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Anti-Pasch optimal packings with triples

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Abstract
It is shown that for $v \neq 6, 7, 10, 11, 12, 13$ there exists an optimal packing with triples on $v$ points that contains no Pasch configurations. Furthermore, for all $v \equiv 5 \pmod{6}$ there exists a pairwise balanced design of order $v$, whose blocks are all triples apart from a single quintuple, and that has no Pasch configurations amongst its triples.

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1 Introduction

The background to this paper is the anti-Pasch problem for Steiner triple systems. A Steiner triple system of order \( v \), \( \text{STS}(v) \), is a pair \((V, \mathcal{B})\) where \( V \) is a set of \( v \) elements (called points) and \( \mathcal{B} \) is a set of 3-element subsets of \( V \) (called blocks or triples) with the property that each 2-element subset of \( V \) is contained in exactly one block. An \( \text{STS}(v) \) exists if and only if \( v \equiv 1 \) or 3 \((\text{mod 6})\) [12], and such values are called admissible. A Pasch configuration, also known as a quadrilateral, is a set of 3-element sets on six points having the form

\[
\{\{a, b, c\}, \{a, y, z\}, \{x, b, z\}, \{x, y, c\}\}.
\]

The anti-Pasch conjecture, originally made by Paul Erdős [7] in a more general form, was that for all sufficiently large admissible \( v \) there exists an \( \text{STS}(v) \) that contains no Pasch configurations among its blocks. The conjecture was finally established in a series of papers [1, 9, 10, 13] culminating in [8]. So it is now known that there exists an \( \text{STS}(v) \) that contains no Pasch configurations provided \( v \) is admissible and \( v \neq 7, 13 \). Our current paper addresses the issue of what can be said about collections of triples when \( v \) is not admissible.

When \( v \) is not admissible, there is no \( \text{STS}(v) \). However, there will still be a maximum partial triple system, or optimal packing with triples, of order \( v \). In the current paper we determine the anti-Pasch result for such systems. A partial triple system of order \( v \), \( \text{PTS}(v) \), is a pair \((V, \mathcal{B})\) and is defined similarly to an \( \text{STS}(v) \), except that each 2-element subset of \( V \) is required to be contained in at most one block. A \( \text{PTS}(v) = (V, \mathcal{B}) \) for which there is no set of triples \( \mathcal{B}' \) with \(|\mathcal{B}'| > |\mathcal{B}| \) and \( \mathcal{B} \subseteq \mathcal{B}' \) is called a maximal partial triple system, \( \text{MPTS}(v) \). An \( \text{MPTS}(v) \) with the largest possible set of blocks is called a maximum maximal partial triple system, \( \text{MMPTS}(v) \). The name is generally shortened to “maximum partial triple system”. Such systems are also known as optimal or maximal packings with triples, and they give rise to optimal constant weight error-correcting codes (see [2, Section VI.40]). In a sense, these systems are as close as it is possible to get to an \( \text{STS}(v) \) when \( v \) is not admissible.

Given an \( \text{MMPTS}(v) = (V, \mathcal{B}) \), the set of 2-element subsets of \( V \) that do not appear in any block of \( \mathcal{B} \) is called the leave of the system (see [2, page 553]). For \( v \equiv 1 \) or 3 \((\text{mod 6})\) an \( \text{MMPTS}(v) \) is an \( \text{STS}(v) \) and the leave is empty. For \( v \equiv 0 \) or 2 \((\text{mod 6})\) an \( \text{MMPTS}(v) \) corresponds to an \( \text{STS}(v + 1) \) in which one point has been deleted. In these cases the leave comprises \( v/2 \) disjoint pairs. The more interesting case is \( v \equiv 5 \) \((\text{mod 6})\), and then it can be shown that the leave of an \( \text{MMPTS}(v) \) is a set of four pairs \( \{\{a, b\}, \{b, c\}, \{c, d\}, \{d, a\}\} \), which may be represented as a 4-cycle \((a, b, c, d)\). For \( v \equiv 4 \) \((\text{mod 6})\) an \( \text{MMPTS}(v) \) corresponds to an
MMPTS\((v + 1)\) from which a point of its leave has been deleted. Thus an MMPTS\((v)\) with \(v \equiv 4 \pmod{6}\) has a leave comprising three intersecting pairs \(\{a, b\}, \{b, c\}, \{b, e\}\) and a further \((v - 4)/2\) disjoint pairs covering the remaining points.

We will denote an STS\((v)\) that contains no Pasch configurations as an APSTS\((v)\) (anti-Pasch). Earlier papers often used the notation QFSTS\((v)\) (quadrilateral-free) for the same property. Similarly an APMMPTS\((v)\) denotes an anti-Pasch MMPTS\((v)\). Anti-Pasch designs have a practical application to the construction of codes for various purposes such as erasure codes for disk arrays and regular low-density parity-check codes, see [3, 11, 15, 16] and [5, page 224].

Given an APSTS\((v + 1)\) for \(v \equiv 0\) or 2 \((\pmod{6})\), the deletion of any point yields an APMMPTS\((v)\). So an APMMPTS\((v)\) exists for any \(v \equiv 0\) or 2 \((\pmod{6})\) apart possibly for \(v = 6\) or 12. Up to isomorphism there is one STS\((7)\) and two STS\((13)\)s [14], and deletion of any single point in each case does not destroy all the Pasch configurations, so there is no APMMPTS\((6)\) and no APMMPTS\((12)\). Given an APMMPTS\((v + 1)\) for \(v \equiv 4 \pmod{6}\), the deletion of any point of its leave yields an APMMPTS\((v)\). We will prove that an APMMPTS\((v)\) exists for all \(v \equiv 5 \pmod{6}\) apart from \(v = 11\), and it immediately follows that an APMMPTS\((v)\) exists for all \(v \equiv 4 \pmod{6}\) apart possibly from \(v = 10\). Up to isomorphism there are two MMPTS\((11)\)s [4], and deletion of any single point of the leave in each case does not destroy all the Pasch configurations, so there is no APMMPTS\((10)\). Hence the following result will be established.

**Theorem 1.1** There exists an anti-Pasch optimal packing with triples on \(v\) points, i.e. an APMMPTS\((v)\), for all \(v\) except for the values \(v = 6, 7, 10, 11, 12\) and 13.

An MMPTS\((v)\) for \(v \equiv 5 \pmod{6}\) is said to be of quintuple type if the leave is \((a, b, c, d)\) and the system has intersecting blocks \(\{a, c, e\}\) and \(\{b, d, e\}\). If these two blocks are removed from such a system and replaced by the quintuple \(\{a, b, c, d, e\}\), the resulting system is a pairwise balanced design of order \(v\) having one block of size 5 and all remaining blocks of size 3. Such a design is denoted by PBD\((v, \{3, 5^*\})\) and its blocks have the property that each pair of points is contained in exactly one block. The results given below produce APMMPTS\((v)\)s of quintuple type for all \(v \equiv 5 \pmod{6}\) with \(v \neq 11\). Furthermore, one of the two MMPTS\((11)\)s is of quintuple type, and the associated PBD\((11, \{3, 5^*\})\) has no Pasch configurations. So we also establish the following result.

**Theorem 1.2** There exists an anti-Pasch PBD\((v, \{3, 5^*\})\) for all \(v \equiv 5 \pmod{6}\).
In the next section, two constructions are presented. These enable us to prove that for \( v \equiv 5 \pmod{6} \) with \( v \neq 11 \) there exists an \( \text{APMMPTS}(v) \).
The first construction produces an \( \text{APMMPTS}(v) \) for \( v = 18s + 5 \) or \( v = 18s - 1 \) with \( s \geq 3 \) from three anti-Pasch Steiner triple systems. The second construction produces an \( \text{APMMPTS}(v) \) for \( v = 18s + 11 \) with \( s \geq 4 \) from three \( \text{APMMPTS}(6s+5)s \) satisfying certain conditions. Starting with a small number of \( \text{APMMPTS}(v)s \) found by computer searches, the two constructions can be used together recursively to establish the general result given in Theorem 1.1.

### 2 Constructions

Our constructions depend on the cycle structure of \( \text{STS}(v) \) and \( \text{MMPTS}(v) \) designs. For such a design \((V, B)\), define the double neighbourhood of \( x, y \in V \) (with \( x \neq y \)) as

\[
N(x, y) = \{ \{z, w\} : \{x, z, w\} \in B \text{ or } \{y, z, w\} \in B, \text{ and } \{z, w\} \cap \{x, y\} = \emptyset \}. 
\]

A double neighbourhood \( N(x, y) \) can be represented as a graph \( G(x, y) \) by taking the pairs of \( N(x, y) \) as edges. In the case of an \( \text{STS}(v) \) the graph \( G(x, y) \) is 2-regular and so it is the union of simple cycles, each of even length at least four. We refer to these as the cycles on the pair \( \{x, y\} \), or as the \( \{x, y\} \) cycles. In the case of an \( \text{MMPTS}(v) \) with \( v \equiv 5 \pmod{6} \), if the pair \( \{x, y\} \) lies in the leaf, so that the leaf has the form \( \{x, y, z, w\} \), then the points \( z \) and \( w \) have degree one in \( G(x, y) \), and therefore the graph contains a path with end points \( z \) and \( w \), which we refer to as the path on the pair \( \{x, y\} \), or as the \( \{x, y\} \) path. If this path has length \( v - 3 \) (i.e. it has \( v - 2 \) vertices) then there will be no cycles on \( \{x, y\} \), but if its length is less than \( v - 3 \), there will also be cycles on \( \{x, y\} \). In all cases, if there is a cycle of length four then the corresponding four blocks form a Pasch configuration, and so an \( \text{APSTS}(v) \) or an \( \text{APMMPTS}(v) \) cannot give rise to a cycle of length four on a pair of points \( \{x, y\} \).

For a positive integer \( n \) denote the set \( \{0, 1, \ldots, n-1\} \) by \( N \). If \( a, b \in N \), define the difference \( d = |a - b| \pmod{n} \) to be the minimum of \( (a - b) \pmod{n} \) and \( (b - a) \pmod{n} \), so that \( d \in \{0, 1, \ldots, \left\lfloor \frac{n}{2} \right\rfloor \} \). Now suppose that \( S = (V, B) \) is an \( \text{STS}(n+2) \) or an \( \text{MMPTS}(n+2) \) on the point set \( V = \{A, B\} \cup N \). If \( \{A, a, b\} \in B \) with \( a, b \in N \) then we say that \( A \) has an associated difference \( d = |a - b| \pmod{n} \) in \( S \) and that \( d \) is a difference associated with \( A \). The set of all differences associated with \( A \) in \( S \) is denoted by \( D^A \). Note that a block \( \{A, B, x\} \) does not generate a difference. The set of all differences associated with \( B \) in \( S \) is defined in a similar fashion and is denoted by \( D^B \).
We will need to combine three STS\((n+2)\)s or three MMPTS\((n+2)\)s. For 
\(n\) a positive integer and for \(i = 0, 1, 2\), we will denote the set \(\{0, 1, \ldots, (n-1)\}\) by \(N_i\). Now suppose that for \(i = 0, 1, 2\), \(\mathcal{S}_i = (V_i, \mathcal{B}_i)\) is an 
STS\((n+2)\) or an MMPTS\((n+2)\), where \(V_i = \{A, B\} \cup N_i\). Then the 
sets of associated differences \(D^A_i\) and \(D^B_i\) are formed as described above 
as subsets of \(N\) (not \(N_i\)), so that \(d \in D^A_i\) if and only if there exists a 
block \(\{A, a, b_i\} \in \mathcal{B}_i\) such that \(|a - b| \equiv d \mod n\). If \(D^A_i \cap D^A_j = \emptyset\) and 
\(D^B_i \cap D^B_j = \emptyset\) for \(i, j = 0, 1, 2\), with \(i \neq j\), then we say that \(S_0, S_1, S_2\) have 
different differences with respect to \(\{A, B\}\).

We are now in a position to describe our two constructions. We will 
often write triples or pairs without set brackets \(\{}\) or commas when no 
confusion is likely to arise.

**Construction 1.** Suppose that for \(i = 0, 1, 2\), \(\mathcal{S}_i = (V_i, \mathcal{B}_i)\) is an 
APSTS\((n+2)\) (so \(n \equiv 1\) or \(5 \mod 6\)) on the point set \(V_i = \{A, B\} \cup N_i\), 
with \(AB_0 \in \mathcal{B}_i\). Suppose also that \(S_0, S_1, S_2\) have different differences with 
respect to \(\{A, B\}\). Then an APMMPTS\(3(n+2)\), say \(\mathcal{S}\), can be formed on 
the point set \(V = \{A, B\} \cup N_0 \cup N_1 \cup N_2\) with block set \(\mathcal{B}\) containing 
the following triples:

- **Horizontal blocks:** All triples from \(\mathcal{B}_0 \cup \mathcal{B}_1 \cup \mathcal{B}_2\), except for the three 
  triples \(AB_0\), \(i = 0, 1, 2\).

- **Vertical blocks:** All triples \(x_0y_1z_2\) where \(x_0 \in N_0\), \(y_1 \in N_1\), \(z_2 \in N_2\) 
  and \(x + y + z \equiv 0 \mod n\), except for the triple \(0_00_10_2\).

- **Mixed blocks:** The two triples \(A0_00_1\) and \(B0_00_2\).

The points and triples from \(\mathcal{S}_i\) will be said to be at level \(i\), so that \(A\) 
and \(B\) are common to all three levels. Note that there are no blocks of \(\mathcal{B}\) 
containing the pairs \(AB, A0_2, B0_1\) and \(0_10_1\).

We start by proving that \(\mathcal{S}\) is an MMPTS\(3(n+2)\). Clearly the pairs 
covered by the horizontal and vertical blocks are all distinct. Each of the six 
pairs appearing in the mixed blocks lies in one of the deleted triples \(AB_0\), 
\(AB_0, AB_0\) or \(0_00_10_2\). So the blocks of \(\mathcal{B}\) do not contain a repeated pair. 
The total number of blocks in \(\mathcal{B}\) is 
\[
3 \left( \frac{(n+2)(n+1)}{6} - 1 \right) + (n^2 - 1) + 2 = \frac{3n^2 + 3n - 2}{2},
\]
which is the number of blocks in an MMPTS\(3(n+2)\). Hence \(\mathcal{S}\) is an 
MMPTS\(3(n+2)\) with leave \((A, 0_2, 0_1, B)\). In fact the design is of quintuple 
type since the blocks containing the pairs \(A0_1\) and \(B0_2\) have a common 
third point, namely \(0_0\). It remains to prove that \(\mathcal{S}\) is anti-Pasch, and to do 
this we consider two cases.

5
Case (a) Consider the possibility of a Pasch configuration $P$ that does not contain either of the mixed blocks.

If $P$ were formed from four distinct vertical blocks, it would have the form $P = \{x_0 y_1 z_2, x_0 u_1 v_2, w_0 y_1 v_2, w_0 u_1 z_2\}$ where $x + y + z \equiv 0$, $x + u + v \equiv 0$, $w + y + v \equiv 0$ and $w + u + z \equiv 0 \pmod{n}$. But since $n$ is odd, these four equivalences give $x = w, y = u, z = v$, a contradiction. So $P$ must contain a horizontal block.

If $P$ contains two horizontal blocks from the same level then five of the six points of $P$, and hence all six of the points of $P$ lie at that level, contradicting the fact that each $S_i$ is anti-Pasch. The remaining possibilities are that $P$ contains just one horizontal block, or that $P$ has two (or three) horizontal blocks from different levels.

If $P$ has just one horizontal block then this cannot contain $A$ or $B$ since all (non-mixed) blocks containing these points are horizontal and there would then have to be two such blocks in $P$. So if the sole horizontal block is at level 0 then $P$ must contain blocks of the form $x_0 y_0 z_0$ and $x_0 u_1 v_2$. Without loss of generality, $P$ then has blocks $y_0 u_1 v_2$ and $z_0 v_2 w_2$. But there is no horizontal or vertical block of this latter type with one point at level 0 and two points at level 2. A similar argument applies if the sole horizontal block is at level 1 or at level 2. Hence $P$ cannot contain just one horizontal block.

Finally in Case (a) suppose that $P$ contains two horizontal blocks from different levels. Since the two horizontal blocks must intersect, they must contain $A$ or $B$. Assume first that they both contain $A$. If these blocks are at levels 0 and 1, they have the form $Ax_0 y_0$ and $Az_1 v_1$. Then, without loss of generality, $P$ must contain two vertical blocks $x_0 z_1 u_2$ and $y_0 w_1 u_2$, where $x + z + u \equiv 0$ and $y + w + u \equiv 0 \pmod{n}$. Hence the differences $|x - y|$ and $|z - w|$ are equivalent modulo $n$. But $S_0$ and $S_1$ have different differences, so this is not possible. A similar argument applies if the two horizontal blocks are at levels 0 and 2, or at levels 1 and 2, or if $A$ is replaced by $B$.

Case (b) Consider the possibility of a Pasch configuration $P$ that contains one of the mixed blocks. There are six subcases.

1. $P$ contains $A_0 y_1 Z$ and $Ax_0 y_0$. Without loss of generality the other two blocks are $x_0 y_0 Z$ and $y_0 x_1 Z$. The fourth block gives $Z \neq B$, so $Z = z_2$, contradicting the third block since there are no blocks other than $B_0 y_0 Z$ with two points at level 0 and one at level 2.

2. $P$ contains $A_0 y_0 Z$ and $Ax_1 y_1$. Without loss of generality the other two blocks are $x_1 y_0 Z$ and $y_1 x_0 Z$. The third block gives $Z \neq B$, so $Z = z_2$, contradicting the fourth block since there are no blocks with two points at level 1 and one at level 2.
3. $P$ contains $A0_00_1$ and $Ax_2y_2$. Without loss of generality the other two blocks are $x_00_0Z$ and $y_20_1Z$. The fourth block gives $Z \neq B$, so $Z = z_0$, contradicting the third block since there are no blocks other than $B0_00_2$ with two points at level 0 and one at level 2.

4. $P$ contains $B0_00_0$ and $Bx_0y_0$. Without loss of generality the other two blocks are $x_00_0Z$ and $y_00_2Z$. The fourth block gives $Z \neq A$, so $Z = z_1$, contradicting the third block since there are no blocks other than $A0_00_1$ with two points at level 0 and one at level 1.

5. $P$ contains $B0_00_0$ and $Bx_1y_1$. Without loss of generality the other two blocks are $x_10_0Z$ and $y_00_2Z$. The fourth block gives $Z \neq A$, so $Z = z_0$, contradicting the third block since there are no blocks other than $A0_00_1$ with two points at level 0 and one at level 1.

6. $P$ contains $B0_00_0$ and $Bx_2y_2$. Without loss of generality the other two blocks are $x_20_0Z$ and $y_20_2Z$. The third block gives $Z \neq A$, so $Z = z_1$, contradicting the fourth block since there are no blocks with two points at level 2 and one at level 1.

It follows from the argument given in Cases (a) and (b) that the design $S$ produced by Construction 1 cannot contain a Pasch configuration, and so it is an APMMPTS($3n + 2$).

In order for Construction 1 to be of any use, it is necessary to prove that there is a ready supply of APSTS($n + 2$) systems $S_0, S_1, S_2$ having different differences. We now show that this is the case.

Given a APSTS($n + 2$), any two points of the system determine cycles $C_i$ of even lengths $\ell_i$ through these two points. Because the system has no Pasch configurations, $\ell_i$ cannot equal 4, so $\ell_i \geq 6$. Furthermore, $\sum \ell_i = n - 1$. We show how to label the points of such a design with $A, B, 0, 1, \ldots, n - 1$ in such a way that one block is $AB0$, all blocks $Axy$ have $|x - y| = 1$ and all blocks $Bxy$ have $|x - y| = 3$ or 5. These will be absolute differences and not just modulo $n$. We start by choosing two points arbitrarily and labelling them as $A$ and $B$. Then label as 0 the third point in the block containing $A$ and $B$. Now consider any cycle $C$ on the pair $\{A, B\}$. As a convention we record the cycle starting with two points lying in a block with $A$. Suppose that $C$ has length $\ell$. How to label the cycle depends on whether $\ell \equiv 0$ or 2 (mod 4). In each case, Table 1 gives the first four possibilities and a general formula.

In every case, a block $Axy$ has $|x - y| = 1$ and a block $Bxy$ has $|x - y| = 3$ or 5. For subsequent purposes we observe that these differences are absolute and not just modulo $n$. Having labelled the first cycle $C_1$ (with length $\ell_1$) in this way, choose another cycle $C_2$ of length $\ell_2$, and label it in a similar fashion but add $\ell_1$ to all the labels. For a third cycle $\ell_1 + \ell_2$ is added to
the labels, and so on until all the cycles, and hence all the points of the system, are labelled.

\[ \ell \equiv 2 \pmod{4} \]
- 6-cycle: \((1, 2, 5, 6, 3, 4)\)
- 10-cycle: \((1, 2, 5, 6, 9, 10, 7, 8, 3, 4)\)
- 14-cycle: \((1, 2, 5, 6, 9, 10, 13, 14, 11, 12, 7, 8, 3, 4)\)
- 18-cycle: \((1, 2, 5, 6, 9, 10, 13, 14, 17, 18, 15, 16, 11, 12, 7, 8, 3, 4)\)
- \(\ell\)-cycle: (pairs \(1 + 4j, 2 + 4j\) for \(0 \leq j \leq j^*\), followed by pairs \(3 + 4(j^* - j), 4 + 4(j^* - j)\) for \(1 \leq j \leq j^*\)), where \(j^* = (\ell - 2)/4\).

\[ \ell \equiv 0 \pmod{4} \]
- 8-cycle: \((1, 2, 7, 8, 5, 6, 3, 4)\)
- 12-cycle: \((1, 2, 7, 8, 11, 12, 9, 10, 5, 6, 3, 4)\)
- 16-cycle: \((1, 2, 7, 8, 11, 12, 15, 16, 13, 14, 9, 10, 5, 6, 3, 4)\)
- 20-cycle: \((1, 2, 7, 8, 11, 12, 15, 16, 19, 20, 17, 18, 13, 14, 9, 10, 5, 6, 3, 4)\)
- \(\ell\)-cycle: (1, 2, followed by pairs \(7 + 4j, 8 + 4j\) for \(0 \leq j \leq j^*\), followed by pairs \(5 + 4(j^* - j), 6 + 4(j^* - j)\) for \(0 \leq j \leq j^*\), followed by \(3, 4\)\), where \(j^* = (\ell - 8)/4\).

Table 1. Labelling an \(\ell\)-cycle.

![Diagram of a 8-cycle](image.png)

Figure 1. The case of an 8-cycle.
Example 1. As an example, in the 8-cycle case the blocks with $A$ and $B$ are
\[ A12, B27, A78, B85, A56, B63, A34, B41. \]

Figure 1 shows this situation.

We will define a **generic** labelling of an APSTS($n + 2$) to be a labelling of its points by $A, B$ and the elements of $N$ with the following properties.

(i) One block is labelled $AB0$,

(ii) every block labelled $Axy$ with $x, y \in N$ has $|x - y| = 1$ (absolute value, not just modulo $n$),

(iii) every block labelled $Bxy$ with $x, y \in N$ has $|x - y| = 3$ or 5 (absolute values, not just modulo $n$),

(iv) each $\{A, B\}$ cycle is labelled with a subset of consecutive integers from $N$.

We have just shown that every APSTS($n + 2$) has a generic labelling.

Using a generic labelling, any APSTS($n + 2$) can be represented on the point set $\{A, B, 0_1, 1_1, \ldots, (n - 1)_1\}$, with a block $AB0_0$, $D^A_0 = \{1\}$ and $D^B_0 \subseteq \{3, 5\}$. Let $S_0$ denote such a system. By reversing the roles of $A$ and $B$ in a generic labelling, we can represent any APSTS($n + 2$) on the point set $\{A, B, 0_1, 1_1, \ldots, (n - 1)_1\}$ with a block $AB0_1$, $D^A_1 \subseteq \{3, 5\}$ and $D^B_1 = \{1\}$. Let $S_1$ denote such a system. By applying the mapping $x \to 2x$ (mod $n$) (with $A$ and $B$ fixed) to a generic labelling we can represent any APSTS($n + 2$) on the point set $\{A, B, 0_2, 1_2, \ldots, (n - 1)_2\}$ with a block $AB0_2$, $D^A_2 = \{2\}$ and $D^B_2 \subseteq \{6, 10\}$. Let $S_2$ denote such a system. If $n \geq 17$, the differences 1, 2, 3, 5 are distinct modulo $n$ so $D^A_i \cap D^A_j = \emptyset$ for $i \neq j$, and the differences 1, 3, 5, 6, 10 are distinct modulo $n$ so $D^B_i \cap D^B_j = \emptyset$ for $i \neq j$. Hence $S_0, S_1$ and $S_2$ have different differences, and these systems may be used in Construction 1.

Example 2. As an example for $n = 17$, there is a APSTS(19) with a pair of points giving a 10-cycle and a 6-cycle. This system can be used to generate $S_i$ for each $i = 0, 1, 2$. In $S_0$ take the cycles as

\[(1_0, 2_0, 3_0, 6_0, 9_0, 10_0, 7_0, 8_0, 4_0, 5_0)\] and \[(11_0, 12_0, 15_0, 16_0, 13_0, 14_0).\]

In $S_1$ take the cycles as

\[(2_1, 5_1, 6_1, 9_1, 10_1, 7_1, 8_1, 3_1, 4_1, 1_1)\] and \[(12_1, 15_1, 16_1, 13_1, 14_1, 11_1).\]

In $S_2$ take the cycles as

\[(2_2, 4_2, 10_2, 12_2, 1_2, 3_2, 14_2, 16_2, 6_2, 8_2)\] and \[(5_2, 7_2, 13_2, 15_2, 9_2, 11_2).\]

Since there exists a APSTS($v$) for every admissible $v \geq 19$, we may now state the following result.
Theorem 2.1  If \( v \equiv 5 \) or \( 17 \pmod{18} \) and \( v \geq 17 \) then there exists an anti-Pasch MMPTS\((v)\) of quintuple type.

Proof. For \( v = 17, 23, 35 \) and \( 41 \) the result follows from a computer search and the designs are given in [6]. For \( s \geq 3 \), Construction 1 may be employed using an APSTS\((6s + 1)\) to give an APMMPTS\((18s - 1)\), and using an APSTS\((6s + 3)\) to give an APMMPTS\((18s + 5)\).

Remark. Construction 1 cannot be used to give an APMMPTS\((v)\) for \( v = 17, 23, 35 \) or \( 41 \) because there is no APSTS\((7)\), no APSTS\((13)\), and the procedure described for obtaining three APSTS\((v)\)s having different differences requires \( v \geq 19 \), and it therefore fails for \( v = 9, 15 \).

Construction 2. Suppose that for \( i = 0, 1, 2, \) \( S_i = (V_i, B_i) \) is an APMMPTS\((n + 2)\) with \( n \equiv 3 \pmod{6} \) on the point set \( V_i = \{ A, B \} \cup N_i \), such that \( S_0, S_1 \) and \( S_2 \) have different differences with respect to \( \{ A, B \} \), and their leaves are respectively \( (A, a_0, b_0, B) \), \( (A, c_1, d_1, B) \) and \( (A, c_2, d_2, B) \). Suppose also that \( c - d \equiv f - e \pmod{n} \). Let \( \delta \) denote the difference \( |c - d| \) modulo \( n \), and let \( g \) be such that \( g + c + e \equiv g + d + f \equiv 0 \pmod{n} \).

Assume that

(i) \( \delta \not\in D_A^0 \cup D_B^0 \), and

(ii) there are no blocks \( g_0x_0(x + \delta)0 \in B_0 \) (where \( x + \delta \) is taken modulo \( n \)).

Then an APMMPTS\((3n + 2)\), say \( S \), can be formed on the point set \( V = \{ A, B \} \cup N_0 \cup N_1 \cup N_2 \) with block set \( B \) containing the following triples:

- **Horizontal blocks:** All triples from \( B_0 \cup B_1 \cup B_2 \).
- **Vertical blocks:** All triples \( x_0y_1z_2 \) where \( x_0 \in N_0, y_1 \in N_1, z_2 \in N_2 \) and \( x + y + z \equiv 0 \pmod{n} \), except for the two triples \( g_0c_1e_2 \) and \( g_0d_1f_2 \).
- **Mixed blocks:** The four triples \( Ac_1e_2, Bd_1f_2, g_0c_1d_1 \) and \( g_0e_2f_2 \).

The points and triples from \( S_i \) will be said to be at level \( i \), so that \( A \) and \( B \) are common to all three levels. Note that there are no blocks of \( B \) containing the pairs \( AB, Aa_0, Bb_0 \) and \( a_0b_0 \).

We start by proving that \( S \) is an MMPTS\((3n + 2)\). Clearly the pairs covered by the horizontal and vertical blocks are all distinct. Each of the 12 pairs appearing in the mixed blocks either lies in the leave of \( S_1 \) or \( S_2 \), or in one of the deleted triples \( g_0c_1e_2, g_0d_1f_2 \). So the blocks of \( B \) do not contain a repeated pair. The total number of blocks in \( B \) is

\[
3 \left( \frac{(n + 2)(n + 1) - 8}{6} \right) + (n^2 - 2) + 4 = \frac{3n^2 + 3n - 2}{2},
\]
which is the number of blocks in an MMPTS\((3n + 2)\). Hence \(S\) is an MMPTS\((3n + 2)\) with leave \((A, a_0, b_0, B)\). It remains to prove that \(S\) is anti-Pasch, and to do this we consider two cases.

**Case (a)** Consider the possibility of a Pasch configuration \(P\) that does not contain any of the four mixed blocks.

The argument that eliminates this possibility is identical with that given for the corresponding case in Construction 1.

**Case (b)** Consider the possibility of a Pasch configuration \(P\) that contains one of the mixed blocks. Altogether there are thirteen subcases and to save endlessly writing “\((\text{mod } n)\)” we state once that all the congruences are taken modulo \(n\). Consider first a possible Pasch configuration \(P\) containing the point \(A\). There are three possibilities for the two blocks containing \(A\).

1. \(Ac_1e_2, Ax_0y_0\). Without loss of generality the other two blocks are \(x_0c_1Z\) and \(y_0e_2Z\). The third block shows that \(Z \neq B\). Examining the third block gives two possibilities. If \(Z = d_1\) then \(x_0 = g_0\) and the fourth block is \(y_0d_1e_2\), which gives \(y + d + e \equiv 0\), so that \(|x - y| \equiv |g + d + e| \equiv |e - f| \equiv \delta\). Thus the block \(Ax_0y_0\) gives \(\delta \in D^3_0\), a contradiction. The other possibility is that \(Z = z_2\), in which case the fourth block is \(y_0e_2z_2\), which implies that \(y_0 = g_0\) and \(z_2 = f_2\). Then the third block gives \(x + c + z \equiv 0\), so that \(|x - y| \equiv |g + c + f| \equiv |c - d| \equiv \delta\). Thus also in this case the block \(Ax_0y_0\) gives \(\delta \in D^3_0\), a contradiction.

2. \(Ac_1e_2, Ax_1y_1\). Without loss of generality the other two blocks are \(x_1c_1Z\) and \(y_1e_2Z\). The fourth block shows that \(Z \neq B\) and so \(Z = z_0\), where \(y + e + z \equiv 0\). Then the third block gives \(z_0 = g_0\), so \(y + e + g \equiv 0\) and hence \(y_1 = c_1\), a contradiction.

3. \(Ac_1e_2, Ax_2y_2\). Without loss of generality the other two blocks are \(x_2c_1Z\) and \(y_2e_2Z\). The third block shows that \(Z \neq B\) and so \(Z = z_0\), where \(x + e + z \equiv 0\). Then the fourth block gives \(z_0 = g_0\), so \(x + e + g \equiv 0\) and hence \(x_2 = e_2\), a contradiction.

Consider next a possible Pasch configuration \(P\) containing the point \(B\). There are three possibilities for the two blocks containing \(B\).

4. \(Bd_1f_2, Bx_0y_0\). Without loss of generality the other two blocks are \(x_0d_1Z\) and \(y_0f_2Z\). The third block shows that \(Z \neq A\). Examining the third block gives two possibilities. If \(Z = c_1\) then \(x_0 = g_0\) and the fourth block is \(y_0c_1f_2\), which gives \(y + c + f \equiv 0\), so that \(|x - y| \equiv |g + f + c| \equiv |c - d| \equiv \delta\). Thus the block \(Bx_0y_0\) gives \(\delta \in D^3_0\), a contradiction. The other possibility is that \(Z = z_2\), in which case the
fourth block is $y_0f_2z_2$, which implies that $y_0 = g_0$ and $z_2 = c_2$. Then the third block gives $x + d + z \equiv 0$, so that $|x - y| \equiv |g + d + e| \equiv |e - f| \equiv \delta$. Thus also in this case the block $Bx_0y_0$ gives $\delta \in D_0^B$, a contradiction.

5. $Bd_1f_2, Bx_1y_1$. Without loss of generality the other two blocks are $x_1d_1Z$ and $y_1f_2Z$. The fourth block shows that $Z \neq A$ and so $Z = z_0$, where $y + f + z \equiv 0$. Then the third block gives $z_0 = g_0$, so $y + f + g \equiv 0$ and hence $y_1 = d_1$, a contradiction.

6. $Bd_1f_2, Bx_2y_2$. Without loss of generality the other two blocks are $x_2d_1Z$ and $y_2f_2Z$. The third block shows that $Z \neq A$ and so $Z = z_0$, where $x + d + z \equiv 0$. Then the fourth block gives $z_0 = g_0$, so $x + d + g \equiv 0$ and hence $x_2 = f_2$, a contradiction.

It follows from the arguments above that there can be no Pasch configurations involving either of the two mixed blocks containing $A$ and $B$ if condition (i) is satisfied. So we next examine the possibility of a Pasch configuration containing one of the other two mixed blocks. First we deal with the case of a possible Pasch configuration $P$ containing both of these mixed blocks.

7. Suppose that $P$ has blocks $g_0c_1d_1, g_0e_2f_2$. The pair $c_1e_2$ lies in a triple with $A$ and the pair $d_1f_2$ lies in a triple with $B$. So suppose the other two blocks of $P$ are $c_1f_2Z$ and $d_1e_2Z$. Then $Z = z_0$ and $c + f + z \equiv d + e + z \equiv 0$. This gives $c - d \equiv e - f$, but we already have $c - d \equiv f - e$, and since $n$ is odd these give $c = d$, a contradiction.

Next consider a possible Pasch configuration $P$ containing just the one mixed block $g_0c_1d_1$. Without loss of generality there are three possibilities.

8. Suppose that $P$ has blocks $g_0c_1d_1, g_0x_0y_0, x_0c_1Z, y_0d_1Z$. The third block shows that $Z \neq A, B$ and since $Z \neq d_1$, we must have $Z = z_2$. So the third and fourth blocks give $x \equiv -(c + z)$ and $y \equiv -(d + z)$. Now the second block may be written as $g_0(-(c + z))_0(-(d + z))_0$ which has the form $g_0w_0(w + \delta)_0$ because $|c + z - (d + z)| = |c - d| \equiv \delta$. But this contradicts the supposition (ii) that there are no such blocks.

9. Suppose that $P$ has blocks $g_0c_1d_1, g_0x_1y_2, x_1c_1Z, y_2d_1Z$. The fourth block shows that $Z \neq A, B$ and so $Z = z_0$. But then the third block gives $z_0 = g_0$ and $x_1 = d_1$, a contradiction.

10. Suppose that $P$ has blocks $g_0c_1d_1, g_0x_1y_2, x_1d_1Z, y_2c_1Z$. The fourth block shows that $Z \neq A, B$ and so $Z = z_0$. But then the third block gives $z_0 = g_0$ and $x_1 = c_1$, a contradiction.
Finally, consider a possible Pasch configuration \( P \) containing just the one mixed block \( g_0e_2f_2 \). Without loss of generality there are three possibilities.

11. Suppose that \( P \) has blocks \( g_0e_2f_2, g_0x_0y_0, x_0e_2Z, y_0f_2Z \). The third block shows that \( Z \neq A, B \) and since \( Z \neq f_2 \), we must have \( Z = z_1 \). Now the second block may be written as \( g_0(-e + z)g_0(-f + z) \) because \( |e + z - (f + z)| \equiv |e - f| \). But this contradicts the supposition (ii) that there are no such blocks.

12. Suppose that \( P \) has blocks \( g_0e_2f_2, g_0x_1y_2, x_1e_2Z, y_2f_2Z \). The third block shows that \( Z \neq A, B \) and since \( Z \neq f_2 \), we must have \( Z = z_0 \). But then the fourth block gives \( z_0 = g_0 \) and \( y_2 = e_2 \), a contradiction.

13. Suppose that \( P \) has blocks \( g_0e_2f_2, g_0x_1y_2, x_1f_2Z, y_2e_2Z \). The third block shows that \( Z \neq A, B \) and so \( Z = z_0 \). But then the fourth block gives \( z_0 = g_0 \) and \( y_2 = f_2 \), a contradiction.

It follows from Cases (a) and (b) that the design \( S \) produced by Construction 2 cannot contain a Pasch configuration, and so it is an APMMPTS(3n + 2).

In order for Construction 2 to be of any use, it is necessary to prove that there is a ready supply of APMMPTS(\( n + 2 \)) systems \( S_0, S_1, S_2 \) having the appropriate properties. We will show that this is the case, but we will do so in stages.

Given an APMMPTS(\( n + 2 \)) with \( n \equiv 3 \) (mod 6) and with the leave \( (X, \alpha, \beta, Y) \), the pair \( \{X, Y\} \) determines a path of even length \( p \) (i.e. having an odd number of vertices) having the form \((\beta, \ldots, \alpha)\). If \( p < n - 1 \) there will also be cycles \( C_i \) having even lengths \( \ell_i \) on the pair \( \{X, Y\} \). Because the system has no Pasch configurations, \( \ell_i \) cannot equal 4, so \( \ell_i \geq 6 \). Furthermore, \( p + \sum \ell_i = n - 1 \). Note that the design is of quintuple type if and only if the path is of length \( p = 2 \), in which case the path is \((\beta, \gamma, \alpha)\) where \( \gamma \) is the point forming blocks \( X\beta\gamma \) and \( Y\alpha\gamma \).

Our earlier definition of a generic labelling of an APSTS(\( n + 2 \)) can be modified for APMMPTS(\( n + 2 \))s with \( n \equiv 3 \) (mod 6) and leave \( (X, \alpha, \beta, Y) \). A generic labelling of such an APMMPTS(\( n + 2 \)) is a labelling of its points by \( A, B \) and the elements of \( N \) with the following properties.

(i) The leave is labelled \((A, 0, 1, B)\),

(ii) every block labelled \( Axy \) with \( x, y \in N \) has \(|x - y| = 1 \) (absolute value, not just modulo \( n \)),

(iii) every block labelled \( Bxy \) with \( x, y \in N \) has \(|x - y| = 2, 3 \) or \( 5 \) (absolute values, not just modulo \( n \)).
(iv) the \{A, B\} path and each \{A, B\} cycle (if any) is labelled with a subset of consecutive integers from \(N\).

Every APMMPTS\((n + 2)\) with \(n \equiv 3 \pmod{6}\) has a generic labelling. To see this consider first the path. How to label this depends whether its length \(p\) has \(p \equiv 0\) or \(2 \pmod{4}\). In each case Table 2 gives the first four cases and a general formula. As a convention we record the path starting with two points lying in a block with \(A\), so the first point represents \(\beta\) and the last point represents \(\alpha\).

\[
p \equiv 2 \pmod{4}
\]
\[
p = 2: \quad (1, 2, 0)
\]
\[
p = 6: \quad (1, 2, 5, 6, 4, 3, 0)
\]
\[
p = 10: \quad (1, 2, 5, 6, 9, 10, 8, 7, 4, 3, 0)
\]
\[
p = 14: \quad (1, 2, 5, 6, 9, 10, 13, 14, 12, 11, 8, 7, 4, 3, 0)
\]
\[
p \equiv 2: \quad (\text{pairs } 1 + 4j, 2 + 4j \text{ for } 0 \leq j \leq j^*, \text{ followed by } 4(j^* - j), 4(j^* - j) - 1 \text{ for } 0 \leq j \leq j^* - 1),
\]
\[
\text{followed by } 0),
\]
\[
\text{where } j^* = (p - 2)/4.
\]

\[
p \equiv 0 \pmod{4}
\]
\[
p = 4: \quad (1, 2, 4, 3, 0)
\]
\[
p = 8: \quad (1, 2, 5, 6, 8, 7, 4, 3, 0)
\]
\[
p = 12: \quad (1, 2, 5, 6, 9, 10, 12, 11, 8, 7, 4, 3, 0)
\]
\[
p = 16: \quad (1, 2, 5, 6, 9, 10, 13, 14, 16, 15, 12, 11, 8, 7, 4, 3, 0)
\]
\[
p \equiv 0: \quad (\text{pairs } 1 + 4j, 2 + 4j \text{ for } 0 \leq j \leq j^*, \text{ followed by } 4 + 4(j^* - j), 3 + 4(j^* - j) \text{ for } 0 \leq j \leq j^*,
\]
\[
\text{followed by } 0),
\]
\[
\text{where } j^* = (p - 4)/4.
\]

Table 2. Labelling the path.

Every block \(A_{xy}\) in the path has \(|x - y| = 1\), and every block \(B_{xy}\) in the path has \(|x - y| = 2\) or \(3\). The cycles (if any) can then be labelled as described previously in connection with Construction 1, adding an appropriate constant to all the labels for each cycle so that every block \(A_{xy}\) in every cycle has \(|x - y| = 1\) and every block \(B_{xy}\) in every cycle has \(|x - y| = 3\) or \(5\). Thus in the complete labelling, the leave is labelled \((A, 0, 1, B)\) and every block \(A_{xy}\) has \(|x - y| = 1\) and every block \(B_{xy}\) has \(|x - y| = 2, 3\) or \(5\), so that \(D^A = \{1\}\) and \(D^B \subseteq \{2, 3, 5\}\). Furthermore the path and each cycle is labelled with a subset of consecutive integers from \(N\).

Now suppose that we have an APMMPTS\((n + 2)\) with \(n \equiv 3 \pmod{6}\) that is generically labelled and has the additional property, which we call property \(G\), that there is some point \(g \neq A, B\) for which there are no blocks.
If \( D \) and \( S \) label systems are suitable for use in Construction 2. regards the sets of differences associated with property \( G \). For small values of \( n \equiv 3 \) (mod \( n \)) (with \( A \) and \( B \) fixed) to any generically labelled APMMMPTS\((n+2)\) and then appending the suffix 1 to all the points other than \( A \) and \( B \). The leave of \( S_1 \) is then \( (A, 0, 1, 4, B) \) so, in the notation of Construction 2, \( c = 0 \) and \( d = 4 \). Form \( S_2 \) from any generically labelled APMMMPTS\((n+2)\) as follows. First exchange \( A \) and \( B \); this can be achieved by taking the first two points in the path and in each cycle to be in a block with \( B \), the second and third points with \( A \), and so on. Then apply the mapping \( x \rightarrow 4x + \lambda \) (mod \( n \)) (with \( A \) and \( B \) fixed and \( \lambda \) a constant specified below), finally append the suffix 2 to all the points other than \( A \) and \( B \). The constant \( \lambda \) is chosen as follows. The leave of \( S_2 \) is \( (A, (4 + \lambda) 2, \lambda_2, B) \) so, in the notation of Construction 2, \( e \equiv 4 + \lambda \) and \( f \equiv \lambda \). We wish to have blocks \( goe_2c_2 \) and \( gof_1f_2 \), so we require \( g + c + e \equiv g + d + f \equiv 0 \), and this can be achieved by setting \( \lambda \equiv -(g + 4) \) (mod \( n \)). This choice gives \( e \equiv -g \) and \( f \equiv -(g + 4) \). Again in the notation of Construction 2, \( \delta = 4 \). As regards the sets of differences associated with \( A \) and \( B \), we have \( D_0^A = \{1\} \), \( D_0^B \subseteq \{2, 3, 5\} \), \( D_2^A = \{4\} \), \( D_2^B \subseteq \{8, 12, 20\} \), \( D_2^A \subseteq \{8, 12, 20\} \), \( D_2^B = \{4\} \). If \( n \geq 27 \), the differences \( 1, 2, 3, 4, 5, 8, 12, 20 \) are all distinct modulo \( n \), and then the systems \( S_0, S_1 \) and \( S_2 \) have different differences with respect to \( \{A, B\} \), \( \delta \notin D_2^A \cup D_2^B \), and there are no blocks \( q_0x_0(x + \delta) \). So the three labelled systems are suitable for use in Construction 2.

There still remains the difficulty of finding an APMMMPTS\((n+2)\) with \( n \equiv 3 \) (mod \( 6 \)) that can be generically labelled in such a way that it has property \( G \). For small values of \( n \geq 27 \) these are easy to find using a computer and a hill-climbing algorithm. We will prove that any system produced by Construction 1 has this property, and that an additional condition on the ingredients will ensure that some systems produced by Construction 2 also have this property. In preparation for this we give the following rather trivial but useful lemma.

**Lemma 2.1** Suppose that \( S \) is a generically labelled APMMMPTS\((n+2)\) with \( n \equiv 3 \) (mod \( 6 \)). Then there exists a point \( g \neq A, B \) for which there are no blocks \( \{g, x, x+4\} \), if and only if there exists a point \( h \) for which there are two blocks \( \{h, z, z+4\} \) and \( \{h, w, w+4\} \) (\( z \neq w \) and arithmetic modulo \( n \)).

**Proof.** The point set of \( S \) has \( n \) points other than \( A \) and \( B \), and consequently \( n \) pairs \( \{x, x+4\} \). None of these pairs appear in a triple with \( A \) or \( B \) because \( 4 \notin D_2^A \cup D_2^B \). If two such pairs appear with a point \( h \) then \( h \neq A, B \), and so there must exist a point \( g \neq A, B \) for which there are no blocks \( \{g, x, x+4\} \). Conversely if there is a point \( g \neq A, B \) for which there are no blocks \( \{g, x, x+4\} \), then some point \( h \) must occur in blocks with two distinct pairs \( \{z, z+4\} \) and \( \{w, w+4\} \).

\[\Box\]
Define property $H$ for a generically labelled APMMPTS($n + 2$) by the requirement that there should exist a point $h$ for which there are two blocks \{h, z, z + 4\} and \{h, w, w + 4\} (\(z \neq w\) and arithmetic modulo $n$). The lemma shows that properties $G$ and $H$ are equivalent. The lemma is useful because property $H$ is easier to establish than property $G$. It is advantageous to consider a stronger requirement than property $H$ in which the points $z$ and $w$ are well away from $n - 1$. So we define property $H^*$ for a generically labelled APMMPTS($n + 2$) by the requirement that there should exist a point $h$ for which there are two blocks \{h, z, z + 4\} and \{h, w, w + 4\} with $z \neq w$ and $0 \leq z, w \leq n - 5$. Property $H^*$ ensures that the differences of four between $z$ and $z + 4$, and between $w$ and $w + 4$, are absolute, and not just modulo $n$. Clearly property $H^*$ implies property $G$.

We now explain how an APMMPTS(3$n + 2$) produced by Construction 1 can be generically labelled by $A, B, 0, 1, \ldots, 3n - 1$ in such a way that it has property $H^*$, i.e. there is a point $h$ for which there are two blocks \{h, z, z + 4\} and \{h, w, w + 4\} with $z \neq w$ and $0 \leq z, w \leq 3n - 5$. Any system produced by Construction 1 comes with the (non-generic) labelling inherited from that construction, but here we specify a relabelling. Obviously no relabelling will create Pasch configurations. But the relabelling will result in the vertical blocks $x_0y_1z_2$ no longer satisfying the condition $x + y + z \equiv 0 \pmod{n}$. A system produced by Construction 1 will be of quintuple type. The relabelling is done in stages.

The original labelling of the constructed system has the $\{A, B\}$ path $(0_1, 0_0, 0_2)$, and has the $\{A, B\}$ cycles of system $S_0$ labelled with $1_0, 2_0, \ldots, (n - 1)_0$. Note that the differences given by the original labellings are absolute (not modulo $n$). In the relabelling, the points $A$ and $B$ retain their original labels. The path is relabelled as $(1, 2, 0)$ so that $0_1$ is relabelled 1, $0_0$ is relabelled 2, and $0_2$ is relabelled 0. For the $\{A, B\}$ cycles of system $S_0$, drop the suffix 0 and add 2 to all the labels, so that the cycles (and hence the points $1_0, 2_0, \ldots, (n - 1)_0$) are now labelled by $3, 4, \ldots, n + 1$. This relabelling does not affect the differences since these are absolute differences. So, up to this point in the argument, the differences generated by the path and relabelled cycles on $A$ are all 1, and those on $B$ are all 2, 3 or 5. Now pick two distinct points from $3, 4, \ldots, n + 1$ with an absolute difference of 4, say $z$ and $z + 4$. These lie in a block with some other point already relabelled (not $A$ or $B$), say $h$.

Next, the $\{A, B\}$ cycles (and hence the points) of system $S_1$, originally labelled with $1_1, 2_1, \ldots, (n - 1)_1$, are relabelled with $n + 2, n + 3, \ldots, 2n$, but we carry out the relabelling by the generic method described above so that the differences on $A$ are all 1 and those on $B$ are all 3 or 5 (again, absolute values).

Now consider the triple containing the pair of points already relabelled.
as \( h \) and \( 2n \). This triple is one of the original vertical triples from the construction. So there is some point from \( S_2 \), say \( u_2 \), that forms a block with the points now relabelled as \( h \) and \( 2n \). Assume for the moment that \( u_2 \neq 0_2 \) so that \( u_2 \) lies in one of the original \( \{A, B\} \) cycles of \( S_2 \). These cycles (and hence the points) of \( S_2 \), originally labelled with \( 1_2, 2_2, \ldots, (n - 1)_2 \), will be relabelled with \( 2n + 1, 2n + 2, \ldots, 3n - 1 \), but we again carry out the relabelling by the generic method described above so that the differences on \( A \) are all 1 and those on \( B \) are all 3 or 5 (again, absolute values). Moreover, it is possible to arrange that the point \( u_2 \) is relabelled as \( 2n + 4 \). To achieve this, take the cycle containing \( u_2 \) as the first cycle to be relabelled. Let \( \ell \) denote the length of this cycle, so that it is relabelled with \( 2n + 1, 2n + 2, \ldots, 2n + \ell \). Thus one of the points in this cycle is relabelled as \( 2n + 4 \), and we can ensure that this point is \( u_2 \) by taking an appropriate equivalent form for listing the cycle (see Example 3 below for an example of what we mean by this). Such a relabelling results in a block \( \{h, w, w + 4\} \) with \( w = 2n \).

In the exceptional case when \( u_2 = 0_2 \), there will be a point \( v_2 \) that forms a block with the points now relabelled as \( h \) and \( 2n - 1 \), and \( v_2 \neq 0_2 \). So in this case we relabel the points \( 1_2, 2_2, \ldots, (n - 1)_2 \), as before, but now arrange for \( v_2 \) to receive the label \( 2n + 3 \). This results in a block \( \{h, w, w + 4\} \) with \( w = 2n - 1 \).

The constructed system is now labelled with a generic labelling, with \( D^A = \{1\} \) and \( D^B \subseteq \{2, 3, 5\} \), and we have two distinct blocks of the form \( \{h, z, z + 4\} \) and \( \{h, w, w + 4\} \). Consequently there must exist a point \( g \neq A, B \) for which there is no block labelled \( \{g, x, x + 4\} \). This system, as now labelled, is suitable for use in an application of Construction 2 (with \( n \) now replaced by \( 3n \)). It is also useful to note that the two blocks \( \{h, z, z + 4\} \) and \( \{h, w, w + 4\} \) have \( 0 \leq z, w \leq 3n - 5 \), i.e. the system has property \( H^* \).

As explained below, this will enable a system produced by such an application of Construction 2 to be itself used in a reapplication of Construction 2. 

**Example 3.** To clarify the relabelling of system \( S_2 \) described above, suppose that \( u_2 \) lies in the 6-cycle \( \{\ldots, x_2, u_2, y_2, \ldots\} \). Note that the “standard form” for a 6-cycle given earlier is \( \{1, 2, 5, 6, 3, 4\} \).

If the block containing the pair \( \{x_2, u_2\} \) is \( Ax_2u_2 \), take the cycle in the equivalent form \( \{y_2, \ldots, x_2, u_2\} \). Then relabel:

\[
y_2 \rightarrow 2n + 1, \ldots, x_2 \rightarrow 2n + 3, u_2 \rightarrow 2n + 4,
\]

so that the cycle is now \( \{2n + 1, 2n + 2, 2n + 5, 2n + 6, 2n + 3, 2n + 4\} \). Thus \( u_2 \) is relabelled as \( 2n + 4 \) and the differences on \( A \) are 1 and those on \( B \) are 3 or 5 (actually 3 in this example).

If the block containing the pair \( \{x_2, u_2\} \) is \( Bx_2u_2 \), first reverse the cycle to get the equivalent form \( \{\ldots, y_2, u_2, x_2, \ldots\} \), and then write it as
(x_2, \ldots, y_2, u_2). Then relabel:

\[ x_2 \to 2n + 1, \ldots, y_2 \to 2n + 3, u_2 \to 2n + 4, \]

so that the cycle is again (2n + 1, 2n + 2, 2n + 5, 2n + 6, 2n + 3, 2n + 4).

Thus u_2 is relabelled as 2n + 4 and the differences on A are 1 and those on B are 3 or 5 (actually 3 in this example).

It should be clear from this example that it is possible to relabel the points 1_2, 2_2, \ldots, (n - 1)_2 of system S_2 with the labels 2n + 1, 2n + 2, \ldots, 3n - 1 in such a way that the \{A, B\} cycles are labelled with subsets of consecutive integers, the differences on A are 1, those on B are 3 or 5, and any specified point x_2 can be relabelled with any specified label y \in \{2n + 1, 2n + 2, \ldots, 3n - 1\}.

Next we show that some systems produced by Construction 2 can also be given a generic relabelling with property H*.

So suppose that S has been produced using the constituent APMMPTS(n + 2) designs S_0, S_1 and S_2, and suppose also that S_0 has property H*. Observe that the \{A, B\} path and cycles of S_0 are retained in S. Hence S, with its original labelling contains two blocks \{h_0, z_0, (z + 4)_0\} and \{h_0, w_0, (w + 4)_0\} with 0 \leq z, w \leq n - 5. In the relabelling, A and B retain their original labels and each point x_0 of S_0 is relabelled as x. We then have two blocks of S labelled as \{h, z, z + 4\} and \{h, w, w + 4\}, and 0 \leq z, w \leq 3n - 5. The relabelling does not affect the differences on the \{A, B\} path and cycles since these are absolute differences. So, up to this point in the argument, the differences generated by the path and relabelled cycles on A are all 1, and those on B are all 2, 3 or 5.

We now relabel all the remaining cycles by the generic method described above, using labels n, n + 1, \ldots, 3n - 1, so that the differences on A are all 1 and the differences on B are all 3 or 5 (again, absolute values). The result is that S is generically labelled and has property H*. Consequently the relabelled system S is suitable for use in a reapplication of Construction 2, taking the role of the new system S_0; the system resulting from such a reapplication may again be reused, and so on.

We can summarise the results of this section as follows.

- Every APMMPTS(3n + 2) \((n \equiv 1 \text{ or } 5 \pmod{6}, n \geq 17)\) produced by Construction 1 is of quintuple type and it can be generically labelled in such a way that it has property H*.

- If there is an APMMPTS(n + 2) \((n \equiv 3 \pmod{6}, n \geq 27)\) that can be generically labelled in such a way that it has property H*, then Construction 2 can be applied to yield an APMMPTS(3n + 2) that can also be generically labelled in such a way that it has property H*.
If the $\text{APMMPTS}(n+2)$ is of quintuple type, then so is the resulting $\text{APMMPTS}(3n+2)$.

3 Recursion

We now show how the results of the previous section can be used to establish that there exists an $\text{APMMPTS}(6s+5)$ for all $s \neq 1$. The case $s = 0$ is trivial. Computer searches based on a hill-climbing algorithm deal with the values $2 \leq s \leq 7$ and $s = 10$. By this method, we have constructed $\text{APMMPTS}(6s+5)$ designs of quintuple type

(a) for $s = 2$ and $3$, and

(b) for $s = 4, 5, 6, 7$ and $10$, and these designs can be generically labelled to have property $H^*$.

These designs are available from the authors [6]. The cases $s = 8$ and $9$ are resolved by Construction 1, using $\text{APSTS}(19)s$ and $\text{APSTS}(21)s$ respectively. As noted in the previous section, the resulting $\text{APMMPTS}(53)$ and $\text{APMMPTS}(59)$ designs are of quintuple type and can be generically labelled to have property $H^*$.

Having established the result for $4 \leq s \leq 10$, the following lemma provides the inductive step.

**Lemma 3.1** If there exists an $\text{APMMPTS}(6s+5)$ of quintuple type that can be generically labelled to have property $H^*$ for each value $s$ satisfying $4 \leq s \leq M$, where $M \geq 10$, then there exists an $\text{APMMPTS}(6(M+1)+5)$ of quintuple type that can be generically labelled to have property $H^*$.

**Proof.** The proof fall into two cases: if $M \equiv 0 \pmod{3}$ then Construction 2 is applied inductively, otherwise Construction 1 is applied directly.

So suppose first that $M = 3t$. Then $t \geq 4$ so, by the hypothesis, there is an $\text{APMMPTS}(6t+5)$ of quintuple type that can be generically labelled to have property $H^*$. Applying Construction 2 with $n = 6t + 3$ gives an $\text{APMMPTS}(6(M+1)+5)$ of quintuple type that can be generically labelled to have property $H^*$.

Next suppose that $M = 3t + 1$. Then $t \geq 3$ and there exists an $\text{APST}(6t+7)$. Applying Construction 1 with $n = 6t + 5$ gives an $\text{APMMPTS}(6(M+1)+5)$ of quintuple type that can be generically labelled to have property $H^*$. Similarly if $M = 3t + 2$ then $t \geq 3$ and there exists an $\text{APST}(6t+9)$. Applying Construction 1 with $n = 6t + 7$ gives an $\text{APMMPTS}(6(M+1)+5)$ of quintuple type that can be generically labelled to have property $H^*$.

Corollary 3.1 follows immediately from the Lemma 3.1. 

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Corollary 3.1 There exists an APMMPTS($6s + 5$) of quintuple type for all $s \neq 1$.

As explained in the Introduction, this result establishes the truth of Theorems 1.1 and 1.2.

References


