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Generalized Q-functions

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Abstract

The modulus squared of a class of wave functions defined on phase space is used to define a generalized family of $Q$ or Husimi functions. A parameter $\lambda$ specifies orderings in a mapping from the operator $|\psi\rangle\langle\sigma|$ to the corresponding phase space wave function, where $\sigma$ is a given fiducial vector. The choice $\lambda = 0$ specifies the Weyl mapping and the $Q$-function so obtained is the usual one when $|\sigma\rangle$ is the vacuum state. More generally, any choice of $\lambda$ in the range $(-1, 1)$ corresponds to orderings varying between standard and anti-standard. For all such orderings the generalized $Q$-functions are non-negative by construction. They are shown to be proportional to the expectation of the system state $\hat{\rho}$ with respect to a generalized displaced squeezed state which depends on $\lambda$ and position $(p, q)$ in phase space. Thus, when a system has been prepared in the state $\hat{\rho}$, a generalized $Q$-function is proportional to the probability of finding it in the generalized squeezed state. Any such $Q$-function can also be written as the smoothing of the Wigner function for the system state $\hat{\rho}$ by convolution with the Wigner function for the generalized squeezed state.

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1 Introduction

The Weyl transform [1, 2] associates operators with functions on phase space. In particular, the Wigner function $\rho(p, q)$ [3] is the Weyl transform of the density matrix divided by $\hbar = 2\pi\hbar$. Although $\rho(p, q)$ does have many features of a classical distribution it can take on negative values, with bounds [5] given by $2/\hbar \geq \rho(p, q) \geq -2/\hbar$. Indeed Hudson [4] showed that the only pure states $\psi(x)$ for which the Wigner function is non-negative are Gaussian in $x$. This carries over to any number of dimensions [6], and also, for odd dimensions at least, to the formulation of discrete Wigner functions [7].

The Weyl correspondence between operators and functions on phase space—of which the Wigner function is an example—is a special case of the class of the correspondences given by Cohen [8]. In particular, if any Wigner function is convoluted, or smeared, by integration with respect to the Wigner function of the vacuum state, itself a gaussian function on phase space, then the smoothed function, called the $Q$-function (or Husimi function), is non-negative and corresponds to an ordering in Cohen’s class different from that of Wigner and Weyl [9]. More generally, if any Wigner function is convoluted with respect to a Gaussian function which is itself the Wigner function of a pure coherent state, then the result is non-negative [6, 9, 10, 11, 12, 13].
The Wigner function is bilinear with respect to wave functions. For instance if the Weyl transform of the pure state $|\psi\rangle\langle\psi|$ is written $(|\psi\rangle\langle\psi|)_{(p,q)}$, then the corresponding Wigner function [3, 5] is

$$
\rho(p,q) = \frac{1}{\hbar} \left( |\psi\rangle\langle\psi| \right)_{(p,q)} = \frac{1}{\hbar} \int_{-\infty}^{\infty} dx \exp \left( \frac{i}{\hbar} p x \right) \psi(q - \frac{x}{2}) \psi^*(q + \frac{x}{2}) , ~ \quad (1)
$$

so the smeared Wigner functions are also bilinear with respect to the wave functions.

It is also possible in a sense to smear the states themselves, for instance by projecting them onto a class of generalized displaced coherent states, defined [14] by

$$
|p,q;\sigma\rangle \equiv \tilde{D}|p,q\rangle |\sigma\rangle ,
$$

where $|\sigma\rangle$ is any reference ‘fiducial’ state, and

$$
\tilde{D}|p,q\rangle = e^{\frac{i}{\hbar}(p\hat{q} - q\hat{p})}
$$

is Weyl’s displacement operator. Then, corresponding to any wave function $|\psi\rangle$ one can define a ‘smoothed’ wave function on phase space by projecting it onto the coherent state:

$$
\tilde{\psi}_\sigma(p,q) \equiv \langle \sigma | \tilde{D}^\dagger |p,q\rangle |\psi\rangle .
$$

These functions and their time dependence when $\psi$ is driven by the Hamiltonian $\hat{p}^2/2m + V(q)$ have been studied for some choices of $|\sigma\rangle$ by Torres-Vega et al, Harriman, and others [15, 16, 17].

In this paper I generalize $\tilde{\psi}_\sigma(p,q)$ to a phase space wave function $\tilde{\psi}_\sigma^{(\lambda)}(p,q)$ by relating it to a class of orderings labelled by a parameter $\lambda \in (-1,+1)$, where $\psi_\sigma^{(0)}(p,q) = \tilde{\psi}_\sigma(p,q)$, equation (4). A given value of $\lambda$ specifies an association between functions on phase space and operators, $A(p,q) \leftrightarrow^{(\lambda)} \hat{A}$, where $\lambda = -1$ gives the standard ordering (eg $p^n q^m \leftrightarrow \hat{q}^m \hat{p}^n$), $\lambda = +1$ gives the anti-standard rule (eg $p^n q^m \leftrightarrow \hat{p}^n \hat{q}^m$), and $\lambda = 0$ gives the symmetric or Weyl association, of which (1) is an example with $\rho(p,q) \leftrightarrow \hat{\rho}/\hbar$. The time-dependence of the Fourier transform of $\tilde{\psi}_\sigma^{(\lambda)}(p,q)$, and therefore effectively of $\tilde{\psi}_\sigma^{(\lambda)}(p,q)$ itself, has been studied in [18].

$\tilde{\psi}_\sigma^{(\lambda)}(p,q)$ relates to the $\lambda$-orderings of the operator $|\psi\rangle\langle\sigma|$, which is linear in the states $|\psi\rangle$ (the reference or fiducial state is held fixed), but the density matrix $\hat{\rho} = |\psi\rangle\langle\psi|$ is bilinear, so a chosen ordering for $|\psi\rangle\langle\sigma|$ will not be expected to apply to the density matrix, indeed it may not even be of the $\lambda$-class. The generalized $Q$-function for a pure state $|\lambda\rangle$, defined as $|\tilde{\psi}_\sigma^{(\lambda)}(p,q)|^2/\hbar$, is normalized with respect to the integral $\int dp dq$ over all of phase space. The main results of this paper are that the generalized $Q$-function corresponding to any state $\hat{\rho}$ is, first, non-negative, second, proportional to the expectation of $\hat{\rho}$ with respect to a certain generalized displaced squeezed state which depends upon $\sigma$, $\lambda$ and $(p,q)$ and, third, proportional to the convolution of the Wigner functions for $\rho$ with the Wigner function for that squeezed state. The field of quantum mechanics in phase space is a large one, perhaps starting with the analysis of Weyl [1, 2] and of Wigner [3]. In the context of this paper Bopp [19] in 1956 considered classical-like implications of that $Q$-function corresponding to the Weyl ordering ($\lambda = 0$) and with fiducial state chosen (as is usually the case) to be the vacuum state.
$|0\rangle \equiv |h_0\rangle$, namely $\langle h_0 | \hat{D} [p, q] \hat{\rho}(t) \hat{D} [p, q] | h_0 \rangle$. That this can be related to the modulus squared of a wave function, here $\tilde{\psi}_{h_0}^{(o)}(p, q)$ was pointed out by Mizrahi [20] who also studied some of its properties. On a different tack, Cahill and Glauber [21, 22] have studied at length a family of orderings (the $s$-family) $\hat{A} \xrightarrow{(s)} A(p, q)$, centered around the annihilation and creation operators $\hat{a}$ and $\hat{a}^\dagger$, where (in my notation) $\hat{a} = \frac{1}{\sqrt{2}}(\alpha q + i \frac{p}{\alpha})$—where $\alpha$ is a real parameter—so that $[\hat{a}, \hat{a}^\dagger] = 1$. Defining the complex numbers $A = \frac{1}{\sqrt{2}}(\alpha q + i \frac{p}{\alpha})$, when $s = -1$ their mapping corresponds to the association (antinormal ordering) $\hat{a}^m \hat{a}^n \rightarrow A^m A^n$, when $s = 1$ the association is $\hat{a}^m \hat{a}^n \rightarrow A^m A^n$ (normal ordering), and when $s = 0$ the ordering is that of Weyl. Thus the $\lambda$ and $s$ mappings complement each other, and overlap at $\lambda = 0 = s$. Among their many interesting results Cahill and Glauber define what is effectively a phase space wave function corresponding to $|\psi\rangle |h_0\rangle$ for their $s$-ordering, but they do not relate its modulus squared to any $s$-ordered $Q$-function. They do, however, express the usual $Q$-function, $\langle h_0 | \hat{D} [p, q] \hat{\rho}(t) \hat{D} [p, q] | h_0 \rangle$, as a smoothed Wigner function.

In this note I start with the modulus squared of wave functions on phase space, of which equation (4) is a special case, and show that it can correspond to smeared Wigner functions, where the smearing functions themselves are Wigner functions of generalized displaced squeezed states. Section 2 discusses wave functions on phase space and generalizes them to the $\lambda$-class of orderings. Section 3 develops expressions for the $Q$-functions based on these wave functions. Section 4 discusses some properties of these $Q$-functions.

## 2 Wave functions on phase space

It is often convenient to work with the Fourier transform of $\tilde{\psi}_\sigma(p, q)$, defined by

$$
\psi_\sigma(p, q) = \int_{-\infty}^{\infty} \frac{dp'}{\hbar} \int_{-\infty}^{\infty} dq' \exp \left[ \frac{i}{\hbar} (p'q - q'p) \right] \tilde{\psi}_\sigma(p', q')
$$

$$
= \text{Tr}(|\psi\rangle\langle \sigma | \hat{\Delta}(p, q)) ,
$$

(5)

where [5]

$$
\hat{\Delta}(p, q) = \int_{-\infty}^{\infty} \frac{dp'}{\hbar} \int_{-\infty}^{\infty} dq' \exp \left[ \frac{-i}{\hbar} (p'q - q'p) \right] \hat{D}[p', q']
$$

$$
= \int_{-\infty}^{\infty} dx \exp \left( \frac{i}{\hbar} px \right) |q + \frac{x}{2}\rangle \langle q - \frac{x}{2}| .
$$

(6)

The wave functions $\psi_\sigma(p, q)$ were defined in reference [18] where many of their properties are discussed. In particular, they are the Weyl transform of the operators $|\psi\rangle\langle \sigma |$. Indeed, the Weyl transform, which I shall write $(\hat{A})(p, q)$ or $A(p, q)$, and its associated operator $\hat{A}$ are related [5] by

$$
\hat{A} = \int_{-\infty}^{\infty} \frac{dp dq}{\hbar} A(p, q) \hat{\Delta}(p, q) ,
$$

(7)

which, by virtue of the relation

$$
\text{Tr}((\hat{\Delta}(p, q) \hat{\Delta}(p', q'))) = h\delta(p - p')\delta(q - q') ,
$$

(8)

can be inverted to give

$$
A(p, q) = \text{Tr}(\hat{A} \hat{\Delta}(p, q)) .
$$

(9)
So $\psi_\sigma(p, q)$ is the Weyl transform $|\psi(\sigma)\rangle_{(p,q)}$, and $\tilde{\psi}_\sigma(p, q)$ is its Fourier transform.

Another property of the Weyl transform which we need \cite{5} is

$$\text{Tr}(\hat{A}\hat{B}) = \int_{-\infty}^{\infty} \frac{dp\ dq}{h} A_{(p,q)} B_{(p,q)}.$$  \hspace{1cm} (10)

Note from (6) that $\text{Tr}(\hat{\Delta}(p, q)) = 1$, so, from (9), $\langle 1 \rangle_{(p,q)} = 1$, and (letting $\hat{B} = \hat{1}$ in (10))

$$\text{Tr} \hat{A} = \int_{-\infty}^{\infty} \frac{dp\ dq}{h} A_{(p,q)}.$$  \hspace{1cm} (11)

The essential characteristic of the Weyl correspondence follows from equations (3) and (9) together with the first of (6). It is

$$(e^{i(\hat{\xi}\hat{q} + \hat{\eta}\hat{p})})_{(p,q)} = e^{i(\xi q + \eta p)}.$$  \hspace{1cm} (12)

Other orderings defined by Cohen \cite{8} can be specified by the generalization of (12) to the form

$$(e^{i(\xi\hat{q} + \eta\hat{p})})^{f}_{(p,q)} = \frac{1}{f(\xi, \eta)} e^{i(\xi q + \eta p)} = f^{-1}(-i\partial_q, -i\partial_p) e^{i(\xi q + \eta p)},$$  \hspace{1cm} (13)

where $f^{-1}$ means $1/f$ and the choice $f = 1$ gives the Wigner-Weyl ordering. Note that when $f(0, \eta) = 1 = f(\xi, 0)$ then the Weyl transform of a function of $\hat{q}$ (or $\hat{p}$) only is the same function of $q$ (or $p$) only. If we particularize to the class of orderings defined by the function

$$f(\xi, \eta; \lambda) = e^{i\frac{\lambda}{2} \xi \eta},$$  \hspace{1cm} (14)

where $\lambda$ is a real parameter lying in the interval $[-1, +1]$, then

$$(e^{i(\xi\hat{q} + \eta\hat{p})})^{(\lambda)}_{(p,q)} = e^{-i\frac{\lambda}{2} \xi \eta} e^{i(\xi q + \eta p)}.$$  \hspace{1cm} (15)

Use of the Baker-Campbell-Hausdorff theorem leads to the equivalent expressions

$$(e^{i\xi \hat{q}} e^{i\eta \hat{p}})^{(\lambda)}_{(p,q)} = e^{-i\frac{\lambda}{2} (\lambda + 1) \xi \eta} e^{i(\xi q + \eta p)}$$

and

$$(e^{i\eta \hat{q}} e^{i\xi \hat{p}})^{(\lambda)}_{(p,q)} = e^{-i\frac{\lambda}{2} (\lambda - 1) \xi \eta} e^{i(\xi q + \eta p)}.$$  \hspace{1cm} (16)

The choice $\lambda = -1$ in the first of these gives the ‘standard’ or ‘$p$’ association ($\hat{p}$ first, then $\hat{q}$),

$$(e^{i\xi \hat{q}} e^{i\eta \hat{p}})^{(-1)}_{(p,q)} = e^{i(\xi q + \eta p)}$$

and the choice $\lambda = 1$ in the second gives the ‘anti-standard association ($\hat{q}$ first, then $\hat{p}$),

$$(e^{i\eta \hat{q}} e^{i\xi \hat{p}})^{(1)}_{(p,q)} = e^{i(\xi q + \eta p)},$$

while the Wigner-Weyl ordering, $\lambda = 0$, puts $\hat{p}$ and $\hat{q}$ on equal footing, equation (12).

The generalization of $\psi_\sigma(p, q)$ to the family of orderings defined by equations (14) and (15) is given \cite{18} by

$$\psi^{(\lambda)}_\sigma(p, q) = \text{Tr}(|\psi\rangle\langle\sigma|\hat{\Delta}^{(\lambda)}(p, q)) = \langle\sigma|\hat{\Delta}^{(\lambda)}(p, q)|\psi\rangle,$$  \hspace{1cm} (16)

where

$$\hat{\Delta}^{(\lambda)}(p, q) = e^{i\frac{\lambda}{2} \partial_q \partial_p} \hat{\Delta}(p, q).$$  \hspace{1cm} (17)

Equations (16) and (17) generalize the phase space wave function $\psi_\sigma(p, q)$, the Weyl transform of $|\psi\rangle\langle\sigma|$, to the class of orderings defined by (14).
3 Q-functions

The functions $\psi_\sigma(p, q)$ are normalized—this follows from the second of equations (5) and (10)—and so too are the $\tilde{\psi}_\sigma(p, q)$ by dint of the Fourier relation, equation (5). Further, by taking matrix elements of the quantities in equations (3) and (7) one finds [23] that

$$\hat{\Delta}(p, q) = 2\hat{D}[2p, 2q] \hat{\Pi} \quad \text{or} \quad \hat{D}[p, q] = \frac{1}{2} \hat{\Delta}(p/2, q/2) \hat{\Pi},$$

where $\hat{\Pi}$ is the parity operator, i.e.

$$\hat{\Pi} = \int_{-\infty}^{\infty} dx |x\rangle\langle-x|.$$  \hspace{1cm} (19)

From these equations we can define a generalized displacement operator as

$$\hat{D}^{(\lambda)}[p, q] = \frac{1}{2} \hat{\Delta}^{(\lambda)}(p/2, q/2) \hat{\Pi}$$

with corresponding generalized ‘coherent state’ $\hat{D}^{(\lambda)}[p, q]|\sigma\rangle$ and phase space wave function (partner and equivalent to $\psi^{(\lambda)}(p, q)$) given by

$$\tilde{\psi}^{(\lambda)}(p, q) = \langle \sigma|\hat{D}^{(\lambda)}\dagger[p, q]|\psi\rangle.$$  \hspace{1cm} (21)

Consider the product

$$(\mu^{(\lambda)}_\sigma(p, q))^* \psi^{(\lambda)}_\sigma(p, q) = \int d\tau' \int d\tau'' e^{\frac{i}{2\pi} p'q'} e^{-\frac{i}{2\pi} p''q''} \times$$

$$\times e^{\frac{i}{\hbar}(p'q'' - q'p)} e^{-\frac{i}{\hbar}(p''q - q'p)} \tilde{\psi}_\sigma(p', q') \langle \mu_\sigma(p'', q'') \rangle^*,$$

where I have used (5) (16) and (17) and $\int d\tau'$ stands for $\int_{-\infty}^{\infty} dp'dq'/\hbar$, etc. By equations (4), (9) and (10) we can write

$$\tilde{\psi}_\sigma(p', q') \langle \mu_\sigma(p'', q'') \rangle^* = \int d\tau \langle |\psi\rangle\langle \mu| \rangle_{(p, q)} \times$$

$$\times \langle \hat{D}(p'', q'')|\langle \sigma|\hat{D}\dagger(p', q')\rangle \hat{\Delta}(p, q)\hat{D}(p'', q'')\rangle_{(p, q)},$$  \hspace{1cm} (23)

which is an integral over the product of two Weyl transformed operators. In particular, by definition (9) the second term is

$$(\hat{D}(p'', q'')|\langle \sigma|\hat{D}\dagger(p', q')\rangle \hat{\Delta}(p, q)\hat{D}(p'', q'')|\sigma\rangle = \langle \sigma|\hat{D}\dagger(p', q')\hat{\Delta}(p, q)\hat{D}(p'', q'')|\sigma\rangle.$$  \hspace{1cm} (24)

To simplify this quantity one can express $\hat{\Delta}$ here in terms of $\hat{D}$ (equation (6)) and then simplify the resulting triple product of displacement operators by means of these useful algebraic properties [14]:

$$\hat{D}\dagger[p, q] = \hat{D}[-p, -q],$$

$$\hat{D}\dagger[p, q] (\hat{p}, \hat{q}) \hat{D}[p, q] = (\hat{p} + p, \hat{q} + q),$$

$$\hat{D}[p_2, q_2] \hat{D}[p_1, q_1] = e^{\frac{i}{\hbar}(q_1 p_2 - q_2 p_1)} \hat{D}[p_1 + p_2, q_1 + q_2].$$

Utilizing the action of the unitary operator $\hat{D}$ on $\hat{\Delta}$ itself can also help. For instance, using the second of (25) with the first of equations (6) one finds

$$\hat{D}\dagger[p', q'] \hat{\Delta}(p, q)\hat{D}[p', q'] = \hat{\Delta}(p - p', q - q').$$  \hspace{1cm} (26)
This can be simplified using the displacement operator \( \hat{D} \), namely, from (9),

\[
\text{This also obeys an equation like (28).}
\]

It is easy to see from this result that

\[
\int d\tau (\mu_{\sigma}^{(\lambda)}(p, q))^* \psi_{\sigma}^{(\lambda)}(p, q) = \langle \mu|\psi \rangle,
\]

as it must [18].

From (27) we can find an analogous expression for the pair \((\tilde{\psi}_{\sigma}^{(\lambda)}, \tilde{\mu}_{\sigma}^{(\lambda)})\). By equations (21), (10) and (20) it is

\[
(\tilde{\mu}_{\sigma}^{(\lambda)}(p, q))^* \tilde{\psi}_{\sigma}^{(\lambda)}(p, q)
= \text{Tr}
\left( |\psi\rangle\langle\mu| \hat{D}^{(\lambda)}[p, q]|\sigma\rangle\langle\sigma| \hat{D}^{(\lambda)}[p, q] \right)
= \frac{1}{4} \int d\tau' (|\psi\rangle\langle\mu|)_{(p', q')}
\times
\langle\sigma|\hat{\Pi} \hat{\Delta}^{(\lambda)} \hat{\Delta}^{(\lambda)} |(p/2, q/2) \hat{\Delta}(p', q') \hat{\Delta}^{(\lambda)}(p/2, q/2)|\sigma\rangle
= \frac{1}{4} \int d\tau' (|\psi\rangle\langle\mu|)_{(p', q')}
\times
\langle\sigma|\hat{\Delta}^{(\lambda)}(-p/2, -q/2) \hat{\Delta}(-p', -q') \hat{\Delta}^{(\lambda)}(-p/2, -q/2)|\sigma\rangle,
\]

where I have recognized (using \( \hat{\Pi} \) with the first of equations (6)) that

\[
\hat{\Pi} \hat{\Delta}^{(\lambda)}(p, q) \hat{\Pi} = \hat{\Delta}^{(\lambda)}(-p, -q).
\]

Similarly (use an analysis based on (16))

\[
(\mu_{\sigma}^{(\lambda)}(p, q))^* \psi_{\sigma}^{(\lambda)}(p, q)
= \int d\tau' (|\psi\rangle\langle\mu|)_{(p', q')}
\times
\langle\sigma|\hat{\Delta}^{(\lambda)}(p, q) \hat{\Delta}(p', q') \hat{\Delta}^{(\lambda)} |(p, q)|\sigma\rangle.
\]

Since \( \hat{\Delta}^{(\lambda)}(p, q) = \hat{\Delta}(-\lambda)(p, q) \) it follows from (29) and (30) that multiplying by \( 1/4 \) and making the substitutions \((p, q, p', q', \lambda) \rightarrow (-p/2, -q/2, -p', -q', -\lambda) \) in (27) gives

\[
(\tilde{\mu}_{\sigma}^{(\lambda)}(p, q))^* \tilde{\psi}_{\sigma}^{(\lambda)}(p, q)
= \left( \frac{1}{1 - \lambda^2} \right) \int d\tau' (|\psi\rangle\langle\mu|)_{(p', q')}
\times
\langle\sigma|\hat{\Delta}^{(\lambda)} |\sigma\rangle
\left( \frac{(1 - \lambda)p' - p}{(1 + \lambda)} \right)
\left( \frac{(1 + \lambda)q' - q}{(1 - \lambda)} \right).
\]

This also obeys an equation like (28).

The second term in the integrand here is the Weyl transform of the pure state \( |\sigma\rangle\langle\sigma| \), namely, from (9),

\[
\langle\sigma|\langle\sigma| \left( \frac{(1 - \lambda)p' - p}{(1 + \lambda)} \right)
\left( \frac{(1 + \lambda)q' - q}{(1 - \lambda)} \right) = \langle\sigma|\hat{\Delta} \left( \frac{(1 - \lambda)p' - p}{(1 + \lambda)} \right) \left( \frac{(1 + \lambda)q' - q}{(1 - \lambda)} \right) |\sigma\rangle.
\]

This can be simplified using the displacement operator \( \hat{D} \) and the unitary dilation, or squeeze, operator ([24], [25])

\[
\hat{S}(\xi) = e^{i\xi \hat{p}} (\hat{q} + \hat{p}) \hat{S}(\xi) = e^{i\xi \hat{p}} (\hat{q} + \hat{p}),
\]

(32)
which has the properties
\[
\hat{S}^\dagger(\xi) = \hat{S}(-\xi) \quad \text{and} \quad \hat{S}^\dagger(\xi) (\hat{p}, \hat{q}) \hat{S}(\xi) = (e^{\xi \hat{p}}, e^{-\xi \hat{q}}),
\]  
so that (using this with (3) and (6))
\[
\hat{S}^\dagger(\xi) \hat{\Delta}(p, q) \hat{S}(\xi) = \hat{\Delta}(e^{\xi p}, e^{-\xi q}).
\]  

Then
\[
\langle |\sigma\rangle \langle \sigma| \frac{1}{1-\lambda^2} \mu_{\sigma}(p', q') \rangle = \langle |p, q, \lambda; \sigma| \Delta(p', q')|p, q, \lambda; \sigma\rangle = \langle |p, q, \lambda; \sigma| \langle p, q, \lambda; \sigma| \rangle \langle p', q' \rangle,
\]
where
\[
|p, q, \lambda; \sigma\rangle = \hat{D}\left[\frac{p}{1-\lambda}, \frac{q}{1+\lambda}\right] \hat{S}(\ln \frac{1+\lambda}{1-\lambda}) |\sigma\rangle
\]
is a displaced squeezed state [14, 24, 25] generalized to an arbitrary fiducial state $|\sigma\rangle$. And so
\[
(\mu_{\sigma}(p, q))^\dagger \psi_{\sigma}(p, q) = \frac{1}{1-\lambda^2} \int dp' dq' \langle |\psi\rangle \langle \mu| \rangle \langle p', q'| |p, q, \lambda; \sigma\rangle \langle p, q, \lambda; \sigma\rangle \rangle \langle p', q' \rangle \langle p, q, \lambda; \sigma\rangle \langle p, q, \lambda; \sigma\rangle.
\]  

By a slight rearrangement we can also write
\[
(\mu_{\sigma}(p, q))^\dagger \psi_{\sigma}(p, q) = \frac{1}{1-\lambda^2} \int dp' dq' \langle |\psi\rangle \langle \mu| \rangle \langle p', q'| |p, q, \lambda; \sigma\rangle \langle p, q, \lambda; \sigma\rangle \rangle \langle p', q' \rangle \langle p, q, \lambda; \sigma\rangle \langle p, q, \lambda; \sigma\rangle.
\]
Setting $|\mu\rangle = |\psi\rangle$, generalizing from $|\psi\rangle \langle \psi|$ to the density matrix $\hat{\rho} = \sum w_\psi |\psi\rangle \langle \psi|$, and dividing by $h$ gives the ‘diagonal’ component of this sesquilinear form, the generalized $Q$-function. Non-negative by construction, from (37) and (38) it is
\[
Q_{\sigma}(p, q; \rho) = \frac{1}{h} \sum_\psi w_\psi |\psi_{\sigma}(p, q)|^2
\]
\[
= \frac{1}{h} \left(\frac{1}{1-\lambda^2}\right) \langle p, q, \lambda; \sigma|\hat{\rho}|p, q, \lambda; \sigma\rangle
\]
\[
= \frac{1}{1-\lambda^2} \int dp' dq' \rho(p', q') \rho_{\sigma}(p; p' - p, q' - q),
\]
where
\[
\rho(p, q) = \frac{1}{h} \langle \hat{\rho}|p, q\rangle = \frac{1}{h} \text{Tr}(\hat{\rho} \hat{\Delta}(p, q))
\]
is the Wigner function for the state $\hat{\rho}$, $|p, q, \lambda; \sigma\rangle$ is given by (36), and $\rho_{\sigma}(p, q)$ is the Wigner function corresponding to the $p$ and $q$ dependent squeezed state $|\lambda p, -\lambda q, \lambda; \sigma\rangle$:
\[
\rho_{\sigma}(p; p' - p, q' - q) = \frac{1}{h} \langle |\lambda p, -\lambda q, \lambda; \sigma\rangle \langle \lambda p, -\lambda q, \lambda; \sigma\rangle \rangle \langle p' - p, q' - q \rangle.
\]  
The multiplier $1/h$ is chosen by convention so that $Q_{\sigma}(p'; q'; \rho)$, $\rho(p', q')$ and $\rho_{\sigma}(p; p' - p, q' - q)$ are all normalized with respect to the integral $\int dp' dq'$.  


4 Discussion

When there is no squeezing of the fiducial state then $\lambda \to 0$, $|\lambda p, -\lambda q, \lambda; \sigma\rangle \to |\sigma\rangle$, and $|p, q, \lambda; \sigma\rangle \to |p, q; \sigma\rangle$ (defined in equation (2)). In that case

$$
\tilde{Q}^{(0)}_\sigma(p, q; \rho) \equiv \tilde{Q}_\sigma(p, q; \rho) = \frac{1}{\hbar} \langle p, q; \sigma| \hat{p}|p, q; \sigma\rangle
$$

$$
= \int dp'dq' \rho(p', q') \rho_\sigma(p'-p, q'-q).
$$

(42)

where

$$
\rho(p, q) = \frac{1}{\hbar} \text{Tr}(\hat{\rho} \Delta(p, q))
$$

and

$$
\rho_\sigma(p, q) = \frac{1}{\hbar} \text{Tr}(\langle \sigma| \Delta(p, q) |\sigma\rangle)
$$

are the Wigner functions corresponding to the states $\hat{\rho}$ and $|\sigma\rangle\langle \sigma|$.

When $|\sigma\rangle$ is the vacuum state, $Q_\sigma(p, q; \sigma)$ is the well-known Husimi or Q-function and the content of equations (42) is well known, [9, 11, 20, 22]. Generalized to an arbitrary fiducial state $|\sigma\rangle$ the first of equations (42) says that the Q-function is the expectation of state $\hat{\rho}$ with respect to the coherent state of equation (2) while the second expresses it as the Wigner function for $\hat{\rho}$ smeared with respect to the Wigner function for $|\sigma\rangle\langle \sigma|$. The two Weyl transformed functions which are convoluted in equation (42) each separately corresponds to the scheme of equation (13) with $f(\xi, \eta) = 1$, and $\tilde{Q}_\sigma(p, q; \sigma)$ itself is also a member of that class of correspondences. This follows the convolution theorem. For instance, we can define the Fourier component, $[\hat{A}]_{(p,q)}$, of the Weyl transform $\tilde{A}_{(p,q)}$ of an operator $\hat{A}$, as

$$
[\hat{A}]_{(p,q)} = \int \frac{dp'dq'}{\hbar} e^{i(p'q+q'p)}(\hat{A})_{(p',q')},
$$

and using this in equation (42) gives

$$
\tilde{Q}_\sigma(p, q; \sigma) = f^{-1}(-i\partial_q, -i\partial_p)\rho(p', q')
$$

(43)

where

$$
f^{-1}(\xi, \eta) = ||\sigma\rangle\langle \sigma||_{(\xi, \eta)}.\]

The customary choice for the fiducial state is the vacuum [11, 9]. In particular, for an harmonic oscillator in the ground state $|\sigma\rangle = |0\rangle$, where

$$
\langle x|0\rangle = \frac{\alpha^{1/2}}{\sqrt{\pi}} e^{-\frac{1}{2}a^2x^2}, \text{ and } \alpha^2 = \frac{m\omega}{\hbar},
$$

which gives for the Weyl transform of $|0\rangle\langle 0|$ and its Fourier component

$$
(|0\rangle\langle 0|)_{(p,q)} = 2e^{-\alpha^2q^2}e^{-\frac{a^2x^2}{4}} \text{ and } [0\rangle\langle 0]_{(\xi, \eta)} = e^{-\frac{\xi^2}{4a^2}} e^{-\frac{a^2x^2\eta^2}{4}}.
$$

Thus, even when there is no squeezing (i.e. $\lambda = 0$) what was a Weyl association $f = 1$ (equation (12)) for the phase space wave function $|\psi\rangle\langle h_0|$ becomes an association

$$
f(\xi, \eta) = e^{\frac{\xi^2}{4a^2}} e^{-\frac{a^2x^2\eta^2}{4}}
$$

(44)
for the $Q$-function, equation (43). Although the function $f(\xi, \eta)$ of equation (44) does not have the properties $f(0, \eta) = 1 = f(\xi, 0)$ the distribution $\tilde{Q}_\sigma(p, q; \sigma)$ which it generates is non-negative. It is a positive operator-valued measure (POM) [26]. This association is a special case of the $s$-family of orderings considered by Cahill and Glauber [21, 22], which in the notation of this paper can be written

$$f^{(s)}(\xi, \eta) = e^{s^2 \sigma^2_{\xi\eta}} e^{s^2 \sigma^2_{\xi\xi} + s^2 \sigma^2_{\eta\eta}}.$$

For $\lambda \neq 0$ the functions $\tilde{Q}_\sigma^{(\lambda)}(p, q; \rho)$, equation (39), are also a POMs, but owing to the extra $p$-dependence of the smoothing function they do not have corresponding functions $f(\xi, \eta)$. The form of equation (39) shows that $\tilde{Q}_\sigma^{(\lambda)}$ is proportional to the average of $\hat{\rho}$ with respect to the state $|p, q, \lambda; \sigma\rangle$, equation (36). In other words, $\tilde{Q}_\sigma^{(\lambda)}$ is proportional to the probability of finding the system in the state $|p, q, \lambda; \sigma\rangle$ when it has been prepared in the state $\hat{\rho}$. The state $|p, q, \lambda; \sigma\rangle$ is a minimum uncertainty squeezed state when the fiducial state $|\sigma\rangle$ is the vacuum state. To see this using (25), (33) and (36) it is easy to show that, for any choice of $|\sigma\rangle$,

$$\langle p, q, \lambda; \sigma|\hat{\rho}|p, q, \lambda; \sigma\rangle = \frac{p}{1 - \lambda} + \left(\frac{1 + \lambda}{1 - \lambda}\right)\langle \sigma|\hat{\rho}|\sigma\rangle,$$

$$\langle p, q, \lambda; \sigma|\hat{q}|p, q, \lambda; \sigma\rangle = \frac{q}{1 + \lambda} + \left(\frac{1 - \lambda}{1 + \lambda}\right)\langle \sigma|\hat{q}|\sigma\rangle,$$

and that for momentum and position the standard deviations for this state are

$$\Sigma_{\sigma}^{(\lambda)}(p) = \left|\frac{1 + \lambda}{1 - \lambda}\right| \Sigma_{\sigma}(p) \quad \text{and} \quad \Sigma_{\sigma}^{(\lambda)}(q) = \left|\frac{1 - \lambda}{1 + \lambda}\right| \Sigma_{\sigma}(q),$$

where $\Sigma_{\sigma}(p)$ and $\Sigma_{\sigma}(q)$ are the corresponding standard deviations for state $|\sigma\rangle$. So, whatever the degree of squeezing, $\Sigma_{\sigma}^{(\lambda)}(p)\Sigma_{\sigma}^{(\lambda)}(q) = \Sigma_{\sigma}(p)\Sigma_{\sigma}(q)$, and for the vacuum state this product is the minimum value $\hbar/2$.

We could equally choose to work with $\psi^{(\lambda)}_\sigma$ instead of $\tilde{\psi}^{(\lambda)}_\sigma$. For instance

$$Q^{(\lambda)}_{\sigma}(p, q; \rho) \equiv \frac{1}{h} \sum_{\psi} w_\psi |\psi^{(\lambda)}_\sigma(p, q)|^2,$$  \hspace{1cm} (45)

is directly related to $\tilde{Q}^{(\lambda)}_{\sigma}(p, q; \rho)$, for from equations (16), (17), (20) and (21) we have

$$\tilde{\psi}^{(\lambda)}_\sigma(p, q) = \frac{1}{2} \langle \sigma| \hat{\Pi} \hat{\Delta}^{(\lambda)\dagger}(p/2, q/2)|\psi\rangle = \frac{1}{2} \psi^{(-\lambda)}_{\sigma'}(p/2, q/2),$$  \hspace{1cm} (46)

so that

$$\psi^{(\lambda)}_\sigma(p, q) = 2\psi^{(-\lambda)}_{\sigma'}(2p, 2q)$$  \hspace{1cm} (47)

where $|\sigma'\rangle = \hat{\Pi}|\sigma\rangle$ is a reflected fiducial state.

The time-dependence of $\tilde{Q}^{(\lambda)}_{\sigma}(p, q; \rho)$—or of $Q^{(\lambda)}_{\sigma}(p, q; \rho)$—enters through the time-dependence of $\hat{\rho}$, for instance via its Weyl transform $h \times \rho(p', q')$ in the third of equations (39). The equation of motion of Wigner functions is well known [5] and can be transferred to $\tilde{Q}^{(\lambda)}_{\sigma}(p, q; \rho)$ itself by partial integration in equation (39). Another way would be to find the time-dependence of $\tilde{\psi}^{(\lambda)}_\sigma(p, q)$ itself which enters through the time-dependence of $|\psi\rangle$. In [18] it was chosen to study the time variation of $\psi^{(\lambda)}_\sigma(p, q)$—as it is itself a Weyl transform and, from that standpoint, basic—when driven by a Hamiltonian $\hat{H} = \hat{p}^2/2m + V(\hat{q})$. According to equations (46) and (47), this knowledge transfers to $\tilde{\psi}^{(\lambda)}_\sigma(p, q)$. 

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References

[1] Weyl H 1927 Z. physik 46 1
[25] Loudon R and Knight P L 1987 J. Mod. Optics 34 (617) 709