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Quantum quenches in two spatial dimensions using chain array matrix product states

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We describe a method for simulating the real time evolution of extended quantum systems in two dimensions (2D). The method combines the benefits of integrability and matrix product states in one dimension to avoid several issues that hinder other applications of tensor based methods in 2D. In particular, it can be extended to infinitely long cylinders. As an example application we present results for quantum quenches in the 2D quantum [\( (2+1) \)-dimensional] Ising model. In quenches that cross a phase boundary we find that the return probability shows nonanalyticities in time.

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Introduction. The advent of ultracold atomic gas experiments has led to a surge of interest in the time evolution and out-of-equilibrium behavior of many-body quantum systems. Much effort has been focused on one-dimensional (1D) problems because these can be tackled by analytically tractable or highly accurate numerical methods. Key questions that these studies have sought to elucidate are whether and how such systems thermalize after a sudden change, or “quantum quench” of a system’s Hamiltonian, with particular emphasis on the role played by conserved charges in 1D integrable systems [1–11].

Experiments, however, are not limited to 1D and it is interesting to explore similar questions in two dimensions (2D) and above [12]. Unfortunately there is no analog in 2D of the aforementioned analytically exact 1D methods. Numerical approaches using matrix product state (MPS) representations, so successful in 1D, suffer in 2D due to the “area law” growth of entanglement [13,14]. This growth reduces the efficiency of MPS (and related “tensor”) algorithms and limits them to smaller system sizes.

Nonetheless, MPS algorithms can be applied in 2D, by labeling lattice sites (usually in a zigzag fashion) to map to a 1D system [15]. The cost is that nearest neighbor interactions in 2D are mapped to increasingly long ranged 1D interactions, imposing an increasing numerical burden. Recently, progress has been made in performing real time evolution on MPS with such long ranged Hamiltonians by two different routes [16,17]. Algorithms based on generalizations of MPS to higher dimensions, such as projected entangled pair states (PEPS) [18,19], make use of imaginary time evolution to find ground states [20]. However, these higher dimensional tensor methods have not been applied to real time evolution.

In this Rapid Communication we demonstrate that real time evolution is possible for large 2D systems by combining information coming from exactly solvable models with a highly anisotropic MPS formulation. Such an approach retains the contraction efficiency of matrix product states over other tensor methods, while avoiding the build up of long ranged interactions. Our setup will be similar to that used in the density matrix renormalization group (DMRG) studies described in Refs. [21,22] except that here we are explicit in our use of MPS. This change allows for straightforward implementation of algorithms other than DMRG, including those for time evolution and for accurately working with the thermodynamic limit. In particular, using time evolving block decimation (TEBD) [23] we demonstrate that we can study the time evolution after a quench of infinitely long cylinders, with sufficient circumstance that we approach the 2D thermodynamic limit. This includes strong quenches where we cross phase boundaries of a 2D quantum system.

Method. At the core of our method is the wish to maximize the analytically exact input going into our MPS algorithm, while simultaneously controlling the growth of entanglement entropy. The construction we use is depicted in Fig. 1: a coupled array of exactly solvable 1D subunits. For each subunit, we have exact knowledge of the spectrum and matrix elements. This exact knowledge means that we begin with the numerics already having accounted for much of the strong correlations of the system. We emphasize our use of exactly solvable models as a building block is not much of a limitation to the method. Such models are ubiquitous in 1D, including Heisenberg spin chains, Luttinger liquids, and Hubbard models to name but a few [24,25]. In this framework, a state of a system of \(N\) chains is written in MPS form via

\[
|\Psi\rangle_{2D} = \sum_\sigma A^{\sigma_1[1]} \cdots A^{\sigma_N[N]} |\sigma_1 \cdots \sigma_N\rangle,
\]

where each matrix \(A^{\sigma,[i]}\) is labeled by a chain \(i\) and an eigenstate of that individual chain \(\sigma_i\). Like the single sites used in 1D MPS algorithms, we are able to manipulate these chain eigenstates because we know their energies and matrix elements for any relevant operator.

For ground and low-lying states of the system the entanglement entropy \(S_E\) scales as the boundary “area,” that is to say, the chain length. By keeping the chain length finite we can throttle the growth of \(S_E\). By partnering this with the fact that for the systems that we will study, finite size effects are exponentially suppressed, we are able to keep \(S_E\) small while remaining in the 2D thermodynamic limit.

We have previously demonstrated the effectiveness of this...
In this Rapid Communication we present results for quenches in the 2D quantum Ising model:

$$H_{2DQI} = \sum_i \left[ H_{1D,i} + J_\perp \int_0^R dx \; \sigma_i^x(x) \sigma_{i+1}^x(x) \right].$$

We represent the model as 1D Ising chains (of index $i$ and length $R$) coupled together with a longitudinal spin-spin interaction. We take each chain $H_{1D,i}$ to be the continuum limit of the 1D lattice quantum Ising model—or transverse field Ising model (TFIM)—with Hamiltonian, $-J_1 \sum_i [\sigma_i^z \sigma_{i+1}^z + (1 + g)\sigma_i^+ \sigma_{i+1}^+]$ with $l$ an index along the chain. In the continuum limit this reduces to a theory of a 1D Majorana field with mass $\Delta = gJ_i$. Analytic expressions for the spectrum of this theory and the spin matrix elements are detailed in Ref. [38]; we summarize the salient features in [30].

Expanding the Majorana field in terms of fermionic modes $\psi_{k_\perp}$ and $\psi_{k_\parallel}$ (the continuum versions of the usual Jordan-Wigner lattice fermions) yields a quadratic chain Hamiltonian $H_{1D,i} = \sum_k \epsilon_k \psi_k^\dagger \psi_k$, with dispersion $\epsilon_k = \sqrt{2^2 + k^2}$. We work in units such that the intrachain velocity, $v$, is dimensionless and equal to unity. We also define a dimensionless interchain coupling $j_\perp = J_\perp/\Delta_{1D}$, for disordered ($\Delta < 0$) chains a finite value of the interchain coupling $j_\perp$ leads to a 2D quantum $(d = 2 + 1)$ order-disorder transition at a critical value $j_{\perp c} = j_\perp = 0.185$ [22].

We compute the evolution of the postquench state using iTEBD and TEBD, with first and second order Trotter decompositions of the time evolution operator, and time steps $\tau$. The error associated with such decompositions is dependent on $j_\perp$ and $\tau$, but even for the strongest quenches presented in this work we can choose $\tau$ small enough for convergence (see the Supplemental Material [30]). For each set of parameters, we first establish that the numerical results are converged in $s_{\min}$ or $\chi$ before increasing the cutoff $E_c$. Convergence of the method in $s_{\min}$ is demonstrated in [30]. We have also checked the algorithm for two analytically tractable cases: the limit ($j_\perp \ll 1$) using unitary perturbation theory (see Ref. [39]) and a model of free fermionic chains with interchain hopping. In both cases we find excellent agreement with our numerical results [30].

Results. In the following we present results of quantum quenches where the initial state of the system corresponds to the $j_\perp = 0$ ground state, whereupon at $t = 0$ we turn on a finite interchain coupling $j_\perp$. We focus mainly on results for infinitely long cylinders, leaving a discussion of the effect of finite chain number, $N$, until the end. We first address the question of what time scales we expect to feature in the quench. To provide a partial answer we turn to the quasiparticle causality picture of Refs. [1,2,35]. The energy imparted by the quench produces quasiparticle excitations which are entangled on a length scale $|\Delta|^{-1}$ along the chain. Intrachain scattering then only has an effect after a time, $t_{\Delta} = (2|\Delta|)^{-1}$. On the other hand, the time scale governing interchain scattering can be estimated using Fermi’s golden rule to be $t_{j_\perp} = \sqrt{|\Delta|^{-1}}^2 (J_\perp R)^{-2}$. The final time scale of import is that encoding the chain length, $R$. This scale, given by $t_R \sim R/2v = |\Delta| R \Delta$, describes the time for two quasiparticles, created at the same point and moving in opposite directions, to travel around a chain and then meet again. Hence there
is a region, \( t_{\Delta} < t < t_R \), where we may expect the time evolution to be representative of the 2D thermodynamic limit. But for \( t > t_R \) the finite nature of the chains’ circumferences will play a role. We stress that \( t_R \) does not govern the time scale for revivals in the system. Instead these occur on a much longer time scale, \( t_{\text{revival}} \sim N t_J \), where \( N \) is the number of chains in the system. Thus in our iTEBD simulations, we never expect to see strict revivals.

To illustrate these time scales in operation, we consider the occupation number, \( n_i(x) = \psi_i^\dagger(x)\psi_i(x) \), for a fermionic mode on chain \( i \), a simple measure of how the system departs from the initial state, for which \( n_i(x) = 0 \). In Fig. 2 we present how \( n_i(x) \) evolves with time for a quench to \( j_{\perp} = 0.1 \). On the basis of our perturbative results for very small \( j_{\perp} \) [30], we plot \( n(x) \) in units of \( j_{\perp}^2 \) for all four quenches presented. These four quenches correspond to four different chain lengths, \( R \).

We see that at short times, the results for \( n(x)/j_{\perp}^2 \) collapse onto a single curve as a function of \( t/t_{\Delta} \). As time increases, the curves cease to track one another. The first to do this is the \( R = 4 \) curve, then the \( R = 6 \) curve, and then finally the \( R = 8 \) curve. The time at which this happens corresponds, approximately, to \( t_R \); the scale on which the quench explores the finite length of the chain. We expect small departures from this time scale because a finite \( j_{\perp} \) will renormalize the quasiparticle velocity \( v \) to 1 in \( t_R \). We also see from the inset of Fig. 2 that the evolution at longer times is no longer described by perturbation theory.

In Fig. 3 we explore a quench to a \( j_{\perp} \) which exceeds \( j_c \), the critical coupling for the \((2 + 1)\)-dimensional system. Such a quench is among the most challenging numerically as the population of higher energy chain states becomes significant. Concomitantly, the time evolution is most dependent on \( E_c \) in this case. Ramped, rather than sudden, quenches can be implemented with some possible advantages in this regard [40], though we have not yet explored this possibility. Nonetheless in Fig. 3 we see that for a given chain length, \( R \), we can find cutoffs, \( E_c \) such that the time evolution is converged.

It is also possible to calculate postquench correlations between the chains. We show the nearest neighbor spin-spin correlation function as a function of time, \([\sigma^z_i(x,t)\sigma^z_{i+1}(x,t)]\), for a selection of \( R \) and \( j_{\perp} \) in the inset of Fig. 3. Our choice of \( j_{\perp} > 0 \) favors antiferromagnetic correlations, producing the overall negative sign. An expansion in small \( t \) shows that this quantity is proportional to \( j_{\perp} t^2 \) allowing us to collapse the results onto a single curve at short times. Here we see signatures of both the \( t_{\Delta} \) and \( t_R \) scales. In the inset we have marked the intrachain scattering time \( t_{iJ} \), for the system with \( R = 8 \) and \( j_{\perp} = 0.1 \). It is visible as the time that the \( j_{\perp} = 0.1 \) and \( j_{\perp} = 0.01 \) data begin to diverge. We also mark the time scale \( t_R \) at which the data for chains with \( R = 8 \), \( j_{\perp} = 0.01 \) begins to diverge from that of \( R = 10 \), \( j_{\perp} = 0.01 \).

To show that our method can handle nontrivial aspects of quenching through the critical coupling of the coupled chain system, we search for nonanalyticities in the Loschmidt echo as a function in time. The “Loschmidt echo” or overlap probability at a particular \( t \) is the modulus squared of the overlap between the initial and time evolved state:

\[
G(t) = \left| \langle \Psi_0 | e^{-itH_{\text{quad}}} | \Psi_0 \rangle \right|^2 ,
\]

where \( \Psi_0 \) is the ground state of the uncoupled chain system. In 1D it is useful to define a per site rate function, \( \ell(t) \) via \( G(t) = \exp[-N\ell(t)] \). Nonanalyticities in \( \ell(t) \) have been interpreted as “dynamical phase transitions,” following an exact calculation of this quantity for the 1D TFIM [41–43]. The general association of such nonanalytic points with equilibrium critical phenomena is contested [44,45], but we demonstrate analytically in low order perturbation theory [30] that for quenches to \( j_{\perp} > 0.27 \) we expect nonanalyticities in \( G(t) \). While this estimate for the value of \( j_{\perp} \) is larger than \( j_c \)—because of the low order to which we took the computation—it does suggest that simple perturbation theory for the quantity \( G(t) \) can be used to estimate the phase boundaries in some 2D quantum systems.
corresponding quench to \( t/D_1 \) for a quench to \( j_R = 0.5 \). We find no nonanalytic points for the corresponding quench to \( j_L = 0.1 \), even at longer times up to \( t = t_\Delta = 10.0 \) (not shown). Inset: comparison of the infinite chain number system data with a system with \( N = 100 \) chains computed using TEBD. The first nonanalytic point for the infinite cylinder forms the edge of a plateau, whereas for a finite number of chains it takes the form of a peak.

In Fig. 4 we plot \( \log G(t) \) for a quench to \( j_L = 0.5 \)—a value of \( j_L \) where we should see nonanalyticities. In 2D this quantity scales with system volume \( RN \), as does its 1D counterpart [41]. We also observe that it scales with \( j_L^2 \). As expected we find nonanalytic behavior for this quench, within the time window we are able to simulate, and see that the nonanalyticity has the same qualitative structure for both \( E_c = 7|\Delta| \) and \( 8|\Delta| \). For comparison we plot \( \log G(t) \) for a quench to \( j_L = 0.1 \), where in contrast we find that this quantity is smooth within our simulation window. We remark that nonanalyticities appear for the same quantity with \( j_L = 0.2 \) (not plotted), just above \( j_L = 0.185 \), but they first occur only at the edge of the attainable times with iTEBD.

Finally we consider the case of finite length and open boundary conditions. The TEBD algorithm is slower by approximately a factor of \( N \) due to the loss of translational invariance along the cylinder. We find negligible effect, for finite \( N \gtrsim 10 \) and \( i \) away from the ends of the cylinder, on the results for local quantities such as \( n_i(x) \) (up to the time scales we reach). However, this is not true for the Loschmidt echo (a global measure), especially when \( |j_L| > j_R \). The inset of Fig. 4 shows the difference between the iTEBD and \( N = 100 \) results for \( R = 6, j_L = 0.5 \). While there is excellent agreement up to \( t \sim t_\Delta \) (not shown), afterwards there is a clear change in the nonanalytic point structure. We also find that this effect is even more pronounced for very small \( R \) and large \( N \) (where our model reduces to a single 1D TFIM), suggesting that boundary conditions have a non-negligible effect on the Loschmidt echo even for large systems. This last result has important consequences for possible experimental investigations.

**Conclusions.** We have demonstrated a robust method to compute dynamical behavior in 2D quantum (\( d = 2 + 1 \)) systems after a quench, which we intend to use to study other systems including coupled quantum wires (i.e., coupled Luttinger liquids) and Heisenberg chains. The algorithm should prove especially useful when interpreting nonequilibrium cold atom [46,47] and pump-probe experiments in the cuprates [48,49].

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[40] A sudden quench pumps more energy into the system relative to a ramped quench, and therefore will populate higher energy modes, reducing the efficiency of the algorithm.