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Abstract

The modulus squared of a class of wave functions defined on phase space is used to define a generalized family of $Q$ or Husimi functions. A parameter $\lambda$ specifies orderings in a mapping from the operator $|\psi\rangle\langle\sigma|$ to the corresponding phase space wave function, where $\sigma$ is a given fiducial vector. The choice $\lambda = 0$ specifies the Weyl mapping and the $Q$-function so obtained is the usual one when $|\sigma\rangle$ is the vacuum state. More generally, any choice of $\lambda$ in the range $(-1, 1)$ corresponds to orderings varying between standard and anti-standard. For all such orderings the generalized $Q$-functions are non-negative by construction. They are shown to be proportional to the expectation of the system state $\hat{\rho}$ with respect to a generalized displaced squeezed state which depends on $\lambda$ and position $(p, q)$ in phase space. Thus, when a system has been prepared in the state $\hat{\rho}$, a generalized $Q$-function is proportional to the probability of finding it in the generalized squeezed state. Any such $Q$-function can also be written as the smoothing of the Wigner function for the system state $\hat{\rho}$ by convolution with the Wigner function for the generalized squeezed state.

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1 Introduction

The Weyl transform [1, 2] associates operators with functions on phase space. In particular, the Wigner function $\rho(p, q)$ [3] is the Weyl transform of the density matrix divided by $h = 2\pi \hbar$. Although $\rho(p, q)$ does have many features of a classical distribution it can take on negative values, with bounds [5] given by $2/h \geq \rho(p, q) \geq -2/h$. Indeed Hudson [4] showed that the only pure states $\psi(x)$ for which the Wigner function is non-negative are Gaussian in $x$. This carries over to any number of dimensions [6], and also, for odd dimensions at least, to the formulation of discrete Wigner functions [7].

The Weyl correspondence between operators and functions on phase space—of which the Wigner function is an example—is a special case of the class of the correspondences given by Cohen [8]. In particular, if any Wigner function is convoluted, or smeared, by integration with respect to the Wigner function of the vacuum state, itself a gaussian function on phase space, then the smoothed function, called the $Q$-function (or Husimi function), is non-negative and corresponds to an ordering in Cohen’s class different from that of Wigner and Weyl [9]. More generally, if any Wigner function is convoluted with respect to a Gaussian function which is itself the Wigner function of a pure coherent state, then the result is non-negative [6, 9, 10, 11, 12, 13].
The Wigner function is bilinear with respect to wave functions. For instance if the Weyl transform of the pure state \( |\psi\rangle \langle \psi| \) is written \(( |\psi\rangle \langle \psi| \rangle_{p,q})\), then the corresponding Wigner function \([3, 5]\) is
\[
\rho(p, q) = \frac{1}{\hbar} (|\psi\rangle \langle \psi| \rangle_{p,q}) = \frac{1}{\hbar} \int_{-\infty}^{\infty} dx \exp \left( \frac{i}{\hbar} px \right) \psi(q-x/2) \psi^\ast(q+x/2), \tag{1}
\]
so the smeared Wigner functions are also bilinear with respect to the wave functions.

It is also possible in a sense to smear the states themselves, for instance by projecting them onto a class of generalized displaced coherent states, defined \([14]\) by
\[
|p, q; \sigma\rangle \equiv \hat{D}[p, q]|\sigma\rangle,
\]
where \(|\sigma\rangle\) is any reference ‘fiducial’ state, and
\[
\hat{D}[p, q] = e^{\frac{i}{\hbar}(p\hat{q} - q\hat{p})} \tag{3}
\]
is Weyl’s displacement operator. Then, corresponding to any wave function \( |\psi\rangle\) one can define a ‘smoothed’ wave function on phase space by projecting it onto the coherent state:
\[
\tilde{\psi}_\sigma(p, q) \equiv \langle \sigma| \hat{D}^\dagger[p, q]|\psi\rangle. \tag{4}
\]
These functions and their time dependence when \( \psi \) is driven by the Hamiltonian \( \hat{p}^2/2m + V(q) \) have been studied for some choices of \(|\sigma\rangle\) by Torres-Vega et al., Harriman, and others \([15, 16, 17]\).

In this paper I generalize \( \tilde{\psi}_\sigma(p, q) \) to a phase space wave function \( \tilde{\psi}_\sigma^{(\lambda)}(p, q) \) by relating it to a class of orderings labelled by a parameter \( \lambda \in (-1, +1) \), where \( \tilde{\psi}_\sigma^{(0)}(p, q) = \tilde{\psi}_\sigma(p, q) \), equation (4). A given value of \( \lambda \) specifies an association between functions on phase space and operators, \( A(p, q) \leftrightarrow A \), where \( \lambda = -1 \) gives the standard ordering (eg \( p^nq^m \leftrightarrow \hat{q}^m\hat{p}^n \)), \( \lambda = +1 \) gives the anti-standard rule (eg \( p^nq^m \leftrightarrow \hat{p}^n\hat{q}^m \)), and \( \lambda = 0 \) gives the symmetric or Weyl association, of which (1) is an example with \( \rho(p, q) \leftrightarrow \hat{p}/\hbar \). The time-dependence of the Fourier transform of \( \tilde{\psi}_\sigma^{(\lambda)}(p, q) \), and therefore effectively of \( \tilde{\psi}_\sigma^{(\lambda)}(p, q) \) itself, has been studied in \([18]\).

\( \tilde{\psi}_\sigma^{(\lambda)}(p, q) \) relates to the \( \lambda \)-orderings of the operator \(|\psi\rangle \langle \sigma|\), which is linear in the states \(|\psi\rangle\) (the reference or fiducial state is held fixed), but the density matrix \( \hat{\rho} = |\psi\rangle \langle \psi| \) is bilinear, so a chosen ordering for \(|\psi\rangle \langle \sigma|\) will not be expected to apply to the density matrix, indeed it may not even be of the \( \lambda \)-class. The generalized Q-function for a pure state \(|\lambda\rangle\), defined as \( |\tilde{\psi}_\sigma^{(\lambda)}(p, q)|^2/\hbar \), is normalized with respect to the integral \( \int dp dq \) over all of phase space. The main results of this paper are that the generalized Q-function corresponding to any state \( \hat{\rho} \) is, first, non-negative, second, proportional to the expectation of \( \rho \) with respect to a certain generalized displaced squeezed state which depends upon \( \sigma \), \( \lambda \) and \( (p, q) \) and, third, proportional to the convolution of the Wigner functions for \( \rho \) with the Wigner function for that squeezed state.

The field of quantum mechanics in phase space is a large one, perhaps starting with the analysis of Weyl \([1, 2]\) and of Wigner \([3]\). In the context of this paper Bopp \([19]\) in 1956 considered classical-like implications of that Q-function corresponding to the Weyl ordering \( (\lambda = 0) \) and with fiducial state chosen (as is usually the case) to be the vacuum state
|0⟩ ≡ |h₀⟩, namely ⟨h₀|\hat{D}[p,q]|\hat{\rho}(t)\hat{D}[p,q]|h₀⟩. That this can be related to the modulus squared of a wave function, here \(\tilde{\psi}^{(0)}_{h₀}(p,q)\) was pointed out by Mizrahi [20] who also studied some of its properties. On a different tack, Cahill and Glauber [21, 22] have studied at length a family of orderings (the s-family) \(\hat{A} \overset{\mathrm{(s)}}{\longrightarrow} A(p,q)\), centered around the annihilation and creation operators \(\hat{a}\) and \(\hat{a}^\dagger\), where (in my notation) \(\hat{a} = \frac{1}{\sqrt{2}}(\alpha q + i \frac{p}{\hbar})\)—where \(\alpha\) is a real parameter—so that \([\hat{a}, \hat{a}^\dagger] = 1\). Defining the complex numbers \(A = \frac{1}{\sqrt{2}}(\alpha q + i \frac{p}{\hbar})\), when \(s = -1\) their mapping corresponds to the association (antinormal ordering) \(\hat{a}^m \hat{a}^\dagger n \longmapsto A^m A^n\), when \(s = 1\) the association is \(\hat{a}^m \hat{a}^\dagger n \longmapsto A^m A^n\) (normal ordering), and when \(s = 0\) the ordering is that of Weyl. Thus the \(\lambda\) and \(s\) mappings complement each other, and overlap at \(\lambda = 0 = s\). Among their many interesting results Cahill and Glauber define what is effectively a phase space wave function corresponding to \(|\psi⟩⟨h₀|\) for their \(s\)-ordering, but they do not relate its modulus squared to any \(s\)-ordered \(Q\)-function. They do, however, express the usual \(Q\)-function, \(⟨h₀|\hat{D}[p, q]|\hat{\rho}(t)\hat{D}[p, q]|h₀⟩\), as a smoothed Wigner function.

In this note I start with the modulus squared of wave functions on phase space, of which equation (4) is a special case, and show that it can correspond to smeared Wigner functions, where the smearing functions themselves are Wigner functions of generalized displaced squeezed states. Section 2 discusses wave functions on phase space and generalizes them to the \(\lambda\)-class of orderings. Section 3 develops expressions for the \(Q\)-functions based on these wave functions. Section 4 discusses some properties of these \(Q\)-functions.

2 Wave functions on phase space

It is often convenient to work with the Fourier transform of \(\tilde{\psi}_\sigma(p,q)\), defined by

\[
\tilde{\psi}_\sigma(p,q) = \int_{-\infty}^{\infty} \frac{dp'}{h} \frac{dq'}{h} \exp \left[ i \frac{p'q - q'p}{\hbar} \right] \tilde{\psi}(p',q')
\]  

\[
= \text{Tr}(|\psi⟩⟨\sigma| \hat{\Delta}(p,q)) , 
\]  

(5)

where [5]

\[
\hat{\Delta}(p,q) = \int_{-\infty}^{\infty} \frac{dp'}{h} \frac{dq'}{h} \exp \left[ - i \frac{p'q - q'p}{\hbar} \right] \hat{D}[p', q']
\]  

\[
= \int_{-\infty}^{\infty} dx \exp \left( i \frac{px}{\hbar} \right) |q + \frac{x}{2}⟩⟨q - \frac{x}{2}| .
\]  

(6)

The wave functions \(\psi_\sigma(p,q)\) were defined in reference [18] where many of their properties are discussed. In particular, they are the Weyl transform of the operators \(|\psi⟩⟨\sigma|\). Indeed, the Weyl transform, which I shall write \((\hat{A})_{(p,q)}\) or \(A_{(p,q)}\), and its associated operator \(\hat{A}\) are related [5] by

\[
\hat{A} = \int_{-\infty}^{\infty} \frac{dp}{h} A(p,q) \hat{\Delta}(p,q) , 
\]  

(7)

which, by virtue of the relation

\[
\text{Tr}(\hat{\Delta}(p,q)\hat{\Delta}(p', q')) = h\delta(p - p')\delta(q - q') , 
\]  

(8)

can be inverted to give

\[
A_{(p,q)} = \text{Tr}(\hat{A} \hat{\Delta}(p,q)) .
\]  

(9)
So $\psi_\sigma(p, q)$ is the Weyl transform $(|\psi\rangle\langle\sigma|)(p, q)$, and $\tilde{\psi}_\sigma(p, q)$ is its Fourier transform.

Another property of the Weyl transform which we need [5] is

$$\text{Tr}(\hat{A} \hat{B}) = \int_{-\infty}^{\infty} \frac{dp\,dq}{\hbar} A(p, q)B(p, q).$$  \hspace{1cm} (10)

Note from (6) that $\text{Tr}(\hat{\Delta}(p, q)) = 1$ so, from (9), $(\hat{1})(p, q) = 1$, and (letting $\hat{B} = \hat{1}$ in (10))

$$\text{Tr} \hat{A} = \int_{-\infty}^{\infty} \frac{dp\,dq}{\hbar} A(p, q).$$  \hspace{1cm} (11)

The essential characteristic of the Weyl correspondence follows from equations (3) and (9) together with the first of (6). It is

$$(e^{i(\xi \hat{q} + \eta \hat{p})})_{(p, q)} = e^{i(\xi q + \eta p)}. \hspace{1cm} (12)$$

Other orderings defined by Cohen [8] can be specified by the generalization of (12) to the form

$$(e^{i(\xi \hat{q} + \eta \hat{p})})^f_{(p, q)} = \frac{1}{f(\xi, \eta)} e^{i(\xi q + \eta p)} = f^{-1}(-i\partial_q, -i\partial_p) e^{i(\xi q + \eta p)},$$  \hspace{1cm} (13)

where $f^{-1}$ means $1/f$ and the choice $f = 1$ gives the Wigner-Weyl ordering. Note that when $f(0, \eta) = 1 = f(\xi, 0)$ then the Weyl transform of a function of $\hat{q}$ (or $\hat{p}$) only is the same function of $q$ (or $p$) only. If we particularize to the class of orderings defined by the function

$$f(\xi, \eta; \lambda) = e^{\frac{i\lambda}{2} \xi \eta}, \hspace{1cm} (14)$$

where $\lambda$ is a real parameter lying in the interval $[-1, +1]$, then

$$(e^{i(\xi \hat{q} + \eta \hat{p})})_{(p, q)}^{(\lambda)} = e^{-\frac{i\lambda}{2} \xi \eta} e^{i(\xi q + \eta p)}. \hspace{1cm} (15)$$

Use of the Baker-Campbell-Hausdorff theorem leads to the equivalent expressions

$$(e^{i\xi \hat{q}} e^{i\eta \hat{p}})_{(p, q)}^{(\lambda)} = e^{-\frac{i\lambda}{2}(\lambda+1)\xi \eta} e^{i(\xi q + \eta p)},$$

and

$$(e^{i\eta \hat{p}} e^{i\xi \hat{q}})_{(p, q)}^{(\lambda)} = e^{-\frac{i\lambda}{2}(\lambda-1)\xi \eta} e^{i(\xi q + \eta p)}.$$  \hspace{1cm} (13)

The choice $\lambda = -1$ in the first of these gives the ‘standard’ or ‘$p$’ association ($\hat{p}$ first, then $\hat{q}$),

$$(e^{i\xi \hat{q}} e^{i\eta \hat{p}})_{(p, q)}^{(-1)} = e^{i(\xi q + \eta p)}$$

and the choice $\lambda = 1$ in the second gives the ‘anti-standard association’ ($\hat{q}$ first, then $\hat{p}$),

$$(e^{i\eta \hat{p}} e^{i\xi \hat{q}})_{(p, q)}^{(+1)} = e^{i(\xi q + \eta p)},$$

while the Wigner-Weyl ordering, $\lambda = 0$, puts $\hat{p}$ and $\hat{q}$ on equal footing, equation (12).

The generalization of $\psi_\sigma(p, q)$ to the family of orderings defined by equations (14) and (15) is given [18] by

$$\psi_\sigma^{(\lambda)}(p, q) = \text{Tr}(|\psi\rangle\langle\sigma| \hat{\Delta}^{(\lambda)}(p, q)) = \langle\sigma| \hat{\Delta}^{(\lambda)}(p, q)|\psi\rangle,$$  \hspace{1cm} (16)

where

$$\hat{\Delta}^{(\lambda)}(p, q) = e^{i\frac{\lambda}{2} \partial_q \partial_q} \hat{\Delta}(p, q).$$  \hspace{1cm} (17)

Equations (16) and (17) generalize the phase space wave function $\psi_\sigma(p, q)$, the Weyl transform of $|\psi\rangle\langle\sigma|$, to the class of orderings defined by (14).
3 Q-functions

The functions $\psi_\sigma(p,q)$ are normalized—this follows from the second of equations (5) and (10)—and so too are the $\tilde{\psi}_\sigma(p,q)$ by dint of the Fourier relation, equation (5). Further, by taking matrix elements of the quantities in equations (3) and (7) one finds [23] that

$$\hat{\Delta}(p,q) = 2\hat{D}[2p,2q] \hat{\Pi} \quad \text{or} \quad \hat{D}[p,q] = \frac{1}{2} \hat{\Delta}(p/2,q/2) \hat{\Pi},$$

where $\hat{\Pi}$ is the parity operator, i.e.

$$\hat{\Pi} = \int_{-\infty}^{\infty} dx |x\rangle \langle -x|.$$  \hfill (19)

From these equations we can define a generalized displacement operator as

$$\hat{D}^{(\lambda)}[p,q] = \frac{1}{2} \hat{\Delta}^{(\lambda)}(p/2,q/2) \hat{\Pi}$$

with corresponding generalized ‘coherent state’ $\hat{D}^{(\lambda)}[p,q]|\sigma\rangle$ and phase space wave function (partner and equivalent to $\psi^{(\lambda)}(p,q)$) given by

$$\tilde{\psi}^{(\lambda)}(p,q) = \langle \sigma | \hat{D}^{(\lambda)\dagger}[p,q] |\psi\rangle.$$  \hfill (21)

Consider the product

$$(\mu^{(\lambda)}_\sigma(p,q))^* \psi^{(\lambda)}_\sigma(p,q) = \int d\tau' \int d\tau'' e^{\frac{i}{\hbar}p'q'} e^{-\frac{i}{\hbar}p''q''} \times$$

$$\times e^{\frac{i}{\hbar}(p'q'-q'p)} e^{-\frac{i}{\hbar}(p''q''-q''p)} \tilde{\psi}_\sigma(p',q') (\tilde{\mu}_\sigma(p'',q''))^*,$$

where I have used (5) (16) and (17) and $\int d\tau'$ stands for $\int_{-\infty}^{\infty} dp'dq'/\hbar$, etc. By equations (4), (9) and (10) we can write

$$\tilde{\psi}_\sigma(p',q') (\tilde{\mu}_\sigma(p'',q''))^* = \int d\tau (|\psi\rangle \langle \mu|)_{(p,q)} \times$$

$$\times (\hat{D}(p'',q'')|\sigma\rangle \langle \hat{D}^\dagger(p',q')|\sigma\rangle)_{(p,q)},$$

which is an integral over the product of two Weyl transformed operators. In particular, by definition (9) the second term is

$$(\hat{D}(p'',q'')|\sigma\rangle \langle \hat{D}^\dagger(p',q')|\sigma\rangle)_{(p,q)} = \langle \sigma | \hat{D}^\dagger(p',q') \hat{\Delta}(p,q) \hat{D}(p'',q'') |\sigma\rangle.$$  \hfill (24)

To simplify this quantity one can express $\hat{\Delta}$ here in terms of $\hat{D}$ (equation (6)) and then simplify the resulting triple product of displacement operators by means of these useful algebraic properties [14]:

$$\hat{D}^\dagger[p,q] = \hat{D}[-p,-q],$$

$$\hat{D}^\dagger[p,q] (\hat{p}, \hat{q}) \hat{D}[p,q] = (\hat{p} + p, \hat{q} + q),$$

$$\hat{D}[p_2,q_2] \hat{D}[p_1,q_1] = e^{\frac{i}{\hbar}(q_1 p_2 - p_2 q_1)} \hat{D}[p_1 + p_2, q_1 + q_2].$$

Utilizing the action of the unitary operator $\hat{D}$ on $\hat{\Delta}$ itself can also help. For instance, using the second of (25) with the first of equations (6) one finds

$$\hat{D}^\dagger[p',q'] \hat{\Delta}(p,q) \hat{D}[p',q'] = \hat{\Delta}(p - p', q - q').$$  \hfill (26)
This can be simplified using the displacement operator \( \hat{\Delta} \) making the substitutions (21), (10) and (20) it is

\[
(p, q) \rightarrow -\frac{1}{1-\lambda^2} \int d\tau' \langle |\psi\rangle \langle \mu| \rangle_{(p', q')} \times \langle \sigma | \hat{\Delta} (p, q) \hat{\Delta} (p', q') \hat{\Delta} (p, q) | \sigma \rangle.
\]

(27)

It is easy to see from this result that

\[
\int d\tau (\mu^*(p, q) \psi^*(p, q)) = \langle \mu | \psi \rangle,
\]

(28)
as it must [18].

From (27) we can find an analogous expression for the pair \((\tilde{\psi}^*(p, q), \tilde{\mu}^*(p, q))\). By equations (21), (10) and (20) it is

\[
(\tilde{\psi}^*(p, q)) = \hat{\Delta} (p, q) \hat{\Delta} (p', q') \hat{\Delta} (p, q) \hat{\Delta} (p', q') \hat{\Delta} (p, q) \hat{\Delta} (p', q') \hat{\Delta} (p, q) | \sigma \rangle.
\]

(29)

where I have recognized (using \( \hat{\Pi} \) with the first of equations (6)) that

\[
\hat{\Pi} \hat{\Delta} (p, q) \hat{\Pi} = \hat{\Delta} (p, q).
\]

Similarly (use an analysis based on (16))

\[
(\mu^*(p, q)) = \int d\tau' \langle |\psi\rangle \langle \mu| \rangle_{(p', q')} \times \langle \sigma | \hat{\Delta} (p, q) \hat{\Delta} (p', q') \hat{\Delta} (p, q) | \sigma \rangle.
\]

(30)

Since \( \hat{\Delta} (p, q) = \hat{\Delta} (-p, -q) \) it follows from (30) that multiplying by 1/4 and making the substitutions \((p, q, p', q', -\lambda) \rightarrow (p', q', -p', -q', -\lambda) \) in (27) gives

\[
(\tilde{\psi}^*(p, q)) = \hat{\Delta} (-p, -q) \hat{\Delta} (p', q') \hat{\Delta} (p, q) | \sigma \rangle.
\]

(31)

This also obeys an equation like (28).

The second term in the integrand here is the Weyl transform of the pure state \(|\sigma\rangle\langle \sigma|\), namely, from (9),

\[
(|\sigma\rangle\langle \sigma|) = \hat{\Delta} (p', q') | \sigma \rangle.
\]

This can be simplified using the displacement operator \( \hat{D} \) and the unitary dilation, or squeeze, operator ([24] [25])

\[
\hat{S}(\xi) = e^{i\xi (\hat{p}\hat{q} + \hat{q}^2)}.
\]

(32)
which has the properties
\[ \hat{S}^\dagger(\xi) = \hat{S}(-\xi) \text{ and } \hat{S}^\dagger(\xi) \hat{S}(\xi) = (e^{\xi\hat{p}}, e^{-\xi\hat{q}}), \]
so that (using this with (3) and (6))
\[ \hat{S}^\dagger(\xi) \hat{\Delta}(p, q) \hat{S}(\xi) = \hat{\Delta}(e^{-\xi}p, e^{\xi}q). \]

Then
\[
\begin{align*}
\langle [\sigma] \rangle \langle [\sigma] \rangle (\frac{1+\lambda}{1+\lambda}, \frac{1+\lambda}{1+\lambda}) \times (\frac{1+\lambda}{1+\lambda}) = & \langle [p, q, \lambda; \sigma] \Delta(p', q') \mid p, q, \lambda; \sigma \rangle \\
= & \langle [p, q, \lambda; \sigma] \langle [p, q, \lambda; \sigma] \rangle (p', q') \rangle.
\end{align*}
\]

where
\[ |p, q, \lambda; \sigma \rangle = \hat{D}(\frac{p}{1+\lambda}, \frac{q}{1+\lambda}) \hat{S}(\ln \frac{1+\lambda}{1-\lambda}) |\sigma \rangle \]
is a displaced squeezed state [14, 24, 25] generalized to an arbitrary fiducial state $|\sigma \rangle$. And so
\[
\langle [\mu] \rangle \langle [\mu] \rangle (p, q) = \langle [\mu] \rangle \langle [\mu] \rangle (\hat{\Delta}(p', q') |p, q, \lambda; \sigma \rangle \langle [p, q, \lambda; \sigma] \rangle (p', q') \rangle (p', q'),
\]

By a slight rearrangement we can also write
\[
\langle [\mu] \rangle \langle [\mu] \rangle (p, q) = \langle [\mu] \rangle \langle [\mu] \rangle (\hat{\Delta}(p', q') |p, q, \lambda; \sigma \rangle \langle [p, q, \lambda; \sigma] \rangle (p', q') \rangle (p', q) \rangle.
\]

Setting $|\mu \rangle = |\psi \rangle$, generalizing from $|\psi \rangle \langle \psi |$ to the density matrix $\hat{\rho} = \sum w_\psi |\psi \rangle \langle \psi |$, and dividing by $\hbar$ gives the ‘diagonal’ component of this sesquilinear form, the generalized $Q$-function. Non-negative by construction, from (37) and (38) it is
\[
\tilde{Q}_\sigma(p, q; \rho) = \frac{1}{\hbar} \sum w_\psi |w_\sigma(\lambda \sigma)(p, q)|^2
\]
\[
= \frac{1}{\hbar} \left( \frac{1}{1-\lambda^2} \right) \langle [p, q, \lambda; \sigma] \rho(p, q, \lambda; \sigma) \rangle
\]
\[
= \left( \frac{1}{1-\lambda^2} \right) \int dp' dq' \rho(p', q') \rho^{\dagger}(p; p' - p, q' - q),
\]

where
\[
\rho(p, q) = \frac{1}{\hbar} \langle \rho(p, q) \rangle = \frac{1}{\hbar} \text{Tr}(\hat{\rho} \hat{\Delta}(p, q))
\]
is the Wigner function for the state $\hat{\rho}$, $|p, q, \lambda; \sigma \rangle$ is given by (36), and $\rho^{\dagger}(\lambda \sigma)$ is the Wigner function corresponding to the $p$ and $q$ dependent squeezed state $|\lambda p, -\lambda q, \lambda; \sigma \rangle$:
\[
\rho^{\dagger}(p, q) = \frac{1}{\hbar} \langle \rho^{\dagger}(p, q) \rangle = \frac{1}{\hbar} \text{Tr}(\hat{\rho} \hat{\Delta}(p, q))
\]

The multiplier $1/\hbar$ is chosen by convention so that $\tilde{Q}_\sigma(p', q'; \rho)$, $\rho^{\dagger}(p', q')$ and $\rho^{\dagger}(p; p' - p, q' - q)$ are all normalized with respect to the integral $\int dp' dq'$. 


4 Discussion

When there is no squeezing of the fiducial state then $\lambda \to 0$, $|\lambda p, -\lambda q, \lambda; \sigma \rangle \to |\sigma \rangle$, and $|p, q, \lambda; \sigma \rangle \to |p, q; \sigma \rangle$ (defined in equation (2)). In that case

$$\tilde{Q}_\sigma^{(0)}(p, q; \rho) \equiv \tilde{Q}_\sigma(p, q; \rho) = \frac{1}{\hbar} \langle p, q; \sigma | \hat{\rho} | p, q; \sigma \rangle$$

$$= \int dp' dq' \rho(p', q') \rho_\sigma(p'-p, q'-q). \tag{42}$$

where

$$\rho(p, q) = \frac{1}{\hbar} \text{Tr}(\hat{\rho} \Delta(p, q))$$

and

$$\rho_\sigma(p, q) = \frac{1}{\hbar} \text{Tr}(|\sigma \rangle \langle \sigma | \Delta(p, q))$$

are the Wigner functions corresponding to the states $\hat{\rho}$ and $|\sigma \rangle \langle \sigma |$.

When $|\sigma \rangle$ is the vacuum state, $Q_\sigma(p, q; \sigma)$ is the well-known Husimi or Q-function and the content of equations (42) is well known, [9, 11, 20, 22]. Generalized to an arbitrary fiducial state $|\sigma \rangle$ the first of equations (42) says that the Q-function is the expectation of state $\hat{\rho}$ with respect to the coherent state of equation (2) while the second expresses it as the Wigner function for $\hat{\rho}$ smeared with respect to the Wigner function for $|\sigma \rangle \langle \sigma |$. The two Weyl transformed functions which are convoluted in equation (42) each separately corresponds to the scheme of equation (13) with $f(\xi, \eta) = 1$, and $\tilde{Q}_\sigma(p, q; \sigma)$ itself is also a member of that class of correspondences. This follows the convolution theorem. For instance, we can define the Fourier component, $[\hat{A}]_{(p, q)}$, of the Weyl transform $(\hat{A})_{(p, q)}$ of an operator $\hat{A}$, as

$$[\hat{A}]_{(p, q)} = \int \frac{dp' dq'}{\hbar} e^{\frac{i}{\hbar}(p'q+q'p)} (\hat{A})_{(p', q')} ,$$

and using this in equation (42) gives

$$\tilde{Q}_\sigma(p, q; \sigma) = f^{-1}(-i\partial_q, -i\partial_p) \rho(p', q') \tag{43}$$

where

$$f^{-1}(\xi, \eta) = [|\sigma \rangle \langle \sigma |]_{(\xi, \eta)} .$$

The customary choice for the fiducial state is the vacuum [11, 9]. In particular, for an harmonic oscillator in the ground state $|\sigma \rangle = |0 \rangle$, where

$$\langle x|0 \rangle = \frac{\alpha^{1/2}}{\pi^{1/4}} e^{-\frac{1}{2} \alpha^2 x^2}, \quad \alpha^2 = \frac{m\omega}{\hbar} ,$$

which gives for the Weyl transform of $|0 \rangle \langle 0 |$ and its Fourier component

$$\langle 0 \rangle \langle 0 |(p, q) = 2 e^{-\alpha^2 q^2} e^{-\frac{q^2}{4\alpha^2}} \quad \text{and} \quad [0 \rangle \langle 0 |]_{(\xi, \eta)} = e^{-\frac{\xi^2}{4\alpha^2}} e^{-\frac{\alpha^2 \xi^2 \eta^2}{4}} .$$

Thus, even when there is no squeezing (i.e. $\lambda = 0$) what was a Weyl association $f = 1$ (equation (12)) for the phase space wave function $|\psi \rangle \langle h_0 |$ becomes an association

$$f(\xi, \eta) = e^{\frac{\xi^2}{4\alpha^2}} e^{\frac{\alpha^2 \xi^2 \eta^2}{4}} \tag{44}$$
for the \( Q \)-function, equation (43). Although the function \( f(\xi, \eta) \) of equation (44) does not have the properties \( f(0, \eta) = 1 = f(\xi, 0) \) the distribution \( \tilde{Q}_\sigma(p, q; \sigma) \) which it generates is non-negative. It is a positive operator-valued measure (POM) \([26]\). This association is a special case of the \( s \)-family of orderings considered by Cahill and Glauber \([21, 22]\), which in the notation of this paper can be written

\[
f^{(s)}(\xi, \eta) = e^{s \frac{\xi^2}{4\sigma^2}} e^{s \alpha^2 - \eta^2 / 4}.
\]

For \( \lambda \neq 0 \) the functions \( \tilde{Q}_\sigma^{(\lambda)}(p, q; \rho) \), equation (39), are also a POMs, but owing to the extra \( p \)-dependence of the smoothing function they do not have corresponding functions \( f(\xi, \eta) \). The form of equation (39) shows that \( \tilde{Q}_\sigma^{(\lambda)} \) is proportional to the average of \( \hat{\rho} \) with respect to the state \(|p, q, \lambda; \sigma\rangle \), equation (36). In other words, \( \tilde{Q}_\sigma^{(\lambda)} \) is proportional to the probability of finding the system in the state \(|p, q, \lambda; \sigma\rangle \) when it has been prepared in the state \( \hat{\rho} \). The state \(|p, q, \lambda; \sigma\rangle \) is a minimum uncertainty squeezed state when the fiducial state \(|\sigma\rangle \) is the vacuum state. To see this using (25), (33) and (36) it is easy to show that, for any choice of \(|\sigma\rangle \),

\[
\langle p, q, \lambda; \sigma|\hat{\rho}|p, q, \lambda; \sigma\rangle = \frac{p}{1-\lambda} + \left( \frac{1+\lambda}{1-\lambda} \right) \langle \sigma|\hat{\rho}|\sigma\rangle,
\]

and that for momentum and position the standard deviations for this state are

\[
\Sigma^{(\lambda)}(p) = \left| \frac{1+\lambda}{1-\lambda} \right| \Sigma(p) \quad \text{and} \quad \Sigma^{(\lambda)}(q) = \left| \frac{1-\lambda}{1+\lambda} \right| \Sigma(q),
\]

where \( \Sigma(p) \) and \( \Sigma(q) \) are the corresponding standard deviations for state \(|\sigma\rangle \). So, whatever the degree of squeezing, \( \Sigma^{(\lambda)}(p)\Sigma^{(\lambda)}(q) = \Sigma(p)\Sigma(q) \), and for the vacuum state this product is the minimum value \( \hbar / 2 \).

We could equally choose to work with \( \psi^{(\lambda)}_\sigma \) instead of \( \tilde{\psi}^{(\lambda)}_\sigma \). For instance

\[
Q^{(\lambda)}_\sigma(p, q; \rho) \equiv \frac{1}{\hbar} \sum_\psi w_\psi |\psi^{(\lambda)}_\sigma(p, q)|^2,
\]

is directly related to \( \tilde{Q}^{(\lambda)}_\sigma(p, q; \rho) \), for from equations (16), (17), (20) and (21) we have

\[
\tilde{\psi}^{(\lambda)}_\sigma(p, q) = \frac{1}{2} \langle \sigma|\hat{\Pi}\hat{\Delta}^{(\lambda)\dagger}(p/2, q/2)|\psi\rangle = \frac{1}{2} \psi^{(-\lambda)}_{\sigma^r}(p/2, q/2),
\]

so that

\[
\psi^{(\lambda)}_\sigma(p, q) = 2\tilde{\psi}^{(-\lambda)}_{\sigma^r}(2p, 2q)
\]

where \(|\sigma^r\rangle = \hat{\Pi}|\sigma\rangle \) is a reflected fiducial state.

The time-dependence of \( \tilde{Q}^{(\lambda)}_\sigma(p, q; \rho) \)—or of \( Q^{(\lambda)}_\sigma(p, q; \rho) \)—enters through the time-dependence of \( \hat{\rho} \), for instance via its Weyl transform \( \hbar \times \rho(p', q') \) in the third of equations (39). The equation of motion of Wigner functions is well known \([5]\) and can be transferred to \( \tilde{Q}^{(\lambda)}_\sigma(p, q; \rho) \) itself by partial integration in equation (39). Another way would be to find the time-dependence of \( \tilde{\psi}^{(\lambda)}_\sigma(p, q) \) itself which enters through the time-dependence of \(|\psi\rangle \). In \([18]\) it was chosen to study the time variation of \( \psi^{(\lambda)}_\sigma(p, q) \)—as it is itself a Weyl transform and, from that standpoint, basic—when driven by a Hamiltonian \( \hat{H} = \hat{\rho}^2 / 2m + V(\hat{q}) \). According to equations (46) and (47), this knowledge transfers to \( \tilde{\psi}^{(\lambda)}_\sigma(p, q) \).
References

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