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# Substitution-based structures with absolutely continuous spectrum

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**Abstract**

By generalising Rudin's construction of an aperiodic sequence, we derive new substitution-based structures which have a purely absolutely continuous diffraction measure and a mixed dynamical spectrum, with absolutely continuous and pure point parts. We discuss several examples, including a construction based on Fourier matrices which yields constant-length substitutions for any length.

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## 1. Introduction

Substitution dynamical systems are widely used as toy models for aperiodic order in one dimension [1, 2]. By an argument of Dworkin [3], the diffraction spectrum of these systems is related to part of the dynamical spectrum, which is the spectrum of a unitary operator acting on a Hilbert space, as induced by the shift action. We refer the readers to [4] and references therein for recent developments and the current knowledge of the relationship between these different spectral characterisations. Here we are interested in systems that feature absolutely continuous spectra, in spite of being perfectly ordered.

A paradigm of such a system is the (binary) Rudin–Shapiro or Golay–Shapiro sequence.[1] It was introduced in [5, 6, 7] in answer to a question raised by Salem [6] in the context of harmonic analysis; see also [8, Sec. 4.7.1] and [9]. This sequence, represented by a Dirac comb with balanced weights ($\pm 1$), is a substitution-based structure with purely absolutely continuous diffraction spectrum, so it has a mixed dynamical spectrum, with a pure point part arising from the underlying constant-length substitution structure. Indeed, this deterministic sequence has the stronger property that its two-point correlations vanish exactly for *any* non-zero distance; a direct proof of this property can be found in [8,

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[1]For simplicity, we will refer to this sequence as the Rudin–Shapiro sequence, as this is the more commonly used term.

Sec. 10.2]; see also [1]. Recall that the diffraction measure is the Fourier transform of the autocorrelation measure, which in this case is just $\delta_0$ (a point measure located at the origin), so the diffraction measure is Lebesgue measure. Some generalisations of the Rudin–Shapiro sequence were provided in [10], but to date relatively few examples of substitution-based sequences of this type are known explicitly. There are good reasons for this, as one would expect a generic substitution-based structure to produce a singular continuous spectrum [11].

In [12], a systematic generalisation of the Rudin–Shapiro system to higher-dimensional substitutions was derived. It employs Hadamard matrices (matrices with elements $\pm 1$ whose rows are mutually orthogonal). The underlying systems are symbolic constant-length substitutions on a finite alphabet $\mathcal{A}$, based on arrangements of letters on the (hyper)cubic lattice $\mathbb{Z}^d$. Letters in the alphabet are paired, so for each letter $a \in \mathcal{A}$ there is a twin letter $\overline{a} \in \mathcal{A}$, with $\overline{a} \neq a$ and $\overline{\overline{a}} = a$. In particular, the author proved the following result, where $\mathbb{X}$ denotes the hull of the substitution, $\mu$ the corresponding invariant measure, $H_D$ the discrete spectrum, and $Z(f)$ the cyclic subspace associated to a function $f \in L^2(\mathbb{X}, \mu)$.

**Theorem 1.1** ([12]). *Let $(\mathbb{X}, \mathbb{Z}^d, \mu)$ be a dynamical system associated to an aperiodic $\mathbb{Z}^d$-substitution subject to the conditions that*[2]

- *each letter in the alphabet $\mathcal{A}$ is only allowed to appear in the position given by its underlying number (so the images of letters under the substitution differ only in the number and/or position of the bars that distinguish paired letters);*
- *paired letters are substituted by corresponding paired blocks;*
- *the symbol matrix of the corresponding substitution is a Hadamard matrix.*

*Then there exist functions $f_1, \ldots, f_K \in L^2(\mathbb{X}, \mu)$, each with spectral measure equal to Lebesgue measure, such that*

$$L^2(\mathbb{X}, \mu) \; = \; H_D \oplus Z(f_1) \oplus \cdots \oplus Z(f_K).$$

In this article we provide further examples of substitution-based structures with Lebesgue spectrum. These systems do not satisfy the last condition of Theorem 1.1, although they still appear to have a close relationship to Hadamard matrices and their complex analogues. Our approach is based on modifying and extending the original construction of Rudin [6]. As a consequence, our examples are sequences $(\varepsilon_n)_{n\in\mathbb{N}}$ that satisfy the property

$$\sup_{|x|=1} \left| \sum_{n=1}^{N} \varepsilon_n x^n \right| \; \leq \; C\, N^{\frac{1}{2}} \tag{1}$$

for some positive constant $C$, where the supremum is taken over complex numbers of unit modulus. We shall refer to this property as the *root-N property.* This bound on the

[2] We refer to the original article for more details on these conditions.

growth of the exponential sums implies that the corresponding diffraction measure is purely absolutely continuous; compare [10, 13]. Similar attempts to generalise the Rudin–Shapiro system, albeit from a different viewpoint, can be found in [14, 15].

We start by revisiting the recurrence relations that give rise to the Rudin–Shapiro sequence. We then generalise this approach in Section 3 by introducing a sequence of signs in the recurrence relations. This results in substitution-based structures in which the underlying substitutions are of length $2^k$ for $k \in \mathbb{N}$. In Section 4, we go a step further by considering recurrence relations with complex coefficients, which relate to Fourier matrices. In this case, we obtain new substitutions for *any* length $n \geq 3$, which give rise to weighted Dirac combs with (in the balanced weight case) purely absolutely continuous diffraction measure.

## 2. The Rudin–Shapiro sequence revisited

Let us briefly review Rudin's original construction of the Rudin–Shapiro (RS) sequence. For details of the construction, see [6].

We start by defining two sequences of polynomials, $(P_k(x))_{k\in\mathbb{N}_0}$ and $(Q_k(x))_{k\in\mathbb{N}_0}$, where $P_k$ and $Q_k$ both have degree $2^k$. They are determined by the initial choices $P_0(x) = Q_0(x) = x$ together with the recurrence relations

$$\begin{aligned} P_{k+1}(x) &= P_k(x) + x^{2^k} Q_k(x), \\ Q_{k+1}(x) &= P_k(x) - x^{2^k} Q_k(x). \end{aligned} \tag{2}$$

It is clear from Eq. (2) that the first $2^k$ terms of $P_{k+1}(x)$ and of $Q_{k+1}(x)$ coincide with those of $P_k(x)$, and that their remaining terms differ by a sign. By construction, $P_k(x)$ is of the form

$$P_k(x) = \sum_{n=1}^{2^k} \varepsilon_n \, x^n, \tag{3}$$

where each coefficient $\varepsilon_i$ is either $-1$ or 1, so we can define a binary sequence $(\varepsilon_n)_{n\in\mathbb{N}} \in \{\pm 1\}^{\mathbb{N}}$ from the coefficients. This is the binary RS sequence. For example, for $k = 3$, we have the polynomial

$$P_3(x) = x + x^2 + x^3 - x^4 + x^4(x + x^2 - x^3 + x^4),$$

from which we read off the sequence $111\bar{1}11\bar{1}1$, where here (and henceforth) we use the convention that $\bar{1} = -1$, with the implied assumption that $\overline{uv} = \bar{u}\,\bar{v}$ holds for all finite sequences (or words) $u$ and $v$. Here and in what follows, we will often switch between considering $(\varepsilon_n)_{n\in\mathbb{N}}$ as a binary sequence or as a word in the two-letter alphabet $\{1, \bar{1}\}$.

If $a_k = \varepsilon_1\varepsilon_2 \cdots \varepsilon_{2^k} \in \{\pm 1\}^{2^k}$ denotes the word of length $2^k$ of coefficients of $P_k(x)$, and $b_k$ denotes the corresponding word for $Q_k(x)$, then the recurrence relations of Eq. (2) correspond to the concatenation relations

$$\begin{aligned} a_{k+1} &= a_k b_k, \\ b_{k+1} &= a_k \overline{b_k}, \end{aligned} \tag{4}$$

on words in the two-letter alphabet $\{1, \overline{1}\}$, with initial values $a_0 = b_0 = 1$.

The concatenation relations (4) can be seen to correspond to the substitution rule $A \mapsto AB$, $B \mapsto A\overline{B}$ on the four-letter alphabet $\{A, B, \overline{A}, \overline{B}\}$, which upon completion to a four-letter substitution rule becomes

$$S_+\colon \quad A \mapsto AB, \quad B \mapsto A\overline{B}, \quad \overline{A} \mapsto \overline{AB}, \quad \overline{B} \mapsto \overline{A}B, \tag{5}$$

so that the 'bar' operation is compatible with the substitution; see [16] for more on substitutions that feature a 'bar-swap symmetry' of this kind. This substitution is often referred to as the four-letter RS substitution rule. Clearly, by induction, this rule gives rise to the concatenation relations

$$\begin{aligned} A_{k+1} &= A_k B_k, \\ B_{k+1} &= A_k \overline{B_k}, \end{aligned}$$

which have the same structure as Eq. (4), but work on the four-letter alphabet $\{A, B, \overline{A}, \overline{B}\}$ instead of the two-letter alphabet $\{1, \overline{1}\}$. The connection between the two is provided by the map, often referred to as a coding,

$$\varphi\colon \begin{cases} A, B \mapsto 1, \\ \overline{A}, \overline{B} \mapsto \overline{1}, \end{cases} \tag{6}$$

which defines a homomorphism from $\{A, B, \overline{A}, \overline{B}\}^{\mathbb{N}}$ to $\{1, \overline{1}\}^{\mathbb{N}}$.

Iterating the substitution rule $S_+$ (defined by Eq. (5)) on the initial letter $A$ (which turns out to be the relevant choice[3] in our case) gives

$$A \mapsto AB \mapsto ABA\overline{B} \mapsto ABA\overline{B}AB\overline{A}B \mapsto ABA\overline{B}AB\overline{A}BABA\overline{B}\overline{A}\overline{B}A\overline{B} \mapsto \cdots \longrightarrow w_+,$$

which converges (in the local topology) to an infinite fixed point word $w_+$. We denote the corresponding one-sided hull, which is the closure of the orbit of $w_+$ under the shift map, by $\mathbb{X}_+$. Here, the (left) shift map $T$ acts on an infinite symbolic sequence $w = w_1 w_2 w_3 w_4 \cdots$ as $w \mapsto Tw$ with

$$(Tw)_n = w_{n+1} \quad \text{for all } n \in \mathbb{N}. \tag{7}$$

The binary RS sequence is then recovered as the image of $w_+$ under the factor map $\varphi$ of Eq. (6), which reproduces the sequence $(\varepsilon_n)_{n\in\mathbb{N}} \in \{\pm 1\}^{\mathbb{N}}$. Note that there is no two-letter substitution rule for this sequence, unless you work with a staggered substitution with different rules for even and odd positions along the word; see [8, Sec. 4.7.1].

The main ingredient in Rudin's proof [6] of the root-$N$ property (1) for the binary sequence is the parallelogram law,

$$|\alpha + \beta|^2 + |\alpha - \beta|^2 \;=\; 2|\alpha|^2 + 2|\beta|^2, \tag{8}$$

[3] Here and below we use the initial letter $A$ to construct a fixed point sequence $w$. There will always be a second fixed point sequence, which due to the bar-swap symmetry of our substitutions is just $\overline{w}$, which can be obtained by iterating $S_+$ on the initial letter $\overline{A}$.

where $\alpha, \beta \in \mathbb{C}$, and this will also be the case in our generalisations discussed below. It follows that the consequences for spectral properties specifically apply to the binary sequence $(\varepsilon_n)_{n\in\mathbb{N}} \in \{\pm 1\}^{\mathbb{N}}$; the argument does not directly provide information about the spectral properties of the underlying four-letter sequence obtained from the substitution rule of Eq. (5).

## 3. Modifying Rudin's construction

Let us now introduce some modifications to the original construction of Rudin, and show that our newly derived recurrence relations still satisfy the root-$N$ property of Eq. (1). Following this, we compute some concrete examples and derive the corresponding substitution systems, in the same way as for the RS sequence above.

Our approach to modifying the RS sequence is somewhat similar to that of [14, 15], but framed in a different context.

We again work with two sequences of polynomials $(P_k(x))_{k\in\mathbb{N}_0}$ and $(Q_k(x))_{k\in\mathbb{N}_0}$, with $P_0(x) = Q_0(x) = x$. By introducing additional signs $\sigma_k \in \{\pm 1\}$ in the recurrence relations of Eq. (2), we consider

$$\begin{aligned} P_{k+1}(x) &= P_k(x) + \sigma_k\, x^{2^k} Q_k(x), \\ Q_{k+1}(x) &= P_k(x) - \sigma_k\, x^{2^k} Q_k(x), \end{aligned} \tag{9}$$

for $k \in \mathbb{N}_0$. At this stage, we do not yet specify the values of $\sigma_k$.

Clearly, the RS case corresponds to the choice $\sigma_k = 1$ for all $k \in \mathbb{N}_0$. If instead one chooses $\sigma_k = -1$ for all $k \in \mathbb{N}_0$, the recurrence relations correspond to the substitution

$$S_-: \quad A \mapsto A\overline{B}, \quad B \mapsto AB, \quad \overline{A} \mapsto \overline{A}B, \quad \overline{B} \mapsto \overline{AB}. \tag{10}$$

Its one-sided fixed point $w_-$, obtained by iterating $S_-$ on the letter $A$,

$$A \mapsto A\overline{B} \mapsto A\overline{BAB} \mapsto A\overline{BABA}B\overline{AB} \mapsto A\overline{BABA}B\overline{ABA}BAB\overline{A}B\overline{AB} \mapsto \cdots \longrightarrow w_-,$$

gives rise to the (one-sided) hull $\mathbb{X}_-$. It is easy to verify that $\mathbb{X}_+ \neq \mathbb{X}_-$, since there are subwords of length six in $w_+$ (such as $BABA\overline{BA}$ or $BAB\overline{A}BA$) which do not occur as subwords of $w_-$, and vice versa. Indeed, the same holds true for the corresponding binary sequences and their hulls $\mathbb{Y}_+ := \varphi(\mathbb{X}_+)$ and $\mathbb{Y}_- := \varphi(\mathbb{X}_-)$. For example, $\varphi(BABA\overline{BA}) = 1111\overline{11}$ is a subword of $\varphi(w_+)$ but not of $\varphi(w_-)$, as $ABAB$ does not appear in either $w_+$ or $w_-$. Observe that, in fact, $\varphi$ induces a bijection between the hulls, so $\mathbb{X}_+$ and $\mathbb{Y}_+$ (and also $\mathbb{X}_-$ and $\mathbb{Y}_-$) are mutually locally derivable, and the corresponding four-letter and two-letter dynamical systems (under the shift action) are topologically conjugate; compare [8, Rem. 4.11]. This can, for instance, be seen by realising that the subword 1111, which occurs in both $\varphi(w_+)$ and $\varphi(w_-)$ with bounded gaps, has the unique preimage $BABA$ in both $w_+$ and $w_-$.

One can also verify that the substitution matrices for $S_+$ and $S_-$ have different eigenvalues ($2$, $\pm\sqrt{2}$ and $0$ for $S_+$ and $2$, $1 \pm \mathrm{i}$ and $0$ for $S_-$), so the corresponding substitution dynamical systems cannot be conjugate (as they have different dynamical zeta functions). However, this does not answer the question whether $\mathbb{X}_+$ and $\mathbb{X}_-$ are mutually locally derivable or not, because this difference vanishes if you look at the eighth power of the substitutions.

**Proposition 3.1.** *The sequence of coefficients* $(\varepsilon_n)_{n\in\mathbb{N}}$ *of the functions* $P_k$, $k \in \mathbb{N}_0$, *defined by the recurrence relations of Eq.* (9), *satisfies the root-*$N$ *property of Eq.* (1).

*Proof.* The proof proceeds by induction. Consider the case $|x| = 1$. By the recurrence relations (9), we have

$$|P_{k+1}(x)|^2 + |Q_{k+1}(x)|^2 \;=\; |P_k(x) + \sigma_k\, x^{2^k} Q_k(x)|^2 + |P_k(x) - \sigma_k\, x^{2^k} Q_k(x)|^2.$$

Applying the parallelogram law (8), we find that

$$|P_{k+1}(x)|^2 + |Q_{k+1}(x)|^2 \;=\; 2\big(|P_k(x)|^2 + |Q_k(x)|^2\big).$$

Since $|P_0(x)|^2 + |Q_0(x)|^2 = 2$, we can conclude that

$$|P_k(x)|^2 + |Q_k(x)|^2 \;=\; 2^{k+1},$$

and hence

$$|P_k(x)| \;\le\; 2^{\frac{k+1}{2}}. \tag{11}$$

This proves the root-$N$ property for $N = 2^k$.

In order to tackle the case when $N$ is not necessarily a power of 2, we define partial sums of $P_k$ and $Q_k$ as follows,

$$P_{k|m}(x) \;=\; \sum_{n=1}^{m} \varepsilon_n x^n, \quad Q_{k|m}(x) \;=\; \sum_{n=1}^{m} \gamma_n x^n,$$

where $2^{k-1} < m \le 2^k$, $k \in \mathbb{N}_0$, and where $\varepsilon_n, \gamma_n \in \{\pm 1\}$ are the corresponding coefficients. We now show that these satisfy

$$\big|P_{k|m}(x)\big| \;\le\; G\,2^{\frac{k}{2}} \quad \text{and} \quad \big|Q_{k|m}(x)\big| \;\le\; G\,2^{\frac{k}{2}} \tag{12}$$

for all $|x| = 1$ and $k \in \mathbb{N}_0$, where $G = 2 + 2^{1/2}$.

The above estimates are obviously true for $k = 0$. Suppose that they hold for some $k \in \mathbb{N}_0$, and consider an integer $m$ with $2^k < m \le 2^{k+1}$. By using the triangle inequality together with Eqs. (11) and (12), we obtain

$$\big|P_{k+1|m}(x)\big| \;\le\; \big|P_k(x)\big| + \big|Q_{k|m-2^k}(x)\big| \;\le\; 2^{\frac{k+1}{2}} + G\,2^{\frac{k}{2}} \;=\; G\,2^{\frac{k+1}{2}},$$

which establishes Eq. (12) for $k + 1$. The same argument clearly works for $Q_{k+1|m}(x)$.

To complete the proof, suppose that $2^{k-1} < N \le 2^k$. By Eq. (12), we have

$$\big|P_{k|N}(x)\big| \;\le\; (2 + 2^{\frac{1}{2}})2^{\frac{k}{2}} \;\le\; 2(1 + 2^{\frac{1}{2}})N^{\frac{1}{2}},$$

which shows that the root-$N$ property holds. □

**Corollary 3.2.** *Whatever the choice of the signs* $\sigma_k \in \{\pm 1\}$ *in Eq.* (9)*, the corresponding sequence* $(\varepsilon_n)_{n\in\mathbb{N}}$ *is balanced.*

*Proof.* The average value of the first $N$ coefficents is given by

$$\frac{1}{N}\sum_{n=1}^{N}\varepsilon_n \;=\; \frac{1}{N}P_{k|N}(1)$$

for $2^{k-1} < N \leq 2^k$. By Proposition 3.1, this satisfies

$$\left|\frac{1}{N}\sum_{n=1}^{N}\varepsilon_n\right| \;\leq\; 2(1+2^{\frac{1}{2}})N^{-\frac{1}{2}}$$

and therefore the average value tends to 0 as $N \to \infty$. $\square$

As mentioned earlier, the root-$N$ property implies absolute continuity of the diffraction measure for the binary sequence.

**Corollary 3.3.** *For any sequence of coefficients* $(\varepsilon_n)_{n\in\mathbb{N}}$ *as in Proposition* 3.1*, the corresponding Dirac comb* $\sum_{n\in\mathbb{N}}\varepsilon_n\delta_n$ *has purely absolutely continuous diffraction measure.* $\square$

We now consider some examples.

**Example 3.1.** Let us start with the choice $\sigma_k = (-1)^{k+1}$, so the signs in the recurrence relations for the polynomials alternate, and we have

$$\begin{aligned} P_{k+1}(x) \;&=\; P_k(x) + (-1)^{k+1}x^{2^k}Q_k(x), \\ Q_{k+1}(x) \;&=\; P_k(x) - (-1)^{k+1}x^{2^k}Q_k(x), \end{aligned}$$

for $k \in \mathbb{N}_0$. We could now read off the corresponding substitution rule just as we did for the RS substitution, but this case is more complicated because of the alternating signs. One way to overcome this problem is to look at two consecutive steps at once,

$$\begin{aligned} P_{k+2}(x) \;&=\; P_k(x) + (-1)^{k+1}x^{2^k}Q_k(x) + (-1)^{k+2}x^{2\cdot 2^k}P_k(x) + x^{3\cdot 2^k}Q_k(x), \\ Q_{k+2}(x) \;&=\; P_k(x) + (-1)^{k+1}x^{2^k}Q_k(x) - (-1)^{k+2}x^{2\cdot 2^k}P_k(x) - x^{3\cdot 2^k}Q_k(x). \end{aligned} \tag{13}$$

Choosing $k$ to be even (which corresponds to the case we are interested in, since our recursion starts with $k = 0$) and associating letters $A$ and $B$, and their counterparts $\overline{A}$ and $\overline{B}$, to the sequences corresponding to $P$ and $Q$, we obtain the substitution rule

$$S_{-+}\colon \quad A \mapsto A\overline{B}AB, \quad B \mapsto A\overline{BAB}, \quad \overline{A} \mapsto \overline{A}B\overline{AB}, \quad \overline{B} \mapsto \overline{A}BAB. \tag{14}$$

This is a substitution of constant length four, because we used a double step of the recursion, and Eq. (13) corresponds to concatenation of four sets of coefficients. As before, a one-sided fixed point sequence $w_{-+}$ is obtained from iterating the substitution on the initial letter $A$,

$$A \mapsto A\overline{B}AB \mapsto A\overline{B}AB\overline{A}BABA\overline{B}ABA\overline{BAB} \mapsto \cdots \longrightarrow w_{-+}.$$

By mapping $A$, $B$ to $1$ and $\overline{A}$, $\overline{B}$ to $\overline{1} = -1$ using the map $\varphi$ of Eq. (6), we obtain the binary sequence $v_{-+} = \varphi(w_{-+}) = 1\overline{1}11\overline{1}1111\overline{1}111\overline{1}\overline{1}\overline{1}\cdots$ as our new RS-type sequence.

Alternatively, one can see the substitution $S_{-+}$ as the composition of the two substitutions $S_+$ and $S_-$ from Eqs. (5) and (10), in the sense that $S_{-+} = S_- \circ S_+$. To see this explicitly, let us verify the composition on the letters $A$ and $B$,

$$\begin{aligned} A &\xrightarrow{S_+} AB \xrightarrow{S_-} A\overline{B}AB, \\ B &\xrightarrow{S_+} A\overline{B} \xrightarrow{S_-} A\overline{BAB}. \end{aligned}$$

The images of $\overline{A}$ and $\overline{B}$ can be computed using the bar-swap symmetry.

From Proposition 3.1, we conclude that the binary sequence $v_{-+} = \varphi(w_{-+})$ satisfies the root-$N$ property, and hence the corresponding diffraction measure is absolutely continuous.

Our next example is closely related. We again alternate the signs in the recursion, but shifted by one. Maybe surprisingly, this produces a different sequence of coefficients.

**Example 3.2.** Here we choose $\sigma_k = (-1)^k$. The recurrence relations are now

$$\begin{aligned} P_{k+1}(x) &= P_k(x) + (-1)^k x^{2^k} Q_k(x), \\ Q_{k+1}(x) &= P_k(x) - (-1)^k x^{2^k} Q_k(x), \end{aligned}$$

for $k \in \mathbb{N}_0$. Using the same approach as above, this gives rise to the substitution rule

$$S_{+-}\colon \quad A \mapsto AB\overline{A}B, \quad B \mapsto ABA\overline{B}, \quad \overline{A} \mapsto \overline{AB}A\overline{B}, \quad \overline{B} \mapsto \overline{ABA}B. \tag{15}$$

This rule can also be expressed as the composition of the two substitution systems $S_+$ and $S_-$, this time as $S_{+-} = S_+ \circ S_-$, because

$$\begin{aligned} A &\xrightarrow{S_-} A\overline{B} \xrightarrow{S_+} AB\overline{A}B, \\ B &\xrightarrow{S_-} AB \xrightarrow{S_+} ABA\overline{B}, \end{aligned}$$

and the relations for the barred letters follow by bar-swap symmetry. Again, Proposition 3.1 shows that the corresponding binary sequence $v_{+-} = \varphi(w_{+-})$, where $w_{+-}$ denotes the fixed point of $S_{+-}$ obtained by iterating $S_{+-}$ on the letter $A$, satisfies the root-$N$ property and hence gives rise to a Dirac comb with absolutely continuous diffraction measure.

The observations of Examples 3.1 and 3.2 show that alternating the sign in the recurrence relations corresponds to the composition of the two substitutions $S_+$ and $S_-$. Clearly, the two substitutions $S_{-+}$ and $S_{+-}$ and their respective one-sided hulls are closely related.

**Lemma 3.4.** *The hulls $\mathbb{X}_{-+}$ and $\mathbb{X}_{+-}$ of the substitutions $S_{-+}$ and $S_{+-}$ defined by Eqs.* (14) *and* (15) *satisfy the relations*

$$\begin{aligned}\mathbb{X}_{+-} &= S_{+}(\mathbb{X}_{-+}) \cup TS_{+}(\mathbb{X}_{-+}),\\ \mathbb{X}_{-+} &= S_{-}(\mathbb{X}_{+-}) \cup TS_{-}(\mathbb{X}_{+-}),\end{aligned}$$

*where $T$ denotes the shift map on $\{A, \overline{A}, B, \overline{B}\}^{\mathbb{N}}$; see Eq.* (7)*.*

*Proof.* Let $w_{-+}$ and $w_{+-}$ denote fixed point sequences for $S_{-+}$ and $S_{+-}$. Then the (one-sided) hulls $\mathbb{X}_{-+}$ and $\mathbb{X}_{+-}$ are the closures of the orbits of the fixed points under the shift map $T$. Now, $S_{-+} = S_{-} \circ S_{+}$ implies that

$$S_{-+}(S_{-}w_{+-}) \;=\; \big(S_{-} \circ S_{+} \circ S_{-}\big)\, w_{+-} \;=\; S_{-}(S_{+-}w_{+-}) \;=\; S_{-}w_{+-},$$

which shows that $S_{-}w_{+-}$ is a fixed point of $S_{-+}$. Similarly, since $S_{+-} = S_{+} \circ S_{-}$, we have

$$S_{+-}(S_{+}w_{-+}) \;=\; \big(S_{+} \circ S_{-} \circ S_{+}\big)\, w_{-+} \;=\; S_{+}(S_{-+}w_{-+}) \;=\; S_{+}w_{-+},$$

and consequently $S_{+}w_{-+}$ is a fixed point of $S_{+-}$.

Since the substitutions $S_{+}$ and $S_{-}$ have constant length two, we have

$$S_{+} \circ T \;=\; T^{2} \circ S_{+} \quad \text{and} \quad S_{-} \circ T \;=\; T^{2} \circ S_{-}, \tag{16}$$

which implies that $S_{+}(\mathbb{X}_{-+})$ is the subset of $\mathbb{X}_{+-}$ of all sequences starting with a letter $A$ or $\overline{A}$, since only even shifts are included. By continuity of the action, limits are included, so the closure does not add any additional elements. Hence the union $S_{+}(\mathbb{X}_{-+}) \cup TS_{+}(\mathbb{X}_{-+})$ gives the complete hull $\mathbb{X}_{+-}$, and the analogous result holds for the case where the signs are interchanged. □

Notice that, despite this close connection, the two hulls $\mathbb{X}_{-+}$ and $\mathbb{X}_{+-}$ are indeed different, as can be verified by considering words of length six. Note also that the eigenvalues of the substitution matrices of $S_{-+}$ and $S_{+-}$ are again different; they are 4, 2 (twice) and 0 for $S_{-+}$, and 4, $\pm 2$ and 0 for $S_{+-}$. The question of whether the two hulls are mutually locally derivable remains open.

Still, the following result on induced systems [17, 18] shows that the two systems are intimately linked.

**Proposition 3.5.** $(\mathbb{X}_{-+}, T)$ *is conjugate to the induced system of* $(\mathbb{X}_{+-}, T)$ *on the subset* $S_{+}(\mathbb{X}_{-+})$*, and* $(\mathbb{X}_{+-}, T)$ *is conjugate to the induced system of* $(\mathbb{X}_{-+}, T)$ *on the subset* $S_{-}(\mathbb{X}_{+-})$*.*

*Proof.* Here we prove the first claim; the second follows analogously. As mentioned above, $S_{+}(\mathbb{X}_{-+}) = [A] \cup [\,\overline{A}\,] \subset \mathbb{X}_{+-}$, where the brackets denote cylinder sets of words starting with the given letter. Now, consider the return time function [17, Sec. 2.2], that is, the return time of the fixed point generated by $S_{+-}$ to the clopen set $[A] \cup [\,\overline{A}\,]$,

$$r_{[A]\cup[\overline{A}]} = \inf\{n > 0 : T^{n}(w_{+-}) \in [A] \cup [\,\overline{A}\,]\}.$$

As $S_+$ is a substitution of length two, each letter is mapped into a length two word starting with $A$ or $\overline{A}$ and it follows that $r_{[A]\cup[\overline{A}]} = 2$. The induced map is then given by $T^2$, which maps the set $[A] \cup [\,\overline{A}\,]$ onto itself; compare Eq. (16). Hence, $([A] \cup [\,\overline{A}\,], T^2)$ is the induced system. As $S_+$ is an injective map from $\mathbb{X}_{-+}$ to $S_+(\mathbb{X}_{-+})$, the claimed conjugacy follows [18, Sec. 2.1]. □

By using the following result, the induced systems inherit the spectral properties of the conjugated systems.

**Theorem 3.6** ([19, Thm. 2.9]). *Let $T_i$ with $i \in \{1, 2\}$ be measure-preserving transformations of probability spaces. If $T_1$ and $T_2$ are conjugate, then they are spectrally isomorphic.*

Note that, as previously, the four-letter hull $\mathbb{X}_{-+}$ and two-letter hull $\varphi(\mathbb{X}_{-+})$ are mutually locally derivable (as are $\mathbb{X}_{+-}$ and $\varphi(\mathbb{X}_{+-})$), and the corresponding dynamical systems are hence topologically conjugate. The argument is the same as above; the subword 1111, which occurs in both $\varphi(w_{-+})$ and $\varphi(w_{+-})$ with bounded gaps, has the unique preimage $BABA$ in both $w_{-+}$ and $w_{+-}$.

The observations of Examples 3.1 and 3.2 suggest the following general picture.

**Proposition 3.7.** *Let $(\sigma_k)_{k\in\mathbb{N}} \in \{\pm 1\}^{\mathbb{N}_0}$ be a given sequence. Then, for any $k \in \mathbb{N}_0$, the sequence of coefficients of the polynomial $P_k$ defined by Eq.* (9) *is the image under the map $\varphi$ of $A_k = S_{\sigma_0} \circ S_{\sigma_1} \circ \cdots \circ S_{\sigma_{k-1}} A$.*

*Proof.* Let $a_k := \varepsilon_1 \varepsilon_2 \cdots \varepsilon_{2^k} \in \{\pm 1\}^{2^k}$ denote the word of length $2^k$ of coefficients of $P_k(x)$, and let $b_k$ denote the corresponding word for $Q_k(x)$. Then the recurrence relations of Eq. (9) correspond to the concatenation relations

$$a_{k+1} = \begin{cases} a_k b_k, & \sigma_k = 1, \\ a_k \overline{b_k}, & \sigma_k = -1, \end{cases} \qquad b_{k+1} = \begin{cases} a_k \overline{b_k}, & \sigma_k = 1, \\ a_k b_k, & \sigma_k = -1, \end{cases}$$

with initial values $a_0 = b_0 = 1$. These recurrence relations correspond to the substitution rule $S_{\sigma_k}$, and by induction we obtain $a_k = \varphi(A_k)$ with

$$A_k = S_{\sigma_0} \circ \cdots \circ S_{\sigma_{k-1}} A$$

for any $k \in \mathbb{N}$. □

Clearly, if we choose $\sigma_k = 1$ for all $k \in \mathbb{N}_0$, we are back at the RS case with substitution $S_+$. More generally, for any periodic sequence we have the following result.

**Corollary 3.8.** *Let $(\sigma_k)_{k\in\mathbb{N}_0} \in \{\pm 1\}^{\mathbb{N}_0}$ be a periodic sequence of period $p$, so $\sigma_{k+p} = \sigma_k$ for all $k \in \mathbb{N}_0$. Then, the sequence of coefficients of the polynomials $P_k$ defined by Eq.* (9) *is the image under the map $\varphi$ of the fixed point of the substitution*

$$S_{\sigma_0\sigma_1\cdots\sigma_{p-1}} := S_{\sigma_0} \circ S_{\sigma_1} \circ \cdots \circ S_{\sigma_{p-1}}$$

*with initial letter $A$.*

*Proof.* As the sequence of signs $\sigma_k$ is periodic with period $p$, Proposition 3.7 implies that

$$A_{np} = (S_{\sigma_0} \circ \cdots \circ S_{\sigma_{p-1}})^n A$$

holds for $n \in \mathbb{N}$, and the assertion follows. $\square$

According to the terminology of S. Ferenczi [20] (see also [21, 22]), infinite sequences obtained in the manner of Proposition 3.7 are called '$S$-adic expansions'. Even in this framework, our construction still gives convergence in the local topology to well-defined binary or quaternary sequences. Our quaternary sequence is an $S$-adic limit word of the 'directive sequence' $(S_{\sigma_k})_{k\in\mathbb{N}_0}$. However, it is no longer a fixed point of a primitive substitution of finite length, so we do not know much about the corresponding one-sided hull. Nevertheless, the root-$N$ property and hence absolute continuity of the spectral measure also holds in this case. Note that this construction is different from the notion of random substitutions introduced in [23] and recently considered in [24, 25], where several substitution rules are mixed at a *local* level, in the sense that the substitution rule is chosen independently for each letter of a sequence.

**Example 3.3.** Let us consider one more example, with

$$\sigma_k = \begin{cases} 1, & \text{if } k \equiv 0, 1 \bmod 3, \\ -1, & \text{if } k \equiv 2 \bmod 3. \end{cases}$$

From Proposition 3.7, we know that the corresponding substitution is $S_{++-} = S_+ \circ S_+ \circ S_-$, which turns out to be

$$A \xrightarrow{S_-} A\overline{B} \xrightarrow{S_+} AB\overline{A}B \xrightarrow{S_+} ABA\overline{B}\overline{A}\overline{B}A\overline{B},$$
$$B \xrightarrow{S_-} AB \xrightarrow{S_+} ABA\overline{B} \xrightarrow{S_+} ABA\overline{B}AB\overline{A}B,$$

together with the corresponding relations for the barred letters.

**Proposition 3.9.** *Let $(\sigma_k)_{k\in\mathbb{N}} \in \{\pm 1\}^{\mathbb{N}}$ be a periodic sequence of period $p$ and $S_{\sigma_1\sigma_2\cdots\sigma_p}$ be the corresponding substitution according to Corollary 3.8. Its hull $\mathbb{X}_{\sigma_1\sigma_2\cdots\sigma_p}$ is then mutually locally derivable with $\varphi(\mathbb{X}_{\sigma_1\sigma_2\cdots\sigma_p})$.*

*Proof.* Local derivability of the two-letter sequence from the four-letter sequence is clear, as $\varphi$ acts locally.

To show local derivability of the four-letter sequence, note that $BABA$ is a legal four-letter word for $S_+$ and $S_-$ as well as for $S_{-+} = S_- \circ S_+$ and $S_{+-} = S_+ \circ S_-$. Hence, it is also legal for $S_{\sigma_1\sigma_2\cdots\sigma_p} = S_{\sigma_1} \circ S_{\sigma_2} \circ \cdots \circ S_{\sigma_p}$, and occurs with bounded gaps in any element of the hull $\mathbb{X}_{\sigma_1\sigma_2\cdots\sigma_p}$ by repetitivity of the hull. Since $\varphi(BABA) = 1111$, the latter also occurs with bounded gaps in any element of the two-letter hull.

Now, observe that $ABAB$ is not a legal word for $S_+$ or $S_-$ (as its pre-image would have to be $AA$ or $BB$), or for $S_{-+} = S_- \circ S_+$ or $S_{+-} = S_+ \circ S_-$. As a consequence, it cannot occur as a legal word for $S_{\sigma_1\sigma_2\cdots\sigma_p} = S_{\sigma_1} \circ S_{\sigma_2} \circ \cdots \circ S_{\sigma_p}$ either.

Hence $BABA$ is the unique pre-image of 1111, and local derivability follows. $\square$

## 4. Generalizing Rudin's argument to Fourier matrices

We are now going to generalise Rudin's argument further by considering *complex* coefficients in our polynomials, which will naturally lead us to look at Fourier matrices. Here, the Fourier matrix of order $n$ is the unitary $n\times n$ matrix with elements $\frac{1}{\sqrt{n}}\exp\bigl(\frac{2\pi\mathrm{i}(j-1)(k-1)}{n}\bigr)$, where $1 \le j,k \le n$. The matrices that are going to enter below will be $x$-dependent generalisations of these Fourier matrices, without the normalisation factor $1/\sqrt{n}$.

Our main result in this section, Theorem 4.1, is similar to results of [15], but framed in the context of substitution dynamics rather than Fourier analysis.

It will be convenient to express the recurrence relations (2) in terms of matrices as follows,

$$\begin{pmatrix} P_{k+1}(x) \\ Q_{k+1}(x) \end{pmatrix} \;=\; \begin{pmatrix} 1 & x^{2^k} \\ 1 & -x^{2^k} \end{pmatrix} \begin{pmatrix} P_k(x) \\ Q_k(x) \end{pmatrix} \;=\; A^{(2,k)} \begin{pmatrix} P_k(x) \\ Q_k(x) \end{pmatrix}.$$

Now, for $n > 2$, consider a vector of $n$ polynomials

$$v_k \;=\; \begin{pmatrix} P^{(1)}_k(x) \\ \vdots \\ P^{(n)}_k(x) \end{pmatrix}$$

satisfying the recurrence relation

$$v_{k+1} \;=\; A^{(n,k)} v_k, \tag{17}$$

with initial condition $v_0 = (x,\dots,x)^t$. Here, $A^{(n,k)}$ is the $n\times n$ matrix

$$A^{(n,k)} \;=\; \begin{pmatrix} 1 & x^{n^k} & \cdots & x^{(n-1)n^k} \\ 1 & \omega x^{n^k} & \cdots & \omega^{n-1} x^{(n-1)n^k} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \omega^{n-1} x^{n^k} & \cdots & \omega^{(n-1)^2} x^{(n-1)n^k} \end{pmatrix},$$

where $\omega = \exp(2\pi\mathrm{i}/n)$. For $x = 1$, $A^{(n,k)}$ reduces to the $n\times n$ Fourier matrix, apart from the normalisation factor $1/\sqrt{n}$. As a consequence, for $|x| = 1$, the matrix satisfies $(A^{(n,k)})^{\dagger} A^{(n,k)} = n\,\mathbf{1}^{(n)}$, where $\mathbf{1}^{(n)}$ denotes the $n\times n$ identity matrix and $M^{\dagger} = \overline{M}^{t}$ denotes the Hermitian adjoint of the matrix (or vector) $M$.

Generalising Eq. (3), we can now define a sequence $\bigl(\varepsilon_m\bigr)_{m\in\mathbb{N}}$ of complex coefficients $\varepsilon_m \in \bigl\{\omega^j \mid 0 \le j \le n-1\bigr\}$ by

$$P^{(1)}_k(x) \;=\; \sum_{m=1}^{n^k} \varepsilon_m x^m. \tag{18}$$

We will show that these sequences also satisfy the root-$N$ property of Eq. (1).

**Theorem 4.1.** *The sequence of coefficicents $\bigl(\varepsilon_m\bigr)_{m\in\mathbb{N}}$ of the functions $P^{(1)}_k$ satisfies the root-$N$ property of Eq. (1).*

*Proof.* The proof proceeds by induction. We want to derive a bound for $|P_{k+1}^{(1)}(x)|$. To do so, we express the sum of the squared norms of the polynomials $P_{k+1}^{(1)}(x), \ldots, P_{k+1}^{(n)}(x)$ as

$$v_{k+1}^{\dagger} v_{k+1} \;=\; \sum_{j=1}^{n} \left|P_{k+1}^{(j)}(x)\right|^2.$$

Using the recurrence relation and the identity $(A^{(n,k)})^{\dagger} A^{(n,k)} = n\, \mathbf{1}^{(n)}$, we obtain

$$v_{k+1}^{\dagger} v_{k+1} \;=\; v_k^{\dagger} (A^{(n,k)})^{\dagger} A^{(n,k)} v_k \;=\; n v_k^{\dagger} v_k \;=\; n \left( \sum_{j=1}^{n} \left|P_k^{(j)}(x)\right|^2 \right).$$

This shows that

$$\sum_{j=1}^{n} \left|P_{k+1}^{(j)}(x)\right|^2 = n \left( \sum_{j=1}^{n} \left|P_k^{(j)}(x)\right|^2 \right).$$

Since we have $\sum_{j=1}^{n} |P_0^{(j)}(x)|^2 = n$ by the initial conditions, we conclude by induction that

$$\sum_{j=1}^{n} \left|P_k^{(j)}(x)\right|^2 \;=\; n^{k+1}.$$

Hence we get the bound

$$\left|P_k^{(j)}(x)\right| \;\leq\; n^{\frac{k+1}{2}}, \tag{19}$$

and in particular $\left|P_k^{(1)}(x)\right| \;\leq\; n^{\frac{1}{2}} n^{\frac{k}{2}}$, which proves the root-$N$ property for $N = n^k$.

It remains to prove the property for other values of $N$. The argument is similar to that used in the proof of Proposition 3.1. Let $P_{k|m}^{(j)}$ denote the $m$-th partial sum of $P_k^{(j)}$ for $1 \leq j \leq n$, where $n^{k-1} < m \leq n^k$. We will prove by induction that these functions satisfy

$$\left|P_{k|m}^{(j)}(x)\right| \;\leq\; G\, n^{\frac{k}{2}} \tag{20}$$

for all $|x| = 1$ and $k \in \mathbb{N}_0$, where $G = n + n^{\frac{1}{2}}$.

Clearly, this estimate is true if $k = 0$. Suppose now that Eq. (20) holds for some $k \in \mathbb{N}_0$, and consider an integer $m$ with $n^k < m \leq n^{k+1}$. For $n^k < m \leq 2n^k$, by using the recursion (17) as well as the triangle inequality together with Eqs. (19) and (20), we obtain

$$\left|P_{k+1|m}^{(j)}(x)\right| \;\leq\; \left|P_k^{(1)}(x)\right| + \left|\omega^{j-1} x^{n^k} P_{k|m-n^k}^{(2)}(x)\right| \;\leq\; n^{\frac{k+1}{2}} + G\, n^{\frac{k}{2}} \;\leq\; G\, n^{\frac{k+1}{2}}$$

for all $1 \leq j \leq n$.

Similarly, we can derive bounds for the cases where $\ell n^k < m \leq (\ell+1)n^k$ for all $1 \leq \ell \leq n-1$, where more and more terms contribute. We obtain

$$\begin{aligned}
\left|P^{(j)}_{k+1|m}(x)\right| &= \left|\sum_{r=1}^{\ell} \omega^{(r-1)(j-1)} x^{(r-1)n^k} P^{(r)}_{k}(x) \,+\, \omega^{\ell(j-1)} x^{\ell n^k} P^{(\ell+1)}_{k|m-\ell n^k}(x)\right| \\
&\leq \sum_{r=1}^{\ell} \left|\omega^{(r-1)(j-1)} x^{(r-1)n^k} P^{(r)}_{k}(x)\right| \,+\, \left|\omega^{\ell(j-1)} x^{\ell n^k} P^{(\ell+1)}_{k|m-\ell n^k}(x)\right| \\
&\leq \sum_{r=1}^{\ell} \left|P^{(r)}_{k}(x)\right| \,+\, \left|P^{(\ell+1)}_{k|m-\ell n^k}(x)\right| \\
&\leq \ell\, n^{\frac{k+1}{2}} + G\, n^{\frac{k}{2}} \\
&\leq \left((n-1) + (n^{\frac{1}{2}}+1)\right) n^{\frac{k+1}{2}} \,=\, G\, n^{\frac{k+1}{2}} ,
\end{aligned}$$

which completes the induction argument.

To finish the proof, suppose that $n^{k-1} < N \leq n^k$. By Eq. (20), we have

$$\left|P^{(1)}_{k|N}(x)\right| \,\leq\, (n + n^{\frac{1}{2}}) n^{\frac{k}{2}} \,\leq\, n(n^{\frac{1}{2}}+1) N^{\frac{1}{2}},$$

which shows that the root-$N$ property holds. □

**Corollary 4.2.** *For any series of coefficients $(\varepsilon_n)_{n\in\mathbb{N}}$ as in Theorem 4.1, the corresponding Dirac comb $\sum_{n\in\mathbb{N}} \varepsilon_n \delta_n$ has purely absolutely continuous diffraction measure.* □

Note that the case $n = 2$ corresponds to Eq. (2), which is the RS case. Let us now look at a couple of examples.

**Example 4.1.** Consider the case $n = 3$. We start by setting $P_0(x) = Q_0(x) = R_0(x) = x$ and define polynomials $P_k$, $Q_k$ and $R_k$ recursively by

$$\begin{pmatrix} P_{k+1}(x) \\ Q_{k+1}(x) \\ R_{k+1}(x) \end{pmatrix} = \begin{pmatrix} 1 & x^{3^k} & x^{2\cdot 3^k} \\ 1 & \omega x^{3^k} & \omega^2 x^{2\cdot 3^k} \\ 1 & \omega^2 x^{3^k} & \omega x^{2\cdot 3^k} \end{pmatrix} \begin{pmatrix} P_k(x) \\ Q_k(x) \\ R_k(x) \end{pmatrix},$$

where $k \in \mathbb{N}_0$ and $\omega = \exp(2\pi\mathrm{i}/3)$. From Theorem 4.1, we know that the corresponding sequence of coefficients satisfies the root-$N$ property. This is now a ternary sequence in the alphabet $\{1, \omega, \omega^2\}$.

As above, we can connect this to a substitution rule, where we now need nine letters. We denote these by $A$, $B$ and $C$ as well as the corresponding letters with a single or double bar. Here, $A$, $B$ and $C$ correspond to the coefficients of the polynomials $P$, $Q$ and $R$, respectively, while the barred versions describe the multiplication by $\omega$ (single bar) and $\omega^2$ (double bar). Accordingly, we have $\overline{\overline{\overline{A}}} = A$ and similarly for the other letters.

The structure of the matrix $A^{(3,k)}$ yields the following substitution rule

$$A \mapsto A\,B\,C, \qquad B \mapsto A\,\overline{B}\,\overline{\overline{C}}, \qquad C \mapsto A\,\overline{\overline{B}}\,\overline{C},$$

and the corresponding rules for the barred letters, leading to the nine-letter substitution

$$\begin{array}{lllllllll}
A & \mapsto & A\,B\,C, & \overline{A} & \mapsto & \overline{A}\,\overline{B}\,\overline{C}, & \overline{\overline{A}} & \mapsto & \overline{\overline{A}}\,\overline{\overline{B}}\,\overline{\overline{C}}, \\
B & \mapsto & A\,\overline{B}\,\overline{\overline{C}}, & \overline{B} & \mapsto & \overline{A}\,\overline{\overline{B}}\,C, & \overline{\overline{B}} & \mapsto & \overline{\overline{A}}\,B\,\overline{C}, \\
C & \mapsto & A\,\overline{\overline{B}}\,\overline{C}, & \overline{C} & \mapsto & \overline{A}\,B\,\overline{\overline{C}}, & \overline{\overline{C}} & \mapsto & \overline{\overline{A}}\,\overline{B}\,C.
\end{array}$$

This is a substitution of length 3, which has a fixed point obtained by iteration on the initial letter $A$,

$$\begin{aligned}
A &\mapsto A\,B\,C \mapsto A\,B\,C\,A\,\overline{B}\,\overline{\overline{C}}\,A\,\overline{\overline{B}}\,\overline{C} \\
&\mapsto A\,B\,C\,A\,\overline{B}\,\overline{\overline{C}}\,A\,\overline{\overline{B}}\,\overline{C}\,A\,B\,C\,\overline{A}\,\overline{\overline{B}}\,C\,\overline{\overline{A}}\,\overline{B}\,C\,A\,B\,C\,\overline{\overline{A}}\,B\,\overline{C}\,\overline{A}\,B\,\overline{\overline{C}} \mapsto \cdots
\end{aligned}$$

This fixed point is mapped to the ternary sequence of coefficients of $P_k(x)$ by the factor map

$$\varphi^{(3)}\colon \begin{cases} A, B, C \mapsto 1, \\ \overline{A}, \overline{B}, \overline{C} \mapsto \omega, \\ \overline{\overline{A}}, \overline{\overline{B}}, \overline{\overline{C}} \mapsto \omega^2. \end{cases}$$

By Corollary 4.2, we know that the weighted Dirac comb corresponding to this sequence has absolutely continuous diffraction measure.

Reasoning as before, we see that the three and nine-letter sequences are in fact mutually locally derivable. Here it again suffices to consider words of length four, many of which only have a single pre-image under $\varphi^{(3)}$. An example is $111\omega^2$, for which the only pre-image is $ABC\overline{\overline{A}}$. Due to repetitiveness, we can therefore determine the three sublattices locally, and hence locally derive the nine-letter sequence.

**Example 4.2.** For our final example, we consider the case $n = 4$. Our recurrence relations are given by

$$\begin{pmatrix} P_{k+1}(x) \\ Q_{k+1}(x) \\ R_{k+1}(x) \\ S_{k+1}(x) \end{pmatrix} = \begin{pmatrix} 1 & x^{4^k} & x^{2\cdot 4^k} & x^{3\cdot 4^k} \\ 1 & \mathrm{i}x^{4^k} & -x^{2\cdot 4^k} & -\mathrm{i}x^{3\cdot 4^k} \\ 1 & -x^{4^k} & x^{2\cdot 4^k} & -x^{3\cdot 4^k} \\ 1 & -\mathrm{i}x^{4^k} & -x^{2\cdot 4^k} & +\mathrm{i}x^{3\cdot 4^k} \end{pmatrix} \begin{pmatrix} P_k(x) \\ Q_k(x) \\ R_k(x) \\ S_k(x) \end{pmatrix}$$

with initial conditions $P_0(x) = Q_0(x) = R_0(x) = S_0(x) = x$. In this case, we obtain the 16-letter substitution

$$A \mapsto A\,B\,C\,D, \qquad B \mapsto A\,\overline{B}\,\overline{\overline{C}}\,\overline{\overline{\overline{D}}}, \qquad C \mapsto A\,\overline{\overline{B}}\,C\,\overline{\overline{D}}, \qquad D \mapsto A\,\overline{\overline{\overline{B}}}\,\overline{\overline{C}}\,\overline{D},$$

together with the corresponding rules for the barred letters. The factor map becomes

$$\varphi^{(4)}\colon\ \begin{cases} A,B,C,D \mapsto 1, \\ \overline{A},\overline{B},\overline{C},\overline{D} \mapsto \mathrm{i}, \\ \overline{\overline{A}},\overline{\overline{B}},\overline{\overline{C}},\overline{\overline{D}} \mapsto -1, \\ \overline{\overline{\overline{A}}},\overline{\overline{\overline{B}}},\overline{\overline{\overline{C}}},\overline{\overline{\overline{D}}} \mapsto -\mathrm{i}. \end{cases}$$

As before, the four-letter and 16-letter sequences are mutually locally derivable, and again it is possible to find words of length 4 that only have a single ancestor under $\varphi^{(4)}$. One example is $111\overline{1}$ (where $\overline{1} = -1$) whose ancestor is $BCD\overline{\overline{A}}$.

In the same way, starting from the $n\times n$ Fourier matrix, we can construct substitution rules for any $n > 1$, which all have absolute continuous components in their spectra. The general structure is clear from the examples above. The substitutions act on $n^2$ letters, with $n$ 'basic' letters that appear in $n$ different 'flavours', each distinguished by the number of bars, from 0 to $n-1$. The distribution of bars in the image of the four basic letters can be read off directly from the Fourier matrix, and the remainder of the substitution is then fixed by cyclic symmetry under the bar operation. The corresponding factor map $\varphi^{(n)}$ identifies all basic letters and the image only depends on the number of bars, giving the corresponding power of $\exp(2\pi\mathrm{i}/n)$.

Theorem 4.1 shows that the sequences of complex numbers obtained by applying the factor map $\varphi^{(n)}$ satisfy the root-$N$ property, and hence the corresponding Dirac comb has purely absolutely continuous diffraction measure. We conjecture that the dynamical spectrum of the $n^2$-letter hull of the substitution just contains the absolute continuous component and the pure point component corresponding to the maximum equicontinuous factor, which is the corresponding solenoid.

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