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Generalized Q-functions

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Abstract

The modulus squared of a class of wave functions defined on phase space is used to define a generalized family of $Q$ or Husimi functions. A parameter $\lambda$ specifies orderings in a mapping from the operator $|\psi\rangle\langle\sigma|$ to the corresponding phase space wave function, where $\sigma$ is a given fiducial vector. The choice $\lambda = 0$ specifies the Weyl mapping and the $Q$-function so obtained is the usual one when $|\sigma\rangle$ is the vacuum state. More generally, any choice of $\lambda$ in the range $(-1, 1)$ corresponds to orderings varying between standard and anti-standard. For all such orderings the generalized $Q$-functions are non-negative by construction. They are shown to be proportional to the expectation of the system state $\hat{\rho}$ with respect to a generalized displaced squeezed state which depends on $\lambda$ and position $(p, q)$ in phase space. Thus, when a system has been prepared in the state $\hat{\rho}$, a generalized $Q$-function is proportional to the probability of finding it in the generalized squeezed state. Any such $Q$-function can also be written as the smoothing of the Wigner function for the system state $\hat{\rho}$ by convolution with the Wigner function for the generalized squeezed state.

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1 Introduction

The Weyl transform [1, 2] associates operators with functions on phase space. In particular, the Wigner function $\rho(p, q)$ [3] is the Weyl transform of the density matrix divided by $\hbar = 2\pi \hbar$. Although $\rho(p, q)$ does have many features of a classical distribution it can take on negative values, with bounds [5] given by $2/h \geq \rho(p, q) \geq -2/h$. Indeed Hudson [4] showed that the only pure states $\psi(x)$ for which the Wigner function is non-negative are Gaussian in $x$. This carries over to any number of dimensions [6], and also, for odd dimensions at least, to the formulation of discrete Wigner functions [7].

The Weyl correspondence between operators and functions on phase space—of which the Wigner function is an example—is a special case of the class of the correspondences given by Cohen [8]. In particular, if any Wigner function is convoluted, or smeared, by integration with respect to the Wigner function of the vacuum state, itself a gaussian function on phase space, then the smoothed function, called the $Q$-function (or Husimi function), is non-negative and corresponds to an ordering in Cohen’s class different from that of Wigner and Weyl [9]. More generally, if any Wigner function is convoluted with respect to a Gaussian function which is itself the Wigner function of a pure coherent state, then the result is non-negative [6, 9, 10, 11, 12, 13].
The Wigner function is bilinear with respect to wave functions. For instance if the Weyl transform of the pure state $|\psi\rangle\langle\psi|$ is written $(|\psi\rangle\langle\psi|)_{(p,q)}$, then the corresponding Wigner function $[3, 5]$ is

$$\rho(p, q) = \frac{1}{\hbar} (|\psi\rangle\langle\psi|)_{(p,q)} = \frac{1}{\hbar} \int_{-\infty}^{\infty} dx \exp \left( \frac{i}{\hbar} px \right) \psi(q - \frac{x}{2}) \psi^{*}(q + \frac{x}{2}),$$  \hspace{1cm} (1)$$

so the smeared Wigner functions are also bilinear with respect to the wave functions.

It is also possible in a sense to smear the states themselves, for instance by projecting them onto a class of generalized displaced coherent states, defined $[14]$ by $$|p, q; \sigma\rangle \equiv \hat{D}|p, q|\sigma\rangle,$$ (2)

where $|\sigma\rangle$ is any reference ‘fiducial’ state, and

$$\hat{D}[p, q] = e^{\frac{i}{\hbar}(p\hat{q} - q\hat{p})}$$ (3)

is Weyl’s displacement operator. Then, corresponding to any wave function $|\psi\rangle$ one can define a ‘smoothed’ wave function on phase space by projecting it onto the coherent state:

$$\tilde{\psi}_{\sigma}(p, q) \equiv \langle\sigma|\hat{D}[p, q]|\psi\rangle. $$ (4)

These functions and their time dependence when $\psi$ is driven by the Hamiltonian $\hat{p}^{2}/2m + V(q)$ have been studied for some choices of $|\sigma\rangle$ by Torres-Vega et al, Harriman, and others $[15, 16, 17]$.

In this paper I generalize $\tilde{\psi}_{\sigma}(p, q)$ to a phase space wave function $\tilde{\psi}_{\sigma}^{(\lambda)}(p, q)$ by relating it to a class of orderings labelled by a parameter $\lambda \in (-1, +1)$, where $\tilde{\psi}_{\sigma}^{(0)}(p, q) = \tilde{\psi}_{\sigma}(p, q)$, equation (4). A given value of $\lambda$ specifies an association between functions and operators, $A(p, q) \leftrightarrow A$, where $\lambda = -1$ gives the standard ordering (eg $p^{n}q^{m} \leftrightarrow \hat{q}^{m}\hat{p}^{n}$), $\lambda = +1$ gives the anti-standard rule (eg $p^{n}q^{m} \leftrightarrow \hat{p}^{n}\hat{q}^{m}$), and $\lambda = 0$ gives the symmetric or Weyl association, of which (1) is an example with $\rho(p, q) \leftrightarrow \hat{\rho}/h$.

The time-dependence of the Fourier transform of $\tilde{\psi}_{\sigma}^{(\lambda)}(p, q)$, and therefore effectively of $\tilde{\psi}_{\sigma}^{(\lambda)}(p, q)$ itself, has been studied in $[18]$.

$\tilde{\psi}_{\sigma}^{(\lambda)}(p, q)$ relates to the $\lambda$-orderings of the operator $|\psi\rangle\langle\sigma|$, which is linear in the states $|\psi\rangle$ (the reference or fiducial state is held fixed), but the density matrix $\hat{\rho} = |\psi\rangle\langle\psi|$ is bilinear, so a chosen ordering for $|\psi\rangle\langle\sigma|$ will not be expected to apply to the density matrix, indeed it may not even be of the $\lambda$-class. The generalized $Q$-function for a pure state $|\lambda\rangle$, defined as $|\tilde{\psi}_{\sigma}^{(\lambda)}(p, q)|^{2}/\hbar$, is normalized with respect to the integral $\int dp dq$ over all of phase space. The main results of this paper are that the generalized $Q$-function corresponding to any state $\hat{\rho}$ is, first, non-negative, second, proportional to the expectation of $\rho$ with respect to a certain generalized displaced squeezed state which depends upon $\sigma$, $\lambda$ and $(p, q)$ and, third, proportional to the convolution of the Wigner functions for $\rho$ with the Wigner function for that squeezed state.

The field of quantum mechanics in phase space is a large one, perhaps starting with the analysis of Weyl $[1, 2]$ and of Wigner $[3]$. In the context of this paper Bopp $[19]$ in 1956 considered classical-like implications of that $Q$-function corresponding to the Weyl ordering ($\lambda = 0$) and with fiducial state chosen (as is usually the case) to be the vacuum state.
\( |0\rangle \equiv |h_0\rangle \), namely \( \langle h_0 | \hat{D}[p,q]^{\dagger} \hat{\rho}(t) \hat{D}[p,q] | h_0 \rangle \). That this can be related to the modulus squared of a wave function, here \( \tilde{\psi}_{h_0}^{(o)}(p,q) \) was pointed out by Mizrahi [20] who also studied some of its properties. On a different tack, Cahill and Glauber [21, 22] have studied at length a family of orderings (the \( s \)-family) \( \hat{A} \overset{(s)}{\longrightarrow} A(p,q) \), centered around the annihilation and creation operators \( \hat{a} \) and \( \hat{a}^{\dagger} \), where (in my notation) \( \hat{a} = \frac{1}{\sqrt{2}} (\alpha q + i \frac{p}{\alpha h}) \)—where \( \alpha \) is a real parameter—so that \( [\hat{a}, \hat{a}^{\dagger}] = 1 \). Defining the complex numbers \( \lambda = \frac{1}{\sqrt{2}} (\alpha q + i \frac{p}{\alpha h}) \), when \( s = -1 \) their mapping corresponds to the association (antinormal ordering) \( \hat{a}^m \hat{a}^{\dagger n} \longmapsto A^m A^n \), when \( s = 1 \) the association is \( \hat{a}^m \hat{a}^{\dagger n} \longmapsto A^{m^*} A^n \) (normal ordering), and when \( s = 0 \) the ordering is that of Weyl. Thus the \( \lambda \) and \( s \) mappings complement each other, and overlap at \( \lambda = 0 = s \). Among their many interesting results Cahill and Glauber define what is effectively a phase space wave function corresponding to \( |\psi\rangle |h_0\rangle \) for their \( s \)-ordering, but they do not relate its modulus squared to any \( s \)-ordered \( Q \)-function. They do, however, express the usual \( Q \)-function, \( \langle h_0 | \hat{D}[p,q]^{\dagger} \hat{\rho}(t) \hat{D}[p,q] | h_0 \rangle \), as a smoothed Wigner function. In this note I start with the modulus squared of wave functions on phase space, of which equation (4) is a special case, and show that it can correspond to smeared Wigner functions, where the smearing functions themselves are Wigner functions of generalized displaced squeezed states. Section 2 discusses wave functions on phase space and generalizes them to the \( \lambda \)-class of orderings. Section 3 develops expressions for the \( Q \)-functions based on these wave functions. Section 4 discusses some properties of these \( Q \)-functions.

## 2 Wave functions on phase space

It is often convenient to work with the Fourier transform of \( \tilde{\psi}_\sigma(p,q) \), defined by

\[
\psi_\sigma(p,q) = \int_{-\infty}^{\infty} \frac{dp'}{h} \frac{dq'}{h} \exp \left[ \frac{i}{h} (p'q - q'p) \right] \tilde{\psi}(p',q') = \text{Tr}(|\psi\rangle \langle \sigma | \hat{\Delta}(p,q)) ,
\]

where [5]

\[
\hat{\Delta}(p,q) = \int_{-\infty}^{\infty} \frac{dp'}{h} \frac{dq'}{h} \exp \left[ -\frac{i}{h} (p'q - q'p) \right] \hat{D}[p',q']
= \int_{-\infty}^{\infty} dx \exp \left( \frac{i}{\hbar} px \right) |q + \frac{x}{2} \rangle \langle q - \frac{x}{2} | .
\]

The wave functions \( \psi_\sigma(p,q) \) were defined in reference [18] where many of their properties are discussed. In particular, they are the Weyl transform of the operators \( |\psi\rangle \langle \sigma | \). Indeed, the Weyl transform, which I shall write \( (\hat{A})_{(p,q)} \) or \( A_{(p,q)} \), and its associated operator \( \hat{A} \) are related [5] by

\[
\hat{A} = \int_{-\infty}^{\infty} \frac{dp}{h} \frac{dq}{h} A(p,q) \hat{\Delta}(p,q) ,
\]

which, by virtue of the relation

\[
\text{Tr}(\hat{\Delta}(p,q)\hat{\Delta}(p',q')) = h\delta(p-p')\delta(q-q') ,
\]

can be inverted to give

\[
A_{(p,q)} = \text{Tr}(\hat{A} \hat{\Delta}(p,q)) .
\]
So $\psi_\sigma(p, q)$ is the Weyl transform $\langle |\psi\rangle (|\sigma\rangle(p, q))$, and $\tilde{\psi}_\sigma(p, q)$ is its Fourier transform.

Another property of the Weyl transform which we need [5] is

$$\text{Tr}(\hat{A} \hat{B}) = \int_{-\infty}^{\infty} \frac{dp dq}{\hbar} A(p, q) B(p, q).$$

(10)

Note from (6) that $\text{Tr}(\hat{A}(p, q)) = 1$ so, from (9), $(1)_{(p, q)} = 1$, and (letting $\hat{B} = \hat{1}$ in (10))

$$\text{Tr} \hat{A} = \int_{-\infty}^{\infty} \frac{dp dq}{\hbar} A(p, q).$$

(11)

The essential characteristic of the Weyl correspondence follows from equations (3) and (9) together with the first of (6). It is

$$\langle e^{i(\xi \hat{q} + \eta \hat{p})}\rangle_{(p, q)} = e^{i(\xi q + \eta p)}.$$  

(12)

Other orderings defined by Cohen [8] can be specified by the generalization of (12) to the form

$$\langle e^{i(\xi \hat{q} + \eta \hat{p})}\rangle_{f(p, q)}^f = \frac{1}{f(\xi, \eta)} e^{i(\xi q + \eta p)} = f^{-1}(-i\partial_q, -i\partial_p) e^{i(\xi q + \eta p)} ,$$

(13)

where $f^{-1}$ means $1/f$ and the choice $f = 1$ gives the Wigner-Weyl ordering. Note that when $f(0, \eta) = 1 = f(\xi, 0)$ then the Weyl transform of a function of $\hat{q}$ (or $\hat{p}$) only is the same function of $q$ (or $p$) only. If we particularize to the class of orderings defined by the function

$$f(\xi, \eta; \lambda) = e^{i\frac{\lambda}{2} \xi \eta},$$

(14)

where $\lambda$ is a real parameter lying in the interval $[-1, +1]$, then

$$\langle e^{i(\xi \hat{q} + \eta \hat{p})}\rangle_{(p, q)}^{(\lambda)} = e^{-\frac{\lambda}{2} \xi \eta} e^{i(\xi q + \eta p)}.$$  

(15)

Use of the Baker-Campbell-Hausdorff theorem leads to the equivalent expressions

$$\langle e^{i\xi \hat{q}} e^{i\eta \hat{p}}\rangle_{(p, q)}^{(\lambda)} = e^{-\frac{\lambda}{2} (\lambda + 1) \xi \eta} e^{i(\xi q + \eta p)}$$

and

$$\langle e^{i\eta \hat{p}} e^{i\xi \hat{q}}\rangle_{(p, q)}^{(\lambda)} = e^{-\frac{\lambda}{2} (\lambda - 1) \xi \eta} e^{i(\xi q + \eta p)}.$$  

The choice $\lambda = -1$ in the first of these gives the ‘standard’ or ‘$p$’ association ($\hat{p}$ first, then $\hat{q}$),

$$\langle e^{i\xi \hat{q}} e^{i\eta \hat{p}}\rangle_{(p, q)}^{(-1)} = e^{i(\xi q + \eta p)}$$

and the choice $\lambda = 1$ in the second gives the ‘anti-standard association ($\hat{q}$ first, then $\hat{p}$),

$$\langle e^{i\eta \hat{p}} e^{i\xi \hat{q}}\rangle_{(p, q)}^{(1)} = e^{i(\xi q + \eta p)} ,$$

while the Wigner-Weyl ordering, $\lambda = 0$, puts $\hat{p}$ and $\hat{q}$ on equal footing, equation (12).

The generalization of $\psi_\sigma(p, q)$ to the family of orderings defined by equations (14) and (15) is given [18] by

$$\psi_\sigma^{(\lambda)}(p, q) = \text{Tr}(\hat{\psi}\hat{\sigma} \hat{\Delta}^{(\lambda)}(p, q)) = \langle \sigma | \hat{\Delta}^{(\lambda)}(p, q) \hat{\psi} \rangle,$$

(16)

where

$$\hat{\Delta}^{(\lambda)}(p, q) = e^{i\frac{\lambda}{2} \partial_q \partial_p} \hat{\Delta}(p, q).$$

(17)

Equations (16) and (17) generalize the phase space wave function $\psi_\sigma(p, q)$, the Weyl transform of $|\psi\rangle \langle \sigma|$, to the class of orderings defined by (14).
3 Q-functions

The functions $\psi_\sigma(p,q)$ are normalized—this follows from the second of equations (5) and (10)—and so too are the $\tilde{\psi}_\sigma(p,q)$ by dint of the Fourier relation, equation (5). Further, by taking matrix elements of the quantities in equations (3) and (7) one finds [23] that

$$\hat{\Delta}(p,q) = 2\hat{D}[2p,2q] \hat{\Pi} \quad \text{or} \quad \hat{D}[p,q] = \frac{1}{2} \hat{\Delta}(p/2,q/2) \hat{\Pi},$$

where $\hat{\Pi}$ is the parity operator, i.e.

$$\hat{\Pi} = \int_{-\infty}^{\infty} dx|x(-x)|.$$

From these equations we can define a generalized displacement operator as

$$\hat{D}^{(\lambda)}[p,q] = \frac{1}{2} \hat{\Delta}^{(\lambda)}(p/2,q/2) \hat{\Pi}$$

with corresponding generalized ‘coherent state’ $\hat{D}^{(\lambda)}[p,q]|\sigma\rangle$ and phase space wave function (partner and equivalent to $\psi^{(\lambda)}(p,q)$) given by

$$\tilde{\psi}^{(\lambda)}(p,q) = \langle \sigma|\hat{D}^{(\lambda)\dagger}[p,q]|\psi\rangle.$$

Consider the product

$$(\mu^{(\lambda)}_\sigma(p,q))^* \psi^{(\lambda)}_\sigma(p,q) = \int d\tau' \int d\tau'' e^{\frac{i}{\hbar} p'q'} e^{\frac{-i}{\hbar} p''q''} \times$$

$$\times e^{\frac{i}{\hbar}(p'q' - q'p')} e^{-\frac{i}{\hbar}(p''q'' - q''p'')} \tilde{\psi}_\sigma(p',q') (\bar{\mu}_\sigma(p'',q''))^*,$$

where I have used (5) (16) and (17) and $\int d\tau'$ stands for $\int_{-\infty}^{\infty} dp'dq'/\hbar$, etc. By equations (4), (9) and (10) we can write

$$\tilde{\psi}_\sigma(p',q') (\bar{\mu}_\sigma(p'',q''))^* = \int d\tau (|\psi\rangle \langle \mu |)_{(p,q)} \times$$

$$\times (\hat{D}(p'',q'')(\sigma) \langle \sigma|\hat{D}^{\dagger}(p',q') \hat{\Delta}(p,q) \hat{D}(p'',q'')|\sigma\rangle)_{(p,q)},$$

which is an integral over the product of two Weyl transformed operators. In particular, by definition (9) the second term is

$$(\hat{D}(p'',q'')(\sigma) \langle \sigma|\hat{D}^{\dagger}(p',q') \hat{\Delta}(p,q) \hat{D}(p'',q'')|\sigma\rangle)_{(p,q)} = \langle \sigma|\hat{D}^{\dagger}(p',q') \hat{\Delta}(p,q) \hat{D}(p'',q'')|\sigma\rangle.$$

To simplify this quantity one can express $\hat{\Delta}$ here in terms of $\hat{D}$ (equation (6)) and then simplify the resulting triple product of displacement operators by means of these useful algebraic properties [14]:

$$\hat{D}^{\dagger}[p,q] = \hat{D}[-p,-q],$$

$$\hat{D}^{\dagger}[p,q] (\hat{p}, \hat{q}) \hat{D}[p,q] = (\hat{p} + p, \hat{q} + q),$$

$$\hat{D}[p_2, q_2] \hat{D}[p_1, q_1] = e^{\frac{i}{\hbar}(q_2p_1 - p_2q_1)} \hat{D}[p_1 + p_2, q_1 + q_2].$$

Utilizing the action of the unitary operator $\hat{D}$ on $\hat{\Delta}$ itself can also help. For instance, using the second of (25) with the first of equations (6) one finds

$$\hat{D}^{\dagger}[p',q'] \hat{\Delta}(p,q) \hat{D}[p',q'] = \hat{\Delta}(p - p', q - q').$$
The upshot is that by direct calculation equations (22) to (26) can be combined and simplified to give

\[
(\mu_\sigma^{(\lambda)}(p, q))^* \psi_\sigma^{(\lambda)}(p, q) = \left( \frac{4}{1 - \lambda^2} \right) \int d\tau'(|\psi\rangle\langle\mu|)(p', q')(|\sigma\rangle\langle\sigma|) \left( \frac{2p - (1 + \lambda)p' - 2q - (1 - \lambda)q'}{(1 - \lambda)} \right) .
\]

(27)

It is easy to see from this result that

\[
\int d\tau(\mu_\sigma^{(\lambda)}(p, q))^* \psi_\sigma^{(\lambda)}(p, q) = \langle \mu | \psi \rangle ,
\]

(28)
as it must [18].

From (27) we can find an analogous expression for the pair \((\tilde{\psi}_\sigma^{(\lambda)}, \tilde{\mu}_\sigma^{(\lambda)})\). By equations (21), (10) and (20) it is

\[
(\tilde{\mu}_\sigma^{(\lambda)}(p, q))^* \tilde{\psi}_\sigma^{(\lambda)}(p, q) = \text{Tr} \left( |\psi\rangle\langle\mu| \hat{D}^{(\lambda)}(p, q)|\sigma\rangle\langle\sigma| \hat{D}^{(\lambda)\dagger}(p, q) \right)
\]

\[
= \frac{1}{4} \int d\tau'(|\psi\rangle\langle\mu|)(p', q') \times
\]

\[
\times \langle \sigma | \hat{\Pi} \hat{\Delta}^{(\lambda)\dagger}(p/2, q/2) \hat{\Delta}(p', q') \hat{\Delta}^{(\lambda)}(p/2, q/2) \hat{\Pi} | \sigma \rangle
\]

\[
= \frac{1}{4} \int d\tau'(|\psi\rangle\langle\mu|)(p', q') \times
\]

\[
\times \langle \sigma | \hat{\Delta}^{(\lambda)\dagger}(-p/2, -q/2) \hat{\Delta}(-p', -q') \hat{\Delta}^{(\lambda)}(-p/2, -q/2) | \sigma \rangle ,
\]

(29)

where I have recognized (using \(\hat{\Pi}\) with the first of equations (6)) that

\[
\hat{\Pi} \hat{\Delta}^{(\lambda)}(p, q) \hat{\Pi} = \hat{\Delta}^{(\lambda)}(-p, -q) .
\]

Similarly (use an analysis based on (16))

\[
(\mu_\sigma^{(\lambda)}(p, q))^* \psi_\sigma^{(\lambda)}(p, q) = \int d\tau'(|\psi\rangle\langle\mu|)(p', q') \times
\]

\[
\times \langle \sigma | \hat{\Delta}^{(\lambda)}(p, q) \hat{\Delta}(p', q') \hat{\Delta}^{(\lambda)\dagger}(p, q) | \sigma \rangle .
\]

(30)

Since \(\hat{\Delta}^{(\lambda)\dagger}(p, q) = \hat{\Delta}^{(-\lambda)}(p, q)\) it follows from (29) and (30) that multiplying by 1/4 and making the substitutions \((p, q, p', q', \lambda) \rightarrow (-p/2, -q/2, -p', -q', -\lambda)\) in (27) gives

\[
(\tilde{\mu}_\sigma^{(\lambda)}(p, q))^* \tilde{\psi}_\sigma^{(\lambda)}(p, q) = \left( \frac{1}{1 - \lambda^2} \right) \int d\tau'(|\psi\rangle\langle\mu|)(p', q') \times \langle \sigma | \hat{\Delta}^{(\lambda)}(p, q) \hat{\Delta}(p', q') \hat{\Delta}^{(\lambda)\dagger}(p, q) | \sigma \rangle .
\]

(31)

This also obeys an equation like (28).

The second term in the integrand here is the Weyl transform of the pure state \(|\sigma\rangle\langle\sigma|\), namely, from (9),

\[
\langle \sigma | \langle \sigma | \left( \frac{(1 - \lambda)p' - p}{(1 + \lambda)} \right) \left( \frac{1 - \lambda)q' - q}{(1 + \lambda)} \right) = \langle \sigma | \hat{\Delta} \left( \frac{(1 - \lambda)p' - p}{(1 + \lambda)}, \frac{(1 + \lambda)q' - q}{(1 - \lambda)} \right) | \sigma \rangle .
\]

This can be simplified using the displacement operator \(\hat{D}\) and the unitary dilation, or squeeze, operator ([24] [25])

\[
\hat{S}(\xi) = e^{i\frac{\xi}{2}(\hat{p}\hat{q} + \hat{q}\hat{p})},
\]

(32)
which has the properties
\[ \hat{S}^\dagger(\xi) = \hat{S}(-\xi) \quad \text{and} \quad \hat{S}^\dagger(\xi)(\hat{p}, \hat{q}) \hat{S}(\xi) = (e^{\xi \hat{p}}, e^{-\xi \hat{q}}), \]
so that (using this with (3) and (6))
\[ \hat{S}^\dagger(\xi) \hat{\Delta}(p, q) \hat{S}(\xi) = \hat{\Delta}(e^{-\xi}p, e^{\xi}q). \]
Then
\[
\begin{align*}
\langle \sigma | \langle \sigma | \left( \frac{(1-\lambda)\mu_{\sigma}^\prime - p}{(1+\lambda)\mu_{\sigma}^\prime - q} \right) & = \langle p, q, \lambda ; \sigma | \Delta(p', q') | p, q, \lambda ; \sigma \rangle \\
& = \langle p, q, \lambda ; \sigma | \langle p, q, \lambda ; \sigma |(p', q') \rangle,
\end{align*}
\]
where
\[ | p, q, \lambda ; \sigma \rangle = \hat{D}[\frac{p}{1-\lambda}, \frac{q}{1+\lambda}] \hat{S}(\ln \frac{1+\lambda}{1-\lambda}) | \sigma \rangle \]
is a displaced squeezed state [14, 24, 25] generalized to an arbitrary fiducial state $| \sigma \rangle$. And so
\[
\begin{align*}
(\hat{\mu}_{\sigma}^{(\lambda)}(p, q))^* \tilde{\psi}_{\sigma}^{(\lambda)}(p, q) & = \left( \frac{1}{1-\lambda^2} \right) \int d\bar{\tau}'(|\psi\rangle \langle \mu|)(p', q') \langle p, q, \lambda ; \sigma | \langle p, q, \lambda ; \sigma |(p', q') \\
& = \left( \frac{1}{1-\lambda^2} \right) \langle p, q, \lambda ; \sigma | \psi \rangle \langle \mu | p, q, \lambda ; \sigma \rangle.
\end{align*}
\]
By a slight rearrangement we can also write
\[
\begin{align*}
(\hat{\mu}_{\sigma}^{(\lambda)}(p, q))^* \tilde{\psi}_{\sigma}^{(\lambda)}(p, q) & = \\
& = \left( \frac{1}{1-\lambda^2} \right) \int d\tau'(|\psi\rangle \langle \mu|)(\lambda p, -\lambda q, \lambda ; \sigma) \langle \lambda p, -\lambda q, \lambda ; \sigma |(p', q') \rho_{\sigma}^{(\lambda)}(p; p'-p, q'-q),
\end{align*}
\]
Setting $| \mu \rangle = | \psi \rangle$, generalizing from $| \psi \rangle \langle \psi |$ to the density matrix $\hat{\rho} = \sum w_{\psi} | \psi \rangle \langle \psi |$, and dividing by $h$ gives the ‘diagonal’ component of this sesquilinear form, the generalized $Q$-function. Non-negative by construction, from (37) and (38) it is
\[
\begin{align*}
\tilde{Q}_{\sigma}^{(\lambda)}(p, q; \rho) & = \frac{1}{h} \sum_{\psi} w_{\psi} | \tilde{\psi}_{\sigma}^{(\lambda)}(p, q) |^2 \\
& = \frac{1}{h} \left( \frac{1}{1-\lambda^2} \right) \langle p, q, \lambda ; \sigma | \hat{\rho} | p, q, \lambda ; \sigma \rangle \\
& = \left( \frac{1}{1-\lambda^2} \right) \int dp' dq' \rho(p', q') \rho_{\sigma}^{(\lambda)}(p; p'-p, q'-q),
\end{align*}
\]
where
\[
\rho(p, q) = \frac{1}{h} \langle \hat{\rho} | p, q \rangle = \frac{1}{h} \text{Tr}(\hat{\rho} \hat{\Delta}(p, q))
\]
is the Wigner function for the state $\hat{\rho}$, $| p, q, \lambda ; \sigma \rangle$ is given by (36), and $\rho_{\sigma}^{(\lambda)}(p, q)$ is the Wigner function corresponding to the $p$ and $q$ dependent squeezed state $| \lambda p, -\lambda q, \lambda ; \sigma \rangle$:
\[
\rho_{\sigma}^{(\lambda)}(p; p'-p, q'-q) = \frac{1}{h} (| \lambda p, -\lambda q, \lambda ; \sigma \rangle \langle \lambda p, -\lambda q, \lambda ; \sigma |(p'-p, q'-q).\]
The multiplier $1/h$ is chosen by convention so that $\tilde{Q}_{\sigma}^{(\lambda)}(p', q'; \rho)$, $\rho(p', q')$ and $\rho_{\sigma}^{(\lambda)}(p; p'-p, q'-q)$ are all normalized with respect to the integral $\int dp' dq'$. 

7
4 Discussion

When there is no squeezing of the fiducial state then $\lambda \to 0$, $|\lambda p, -\lambda q, \lambda; \sigma \rangle \to |\sigma \rangle$, and $|p, q, \lambda; \sigma \rangle \to |p, q; \sigma \rangle$ (defined in equation (2)). In that case

$$Q^{(0)}_{\sigma} (p, q; \rho) \equiv \tilde{Q}_{\sigma} (p, q; \rho) = \frac{1}{\hbar} \langle p, q; \sigma | \hat{\rho} | p, q; \sigma \rangle$$

$$= \int dp' dq' \rho (p', q') \rho_{\sigma} (p' - p, q' - q).$$

(42)

where

$$\rho (p, q) = \frac{1}{\hbar} \text{Tr} (\hat{\rho} \Delta (p, q))$$

and

$$\rho_{\sigma} (p, q) = \frac{1}{\hbar} \text{Tr} (|\sigma \rangle \langle \sigma | \Delta (p, q))$$

are the Wigner functions corresponding to the states $\hat{\rho}$ and $|\sigma \rangle \langle \sigma |$.

When $|\sigma \rangle$ is the vacuum state, $Q_{\sigma} (p, q; \sigma)$ is the well-known Husimi or $Q$-function and the content of equations (42) is well known, [9, 11, 20, 22]. Generalized to an arbitrary fiducial state $|\sigma \rangle$ the first of equations (42) says that the $Q$-function is the expectation of state $\hat{\rho}$ with respect to the coherent state of equation (2) while the second expresses it as the Wigner function for $\hat{\rho}$ smeared with respect to the Wigner function for $|\sigma \rangle \langle \sigma |$. The two Weyl transformed functions which are convoluted in equation (42) each separately corresponds to the scheme of equation (13) with $f (\xi, \eta) = 1$, and $Q_{\sigma} (p, q; \sigma)$ itself is also a member of that class of correspondences. This follows the convolution theorem. For instance, we can define the Fourier component, $[\hat{A}]_{(p, q)}$, of the Weyl transform $(\hat{A})_{(p, q)}$ of an operator $\hat{A}$, as

$$[\hat{A}]_{(p, q)} = \int \frac{dp' dq'}{\hbar} e^\frac{i}{\hbar} (p'q + q'p) (\hat{A})_{(q', q')} ,$$

and using this in equation (42) gives

$$Q_{\sigma} (p, q; \sigma) = f^{-1} (-i\partial_q, -i\partial_p) \rho (p', q')$$

(43)

where

$$f^{-1} (\xi, \eta) = [|\sigma \rangle \langle \sigma |]_{(\xi, \eta)} .$$

The customary choice for the fiducial state is the vacuum [11, 9]. In particular, for an harmonic oscillator in the ground state $|\sigma \rangle = |0 \rangle$, where

$$\langle x | 0 \rangle = \frac{\alpha^{1/2}}{\pi^{1/4}} e^{-\frac{1}{2} \alpha^2 x^2} , \quad \text{and} \quad \alpha^2 = \frac{m \omega}{\hbar} ,$$

which gives for the Weyl transform of $|0 \rangle \langle 0 |$ and its Fourier component

$$|0 \rangle \langle 0 |_{(p, q)} = 2 e^{-\alpha^2 q^2} e^{-\frac{q^2}{4 \alpha^2}} , \quad \text{and} \quad |0 \rangle \langle 0 |_{(\xi, \eta)} = e^{-\frac{\xi^2}{4 \alpha^2}} e^{-\frac{\alpha^2 \xi^2 \eta^2}{4}} .$$

Thus, even when there is no squeezing (i.e. $\lambda = 0$) what was a Weyl association $f = 1$ (equation (12)) for the phase space wave function $|\psi \rangle \langle h_0 |$ becomes an association

$$f (\xi, \eta) = e^{\frac{\xi^2}{4 \alpha^2}} e^{-\frac{\alpha^2 \xi^2 \eta^2}{4}}$$

(44)
for the $Q$-function, equation (43). Although the function $f(\xi, \eta)$ of equation (44) does not have the properties $f(0, \eta) = 1 = f(\xi, 0)$ the distribution $\tilde{Q}_\sigma(p, q; \sigma)$ which it generates is non-negative. It is a positive operator-valued measure (POM) [26]. This association is a special case of the $s$-family of orderings considered by Cahill and Glauber [21, 22], which in the notation of this paper can be written

$$f^{(s)}(\xi, \eta) = e^{s^2/4\sigma^2} e^{s^2\sigma^2/2}. $$

For $\lambda \neq 0$ the functions $\tilde{Q}_\sigma^{(\lambda)}(p, q; \rho)$, equation (39), are also a POMs, but owing to the extra $p$-dependence of the smoothing function they do not have corresponding functions $f(\xi, \eta)$. The form of equation (39) shows that $\tilde{Q}_\sigma^{(\lambda)}$ is proportional to the average of $\hat{\rho}$ with respect to the state $|p, q, \lambda; \sigma\rangle$, equation (36). In other words, $\tilde{Q}_\sigma^{(\lambda)}$ is proportional to the probability of finding the system in the state $|p, q, \lambda; \sigma\rangle$ when it has been prepared in the state $\hat{\rho}$. The state $|p, q, \lambda; \sigma\rangle$ is a minimum uncertainty squeezed state when the fiducial state $|\sigma\rangle$ is the vacuum state. To see this using (25), (33) and (36) it is easy to show that, for any choice of $|\sigma\rangle$,

$$\langle p, q, \lambda; \sigma|\hat{\rho}|p, q, \lambda; \sigma\rangle = \frac{p}{1 - \lambda} + \frac{(1 + \lambda)}{1 - \lambda} \langle \sigma|\hat{\rho}|\sigma\rangle,$$

$$\langle p, q, \lambda; \sigma|\hat{q}|p, q, \lambda; \sigma\rangle = \frac{q}{1 + \lambda} + \frac{(1 - \lambda)}{1 + \lambda} \langle \sigma|\hat{q}|\sigma\rangle,$$

and that for momentum and position the standard deviations for this state are

$$\Sigma^{(\lambda)}(p) = \left| \frac{1 + \lambda}{1 - \lambda} \right| \Sigma(p) \quad \text{and} \quad \Sigma^{(\lambda)}(q) = \left| \frac{1 - \lambda}{1 + \lambda} \right| \Sigma(q),$$

where $\Sigma(p)$ and $\Sigma(q)$ are the corresponding standard deviations for state $|\sigma\rangle$. So, whatever the degree of squeezing, $\Sigma^{(\lambda)}(p)\Sigma^{(\lambda)}(q) = \Sigma(p)\Sigma(q)$, and for the vacuum state this product is the minimum value $\hbar/2$.

We could equally choose to work with $\psi^{(\lambda)}_\sigma$ instead of $\tilde{\psi}^{(\lambda)}_\sigma$. For instance

$$Q^{(\lambda)}_\sigma(p, q; \rho) \equiv \frac{1}{\hbar} \sum_\psi w_\psi |\psi^{(\lambda)}_\sigma(p, q)|^2,$$

is directly related to $\tilde{Q}^{(\lambda)}_\sigma(p, q; \rho)$, for from equations (16), (17), (20) and (21) we have

$$\tilde{\psi}^{(\lambda)}_\sigma(p, q) = \frac{1}{2} \langle \sigma|\hat{\Pi}\hat{\Delta}^{(\lambda)\dagger}(p/2, q/2)|\psi\rangle = \frac{1}{2} \psi^{(-\lambda)}_\sigma(p/2, q/2),$$

so that

$$\psi^{(\lambda)}_\sigma(p, q) = 2\psi^{(-\lambda)}_\sigma(2p, 2q) \quad \text{(47)}$$

where $|\sigma_r\rangle = \hat{\Pi}|\sigma\rangle$ is a reflected fiducial state.

The time-dependence of $\tilde{Q}^{(\lambda)}_\sigma(p, q; \rho)$—or of $Q^{(\lambda)}_\sigma(p, q; \rho)$—enters through the time-dependence of $\hat{\rho}$, for instance via its Weyl transform $\hbar\times\rho(p', q')$ in the third of equations (39). The equation of motion of Wigner functions is well known [5] and can be transferred to $\tilde{Q}^{(\lambda)}_\sigma(p, q; \rho)$ itself by partial integration in equation (39). Another way would be to find the time-dependence of $\tilde{\psi}^{(\lambda)}_\sigma(p, q)$ itself which enters through the time-dependence of $|\psi\rangle$. In [18] it was chosen to study the time variation of $\psi^{(\lambda)}_\sigma(p, q)$—as it is itself a Weyl transform and, from that standpoint, basic—when driven by a Hamiltonian $\hat{H} = \hat{p}^2/2m + V(\hat{q})$. According to equations (46) and (47), this knowledge transfers to $\tilde{\psi}^{(\lambda)}_\sigma(p, q)$. 


References


