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Developing Subject Knowledge

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Chapter 11 – Developing Subject Knowledge FINAL DRAFT 3

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Developing and extending your own subject knowledge is an essential part of preparing to teach secondary mathematics. Extending your subject knowledge will be crucial in ensuring you continue to appreciate mathematics and its role in society as well as helping you to progress within your chosen profession. Perhaps more importantly teachers with stronger subject knowledge are better equipped to make good choices of task that will engage their pupils, and to organise those tasks better so that their pupils are involved in activities that are mathematically rich.

Three ways of developing subject knowledge

Your subject knowledge will have to develop throughout your career, in order to respond to curriculum change or when you are asked to teach different courses. Working on extending your subject knowledge can also help to keep your ideas fresh and that will help you in engaging your pupils. In fact, the daily task of teaching your pupils in your classroom will develop your mathematical knowledge, helping you see new connections and standpoints. Interacting with your students and planning lessons introduces new questions and perspectives about areas of mathematics that previously seemed familiar but which now need thinking about. Preparing to teach a new topic will require you to make sure you fully understand the concepts and techniques involved, as well as having a full appreciation of the connections that topic has with other areas and how it fits into the wider mathematics scheme.

Extending your subject knowledge involves working in three directions:

- **Deeper**, by looking at how familiar mathematical topics are connected, and exploring the background and applications of some of these areas of mathematics;
- **Higher**, by learning some new mathematics from further on in the curriculum than where you currently teach;
- **Wider**, by exploring some new mathematics from outside the curriculum.

For instance, if you already feel comfortable teaching the core content of A-level mathematics, you might extend your subject knowledge **deeper** by exploring some real-life application of calculus in economics or manufacturing, or by looking for connections between different parts of the A-level curriculum; **higher** by looking at how some topics are developed in Further Mathematics A-level or within undergraduate mathematics; or **wider** into something totally different, such as game theory. Each of these choices would support your practice as a teacher as well as your personal development as a mathematician.

Shulman (1986) suggested that having secure subject knowledge for mathematics teaching involves developing:

- Mathematical knowledge (often called subject content knowledge)
- Pedagogic content knowledge
- Curriculum knowledge

Mathematical knowledge is defined as a strong understanding of the mathematics itself, namely the key facts, concepts, processes and principles of the subject. It also involves an ability to think mathematically and to understand the nature of mathematical reasoning and enquiry.

Pedagogic content knowledge as described by Shulman is “an understanding of what makes the learning of specific topics easy or difficult” (Shulman, 1986, p.9). In other words, this type of knowledge includes understanding the usefulness of different examples, representations, approaches and explanations that might be used to teach mathematical concepts, as well as being aware of common misconceptions. For instance, when teaching two-step equations a teacher with good pedagogic content knowledge might write deliberately write equations such as “ $9 + 2x = 13$ ” instead of “ $2x + 9 = 13$ ” in order to check for understanding and develop their pupils’ fluency.

Finally, *curriculum knowledge* covers the sequence and scope of how topics are taught, as well as the resources used to teach them. Having a thorough understanding of what is expected by current government policies and your school’s schemes of work will ensure that your planning covers the required curriculum content. This aspect would also include knowing the progressions from the primary curriculum through to A-level. For instance, if you are about to teach angles to year 7, it is useful to know that KS2 pupils will have met some

specialised terminology such as acute, obtuse and right angle. For the GCSE they will need to recognise the right angle sign and identify angles that are “opposite” other angles and sides.

The remainder of this chapter outlines ways of extending mathematical knowledge and describes how each has the potential to benefit your curriculum and pedagogic content knowledge. Working in these ways will allow you to say that you have secure subject knowledge for teaching mathematics.

Extending Your Subject Knowledge Deeper

Exploring different contexts and applications of a topic is a way of deepening your subject knowledge, revealing new relationships in the mathematics. For instance, in key stage three (KS3), you will be teaching angles in regular and irregular polygons. Discovering how regular and Archimedean tessellations are used in Islamic art will both extend your knowledge of this subject and give you interesting and engaging contexts for teaching your pupils.

There are just three regular tessellations in the plane: equilateral triangles, squares and regular hexagons. Archimedean, or semi-regular, tessellations are formed from two or more regular polygons with the same pattern of shapes meeting at every vertex.

TASK: How many Archimedean tessellations are there? Can you use angle sums to prove that your list is exhaustive? What is the connection with unit fractions that add to 1? You could follow up these ideas by looking at the ideas on semi-regular tessellations on the Nrich website, <https://nrich.maths.org/4832>

Figure 11.1: ATM MATs are a great resource to use with pupils when investigating tessellations.

Further exploration could take you onto the study of irregular tessellations, or you could investigate how polygons meet to form 3D solids.

TASK: Look at the tiles from inside the Alhambra in Granada shown in figure 11.2.

What is the relationship between the side lengths shown in the second image?

Can you find any other geometric relationships within the tiles?

Figure 11.2: Alhambra tiles

The mathematical study of Archimedean tessellations does not appear in many mathematics degrees and neither is it in the school curriculum. However, by exploring the properties of these tiling patterns, you will have discovered mathematics within an artistic context which may motivate students to explore lengths and angles.

Another way to deepen your understanding of a mathematical topic is to open up tasks by removing constraints and exploring the consequences.

TASK: Two sides of a triangle are 3cm and 7cm. What lengths can the other side be? Can the triangle be isosceles? Right angled? Obtuse? Use dynamic geometry software to test your conjectures.

To define a triangle uniquely you need at least three pieces of information, for instance either three sides or two sides and the angle between them. However, in this task you have only been given two of the three essential facts, opening up the possibilities and allowing for exploration of limits. Many standard tasks can be made richer for students by relaxing or varying one constraint. Using dynamic geometry software to explore tasks such as the one above is an effective way for you to perceive the generality and decide which task adaptations are suitable for your students and, where necessary, to expand your knowledge of and confidence in using this important teaching tool.

Another context that can add depth to your subject knowledge is to look at how mathematics has been used historically to solve real-life problems. For example; mathematicians have changed the way we now understand the spread of disease. Indeed, Florence Nightingale used statistics in her representations to improve nursing care and was an early user of infographics to show how hospitals could be improved.

In the mid-nineteenth century, Dr John Snow suspected that drinking water might have caused the spread of cholera in Soho. He plotted all the cholera cases on a local map, and marked on the position of the water pumps. By 'stacking' data on the map, the number of deaths at an address became more easily visualised. From this, Dr Snow identified a contaminated pump on Broad Street, which was replaced, averting an epidemic. Examples of problem solving such as this real-life context can be used as a 'hook' to introduce a related mathematical task in modern notation, as in figure 11.3.

Figure 11.3: Disease Detective Task.

John Snow's map can be found on the British Library website

Historical applications of mathematics help to show that people from different backgrounds and cultures have used mathematics to make a difference to their societies. Using them in the classroom usually requires 5-10 minutes for discussing the problem context before setting a suitable activity as in the figure above. Exploring the history of mathematics also provides an opportunity to look at a range of mathematical proofs, such as the many proofs of Pythagoras' Theorem (Ward-Penny, 2011). As a teacher, your own understanding of mathematical properties and theorems is deepened if you understand multiple approaches to proving them.

Using such contexts in lessons that you find in your research, will help your pupils appreciate the idea that mathematics is a real, valuable, historical and cross-cultural human endeavour, in which people can get involved. Therefore these contexts will help you to teach lessons that are ALIVE (see chapter 10) as well helping you to extend your appreciation of where the mathematical concepts in use today have come from and the many connections between mathematical ideas.

TASK: Begin with any right-angled triangle and arrange two copies as shown in figure 11.4. Write down an expression for the area of the three resulting right-angled triangles, and

equate this to an expression for the area of the trapezium. Rearrange this equation to get the familiar form of Pythagoras' theorem.

This is an a proof of Pythagoras' Theorem, produced by the twentieth president of the Unites States, James Garfield. (Follow up

<https://www.maa.org/press/periodicals/convergence/mathematical-treasure-james-a-garfields-proof-of-the-pythagorean-theorem>)

Figure 11.4: Garfield's Proof of Pythagoras' Theorem

Developing your subject knowledge deeper can be considered as becoming aware of connections within the subject and also as building a 'general knowledge' of mathematics. Knowing about, and sharing, some contexts and purposes of mathematics can help your teaching come to life. Using questions from your own study can provide engaging 'hooks' at the beginning of or during lessons or to extend your pupils thinking.

TASK: Choose a mathematical topic which you are likely to teach soon. Extend your subject knowledge of this topic using the questions on the prompt sheet shown in table 11.1.

NOW TRY THIS: Once you have completed the prompt sheet, shown in table 11.1, select one element which you feel would be of interest or use to your pupils. Incorporate this element into a forthcoming lesson plan.

Table 11.1 Deepening Subject Knowledge Prompt Sheet

Deepening Subject Knowledge Prompt Sheet

<p>Exploration</p> <ul style="list-style-type: none"> What other areas or topics in mathematics does this topic connect with? Can this topic be drawn as a picture or series of pictures? Are there any patterns or mathematical coincidences related to this topic? 	
<p>Application</p> <ul style="list-style-type: none"> Which other school subjects might use this mathematical topic? How? Which jobs might use this mathematical topic? How? Why do you think this topic is on the syllabus? 	
<p>Context</p> <ul style="list-style-type: none"> Why, and when, was this topic first considered? What problem did it solve? Which cultures were involved in the development of this topic? Did they approach it differently? What are the origins of any vocabulary specific to this topic? 	

Another feature of developing your subject knowledge 'deeper' is to give yourself opportunities to practice sophisticated mathematical processes such as proving. The 2017 mathematics GCSE emphasises mathematical reasoning and problem solving. As a result,

pupils need to have experience of applying the mathematics they know to non-routine problems. They need opportunities to find relationships for themselves and to convince others that their conjectures hold true.

Extending Your Subject Knowledge Higher

In the early years of teaching you will find that your focus is on developing the examples, representations and language of the curriculum topics you are teaching, which is developing your pedagogic content knowledge (Shulman, 1986). As well as deepening your subject knowledge in this way it is also a good idea to develop your knowledge of a higher level of mathematics than that which you are currently teaching. In Shulman's terms you will be developing your own *mathematical content knowledge*, but this is not all. You will start to feel more confident when asked to answer unexpected questions and when differentiating lessons for higher attaining learners. Understanding the systematic development of mathematical concepts, from lower secondary to beyond A-level, will allow you to build extensive *curriculum knowledge*. This will help you to be able to signpost connections and prepare your pupils for the language and imagery they will need at the next stage of their mathematical studies (Smith and Golding, 2018). For example, calculus studied at A-level is based on the earlier study of graphs and gradients, whilst the study of groups at undergraduate level is grounded in an understanding of multiplication, factors and primes.

In exploring these curriculum links, you may also find that there are some ways of understanding mathematics that apply at earlier stages but become unhelpful later. For instance, when studying A-level mathematics in order to extend her knowledge and prepare to teach at that level, one teacher realised that she had never thought about degrees as being a unit of measure for angles. She saw that she would need to change her language in GCSE classes to make this clear. She had to stop asking "How many degrees is x ?" and to ask instead "What is angle x in degrees?" Another teacher learnt about the x -intercept and y -intercept points of cubics, and their relationship with the solutions to $f(x) = 0$ and $= f(0)$. He realised that for linear graphs he spoke of the y -intercept only as the name for the number c in $y = mx + c$ which would need to be "unlearned" later. Becoming aware of your own assumptions and the misconceptions you formed earlier in your study of mathematics allows you to recognise them and will help you teach mathematics better. It will help you to

distinguish between pupils' errors and the systematic and deeper-rooted patterns of misconceptions.

TASK: In response to the image shown in figure 11.5 a pupil has been asked to find the missing angle in Figure 11.5 and has given the answer "angle A = 85° ". What misconception might have led to this incorrect answer and where do you think this could have come from?

Figure 11.5: A mathematical misconception.

You could follow up ideas on misconceptions by exploring misconceptions on this website:

<http://www.mrbartonmaths.com/blog/category/diagnostic-questions/guess-the-misconception/>)

Depending on your previous experience and confidence with mathematics, you may decide how you want to develop your subject knowledge 'upwards' as you progress throughout your teaching career. If you do, you could enrol on an undergraduate or masters level mathematics course, or there are many mathematics MOOCs (Massive On-line Open Courses) available on the web. For example, The Open University offers

As it turns out, the art of teaching mathematics within the classroom is keeping the minds of the students outside of the classroom. Always aim to connect a new topic to something the students can relate to. Just asking your students to investigate a topic and feedback to the class the next day is certainly a productive homework and it helps to further engage your audience. For example, "So can you think of a real world application for circle theorems?" I even gave this next question to a group of year 7 students who gave me a thirty minute discussion! "Can you think of any companies that would collect quantitative data? In what way would they use this data?"

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distance-learning undergraduate-level tuition and free Open Learn courses (<http://www.open.edu/openlearn/>) that can be completed at your own pace, covering topics such as complex numbers and Bayesian statistics. There are also professional development opportunities such as undergraduate or masters Mathematics Education modules, through which you will work on both your own learning and teaching of mathematics. Some schools and academies have university connections that provide the opportunity to study at master's level.

If you are teaching A-level mathematics and want to go beyond the syllabus, you might also enjoy tackling the very challenging materials posed by university admissions papers such as the STEP (Sixth Term Entrance Paper). You can find an archive of past STEP papers and solutions here: <http://www.admissionstesting.org/for-test-takers/step/preparing-for-step/>.

One other source of great mathematical problems for pupils and teachers is the Mathematics Challenge competitions run by the United Kingdom Mathematics Trust. The questions are based on problem solving, rather than learned algorithms, and can usually be approached in several ways. They can be a resource for stimulating discussion around mathematical reasoning, for example showing the merits of starting by looking at a diagram, seeing what you notice and labelling it.

TASK: Look at the problems in figures 11.6 and 11,7 which are similar in style to UKMT Mathematics Challenge questions. For each question, can you find two or more ways of approaching the problem?

Shapes M and N are made out of identical rectangles. Their perimeters are 54 cm and 77 cm respectively.

What is the perimeter of one of the rectangles?

- A 23cm B 25cm C 27cm D 28cm E 31 cm

Figure 11.6: A problem aimed at year 8 pupils

The diagram shows a hexagon. All its interior angles are 120° . The lengths of some of the sides are shown.

What is the area of the hexagon?

- A $13\sqrt{3}$ B $14\sqrt{3}$ C $15\sqrt{3}$ D $16\sqrt{3}$ E $17\sqrt{3}$

Figure 11.7: A problem aimed at year 11 pupils

You could follow up these questions with UKMT mathematics challenge problems and solutions: <https://www.ukmt.org.uk>.

Extending Your Subject Knowledge Wider

The discussion so far has primarily focused on developing your understanding of the mathematics that is taught in school. It can also be beneficial to move sideways, exploring new topics and ideas from outside the regular curriculum.

For instance, the topic of optimisation focuses on systematic problem-solving approaches and proofs that solutions are optimal. It has applications in many areas, including aerospace engineering, molecular modelling and economics.

TASK: The Narrow Bridge

Four friends need to cross a long, narrow bridge. The bridge only supports two people at a time, and it is dark so the friends need to use a torch. They only have one torch between them, with limited battery life.

When two people cross the bridge together, they must move at the slower person's pace. Alice takes 1 minute to cross, Barry takes 2 minutes, Charlie takes 5 minutes and Daniel takes 8 minutes. Can they all get across the bridge if the torch lasts only 15 minutes?

Solving this problem involves reading and understanding constraints, then making the inference that someone must return with the torch. Students will usually suggest making the fastest person travel most often. This strategy does not yield the optimum solution: for all four friends to cross in 15 minutes, the two slowest friends must cross together. This version of the puzzle is thus a rich task to use with pupils as it introduces two competing intuitive approaches. It can be used as a stimulus for discussing what mathematicians do to extend and classify problems that are similar to each other.

Follow on TASK: Find the optimal solution for this problem with the crossing times listed below. When is the optimum strategy to pair the slowest friends together, and when is it better for the fastest friend to travel most?

- 1) 1, 2, 5, 10
- 2) 1, 2, 7, 10
- 3) 5, 10, 20, 25
- 4) 5, 10, 15, 20
- 5) 1, 4, 6, 8

Find your own set of speeds for which both strategies work equally well. Can you find an algebraic generalisation for each case (with speeds $a \leq b \leq c \leq d$)?

Working on these types of optimisation puzzles, where different strategies yield optimum solutions in different cases, can help pupils develop their problem-solving skills of organising, comparing and classifying.

You could follow up this task using an interactive online version of this puzzle in Perplex, a mathematical puzzle app containing eight classic puzzles and 40 challenges: <http://www2.open.ac.uk/openlearn/perplex/>.

TASK: Choose two or three ideas from this chapter. Find out more about them using the internet, reference books or otherwise.

NOW TRY THIS: Starting from one of the items you have researched, devise a starter activity or resource and go on to use it with one of your classes.

Support and Resources

There is a fantastic range of resources currently available to support you in developing your subject knowledge, and many of these are online. It is impossible to provide a comprehensive list, but these sites are listed here as starting points:

- Nrich (<http://nrich.maths.org/public/>) for mathematical investigations for school aged pupils which frequently step out of the standard curriculum.
- Solvemymaths.com – Mathematical problems – lots of geometry
- The website ‘Learn About Operational Research’ (<http://www.learnaboutor.co.uk/>) may help you if you are new to decision or discrete mathematics.
- The publications and websites of the Mathematical Association (<http://www.m-a.org.uk/jsp/index.jsp>) and the Association of Teachers of Mathematics (<http://www.atm.org.uk/>).
- Numberphile videos on YouTube (<https://www.youtube.com/channel/UCoxciq-8xIDTYp3uz647V5A>)
- The Open University’s Open Learn website (<http://www.open.edu/openlearn/free-courses>) is a free source of learning materials ranging from school level to post-graduate mathematics and its application (search “mathematics”). Modules from the distance-learning undergraduate degree *Mathematics and its Learning* are listed at <http://www.open.ac.uk/courses/qualifications/q46>.

Finally, there are a number of popular mathematics books, old and new at the time of writing that will expand your subject knowledge. A few of our current favourites are:

- The Mathematics of Love, by Hannah Fry.
- Things to Make and Do in the Fourth Dimension, by Matt Parker.
- Big Data: does size Matter? by Timandra Harkness.
- Puzzle Ninja by Alex Bellos.
- Geometry Snacks by Ed Southall and Vincent Pantaloni.

Conclusion

Whilst subject knowledge development might not seem as pressing a goal as lesson planning or completing marking whilst on placement, it is an important element of being a secondary mathematics teacher. This chapter argues that the goal of developing your subject knowledge is both important and manageable, as well as being a significant part of your future professional development. Continuing to explore mathematics will help to remind you of the beauty and relevance of the subject; keeping your own interest in mathematics fresh will in turn help you motivate and enthuse your pupils.

Summary:

In this chapter we invited you to think about the process of developing your subject knowledge, by:

- extending your knowledge deeper by considering how mathematical topics are connected; by discovering real-life applications of mathematical concepts as well as historical and cross-cultural roots;
- challenging your mathematical thinking and preparing pupils for progression when extending your knowledge higher;
- thinking about why extending your subject knowledge wider can help you to engage and interest your pupils; and
- showing you where to find resources to help you go higher, deeper and wider.

Further Reading

See support and resources section in the text.

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