{-1}X$. Let Y be a typical member of $\pi^{-1}X$. Then $H \cap G(Y)$ is a subgroup of $G(Y)$ of index less than $2^{\aleph_0}$, and since Y is assumed to have the SIP, this subgroup contains the pointwise stabilizer of a finite subset X_Y of Y. Then the pointwise stabilizer in G of $\bigcup_{Y\in\pi^{-1}X} X_Y$ contains H, establishing the SIP for $\mathbb{Q}_n(Y_0, Y_1, \ldots, Y_{n-1})$. □

COROLLARY 2.4. *The small index property holds for $\mathbb{Q}.\mathbb{Z}$.*

Proof. This follows from the theorem, since the SIP is trivially true for $\mathbb{Z}$ (its automorphism group being countable), and $\mathbb{Q}.\mathbb{Z}$ is obtained from $\mathbb{Q}_1 = \mathbb{Q}$ by replacing each point by $\mathbb{Z}$. □

We say that two sets are *almost disjoint* if their intersection is finite. The existence of almost disjoint families of cardinality $2^{\aleph_0}$ is folklore, but we require the following slight strengthening.

LEMMA 2.5. *There is a family $\{X_\lambda : \lambda \in \Lambda\}$ of $2^{\aleph_0}$ pairwise almost disjoint subsets of ω^2 such that for each $\lambda \in \Lambda$ and $n \in \omega$, $\{m : (m, n) \in X_\lambda\}$ is infinite.*

Proof. Choose distinct elements $t_\sigma \in \omega^2$ for $\sigma \in 2^{<\omega}$ (the set of finite binary sequences) inductively on length σ. Let ω^2 be enumerated as $\{u_n : n \in \omega\}$, and let t_σ for σ of length n be 2^n distinct elements of $\omega \times \{m\}$ where $u_n = (l, m)$, none equal to $t_{\sigma'}$ for any $\sigma' \in 2^{<n}$. This is always possible since only finitely many elements have so far been chosen, and infinitely many are available. We let $\Lambda = 2^\omega$, and for each $\lambda \in 2^\omega$ let $X_\lambda = \{t_{\lambda|n} : n \in \omega\}$. $\square$

LEMMA 2.6. *Let G be the pointwise stabilizer of $\mathbb{Z}$ in $A(\mathbb{Q}_n)$. Then G has the small index property.*

Proof. Suppose that $|G : H| < 2^{\aleph_0}$. Let π_m be the projection from G to $A(\mathbb{Q}_n \cap (m, m+1))$. Then $|\pi_m G : \pi_m H| < 2^{\aleph_0}$, so by the small index property for $A(\mathbb{Q}_n)$, and since $\mathbb{Q}_n \cong \mathbb{Q}_n \cap (m, m+1)$, $\pi_m H$ is equal to the stabilizer in $\pi_m G$ of a finite subset X_m of $\mathbb{Q}_n \cap (m, m+1)$. Furthermore, X_m is equal to the set of fixed points of $\pi_m H$, which in turn is equal to the set of fixed points of H in $\mathbb{Q}_n \cap (m, m+1)$.

Now we note that H cannot have infinitely many fixed points in $\mathbb{Q}_n \backslash \mathbb{Z}$. This follows as in the proof of **[9]** Theorem 3.5 by considering an infinite monotonic subsequence of the set of fixed points. It follows that $X = \bigcup_{m \in \mathbb{Z}} X_m$ is finite. We shall show that H is equal to the stabilizer of X in G.

Now $\mathbb{Z} \cup X \cong \mathbb{Z}$, and if $a < b$ are consecutive members of $\mathbb{Z} \cup X$ then the projection of H to $A(\mathbb{Q}_n \cap (a, b))$ is onto. To ease notation, let us write $\mathbb{Z} \cup X$ as $\{a_m : m \in \mathbb{Z}\}$ where $a_m < a_{m+1}$ for each m.

For any open subset I of $\mathbb{Q}_n$, we write $G(I) = \{g \in G_X : supp\ g \subseteq I\}$, where $G_X =$ the stabilizer of X in G (= the pointwise stabilizer of $\mathbb{Z} \cup X$ in $A(\mathbb{Q}_n)$). As before we see that for each m, $H \cap G((a_m, a_{m+1})) \lhd G((a_m, a_{m+1}))$. Since $G_{(a_m, a_{m+1})} \cong A(\mathbb{Q}_n)$, it has no proper normal subgroup of index $<2^{\aleph_0}$. Therefore $H \cap G((a_m, a_{m+1})) = G((a_m, a_{m+1}))$, and we deduce that $G((a_m, a_{m+1})) \le H$. Hence for any $m_1 < m_2$, the intersection of the stabilizer of X with $G((a_{m_1}, a_{m_2}))$ is contained in H.

This time, a *moiety* is defined to be a subset of $\mathbb{Q}_n$ of the form

$$\bigcup_{m \in \mathbb{Z}} \bigcup_{l \in \mathbb{Z}} (b^m_{2l}, b^m_{2l+1}),$$

where all b^m_l are irrational, $a_m < b^m_l < b^m_{l+1} < a_{m+1}$ for each l, m, and $\{b^m_l : l \in \mathbb{Z}\}$ is unbounded above and below in (a_m, a_{m+1}). Suppose for a contradiction that H is not equal to G_X. Then as in **[9]** there is a moiety M of $\mathbb{Q}_n$ such that $G(M) \not\le H$. Write $\mathbb{Q}_n \backslash (\mathbb{Z} \cup X \cup M)$ as a disjoint union of moieties M_m. By a projection argument, there is some m such that H projects onto $G(M_m)$, and applying the result from the previous paragraph (and Lemma 2.2), $G(M_m) \le H$. Write M' for M_m.

Let $M' = \bigcup_{m \in \mathbb{Z}} \bigcup_{l \in \mathbb{Z}} (b^m_{2l}, b^m_{2l+1})$. Let $\{X_\lambda : \lambda \in \Lambda\}$ be a family of $2^{\aleph_0}$ pairwise almost disjoint subsets of ω^2 as given by Lemma 2.5. Then $M_\lambda = \bigcup_{(|l|, |m|) \in X_\lambda} (b^m_{2l}, b^m_{2l+1})$ is a moiety (which is why we arranged the extra condition on the family over and above pairwise almost disjointness). Since G_X clearly acts transitively on moieties, there is $g_\lambda \in G_X$ taking M_λ to $\mathbb{Q}_n \backslash (\mathbb{Z} \cup X \cup M')$. As $|G_X : H| < 2^{\aleph_0}$, there are $\lambda \neq \mu$ in Λ such that $h = g_\lambda g_\mu^{-1} \in H$. Then $G(hM') = hG(M')h^{-1} \leq H$. We find that

$$\begin{aligned}\mathbb{Q}_n \backslash (X \cup \mathbb{Z} \cup M' \cup hM') &= (\mathbb{Q}_n \backslash (X \cup \mathbb{Z} \cup M')) \cap (\mathbb{Q}_n \backslash (X \cup \mathbb{Z} \cup hM')) \\ &= g_\lambda M_\lambda \cap hg_\mu M_\mu \\ &= g_\lambda M_\lambda \cap g_\lambda M_\mu \\ &= g_\lambda (M_\lambda \cap M_\mu) \\ &\subseteq (a_{-N}, a_N) \text{ for some } N \in \mathbb{N},\end{aligned}$$

since $X_\lambda \cap X_\mu$ is finite.

Let $k \in G(M)\backslash H$ and define k' by

$$k'x = \begin{cases} kx & \text{if } a_{-N} < x < a_N \\ x & \text{otherwise} \end{cases}.$$

Then *supp* $(k')^{-1}k \subseteq (-\infty, a_{-N}) \cup (a_N, \infty) \subseteq M' \cup hM'$. In addition, as *supp* $k \subseteq M$, *supp* $(k')^{-1}k \subseteq M$, which is disjoint from M', so in fact, *supp* $(k')^{-1}k \subseteq hM'$. But $G(hM') \leq H$, so $(k')^{-1}k \in H$.

Furthermore, $k' \in \prod_{-N \leq i < N} G((a_i, a_{i+1})) \leq H$, so this gives

$$k = (k') \cdot (k')^{-1} k \in H,$$

a contradiction. □

We remark that Lemma 2.6 really just says that the cartesian power $(A(\mathbb{Q}_n))^{\aleph_0}$ has the SIP in its natural action on the disjoint union of $\aleph_0$ copies of $\mathbb{Q}_n$. We have formulated the result in terms of its action on $\mathbb{Z}$ copies so that $\mathbb{Z} \cup X$ is order-isomorphic to $\mathbb{Z}$, thus marginally simplifying notation.

THEOREM 2.7. *Suppose that $Y_0, Y_1, \ldots, Y_{n-1}$ are countable coloured linear orders coloured by pairwise disjoint colour sets, each having the SIP. Then the cartesian power $(A(\mathbb{Q}_n(Y_0, Y_1, \ldots, Y_{n-1})))^{\aleph_0}$ has the SIP.*

Proof. For ease let us view $G = (A(\mathbb{Q}_n))^{\aleph_0}$ as acting on $\mathbb{Q}_n$ as the pointwise stabilizer of $\mathbb{Z}$ (so that the '$\aleph_0$ copies' of $\mathbb{Q}_n$ are identified with the rationals in $(m, m+1)$ for $m \in \mathbb{Z}$). We may now deduce the result from Lemma 2.6 using the same ideas as in Theorem 2.3. We omit details. □

COROLLARY 2.8. *$\mathbb{Q}.\mathbb{Z}(1 + \mathbb{Q}.\mathbb{Z})$ has the small index property.*

Proof. The automorphism group of $\mathbb{Z}(1 + \mathbb{Q}.\mathbb{Z})$ has a subgroup of countable index which is isomorphic to $(A(\mathbb{Q}.\mathbb{Z}))^{\aleph_0}$. Since $\mathbb{Z}$ has the SIP, by Theorem 2.7 so does $\mathbb{Z}(1 + \mathbb{Q}.\mathbb{Z})$. By Theorem 2.3, $\mathbb{Q}.\mathbb{Z}(1 + \mathbb{Q}.\mathbb{Z})$ also has the SIP. □

We included this corollary because it was the specific question asked in [**3**]. We are able however in the next section to give a complete list of those coloured orders discussed in [**1**] for which the SIP holds.

3. Negative results.

THEOREM 3.1. *The small index property does not hold for $\mathbb{Z}^2$, or, more generally, for any of the 1-transitive linear orders in Morel's list apart from $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{Q}.\mathbb{Z}$, (or the trivial order with one point).*

Proof. We first do the case of $\mathbb{Z}^2$. Let G be its automorphsim group. Then G has a subgroup K of countable index, comprising those automorphisms which fix each copy of $\mathbb{Z}$, and K is isomorphic to the cartesian product of countably many copies of $\mathbb{Z}$. Now $\mathbb{Z}$ has a subgroup of index 2, namely the set of even integers, and taking the corresponding homomorphism on each copy gives rise to a homomorphism from K onto the cartesian product H of $\aleph_0$ copies of the cyclic group of order 2. We now view H as a vector space over the field with 2 elements, and as such it has dimension $2^{\aleph_0}$, and hence has $2^{\aleph_0}$ subspaces of codimension 1. Hence H has $2^{\aleph_0}$ subgroups of index 2, and lifting to K, the same applies to K. Hence G has $2^{\aleph_0}$ subgroups of index $\aleph_0$. But there are only countably many subgroups of G which contain the pointwise stabilizer of a finite set, and so not every subgroup of countable index can contain such a subgroup, giving the failure of the SIP.

To extend this to the other orders mentioned, we note that they all have a convex subset isomorphic to $\mathbb{Z}^2$, so we may appeal to Lemma 2.1. □

We recall from [**1**] that all the countable 1-transitive coloured linear orders for a finite set of colours may be built up from singletons by three methods:

$\mathbb{Q}_n$-combinations of disjointly coloured countable 1-transitive coloured linear orders,

concatenations of disjointly coloured countable 1-transitive coloured linear orders,

lexicographic products of the form $Y.Z$ where Y is in Morel's list (so is monochromatic) and Z is countable 1-transitive coloured.

Theorem 3.1 told us how to handle orders with $\mathbb{Z}^2$ 'on the inside'. The following lemma covers cases where it is 'outside'.

LEMMA 3.2. (i) *If* $(X, <, F)$ *is a countable 1-transitive coloured linear order of the form* $Y.Z$ *where* Y *is a 1-transitive order embedding* $\mathbb{Z}^2$*, then* $(X, <, F)$ *does not have the small index property.*

(ii) *If* $(X, <, F)$ *is a countable coloured linear order which is the concatenation of disjointly coloured coloured linear orders* $Y_0, Y_1, \ldots, Y_{n-1}$ *then* X *has the SIP if and only if each* Y_i *does.*

Proof. (i) $A(X)$ projects onto $A(Y)$, so we may deduce this from Theorem 3.1.

(ii) X has the SIP $\Rightarrow$ Y_i has the SIP follows from Lemma 2.1. Conversely, if each $A(Y_i)$ has the SIP, let $|A(X) : H| < 2^{\aleph_0}$. Then the projection of H to each $A(Y_i)$ has small index, so contains the stabilizer of a finite set. Taking the union of these finite sets gives a finite set whose stabilizer is contained in H. □

For the final result we recall that in [**1**], 'coding trees' for finitely coloured countable 1-transitive linear orders were introduced, which enabled one to describe the ways in which such orderings could be constructed. Now the possible labels on such trees were $\mathbb{Q}_n$ for a $\mathbb{Q}_n$-combination (of the orders encoded at its children), n for a concatenation of those orders, Z for the lexicographic product of a 1-transitive (monochromatic) linear order Z with the one encoded at its child, and 1 for leaves (together with information about colourings). Now it is easy to see that any coding tree containing consecutive vertices labelled by concatenations can be replaced by one in which those vertices are collapsed to one (with a concatenation over larger orderings), and we shall assume in what follows that this has been done. In other words, we are assuming that our coding trees do not now have consecutive concatenations. In addition, note that it was part of the definition of 'coding tree' that we do not have consecutive lexicographic products either.

COROLLARY 3.3. *A countable 1-transitive coloured linear order with finite colour set has the SIP if and only if no vertex of its coding tree is labelled by a member of Morel's list other than $\mathbb{Q}$, $\mathbb{Z}$, or $\mathbb{Q}.\mathbb{Z}$, and if a vertex is labelled $\mathbb{Z}$, and its child is labelled n (concatenation), then it has no grandchild labelled $\mathbb{Z}$. Consequently, any countable saturated 1-transitive coloured linear order has the SIP.*

Proof. We may see by induction that any countable 1-transitive coloured linear order as described has the SIP. The root of any coding tree in the finitely coloured case must be labelled $\mathbb{Q}_n$, n (concatenation), or Z (lexicographic product). For $\mathbb{Q}_n$ we may appeal at once to Theorem 2.3, and for n, to Lemma 3.2(ii). For the case of Z, by assumption, Z must be $\mathbb{Q}$, $\mathbb{Z}$, or $\mathbb{Q}.\mathbb{Z}$. For $\mathbb{Q}$ and $\mathbb{Q}.\mathbb{Z}$ we may again use Theorem 2.3. For the case of $\mathbb{Z}$, the automorphism group has a subgroup of countable index, namely, those automorphisms fixing each copy of the order encoded at the child x of the root setwise, and this subgroup is isomorphic to the cartesian product of automorphism groups of the orders encoded at the children of x, since x must represent a concatenation. By assumption, the root does not have any vertex labelled $\mathbb{Z}$ as a grandchild, and since the tree does not have consecutive concatenations, the concatenation at x must be over singletons and $\mathbb{Q}_n$-combinations (possibly including the case $n = 1$). The result now follows by Theorem 2.3 and Lemma 3.2(ii).

Conversely, if the stipulations required of the coding tree do not hold, then the SIP fails, by Theorem 3.1 if there is a lexicographic product involving $\mathbb{Z}^2$. If there is a vertex labelled $\mathbb{Z}$ having a grandchild also so labelled, then we again refute the SIP by using the same ideas, together with appeal to Lemma 3.2(ii). The final statement follows, since no saturated coloured order can have a coding tree of either of the prohibited forms. □

We conclude by remarking that since every $\aleph_0$-categorical structure is saturated, Duby's result establishing the SIP for the $\aleph_0$-categorical 1-transitive coloured linear orders is also a consequence of ours.

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