Families of Complementary Distributions


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Families of complementary distributions

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SUPPLEMENTARY MATERIAL: proofs of results in Section 3

Using

\[ E_H(T^2) = 1 - 2 \int_0^1 tH(t) \, dt \quad \text{and} \quad \int_0^1 tH^{-1}(t) \, dt = E_H\{T H(T)\}, \]

gives rise to the following proof:

**Proof of Theorem 3.5**

\[ E_{G_p}(X^2) = 1 - 2 \int_0^1 xG_p(x) \, dx \]

\[ = 1 - 2 \left\{ (1 - p)/3 + (2p - 1) \int_0^1 xM_p^{-1}(x) \, dx \right\} / p \]

\[ = 1 - 2 \left[ (1 - p)/3 + (2p - 1)E_{M_p}\{SM_p(S)\} \right] / p \]

\[ = 1 - 2 \left( (1 - p)/3 + (2p - 1) \times \right. \]

\[ \left. \left[ p^2E_F\{VF(V)\} + p(1 - p)\{1 + E_F(V^2)\}/2 + (1 - p)^2/3 \right] \right] / p \]

\[ = p(2p + 1)/3 - 2p(2p - 1)E_F\{VF(V)\} - (2p - 1)(1 - p)E_F(V^2). \]

The theorem results by elementary methods.

Using

\[ E_H(T^3) = 1 - 3 \int_0^1 t^2H(t) \, dt \quad \text{and} \quad \int_0^1 t^2H^{-1}(t) \, dt = E_H\{T H^2(T)\} \]

the proof of Theorem 3.6 ensues:

**Proof of Theorem 3.6**

\[ E_{G_p}(X^3) = 1 - 3 \int_0^1 x^2G_p(x) \, dx \]

\[ = 1 - 3 \left\{ (1 - p)/4 + (2p - 1) \int_0^1 x^2M_p^{-1}(x) \, dx \right\} / p \]

\[ = 1 - 3 \left[ (1 - p)/4 + (2p - 1)E_{M_p}\{SM_p^2(S)\} \right] / p \]

\[ = 1 - 3 \left( (1 - p)/4 + (2p - 1) \times \right. \]

\[ \left. \left[ (1 - p)(p^2 + 2p + 3)/12 + p^3E_F\{VF^2(V)\} + p^2(1 - p)E_F\{V^2F(V)\} ight. \right. \]

\[ \left. \left. + p(1 - p)^2E_F(V^3)/3 \right) \right] / p \]

\[ = p(1 + p + 2p^2)/4 - (2p - 1) \times \left[ 3p^2E_F\{VF^2(V)\} \right. \]

\[ \left. + 3p(1 - p)E_F\{V^2F(V)\} + (1 - p)^2E_F(V^3) \right]. \]
Therefore,
\[
S_{G_p}(X) = p(1 + p + 2p^2)/4 - (2p - 1) \times [3p^2E_F\{VF^2(V)\} + 3p(1 - p)E_F\{V^2F(V)\} + (1 - p)^2E_F(V^3)] \\
- 3\{p - (2p - 1)E_F(V)\} \times [p(2p + 1)/3 - (2p - 1) \{2pE\{VF(V)\} + (1 - p)E_F(V^2)\}] \\
+ 2 \left[p^3 - 3p^2(2p - 1)E_F(V) + 3p(2p - 1)^2\{E_F(V)\}^2 \\
- (2p - 1)^3\{E_F(V)\}^3\right]
\]

which can be rearranged to give the result of the theorem.

**Proof of Theorem 3.7**
\[
L_{2,G_p} = \int_0^1 \left[p(1 - p)u - (1 - p)^2u^2 - 2(2p - 1)(1 - p)uM_p^{-1}(u) \\
+ p(2p - 1)M_p^{-1}(u) - (2p - 1)^2\{M_p^{-1}(u)\}^2\right]/p^2du \\
= \left[p(1 - p)/2 - (1 - p)^2/3 - 2(2p - 1)(1 - p)E_{M_p}\{SM_p(S)\} \\
+ p(2p - 1)E_{M_p}(S) - (2p - 1)^2E_{M_p}(S^2)\right]/p^2 \\
= \left(p(1 - p)/2 - (1 - p)^2/3 - 2(2p - 1)(1 - p)\times \\
[p^2E_F\{VF(V)\} + p(1 - p)\{1 + E_F(V^2)\}/2 + (1 - p)^2/3] \\
+ p(2p - 1) \{pE_F(V) + 1 - p/2\} \\
- (2p - 1)^2 \{pE_F(V^2) + (1 - p)/3\}\right]/p^2.
\]

Further manipulation completes the proof.

**Proof of Theorem 3.8**
\[
L_{3,G_p} = \int_0^1 \left[3p(1 - p)^2u^2 - 2(1 - p)^3u^3 - p^2(1 - p)u \\
+ 6p(2p - 1)(1 - p)uM_p^{-1}(u) - 6(1 - p)^2(2p - 1)u^2M_p^{-1}(u) \\
- 6(1 - p)(2p - 1)^2u\{M_p^{-1}(u)\}^2 + 3p(2p - 1)^2\{M_p^{-1}(u)\}^2 \\
- 2(2p - 1)^3\{M_p^{-1}(u)\}^3 - p^2(2p - 1)M_p^{-1}(u)\right]/p^3du \\
= \left[p(1 - p)^2 - (1 - p)^3/2 - p^2(1 - p)/2 \\
+ 6p(2p - 1)(1 - p)E_{M_p}\{SM_p(S)\} - 6(1 - p)^2(2p - 1)E_{M_p}\{SM_p^2(S)\} \\
- 6(1 - p)(2p - 1)^2E_{M_p}\{S^2M_p(S)\} + 3p(2p - 1)^2E_{M_p}(S^2) \\
- 2(2p - 1)^3E_{M_p}(S^3) - p^2(2p - 1)E_{M_p}(S)\right]/p^3.
\]
\[
\begin{align*}
&= \frac{(1 - p)(4p - 4p^2 - 1)/2 + 6p(2p - 1)(1 - p)}{p^2 E_F\{V F(V)\}} \\
&\quad + \frac{p(1 - p)(1 + E_F(V^2))/2 + (1 - p)^2/3}{p^2 E_F(V^2)} - \frac{6(1 - p)^2(2p - 1)}{p^2 E_F(V^2)} \\
&\quad - \frac{6(1 - p)(2p - 1)}{p^2 E_F(V^2)} \\
&\quad + \frac{3p^3(2p - 1)}{p^2 E_F(V^2)} - \frac{2p^5(2p - 1)}{p^2 E_F(V^2)} \\
&\quad - \frac{p^3(2p - 1)}{p^2 E_F(V^2)} \\
&= \frac{p^4(1 - p)(1 - 2p)/2 + 6p^3(2p - 1)(1 - p)}{p^2 E_F(V^2)} \\
&\quad + \frac{3p^4(2p - 1)E_F(V^2) - 6p^3(1 - p)^2(2p - 1)}{p^2 E_F(V^2)} \\
&\quad - \frac{6p^4(1 - p)(2p - 1)}{p^2 E_F(V^2)} - \frac{2p^5(2p - 1)}{p^2 E_F(V^2)} \\
&\quad - \frac{p^3(2p - 1)}{p^2 E_F(V^2)} \\
\end{align*}
\]

and the result follows almost immediately.