On the defect of vertex-transitive graphs of given degree and diameter

Journal Item

ON THE DEFECT OF VERTEX-TRANSITIVE GRAPHS OF GIVEN DEGREE AND DIAMETER

Geoffrey Exoo
Indiana State University
ge@cs.indstate.edu

Robert Jajcay
Comenius University
robert.jajcay@fmph.uniba.sk

Martin Mačaj
Comenius University
martin.macaj@fmph.uniba.sk

Jozef Širáň
Open University and Slovak University of Technology
siran@math.sk

November 14, 2017

Abstract

We consider the problem of finding largest vertex-transitive graphs of given degree and diameter. Using two classical number theory results due to Niven and Erdős, we prove that for any fixed degree $\Delta \geq 3$ and any positive integer δ, the order of a largest vertex-transitive Δ-regular graph of diameter D differs from the Moore bound by more than δ for (asymptotically) almost all diameters $D \geq 2$. We also obtain an estimate for the growth of this difference, or defect, as a function of D.

Keywords: degree, diameter, vertex-transitive graphs, Moore bound, order estimates.

1 Introduction

The *Degree/Diameter Problem* is the problem of finding the largest order $n(\Delta, D)$ of a graph of maximum degree Δ and diameter D. The well-known Moore bound, $M(\Delta, D)$, provides a natural upper bound on $n(\Delta, D)$, and graphs that attain this bound are called Moore graphs. To avoid trivialities we will assume $\Delta \geq 3$, in which case Moore graphs are very rare. Unless the parameters (Δ, D) allow for the existence of a Moore graph, $M(\Delta, D) > n(\Delta, D)$ or $M(\Delta, D) - n(\Delta, D) > 0$. In accordance with the survey paper [15], any graph G of maximum degree Δ and diameter D (a (Δ, D)-*graph*) is said to have the *defect* $\delta(G) = M(\Delta, D) - |V(G)|$. If $\Delta \geq 3$, there are no (Δ, D)-graphs of defect 1, and for $\Delta = 2$, the only such graphs are the cycles $\mathcal{C}_{2D}$ (for further results and a summary see [15]).

A closely related degree/girth problem, the *Cage Problem*, calls for finding a smallest k-regular graph of girth g, called a (k, g)-*cage*. The natural lower bound on the order of a k-regular graph of girth g is also called the Moore bound. In parallel with the concept of the defect, the *excess* ϵ of a k-regular graph G of girth g is the difference between its order and the corresponding value of the Moore bound. For more on the Cage Problem, consult [5].

Although the Cage and the Degree/Diameter Problems are often thought of as mutually dual problems tied together through the use of the Moore bound, the study of the relation between the order of the extremal graphs and the Moore bound is more developed for cages ([5], p. 14), and the survey paper [15] specifically states that

> "Finding better (tighter) upper bounds for the maximum possible number of vertices, given the other two parameters, and thus attacking the degree/diameter problem 'from above', remains a largely unexplored area."

One of the aims of this article is to address this issue in the case of vertex-transitive graphs. In the case of cages, the orders of vertex-transitive graphs are known to differ from the Moore bound by an arbitrary large excess for infinitely many degree-girth pairs. More precisely, in [2] Biggs proved the following:

Theorem (Biggs). *For each odd integer $k \geq 3$, there is an infinite sequence of values of g such that the excess ϵ of any vertex-transitive graph with degree k and girth g satisfies $\epsilon > g/k$.*

Inspired by the result of Biggs, we present parallel results for the orders of vertex-transitive (Δ, D)-graphs. First, for any fixed degree Δ and positive integer δ, we prove in Theorem 4.5 that for *asymptotically almost all* diameters D every vertex-transitive (Δ, D)-graph has defect greater than δ. This is achieved in two stages. We begin by finding upper and lower bounds on the number of $(2D + 1)$-cycles rooted in a(ny) vertex in an arbitrary Δ-regular graph of diameter D in Section 3. Then, in Section 4, we apply classical number-theoretic results of Niven [17] and Erdős [4] to determine the asymptotic density of those diameters for which no feasible number of rooted $(2D+1)$-cycles falls within these bounds. Using a slight oversimplification, these results not only show that in the case of vertex-transitive graphs one can get arbitrarily far from the Moore bound, but moreover, that the Moore bound is a poor predictor for the order of the largest vertex-transitive graphs for almost all parameter pairs (Δ, D). Our paper concludes with Theorem 5.3, where we estimate the growth of the defect as a function of D and show that for any fixed degree $\Delta \geq 3$, there exists an infinite sequence of diameters D such that the defect δ of any vertex-transitive (Δ, D)-graph is greater than $D^{1/(2+o(1))}$.

2 Vertex-transitive (Δ, D)-graphs

The *distance* $d_G(u, v)$ between two vertices of a graph G is the length of a shortest path connecting u and v, and any such path between two vertices is called a *geodesic*. The *diameter* of G, $diam(G)$, is the maximum distance between any two vertices: $diam(G) = \max\{d_G(u, v) \mid u, v \in V(G)\}$. Throughout the paper, we will always assume that $\Delta, D \geq 2$.

The following well-known *Moore bound* $M(\Delta, D)$ is an upper bound on the order of (Δ, D)-graphs:

$$M(\Delta, D) = \begin{cases} 1 + \Delta\frac{(\Delta-1)^D-1}{\Delta-2}, & \text{if } \Delta > 2, \\ 2D + 1, & \text{if } \Delta = 2. \end{cases} \tag{1}$$

Those (Δ, D)-graphs whose orders are equal to $M(\Delta, D)$ are called *Moore graphs*. All such graphs are Δ-regular and are known to exist for only few pairs of parameters (see [15]) and for most pairs (Δ, D), the Moore bound is unattainable.

A *vertex-transitive graph* is a graph with an automorphism group that acts transitively on the set of vertices. A *Cayley graph* admits a group of automorphisms acting regularly (transitively but with trivial vertex stabilizers) on its vertex set. Each Cayley graph can be constructed from a group Γ and a set X of generators for Γ that does not contain the identity and is closed under inverses. The Cayley graph $G = Cay(\Gamma, X)$ is then the graph with the vertex set Γ and two vertices $g, f \in \Gamma$ adjacent if and only if $g^{-1}f \in X$.

Vertex-transitive and Cayley graphs play an important role in the Cage and Degree/Diameter Problems. Once again, a disparity exists between the level of our knowledge about the vertex-transitive graphs of given degree and girth, and the vertex-transitive (Δ, D)-graphs. Both vertex-transitive and Cayley graphs are known to exist for any degree-girth pair [16, 13, 10, 7, 1], but until recently no equivalent constructions have been known for given Δ and D. In this section, we settle the question of the existence of vertex-transitive or Cayley graphs for any pair (Δ, D) by constructing (Δ, D)-Cayley graphs for all pairs $\Delta, D \geq 2$. We show that all such graphs can be constructed using Cayley graphs based on cyclic groups, and thus all the graphs we construct are *circulants*. This is in contrast to the Cage Problem for vertex-transitive graphs of given degree and girth. The girths of circulants with degree greater than two do not exceed 4 [3, 6]. Between the time we had announced our result at IWONT 2011 and the time it took to publish our article, independent proofs of Theorem 2.1 appeared in [11, 14]. We present our original proof as it is simpler and only uses circulants. We leave out the case of the 2-regular $(2D+1)$-cycles which are well-known to be Cayley graphs.

Theorem 2.1. *Let $\Delta \geq 3, D \geq 2$. Then there exists a Cayley graph of degree Δ and diameter D.*

Proof. First, consider the case $D = 2$. If Δ is even, $\Delta = 2m$, let $\Gamma = \mathbb{Z}_{2m+2}$ and $X = \mathbb{Z}_{2m+2} \setminus \{0, m + 1\}$, and observe that X generates the group $\mathbb{Z}_{2m+2}$, is closed under inverses, $|X| = 2m = \Delta$, and the diameter of $Cay(\Gamma, X)$ is 2. If Δ is odd, $\Delta = 2m+1$, let $\Gamma = \mathbb{Z}_{2m+4}$ and $X = \mathbb{Z}_{2m+4} \setminus \{0, m+1, m+3\}$, and observe that $Cay(\Gamma, X)$ is a $(\Delta, 2)$-graph.

Next, we consider the case $D \geq 3$. In the case $(3, D)$, take $\Gamma = \mathbb{Z}_{4D}$ and $X = \{1, -1, 2D\}$. Then X is closed under inverses, and it is not hard to verify that $Cay(\Gamma, X)$ is a cubic graph of diameter D.

Finally, suppose that $\Delta \geq 4, D \geq 3$, and consider the Cayley graph $G = Cay(\Gamma, X)$, $\Gamma = \mathbb{Z}_{2(\Delta-1)(D-1)}$, $X = \{1, -1\} \cup \{2k(D-1) \mid 1 \leq k < \Delta - 1\}$. The set X does not contain 0 and is closed under taking inverses, and we claim that $Cay(\Gamma, X)$ is a (Δ, D)-graph. First note that the order of the group Γ is even and thus $1, -1 \notin \{2k(D-1) \mid 1 \leq k < \Delta - 1\}$. Since $|\{2k(D-1) \mid 1 \leq k < \Delta - 1\}| = \Delta - 2$, it follows that $|X| = \Delta$; the degree of this Cayley graph is equal to Δ. To see that the diameter of G is D, note that every element n in $\mathbb{Z}_{2(\Delta-1)(D-1)}$ can be expressed as $2k(D-1) + r$ or $2k(D-1) - r$, for some $0 \leq k < \Delta - 1$ and $0 \leq r \leq D - 1$. Thus, $d_G(0, n) \leq 1 + D - 1 = D$, for all $n \in \mathbb{Z}_{2(\Delta-1)(D-1)}$. Moreover, G is vertex-transitive and therefore the diameter of G is equal to $\max \{d_G(0, n) \mid n \in \mathbb{Z}_{2(\Delta-1)(D-1)}\}$. The rest of our proof follows from the fact that $2(\Delta-1)(D-1) \geq 6(D-1)$ which implies $d_G(0, 3(D-1)) = D$. □

3 Bounds on the number of rooted $(2D+1)$-cycles in Δ-regular graphs of diameter D

It is known that Moore graphs exist only if (Δ, D) belongs to the set

$$\{(\Delta, 1) \mid \Delta \geq 1\} \cup \{(2,2), (3,2), (7,2), (57,2)\} \cup \{(2, D) \mid \text{ odd } D \geq 3\}.$$

Graphs are known to exist for all these cases except $(57, 2)$ [15, 12].

The ultimate aim of our paper is to produce upper bounds on the orders of vertex-transitive (Δ, D)-graphs for certain sets of parameters (Δ, D) that are significantly smaller than the Moore bounds. We accomplish this by considering the relationship between the defect and the number of $(2D+1)$-cycles in Δ-regular graphs of diameter D, and thereby showing that graphs with certain small defects do not exist. The following results concern the structure of all Δ-regular graphs of diameter D - *in this section we do not assume that the graphs considered are vertex-transitive.*

We introduce the following notation. Let G be a graph, let b be a vertex of G, and let $n \geq 3$ be an integer. By $\mathcal{C}_G^b(n)$ we denote *the number of n-cycles in G that contain b.* For all (Δ, D)-Moore graphs and for all pairs of vertices b, b', it is known that $\mathcal{C}_G^b(2D+1) = \mathcal{C}_G^{b'}(2D+1)$. This observation goes back to Friedman [8], who used the following Lemma 3.1 to show the non-existence of Moore (Δ, D)-graphs for certain parameter sets (Δ, D). We reprove his result for all (Δ, D)-graphs whose orders match the Moore bound, even though this family contains no graphs other than those indicated above. Our motivation is that this proof illustrates the cycle counting techniques employed in our paper, and it will also allow us to introduce notation that will be used throughout.

Let G be any graph of diameter D. Take b to be an arbitrary vertex of G, and let

$$N_G(b,i) = \{v \mid v \in V(G) \text{ and } d_G(b,v) = i\}, \quad 0 \le i \le D.$$

The *i-th neighborhood sets* $N_G(b,i)$, $0 \le i \le D$, form a partition of $V(G)$.

Now, let us assume in addition that G is a (Δ, D)-graph of order matching the Moore bound. In this case, any vertex in $N_G(b,i)$, $0 < i < D$, is connected to exactly one vertex in $N_G(b,i-1)$ and $(\Delta-1)$ vertices in $N_G(b,i+1)$, and each vertex in $N_G(b,D)$ is connected to one vertex in $N_G(b,D-1)$ and $(D-1)$ vertices in $N_G(b,D)$. These observations yield that if u is a vertex of G of distance $(D-i)$ from b, $0 < i < D$, the number of 'successors' of u in G (vertices v with the property that the geodesic path between b and v passes through u) is equal to

$$1 + (\Delta-1) + (\Delta-1)^2 + \ldots + (\Delta-1)^i, \tag{2}$$

which we shall denote by $S(\Delta-1,i)$, for all $\Delta \ge 3$. It will prove beneficial for future use to extend this definition and assume in addition $S(\Delta-1,0) = 1$ and $S(\Delta-1,-1) = 0$. Note that exactly $(\Delta-1)^i$ of these successors are of distance D from b, and belong to $N_G(b,D)$. We will call the edges connecting the vertices within $N_G(b,D)$ *horizontal with respect to b*.

Also, a Moore graph G contains no cycles of length smaller than $(2D+1)$ and every $(2D+1)$-cycle in G that includes b consists of a horizontal edge and two uniquely determined D-paths connecting the end-points of this horizontal edge to b. Thus, there is a one-to-one correspondence between the set of $(2D+1)$-cycles through b and the set of edges horizontal with respect to b in G. As the number of horizontal edges is easily seen to be equal to $\frac{\Delta}{2}(\Delta-1)^D$, the result for the number of $(2D+1)$-cycles follows:

Lemma 3.1. *Let $\Delta \ge 3, D \ge 2$, and G be a (Δ, D)-Moore graph. Then $\mathcal{C}_G^b(2D+1) = \frac{1}{2}\Delta(\Delta-1)^D$, for all $b \in V(G)$.*

Much of what follows depends on the fact that the number of $(2D+1)$-cycles in Δ-regular graphs of diameter D with small defect δ cannot significantly differ from the value $\frac{1}{2}\Delta(\Delta-1)^D$ because of the above lemma. We denote the value $\frac{1}{2}\Delta(\Delta-1)^D$ by $M_{\mathcal{C}}(\Delta, D)$ and refer to it informally as the *number of* $(2\Delta+1)$*-cycles in a* (Δ, D)*-Moore graph* (even though the graph may not actually exist).

The main aim of this section is to prove the following theorem. The essential fact is that the number $\gamma(\Delta,\delta)$ *does not depend* on the diameter D of the considered Δ-regular graphs of diameter D.

Theorem 3.2. *Let $\Delta \ge 3$ and $\delta \ge 1$ be fixed integers. Then there exists an integer $\gamma(\Delta,\delta)$ such that any Δ-regular graph G of diameter D with $D > 3\frac{\log(\delta)}{\log(\Delta-1)}$ and defect not exceeding δ satisfies*

$$|\mathcal{C}_G^b(2D+1) - M_{\mathcal{C}}(\Delta, D)| < \gamma(\Delta,\delta),$$

for all $b \in V(G)$.

In order to prove Theorem 3.2, we compare the number of the $(2D+1)$-cycles in a Δ-regular graph G of diameter D and of defect not exceeding δ to the number $M_{\mathcal{C}}(\Delta, D)$ of the $(2D+1)$-cycles in a putative (Δ, D)-Moore graph.

As we have stated already, choosing an arbitrary vertex b in a (Δ, D)-Moore graph, every vertex u whose distance from b is equal to i, with $0 < i < D$, is connected to one vertex in $N_G(b, i-1)$, to $(\Delta-1)$ vertices in $N_G(b, i+1)$, and to no vertices in $N_G(b, i)$. Every vertex u whose distance from b is equal to D has one neighbor in $N_G(b, D-1)$ and $(\Delta-1)$ neighbors in $N_G(b, D)$. It is important to note, that a Δ-regular graph G of diameter D is a (Δ, D)-Moore graph if and only if all the vertices of distance i from some vertex b, $0 < i \leq D$, satisfy the above properties. Thus, if G is a Δ-regular graph of diameter D which is *not* a Moore graph and b is its vertex, G must contain vertices that do not behave as above.

To make this more precise, we introduce the following notation. Given any $u \in N_G(b, i)$, $0 < i \leq D$, we let p_u^b and r_u^b denote the number of neighbors of u in $N_G(b, i-1)$ and $N_G(b, i)$, respectively. We say that $u \in N_G(b, i)$, $0 < i \leq D$ is *vertically defective with respect to* b if $p_u^b > 1$, and $u \in N_G(b, i)$, $0 < i < D$ is *horizontally defective with respect to* b if $r_u^b > 0$. Note that while vertices whose distance from b is larger than 1 but smaller than D may be both vertically and horizontally defective, vertices of distance 1 from b can only be horizontally defective, and vertices of distance D from b can only be vertically defective. Summarily we will call the vertically and horizontally defective vertices *defective vertices*. The edges joining horizontally defective vertices both of which are of the same distance from b will then be called *prematurely horizontal edges with respect to* b.

As pointed out in the discussion preceding the above definitions, a Δ-regular graph G of diameter D which is not a Moore graph necessarily contains defective vertices (regardless of the choice of b). Moreover, these vertices are exactly the vertices that 'contribute' to the defect of G. This is made explicit in the following lemma.

Lemma 3.3. *Let $\Delta \geq 3, D \geq 2$, let G be a Δ-regular graph of diameter D, and let b be a vertex of G. The defect δ of G satisfies the identity:*

$$\delta = \sum_{u \in V(G), u \neq b} (p_u^b - 1) S(\Delta-1, D - d_G(b,u)) + (r_u^b + p_u^b - 1) S(\Delta-1, D - d_G(b,u) - 1), \quad (3)$$

where $S(\Delta-1, D-d_G(b,u))$ and $S(\Delta-1, D-d_G(b,u)-1)$ are the sums defined in (2), in particular, $S(\Delta-1, 0) = 1$ and $S(\Delta-1, -1) = 0$.

Proof. Let G be a Δ-regular graph of diameter D, and b be a vertex of G. The definition of defective vertices yields the following recursive relation between the cardinalities of $N_G(b, i+1)$ and $N_G(b, i)$:

$$|N_G(b, i+1)| = |N_G(b, i)| \cdot (\Delta - 1) - \sum_{u \in N_G(b,i)} (r_u^b + p_u^b - 1) - \sum_{u \in N_G(b,i+1)} (p_u^b - 1). \quad (4)$$

Since the defect of G can be alternately expressed in the form

$$\delta = \sum_{0 < i \leq D} \Delta(\Delta-1)^{i-1} - |N_G(b, i)|,$$

formula (3) follows from repeated applications of (4) starting from the first $N_G(b,i)$ which contains vertices that are defective with respect to b. $\square$

Throughout the rest of this paper, we will refer to the value

$$(p_u^b - 1)S(\Delta-1, D-d_G(b,u)) + (r_u^b + p_u^b - 1)S(\Delta-1, D-d_G(b,u)-1)$$

as the *contribution of u toward the deficit of G* (with respect to b). Also, for any Δ-regular graph G of diameter D and of defect $\delta \geq 1$, we will reserve the symbol ℓ to denote the value

$$\ell = \frac{\log(\delta)}{\log(\Delta-1)}. \tag{5}$$

Corollary 3.4. *Let $\Delta \geq 3, D \geq 2$, let G be a Δ-regular graph of diameter D and of defect $\delta \geq 1$, and let b be any vertex of G. Then the following are satisfied:*

(i) The number of defective vertices with respect to b does not exceed the defect δ of G.

(ii) The number of defective vertices with respect to b whose distance from b does not exceed $D-i$, with i a positive integer, is at most $\frac{\delta}{(\Delta-1)^{i-1}}$.

(iii)

$$d_G(b,u) \geq D - \ell - 1,$$

for all defective vertices u.

Proof. The claim that the number of vertices defective with respect to b cannot exceed δ follows immediately from (3) and the fact that the defect contribution

$$(p_u^b - 1)S(\Delta-1, i) + (r_u^b + p_u^b - 1)S(\Delta-1, i-1) > 0,$$

for any defective vertex of distance $D-i$ from b.

The second claim follows once again from (3), since

$$\delta \geq \sum_{u \in N_G(b, D-i)} (p_u^b - 1)S(\Delta-1, i) + (r_u^b + p_u^b - 1)S(\Delta-1, i-1).$$

The third claim follows from the second, as the number of vertices u whose distance from b is smaller than $D-\ell-1$ is at most $\frac{\delta}{(\Delta-1)^{\ell}}$, which is a value strictly smaller than 1.

$\square$

The above corollary implies, among other things, that the subgraph induced by

$$\bigcup_{0 \leq i < D-\ell-1} N_G(b,i)$$

is a b-rooted tree containing no defective vertices of G. Another consequence concerns the girth of Δ-regular graphs of diameter D and states that the girth of such graphs with a small defect δ must be relatively large.

Corollary 3.5. *Let $\Delta \geq 3, D \geq 2$. The girth of any Δ-regular graph G of diameter D with defect δ is at least $2(D - \ell - 1)$.*

Proof. Let G be a Δ-regular graph of diameter D with defect δ and girth g, $\mathcal{C}$ be a g-cycle in G, and b be one of the vertices of $\mathcal{C}$. Since in the case when $g = 2D + 1$ the corollary certainly holds true, we may assume that $g < 2D + 1$, in which case $\mathcal{C}$ must contain a vertex u that is defective with respect to b. Applying the third part of Corollary 3.4 to b and u yields $d_G(b, u) \geq D - \ell - 1$, which implies the desired inequality $g \geq 2(D - \ell - 1)$. □

The girths of the Δ-regular graphs of diameter D whose orders are smaller than the corresponding (Δ, D)-Moore bound are necessarily smaller than $2D+1$. However, we will show, for any vertex b of G, that the number of b-rooted cycles of odd length less than $2D+1$, and the number of b-rooted cycles of length $2D+1$ that contain a prematurely horizontal edge with respect to b, is limited by a number independent of the parameter D.

Lemma 3.6. *Let $\Delta \geq 3$, let G be a Δ-regular graph of diameter D with defect δ and $D > 3\ell$. Let $1 \leq m \leq D - 1$, and $b \in V(G)$. Then*

$$\mathcal{C}_G^b(2D - 2m + 1) < \frac{1}{2}(\Delta - 1)^2\left(\frac{\delta}{(\Delta - 1)^{m-1}}\right)^2.$$

Moreover, the number of b-rooted cycles of length $(2D+1)$ that contain a prematurely horizontal edge with respect to b is bounded from above by the number $\frac{1}{2}(\Delta - 1)^2\delta^2$.

Proof. Due to Corollary 3.5, G contains no cycles of length smaller than $2(D-\ell-1)$, and hence the inequality claimed in our lemma is vacuously satisfied for cycles of length $(2D - 2m + 1)$ with $m > \ell + 1$. Thus, from now on, we assume $m \leq \ell + 1$.

Any b-based cycle $\mathcal{C}$ of *odd* length smaller than $2D + 1$ must contain at least one prematurely horizontal edge and also contains at least two horizontally defective vertices incident with the prematurely horizontal edge. Thus, after choosing an orientation for any such b-based cycle $\mathcal{C}$, we can refer to the first and the last defective vertex of $\mathcal{C}$ when traveling from b along $\mathcal{C}$ in the chosen direction. Let us denote these vertices by $u_\mathcal{C}$ and $v_\mathcal{C}$ respectively, and recall that $d_G(b, u_\mathcal{C}), d_G(b, v_\mathcal{C}) \geq D - \ell - 1$, by Corollary 3.4. Assume further that the length of $\mathcal{C}$ is $(2D-2m+1)$, for some $1 \leq m \leq \ell + 1$, and note that

$$D - m \geq d_G(b, u_\mathcal{C}), d_G(b, v_\mathcal{C}) \geq D - \ell - 1,$$

where the second inequality implies

$$d(u_\mathcal{C}, v_\mathcal{C}) \leq (2D-2m+1) - 2(D - \ell - 1) = 2\ell - 2m - 1.$$

As every b-based $(2D–2m+1)$-cycle consists of a geodesic path from b to $u_\mathcal{C}$, a geodesic path from b to $v_\mathcal{C}$, and a path from $u_\mathcal{C}$ to $v_\mathcal{C}$ of length at most $2\ell - 2m - 1$, the number $\mathcal{C}_G^b(2D - 2m + 1)$ is bounded from above by the *product* of four numbers.

The first number is a bound on the number of unordered pairs of defective vertices u, v of distance at most $D - m$ from b and of mutual distance at most $2\ell - 2m - 1$. The second number is a universal bound on the number of distinct paths between such u and v of length not exceeding $2\ell - 2m - 1$. The last two numbers consist of a bound on the number of geodesic paths from b to u and the number of geodesic paths from b to v, respectively, which beside u, respectively v, do not contain any other defective vertices and are internally disjoint from a fixed u-v path of length at most $2\ell - 2m - 1$. In the next paragraphs, we determine these bounds.

Let us first observe that the existence of two distinct $(2\ell - 2m - 1)$-paths between any two vertices of G would force the existence of a cycle of length at most $(4\ell - 4m - 2)$. However, our assumption $D > 3\ell$ implies $2(D - \ell - 1) > 4\ell - 2$, which means that $(4\ell - 4m - 2) < 2(D - \ell - 1)$, and hence the existence of two distinct $(2\ell - 2m - 1)$-paths between any two vertices of G would cause the existence of a cycle in G that would violate the girth requirements of Corollary 3.5. Thus, there is at most one path between any two vertices of distance at most $2\ell - 2m - 1$.

As for the number of geodesic paths between b and u or b and v which contain no other defective vertices but u or v, all these paths must begin with an edge connecting u or v with a neighborhood that is closer to b than they are and which is not a part of the u-v path. There are at most $\Delta - 1$ such edges for either u or v, which are afterwards connected to b via a uniquely determined geodesic between the other end-point of one of these edges and b. Therefore, the number of such paths between b and u or b and v is in both cases bounded from above by $\Delta - 1$.

Finally, recalling the second claim of Corollary 3.4, we note that the number of defective vertices with respect to b whose distance from b is at most $D-m$ is bounded from above by $\frac{\delta}{(\Delta-1)^{m-1}}$, and hence, the number of u, v pairs is at most the number of unordered pairs chosen from a $\lfloor \frac{\delta}{(\Delta-1)^{m-1}} \rfloor$-element set.

Putting all the above upper bounds together, we obtain

$$\mathcal{C}_G^b(2D - 2m + 1) \leq (\Delta - 1)^2 \binom{\lfloor \frac{\delta}{(\Delta-1)^{m-1}} \rfloor}{2} < \frac{1}{2}(\Delta - 1)^2 (\frac{\delta}{(\Delta - 1)^{m-1}})^2,$$

as claimed.

A similar argument concerning the b-based $(2D+1)$-cycles containing a prematurely horizontal edge with respect to b is left to the reader. □

We are now ready to obtain an upper bound on the number of b-based cycles of length $2D + 1$.

Lemma 3.7. *Let $\Delta \geq 3$, let G be a Δ-regular graph of diameter D and of defect δ and suppose that $D > 3\ell$. If b is any vertex of G, then*

$$\mathcal{C}_G^b(2D + 1) \quad \leq \quad M_{\mathcal{C}}(\Delta, D) + \delta(\Delta - 1)^2 + (\Delta - 1)^2\delta^2.$$

Proof. As every odd-length b-based cycle must contain a horizontal edge with respect to b, b-based cycles of length $(2D + 1)$ may be divided into two groups depending on whether they contain a horizontal edge that is prematurely horizontal with respect to b or not.

As stated in Lemma 3.6, the number of b-based cycles of length $(2D+1)$ that contain an edge that is prematurely horizontal with respect to b is bounded from above by $\frac{1}{2}(\Delta-1)^2\delta^2$.

Thus, to complete the proof, we need to obtain an upper bound on the number of b-based cycles of length $2D+1$ that do not contain prematurely horizontal edges, but contain a horizontal edge between two vertices $u, v \in N_G(b, D)$. Note that the number of b-based cycles of length $(2D+1)$ passing through a specific horizontal edge uv, $u, v \in N_G(b, D)$, is bounded from above by the product of the number of D-paths between b and u and the number of D-paths between b and v. In view of this observation, we divide the horizontal edges in $N_G(b, D)$ into three separate classes:

- Edges for which both end-vertices are connected to b through unique D-paths and which contribute at most one $(2D+1)$-cycle to the overall count of $\mathcal{C}_G^b(2D+1)$,
- horizontal edges uv for which precisely one of the vertices u, v is connected to b via more than one D-path and which therefore might contribute more than one cycle to $\mathcal{C}_G^b(2D+1)$, and
- edges uv for which both vertices u, v are connected to b via more than one D-path and which also might contribute more than one cycle to $\mathcal{C}_G^b(2D+1)$.

The number of horizontal edges of the first kind (both endpoints are connected to b via a unique D-path) is bounded from above by $M_{\mathcal{C}}(\Delta, D)$, the number of horizontal edges in any (Δ, D)-Moore graph. Since any such edge is a part of at most one b-based $(2D+1)$-cycle, $M_{\mathcal{C}}(\Delta, D)$ is also an upper bound on the number of b-based $(2D+1)$-cycles containing a horizontal edge of the first kind.

As for the b-based $(2D+1)$-cycles containing some horizontal edge uv, $u, v \in N_G(b, D)$, of the second kind, suppose that u is the end-vertex connected to b via more than one D-path. Any such D-path must necessarily contain a vertically defective vertex w having the property that there are no further vertically defective vertices between w and b. The distance between w and b is known to be at least $D-\ell-1$ (Corollary 3.4), which also means that $d_G(w, u) \leq \ell+1$. Following the line of argument from the proof of Lemma 3.6, the number of geodesic paths between b and w is bounded from above by $\Delta-1$. The number of geodesic paths connecting w to some $u \in N_G(b, D)$ is not larger than $(\Delta-1)^{D-d_G(b,w)} \leq S(\Delta-1, D-d_G(b, w))$. Since w is vertically defective, $S(\Delta-1, D-d_G(b, w)) \leq (p_w-1)S(\Delta-1, D-d_G(b, w))$, and therefore, the number of geodesic paths connecting w to some $u \in N_G(b, D)$ does not exceed the contribution of w toward the defect of G. Finally, u is adjacent to at most $\Delta-1$ vertices v in $N_G(b, D)$. Putting all the above bounds together, we obtain an upper bound on the number of b-based $(2D+1)$-cycles containing a horizontal edge of the second kind (with the sum only including defective vertices w):

$$\sum_{i=D-\ell-1}^{D} \sum_{w \in N_G(b,i)} (\Delta-1)S(\Delta-1, D-i)(\Delta-1).$$

Factoring out $(\Delta-1)^2$ shows that this value is bounded from above by the product of $(\Delta-1)^2$ and the total of contributions of all defective vertices. It follows that the number of $(2D+1)$-cycles that contain a horizontal edge, with exactly one end-point connected to b via more than one path, is bounded from above by $(\Delta-1)^2\delta$.

Finally, the number of b-based $(2D+1)$-cycles containing horizontal edges uv with the property that both u and v are connected to b via more than one D-path can again be estimated along the lines of the proof of Lemma 3.6 to be not larger than the product of $\Delta-1$ with $\Delta-1$ and with the number of pairs of defective vertices. This number cannot exceed the product $\frac{1}{2}(\Delta-1)^2\delta^2$.

Summing up all the above obtained upper bounds yields the statement of the lemma. □

In the next lemma, we derive a lower bound on $\mathcal{C}_G^b(2D+1)$, and thus obtain the final piece needed to complete the proof of Theorem 3.2.

Lemma 3.8. *Let $\Delta \geq 3$, let G be a Δ-regular graph of diameter D and of defect δ, $D > 3\ell$, and b be a vertex of G. Then*

$$M_{\mathcal{C}}(\Delta, D) - \delta(\Delta-1) - \delta^2\Delta(\Delta-1)^2 \leq \mathcal{C}_G^b(2D+1).$$

Proof. Since δ is the overall defect of G, necessarily, $|N_G(b,D)| \geq \Delta(\Delta-1)^{D-1} - \delta$. On the other hand, $N_G(b,D)$ contains at most δ defective vertices, and therefore the number of non-defective vertices in $N_G(b,D)$ is at least $\Delta(\Delta-1)^{D-1} - 2\delta$. Each non-defective vertex in $N_G(b,D)$ is incident to $\Delta-1$ horizontal edges, and thus, the number of horizontal edges uv, $u, v \in N_G(b,D)$, is at least

$$\frac{1}{2}(\Delta(\Delta-1)^{D-1} - 2\delta) \cdot (\Delta-1) = M_{\mathcal{C}}(\Delta, D) - \delta(\Delta-1). \qquad (6)$$

However, not every horizontal edge uv, $u, v \in N_G(b,D)$, is necessarily contained in a $(2D+1)$-cycle based at b. To obtain the desired lower bound on $\mathcal{C}_G^b(2D+1)$, we need to estimate the maximum number of edges uv, $u, v \in N_G(b,D)$, that are not a part of a b-based $(2D+1)$-cycle. As any vertex in $N_G(b,D)$ is necessarily connected to b via a D-path, the only way for an edge uv, $u, v \in N_G(b,D)$, not to be contained in a b-based $(2D+1)$-cycle is for any two D-paths connecting b to u and b to v, respectively, to share more than just b. The existence of any two such paths forces the existence of an odd-length cycle containing uv and based at some b' that is of length smaller than $(2D+1)$ (we take b' to be the furthest vertex from b shared by both paths). Recall that we have shown in Lemma 3.6 that $\mathcal{C}_G^{b'}(2D-2m+1) < \frac{1}{2}(\Delta-1)^2(\frac{\delta}{(\Delta-1)^{m-1}})^2$. This means that the number of horizontal edges uv (with respect to b) which do not form a part of a b-based $(2D+1)$-cycle, but form a part of some smaller cycle of odd length $2D-2m+1$ based at some b' is bounded from above by the product of the number of vertices b' of distance m from b and the number $\frac{1}{2}(\Delta-1)^2(\frac{\delta}{(\Delta-1)^{m-1}})^2$. Since we have argued that every horizontal edge that is not contained in a b-based $(2D+1)$-cycle must be contained in some $(2D-2m+1)$-cycle (based at some b'), it follows that the total number of edges horizontal with respect to b which are not contained in a $(2D+1)$-cycle based at b is bounded from above by

$$\begin{aligned}\sum_{m=1}^{\lfloor \ell \rfloor +1} \Delta(\Delta-1)^{m-1}\frac{1}{2}(\Delta-1)^2(\frac{\delta}{(\Delta-1)^{m-1}})^2 < \\ < \frac{1}{2}\delta^2\Delta(\Delta-1)^2 \cdot (\sum_{m=0}^{\infty}\frac{1}{(\Delta-1)^m}) \leq \\ \leq \delta^2\Delta(\Delta-1)^2.\end{aligned}$$

Subtracting this upper bound from the minimal total number of horizontal edges (6) yields the number of horizontal edges that are contained in a $(2D+1)$-cycle based at b which is also the desired lower bound for the number of $(2D+1)$-cycles based in b. □

The bounds in Lemmas 3.7 and 3.8 readily imply the main theorem of this section.

Proof of Theorem 3.2.

Using Lemmas 3.7 and 3.8, we have shown that

$$\begin{aligned}M_{\mathcal{C}}(\Delta, D) - \delta(\Delta-1) - \delta^2\Delta(\Delta-1)^2 \leq \\ \leq \mathcal{C}_G^b(2D+1) \leq \\ \leq M_{\mathcal{C}}(\Delta, D) + \delta(\Delta-1) + (\Delta-1)^2\delta^2.\end{aligned}$$

Taking

$$\gamma(\Delta, \delta) = \max\{\delta(\Delta-1) + \delta^2\Delta(\Delta-1)^2, \delta(\Delta-1) + (\Delta-1)^2\delta^2\},$$

provides us with a bound independent of D and satisfying

$$|\mathcal{C}_G^b(2D+1) - M_{\mathcal{C}}(\Delta, D)| < \gamma(\Delta, \delta).$$

. □

To accommodate our arguments in the next section, it will prove beneficial to simplify the bounds proved above. This is achieved in the last corollary of this section.

Corollary 3.9. *Let $\Delta \geq 3$ and $\delta \geq 1$ be integers. Then any Δ-regular graph G of diameter D with $D > 3\ell$ and defect not exceeding δ satisfies*

$$|\mathcal{C}_G^b(2D+1) - M_{\mathcal{C}}(\Delta, D)| < 2\Delta(\Delta-1)^2\delta^2,$$

for all $b \in V(G)$.

4 The defect of vertex-transitive graphs

In this section, we prove that for any fixed $\Delta \geq 3$, $\delta \geq 1$, and for almost all D, the defect of (Δ, D)-vertex-transitive graphs exceeds δ. The proof involves divisibility arguments based on estimates of the number of cycles of length $2D+1$ obtained in the previous section.

We begin with an easy observation concerning vertex-transitive graphs that has already been used repeatedly in the context of cages; see for example [8, 10].

Lemma 4.1. *Let G be a vertex-transitive graph, and $n \geq 3$ be a positive integer. Then*

(i) $\mathcal{C}^a_G(n) = \mathcal{C}^b_G(n)$*, for all* $a, b \in V(G)$*,*

(ii) n *divides* $\mathcal{C}^a_G(n) \cdot |V(G)|$*, for all* $a \in V(G)$*.*

Proof. Since G is vertex-transitive, $\mathcal{C}^a_G(n) = \mathcal{C}^b_G(n)$, for any two vertices $a, b \in V(G)$. Furthermore, as each n-cycle contains n vertices, the total number of rooted n-cycles in G, i.e., the number $\mathcal{C}^a_G(n) \cdot |V(G)|$, must be divisible by n. □

The following result is a direct consequence of part (ii) of the above lemma. Recall that $\ell = \frac{\log(\delta)}{\log(\Delta-1)}$.

Lemma 4.2. *Let $\Delta \geq 3$, $\delta \geq 1$, $D > 3\ell$, $\gamma \geq 2\Delta(\Delta-1)^2\delta^2$, and let*

$$\mathcal{S} = \{\Delta(\Delta-1)^D + i \mid -2\gamma < i < 2\gamma\}.$$

If $2D+1$ divides none of the integers in

$$\mathcal{S}^2 = \{(\Delta(\Delta-1)^D + i) \cdot (\Delta(\Delta-1)^D + j) \mid -2\gamma < i, j < 2\gamma\},$$

then the defect of every vertex-transitive (Δ, D)-graph is greater than δ.

Proof. We proceed by contradiction. Assume all the arithmetic constraints stated in the lemma, as well as the existence of a vertex-transitive (Δ, D)-graph G with defect not exceeding δ.

By the definition of δ, the order of G must belong to the interval

$$1+\Delta\frac{(\Delta-1)^D-1}{\Delta-2}-\delta, 1+\Delta\frac{(\Delta-1)^D-1}{\Delta-2}-\delta+1, \ldots, 1+\Delta\frac{(\Delta-1)^D-1}{\Delta-2}. \quad (7)$$

According to Corollary 3.9, the number $\mathcal{C}^b_G(2D+1)$ of $(2D+1)$-cycles based at any fixed vertex b belongs to the interval

$$\frac{1}{2}\Delta(\Delta-1)^D-\gamma+1, \frac{1}{2}\Delta(\Delta-1)^D-\gamma+2, \ldots, \frac{1}{2}\Delta(\Delta-1)^D+\gamma-2, \frac{1}{2}\Delta(\Delta-1)^D+\gamma-1. \quad (8)$$

Because of Lemma 4.1(ii), $2D+1$ must divide the product $|V(G)| \cdot \mathcal{C}^b_G(2D+1)$. Therefore, $2D+1$ must divide a product of a number from the interval (7) with a number from the interval (8). Consequently, $2D+1$ must also divide a product of a $(\Delta-2)$-multiple of one of the numbers in the interval (7) with one of the 2-multiples of the numbers in the interval (8). However, all latter products appear on the list $\mathcal{S}^2$, while we assume that $2D+1$ divides none of the numbers in $\mathcal{S}^2$; a contradiction. □

In order to show that the density of the diameters D for which the defect is less than δ is 0, we need to introduce some terminology from number theory. Let $\mathcal{A}$ be a set of positive integers. For any $n \geq 1$, let $\mathcal{A}(n)$ denote the number of members of $\mathcal{A}$ that do not exceed n, $\mathcal{A}(n) = |\{a \in \mathcal{A} \mid a \leq n\}|$. The *lower asymptotic density* of $\mathcal{A}$ is the value

$$\underline{d}(\mathcal{A}) = \liminf_{n\to\infty} \frac{\mathcal{A}(n)}{n}.$$

The *upper asymptotic density* of $\mathcal{A}$ is defined analogously by

$$\overline{d}(\mathcal{A}) = \limsup_{n\to\infty} \frac{\mathcal{A}(n)}{n}.$$

Note that $0 \leq \underline{d}(\mathcal{A}) \leq \overline{d}(\mathcal{A}) \leq 1$. If, in addition, $\underline{d}(\mathcal{A}) = \overline{d}(\mathcal{A})$, we say that $\mathcal{A}$ has the asymptotic density

$$d(\mathcal{A}) = \lim_{n\to\infty} \frac{\mathcal{A}(n)}{n}.$$

For any prime p, let

$$\mathcal{A}_p = \{n \in \mathcal{A} \mid p \| n \},$$

where $p\|n$ indicates that $p|n$ but $p^2 \nmid n$.

We will rely on the following two classical theorems from Number Theory. The first of these is a 1951 result of I. Niven [17].

Theorem (Niven). *Let $\{p_i\}_{i=1}^{\infty}$ be a set of primes such that $\sum_{i=1}^{\infty} 1/p_i = +\infty$. If $\mathcal{A}$ is a set of positive integers such that $\sum \overline{d}(\mathcal{A}_{p_i}) < +\infty$, then $d(\mathcal{A}) = 0$.*

The second result is an immediate consequence of an often cited theorem of Erdős [4].

Theorem (Erdős). *Let $\{a_i\}_{i=1}^{\infty}$ be a set of integers such that $a_i \nmid a_j$, unless $i = j$. Then $\sum_{i=1}^{\infty} \frac{1}{a_i \log a_i}$ converges.*

In what follows, we will only need the following special case of the above theorem.

Lemma 4.3. *If $\{p_i\}_{i=1}^{\infty}$ is the sequence of all primes, the sum $\sum_{i=1}^{\infty} \frac{1}{p_i \log p_i}$ converges.*

In order to investigate the divisibility properties of the products used in the proof of Lemma 4.2, we introduce the following notation. Given integers a, b, c, d and q such that $a \neq 0 \neq c$ and $q > 2$, let $\mathcal{A}$ denote the set

$$\mathcal{A}(q, a, b, c, d) = \{n \in \mathbb{N} \mid n \text{ is odd and } n|(aq^{(n-1)/2} + b)(cq^{(n-1)/2} + d)\}.$$

In addition, let $\left(\frac{q}{p}\right)$ denote the Legendre symbol [9], and $o_p(q)$ be the multiplicative order of q in $\mathbb{Z}_p$.

Lemma 4.4. *Given integers a, b, c, d and q satisfying the property $a \neq 0 \neq c$ and $q > 2$,*

$$d(\mathcal{A}(q, a, b, c, d)) = 0.$$

Proof. Let $p \nmid acq$ be a fixed prime and let $n = rp \in \mathcal{A}$. Then $p|(aq^{(n-1)/2} + b)(cq^{(n-1)/2} + d)$, and since p is a prime, it divides the first or the second factor of this product. It follows that at least one of the factors is equal to 0 in $\mathbb{Z}_p$. Since p is relatively prime to acq, both a and c have inverses in $\mathbb{Z}_p$, and therefore $q^{(n-1)/2}$ (as an element of $\mathbb{Z}_p$) is an element of the two-element set $\{-a^{-1}b, -c^{-1}d\}$. Furthermore, $n - 1 = r(p - 1) + r - 1$. By Euler's criterion (Theorem 83, [9]), in $\mathbb{Z}_p$ we have

$$q^{(p-1)/2} = \left(\frac{q}{p}\right)$$

and

$$q^{(n-1)/2} = \left(\frac{q}{p}\right) q^{(r-1)/2}.$$

Thus,

$$q^{(r-1)/2} \in \{-\left(\frac{q}{p}\right) a^{-1}b, -\left(\frac{q}{p}\right) c^{-1}d\}.$$

Consequently, $\frac{r-1}{2}$ belongs to at most two classes modulo $o_p(q)$.

Next observe that $o_p(q) \geq \frac{\log p}{\log q}$, and therefore the asymptotic density of the multiples rp for which r satisfies the above constraints among all multiples of p is at most $2\frac{\log q}{\log p}$. As the asymptotic density of the multiples of p among all positive integers is at most $\frac{1}{p}$, we conclude that

$$\overline{d}(\mathcal{A}_p) \leq \frac{1}{p} \cdot \frac{2}{o_p(q)} \leq \frac{2 \log q}{p \log p}.$$

Since $\sum_{p>acq} \frac{1}{p}$ diverges, and

$$\sum_{p>acq} \overline{d}(\mathcal{A}_p) \leq \sum_{p>acq} \frac{2 \log q}{p \log p},$$

applying the theorem of Erdős and then applying the theorem of Niven yields the desired result $d(\mathcal{A}(q, a, b, c, d)) = 0$. □

We are ready to prove the main theorem of this section.

Theorem 4.5. *For any $\Delta \geq 3$ and $\delta \geq 1$, the asymptotic density of the set of all $D \geq 2$ for which there exists a vertex-transitive (Δ, D)-graph with defect not exceeding δ is* 0.

Proof. Let $\Delta \geq 3$, $\delta \geq 1$, and $\gamma \geq 2\Delta(\Delta - 1)^2\delta^2$ be fixed. By Lemma 4.2, a vertex-transitive graph of diameter D and defect at most δ may exist only if at least one of the products in

$$\mathcal{S}^2 = \{(\Delta(\Delta - 1)^D + i) \cdot (\Delta(\Delta - 1)^D + j) \mid -2\gamma < i, j < 2\gamma\}$$

is divisible by $(2D + 1)$. Each of the products in $\mathcal{S}^2$ has the form of a product in the definition $\mathcal{A}(q, a, b, c, d)$, where $q = \Delta - 1$, $a = \Delta$, $b = i$, $c = \Delta$, and $d = j$; and therefore for each D such that $(2D+1)$ divides $(\Delta(\Delta-1)^D+i)\cdot(\Delta(\Delta-1)^D+j)$ for some i and j the value $2D+1$ belongs to $\mathcal{A}((\Delta-1),\Delta,i,\Delta,j)$. Thus, the set of all $D\geq 2$ for which there exists a vertex-transitive (Δ,D)-graph with defect not exceeding δ is a subset of the set of all D such that $2D+1$ belongs to

$$\bigcup_{-2\gamma<i,j<2\gamma}\mathcal{A}((\Delta-1),\Delta,i,\Delta,j).$$

This union is a finite union of sets of asymptotic density 0, as proved in Lemma 4.4. As a subset of a set of asymptotic density 0, the set of all $D\geq 2$ for which there exists a vertex-transitive (Δ,D)-graph with defect at most δ must also be of asymptotic density 0. □

5 Lower bounds on the growth of the defect in terms of the diameter

The main result of this section is an analogue of Biggs' result from [2] mentioned in the introduction.

Lemma 5.1. *Let $\Delta\geq 3$ and $\delta\geq 1$. Let r be an odd integer, and let p be a prime such that $p>2\Delta(\Delta-1)^{(r-1)/2}>8\Delta(\Delta-1)^2\delta^2$. If $2D+1=rp$, then any vertex-transitive (Δ,D)-graph has defect greater than δ.*

Proof. Since $2D+1=rp$ and p is a prime, we have (as in the proof of Lemma 4.4) $D=r(p-1)/2+(r-1)/2$ and $(\Delta-1)^D=\left(\frac{\Delta-1}{p}\right)(\Delta-1)^{(r-1)/2}$ modulo p, where $\left(\frac{\Delta-1}{p}\right)$ is again the Legendre symbol. Hence the interval

$$\{\ \Delta(\Delta-1)^D+i\mid\ -4\Delta(\Delta-1)^2\delta^2<i<4\Delta(\Delta-1)^2\delta^2\ \}$$

is equivalent modulo p to

$$\{\ \left(\frac{\Delta-1}{p}\right)\Delta(\Delta-1)^{(r-1)/2}+i\mid\ -4\Delta(\Delta-1)^2\delta^2<i<4\Delta(\Delta-1)^2\delta^2\ \}.$$

Since r satisfies $\Delta(\Delta-1)^{(r-1)/2}>4\Delta(\Delta-1)^2\delta^2$, and $p>2\Delta(\Delta-1)^{(r-1)/2}$, the second resulting interval does not contain a multiple of p, and therefore by Lemma 4.2 the defect of any vertex-transitive (Δ,D)-graph is greater than δ. □

Lemma 5.2. *Let $\Delta\geq 3$ and $\delta\geq 1$. Then there exists a constant K (depending only on Δ) and a diameter $D<K\delta^2(1+\log(\delta))$ such that any vertex-transitive (Δ,D)-graph has defect greater than δ.*

Proof. Let r be the smallest odd integer with the property that $\Delta(\Delta-1)^{(r-1)/2}>4\Delta(\Delta-1)^2\delta^2$. By Bertrand's Postulate [9] there exists a prime p such that

$$2\Delta(\Delta-1)^{(r-1)/2}<p<4\Delta(\Delta-1)^{(r-1)/2}. \tag{9}$$

From the inequality bounding r it follows that

$$\frac{r-1}{2} > 2 + \log_{\Delta-1}(4\delta^2)$$

and

$$\frac{r-1}{2} = 3 + \lfloor \log_{\Delta-1}(4\delta^2) \rfloor \leq 3 + \log_{\Delta-1}(4\delta^2).$$

Substituting the first expression for $(r-1)/2$ on the left side of (9), and the second expression on the right side, we obtain

$$8\Delta(\Delta-1)^2\delta^2 < p < 16\Delta(\Delta-1)^3\delta^2,$$

where the extra $(\Delta-1)$ factor on the right comes from possible rounding. Since we have chosen r and p to satisfy the requirements of Lemma 5.1, if $D = \frac{rp-1}{2}$ then any vertex-transitive (Δ, D)-graph has defect greater than δ. Multiplying the upper bounds on r and p, we finally obtain $D \leq rp < K\delta^2 \log(\delta)$ as claimed. □

The upper bound on the diameter, $D < K\delta^2 \log(\delta)$, used in Lemma 5.2, immediately implies the last theorem of our paper.

Theorem 5.3. *For any degree $\Delta \geq 3$, there exists an infinite sequence of diameters D such that for any vertex-transitive (Δ, D)-graph the defect δ satisfies $\delta > D^{\frac{1}{2+o(1)}}$.*

We point out that the number 2 in the denominator of the exponent of the above result is directly related to the exponent 2 of δ in the bound from Corollary 3.9. Therefore any improvement of Corollary 3.9 would result in an improvement of Theorem 5.3.

Acknowledgments

We thank the referees for their useful corrections and suggestions that made us clarify several parts of the paper which ultimately led to better bounds.

The second author acknowledges support by VEGA 1/0474/15, VEGA 1/0596/17, NSFC 11371307, APVV-15-0220, and by the Slovenian Research Agency research project J1-6720.
The third author acknowledges the support by VEGA 1/0474/15, VEGA 1/0596/17, and APVV-15-0220.
The fourth author acknowledges the support by the projects APVV 0136/12, APVV-15-0220, VEGA 1/0007/14, VEGA 1/0026/16, and VEGA 1/0142/17.

References

[1] M. Abas, Homomorphisms of triangle groups with large injectivity radius, Acta Math. Univ. Comen., New Ser. 72, No. 2 (2003) 253-259.

[2] N.L. Biggs, Excess in vertex-transitive graphs, Bull. London Math. Soc. 14 (1982) 52-54.

[3] M. Conder, G. Exoo and R. Jajcay, On the limitations of the use of solvable groups in Cayley graph cage constructions, Europ. J. of Combinatorics 31 (2010) 1819-1828.

[4] P. Erdős, Note on sequences of integers no one of which is divisible by any other, J. London Math. Soc. 10 (1935) 126-128.

[5] G. Exoo and R. Jajcay, Dynamic cage survey, Electron. J. Combin., Dynamic Survey 16, September 2008.

[6] G. Exoo and R. Jajcay, On the girth of voltage graph lifts, Europ. J. of Combinatorics 32 (2011) 554-562.

[7] G. Exoo, R. Jajcay and J. Širáň, Cayley cages, J. Algebr. Comb. 38, Issue 1 (2013) 209-224.

[8] H.D. Friedman, On the impossibility of certain Moore graphs, J. Comb. Theory 10 (1971) 245-252.

[9] G.H. Hardy and E.M. Wright, *An introduction to the theory of numbers*, 5th Ed., Oxford University Press, New York 1979.

[10] R. Jajcay and J. Širáň, Small vertex-transitive graphs of given degree and girth, Ars Mathematica Contemporanea 4 (2011) 375-384.

[11] M. Knor and J. Širáň, Smallest vertex-transitive graphs of given degree and diameter, J. Graph Theory 75, No. 2 (2014) 137-149.

[12] M. Mačaj and J. Širáň, Search for properties of the missing Moore graph, Linear Algebra Appl. 432, No. 9 (2010) 2381-2398.

[13] M. Mačaj, J. Širáň, and M. Ipolyiová, Injectivity radius of representations of triangle groups and planar width of regular hypermaps, Ars Math. Contemp. 1, No. 2 (2008) 223-241 .

[14] H. Macbeth, J. Šiagiová, J. Širáň and T. Vetrík, Large Cayley graphs and vertex-transitive non-Cayley graphs of given degree and diameter, J. Graph Theory 64, No. 2 (2010) 87-98.

[15] M. Miller and J. Širáň, Moore graphs and beyond: A survey of the degree/diameter problem, Electron. J. Combin., Dynamic Survey 14, 2005.

[16] R. Nedela and M. Škoviera, Regular maps on surfaces with large planar width, European J. Combin. 22 (2001) 243-261.

[17] I. Niven, The asymptotic density of sequences, Bull. Amer. Math. Soc., Vol. 57, Number 6 (1951) 420-434.