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## Families of Complementary Distributions

### Journal Item

How to cite:

Jones, M. C. (2018). Families of Complementary Distributions. *Statistics & Probability Letters*, 141 pp. 74–81.

For guidance on citations see [FAQs](#).

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Version: Supplementary Material

Link(s) to article on publisher's website:  
<http://dx.doi.org/doi:10.1016/j.spl.2018.05.021>

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# Families of complementary distributions

M.C. Jones

## SUPPLEMENTARY MATERIAL: proofs of results in Section 3

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Using

$$E_H(T^2) = 1 - 2 \int_0^1 tH(t) dt \quad \text{and} \quad \int_0^1 tH^{-1}(t) dt = E_H\{TH(T)\},$$

gives rise to the following proof:

**Proof of Theorem 3.5**

$$\begin{aligned} E_{G_p}(X^2) &= 1 - 2 \int_0^1 xG_p(x)dx \\ &= 1 - 2 \left\{ (1-p)/3 + (2p-1) \int_0^1 xM_p^{-1}(x)dx \right\} / p \\ &= 1 - 2 \left[ (1-p)/3 + (2p-1)E_{M_p}\{SM_p(S)\} \right] / p \\ &= 1 - 2 \left( (1-p)/3 + (2p-1) \times \right. \\ &\quad \left. [p^2E_F\{VF(V)\} + p(1-p)\{1 + E_F(V^2)\}/2 + (1-p)^2/3] \right) / p \\ &= p(2p+1)/3 - 2p(2p-1)E_F\{VF(V)\} - (2p-1)(1-p)E_F(V^2). \end{aligned}$$

The theorem results by elementary methods.

Using

$$E_H(T^3) = 1 - 3 \int_0^1 t^2H(t) dt \quad \text{and} \quad \int_0^1 t^2H^{-1}(t) dt = E_H\{TH^2(T)\}$$

the proof of Theorem 3.6 ensues:

**Proof of Theorem 3.6**

$$\begin{aligned} E_{G_p}(X^3) &= 1 - 3 \int_0^1 x^2G_p(x)dx \\ &= 1 - 3 \left\{ (1-p)/4 + (2p-1) \int_0^1 x^2M_p^{-1}(x)dx \right\} / p \\ &= 1 - 3 \left[ (1-p)/4 + (2p-1)E_{M_p}\{SM_p^2(S)\} \right] / p \\ &= 1 - 3 \left( (1-p)/4 + (2p-1) \times \left[ (1-p)(p^2 + 2p + 3)/12 \right. \right. \\ &\quad \left. \left. + p^3E_F\{VF^2(V)\} + p^2(1-p)E_F\{V^2F(V)\} \right. \right. \\ &\quad \left. \left. + p(1-p)^2E_F(V^3)/3 \right] \right) / p \\ &= p(1+p+2p^2)/4 - (2p-1) \times \left[ 3p^2E_F\{VF^2(V)\} \right. \\ &\quad \left. + 3p(1-p)E_F\{V^2F(V)\} + (1-p)^2E_F(V^3) \right]. \end{aligned}$$

Therefore,

$$\begin{aligned}
S_{G_p}(X) &= p(1+p+2p^2)/4 - (2p-1) \times [3p^2 E_F\{VF^2(V)\} \\
&\quad + 3p(1-p)E_F\{V^2F(V)\} + (1-p)^2 E_F(V^3)] \\
&\quad - 3\{p - (2p-1)E_F(V)\} \times [p(2p+1)/3 \\
&\quad\quad - (2p-1)\{2pE\{VF(V)\} + (1-p)E_F(V^2)\}] \\
&\quad + 2[p^3 - 3p^2(2p-1)E_F(V) + 3p(2p-1)^2\{E_F(V)\}^2 \\
&\quad\quad - (2p-1)^3\{E_F(V)\}^3]
\end{aligned}$$

which can be rearranged to give the result of the theorem.

**Proof of Theorem 3.7**

$$\begin{aligned}
L_{2;G_p} &= \int_0^1 [p(1-p)u - (1-p)^2u^2 - 2(2p-1)(1-p)uM_p^{-1}(u) \\
&\quad + p(2p-1)M_p^{-1}(u) - (2p-1)^2\{M_p^{-1}(u)\}^2] / p^2 du \\
&= [p(1-p)/2 - (1-p)^2/3 - 2(2p-1)(1-p)E_{M_p}\{SM_p(S)\} \\
&\quad + p(2p-1)E_{M_p}(S) - (2p-1)^2E_{M_p}(S^2)] / p^2 \\
&= (p(1-p)/2 - (1-p)^2/3 - 2(2p-1)(1-p) \times \\
&\quad [p^2 E_F\{VF(V)\} + p(1-p)\{1 + E_F(V^2)\}/2 + (1-p)^2/3] \\
&\quad + p(2p-1)\{pE_F(V) + 1 - p/2\} \\
&\quad - (2p-1)^2\{pE_F(V^2) + (1-p)/3\}) / p^2.
\end{aligned}$$

Further manipulation completes the proof.

**Proof of Theorem 3.8**

$$\begin{aligned}
L_{3;G_p} &= \int_0^1 [3p(1-p)^2u^2 - 2(1-p)^3u^3 - p^2(1-p)u \\
&\quad + 6p(2p-1)(1-p)uM_p^{-1}(u) - 6(1-p)^2(2p-1)u^2M_p^{-1}(u) \\
&\quad - 6(1-p)(2p-1)^2u\{M_p^{-1}(u)\}^2 + 3p(2p-1)^2\{M_p^{-1}(u)\}^2 \\
&\quad - 2(2p-1)^3\{M_p^{-1}(u)\}^3 - p^2(2p-1)M_p^{-1}(u)] / p^3 du \\
&= [p(1-p)^2 - (1-p)^3/2 - p^2(1-p)/2 \\
&\quad + 6p(2p-1)(1-p)E_{M_p}\{SM_p(S)\} - 6(1-p)^2(2p-1)E_{M_p}\{SM_p^2(S)\} \\
&\quad - 6(1-p)(2p-1)^2E_{M_p}\{S^2M_p(S)\} + 3p(2p-1)^2E_{M_p}(S^2) \\
&\quad - 2(2p-1)^3E_{M_p}(S^3) - p^2(2p-1)E_{M_p}(S)] / p^3
\end{aligned}$$

$$\begin{aligned}
&= ((1-p)(4p-4p^2-1)/2 + 6p(2p-1)(1-p) [p^2 E_F\{VF(V)\} \\
&\quad + p(1-p)\{1 + E_F(V^2)\}/2 + (1-p)^2/3] - 6(1-p)^2(2p-1) \times \\
&\quad [(1-p)(p^2+2p+3)/12 + p^3 E_F\{VF^2(V)\} + p^2(1-p)E_F\{V^2F(V)\} \\
&\quad + p(1-p)^2 E_F(V^3)/3] - 6(1-p)(2p-1)^2 [p^2 E_F\{V^2F(V)\} \\
&\quad + p(1-p)\{1 + 2E_F(V^3)\}/3 + (1-p)^2/4] \\
&\quad + 3p(2p-1)^2 \{(1-p)/3 + pE_F(V^2)\} \\
&\quad - 2(2p-1)^3 \{(1-p)/4 + pE_F(V^3)\} \\
&\quad - p^2(2p-1) \{(1-p)/2 + pE_F(V)\} / p^3 \\
&= [p^4(1-p)(1-2p)/2 + 6p^3(2p-1)(1-p)E_F\{VF(V)\} \\
&\quad + 3p^4(2p-1)E_F(V^2) - 6p^3(1-p)^2(2p-1)E_F\{VF^2(V)\} \\
&\quad - 6p^4(1-p)(2p-1)E_F\{V^2F(V)\} - 2p^5(2p-1)E_F(V^3) \\
&\quad - p^3(2p-1)E_F(V)] / p^3
\end{aligned}$$

and the result follows almost immediately.