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# Enumerations of maximum partial triple systems on 16 and 17 points

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## Abstract

For  $v \equiv 1$  or  $3 \pmod{6}$ , maximum partial triple systems on  $v$  points are Steiner triple systems,  $\text{STS}(v)$ s. The 80 non-isomorphic  $\text{STS}(15)$ s were first enumerated around 100 years ago, but the next case for Steiner triple systems was unresolved until around 2004 when it was established that there are precisely 11 084 874 829 non-isomorphic  $\text{STS}(19)$ s. In this paper we complete enumeration of non-isomorphic maximum partial triple systems for  $v \leq 19$ . It is shown that there are 35 810 097 systems on 17 points and 47 744 568 on 16 points. We also establish that there are precisely 157 151 non-isomorphic pairwise balanced designs,  $\text{PBD}(17, \{3, 5\})$ s, having a single block of size 5. Structural properties of these systems are determined, including their automorphism groups, and the numbers of Pasch configurations, mitres and Fano planes contained in them. The systems themselves are available from the authors.

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**Keywords:** Maximum partial triple system; Steiner triple system; Pairwise balanced design; Pasch configuration; mitre configuration.

## 1 Introduction

We will be concerned with certain types of combinatorial designs, in particular partial triple systems. A *partial triple system* of order  $v$  and index  $\lambda$ ,  $\text{PTS}(v, \lambda)$ , is a pair  $(V, \mathcal{B})$ , where  $V$  is a set of  $v$  elements (the *points*) and  $\mathcal{B}$  is a collection of 3-element subsets of  $V$  (the *triples* or *blocks*) with the property that every 2-element subset of  $V$  occurs at most  $\lambda$  times amongst the triples of  $\mathcal{B}$ . A  $\text{PTS}(v, \lambda) = (V, \mathcal{B})$  is said to be *maximal* if there is no other  $\text{PTS}(v, \lambda) = (V, \mathcal{B}')$  with  $\mathcal{B} \subseteq \mathcal{B}'$ . Informally, a  $\text{PTS}(v, \lambda) = (V, \mathcal{B})$  is maximal if no further triples can be added to  $\mathcal{B}$  without violating the condition that pairs appear at most  $\lambda$  times. A maximal  $\text{PTS}(v, \lambda)$  is denoted as  $\text{MPTS}(v, \lambda)$ . A *maximum*  $\text{MPTS}(v, \lambda)$ , denoted as  $\text{MMPTS}(v, \lambda)$ , is an  $\text{MPTS}(v, \lambda)$  having the greatest number of triples of any  $\text{MPTS}(v, \lambda)$ . Rather than calling an  $\text{MMPTS}(v, \lambda)$  a “maximum maximal partial triple system”, we prefer to use the term “maximum partial triple system”. In the case  $\lambda = 1$ , we will denote these systems as  $\text{PTS}(v)$ ,  $\text{MPTS}(v)$  and  $\text{MMPTS}(v)$  respectively, and note that  $\mathcal{B}$  is then necessarily a *set* of triples, i.e. no triple can be repeated. The *leave* of an  $\text{PTS}(v)$  (or  $\text{MPTS}(v)$ ,  $\text{MMPTS}(v)$ ) is the set of pairs of points that do not appear in any triple of the system.

If  $K$  is a set of positive integers, then a *pairwise balanced design*,  $\text{PBD}(v, K, \lambda)$ , of order  $v$  and index  $\lambda$ , is a pair  $(V, \mathcal{B})$ , where  $V$  is a set of  $v$  elements (the *points*) and  $\mathcal{B}$  is a collection of subsets of  $V$  (the *blocks*) with the properties that every 2-element subset of  $V$  occurs precisely  $\lambda$  times amongst the blocks of  $\mathcal{B}$ , and if  $B \in \mathcal{B}$  then  $|B| \in K$ . The elements of  $K$  are called the block sizes. If  $\lambda = 1$ , such a system is denoted more simply as  $\text{PBD}(v, K)$ . If the leave of a  $\text{PTS}(v)$  is combined with its set of triples, then a pairwise balanced design  $\text{PBD}(v, \{2, 3\})$  is obtained. Thus there is a rough equivalence between pairwise balanced designs with block sizes 2 and 3, and partial triple systems.

A *Steiner triple system* of order  $v$ ,  $\text{STS}(v)$ , is an  $\text{MMPTS}(v)$  in which each pair of points appears in some triple, that is to say the leave is empty. It is well known that the necessary and sufficient condition for the existence of an  $\text{STS}(v)$  is that  $v \equiv 1$  or  $3 \pmod{6}$ , and these values are called *admissible*. When  $v \not\equiv 1$  or  $3 \pmod{6}$ , an  $\text{STS}(v)$  does not exist and an  $\text{MMPTS}(v)$  is, in a sense, the closest we can get to a Steiner triple system. When  $v \equiv 5 \pmod{6}$  an  $\text{MMPTS}(v)$  necessarily has a leave comprising four pairs of the form  $\{a, b\}, \{b, c\}, \{c, d\}, \{d, a\}$  (see [1, page 553]). In this case, taking the pairs as edges of a graph on the four points, the leave can be represented as a 4-cycle  $(a, b, c, d)$ . For the cases  $v \equiv 0$  or  $2 \pmod{6}$ , an  $\text{MMPTS}(v)$  corresponds to an  $\text{STS}(v+1)$  in which one point  $a$  has been deleted, resulting in a leave comprising  $v/2$  disjoint pairs. For the case  $v \equiv 4 \pmod{6}$  a similar correspondence holds between an  $\text{MMPTS}(v)$  and an  $\text{MMPTS}(v+1)$  where a point  $a$  from the leave

of the latter has been deleted giving the former a leave consisting of the three intersecting pairs  $\{b, c\}$ ,  $\{c, d\}$ ,  $\{c, e\}$  and, disjoint from these, a further  $(v-4)/2$  disjoint pairs.

It should be noted that maximum partial triple systems, MMPTS( $v$ )s, are also called *optimal*  $2-(v, 3, 1)$  *packings* (see [1, Section VI.40]). For  $v \equiv 0, 1, 2, 3 \pmod{6}$ , they are additionally known as  $(v_r, b_3)$  *configurations*, where  $b$  is the number of triples and  $r$  is the number of triples containing each point (see [1, Section VI.7]).

Two designs,  $(V_1, \mathcal{B}_1)$  and  $(V_2, \mathcal{B}_2)$  (of the same order  $v$ ) are said to be isomorphic if there is a mapping  $\phi$  from  $V_1$  onto  $V_2$  that takes blocks of  $\mathcal{B}_1$  onto those of  $\mathcal{B}_2$ . Such a mapping  $\phi$  is called an isomorphism. An isomorphism from a design to itself is called an automorphism, and the set of all automorphisms of a given design forms a group under composition. This group is referred to as the automorphism group of the design. In discussing enumerations, it is generally the number of isomorphism classes of each design that is sought, in which case the point set may be fixed, for example as the set of integers  $\{0, 1, \dots, v-1\}$ .

As regards enumeration of these designs for small orders  $v$ , the situation for Steiner triple systems (up to isomorphism) is as follows. There are unique STS( $v$ )s for  $v = 3$ ,  $v = 7$  and  $v = 9$ . There are two STS(13)s and 80 STS(15)s; the latter result [19] dates from 1919 and it was certainly not trivial to obtain this in the days prior to the advent of electronic computers. It took a further 85 years before the STS(19)s were enumerated in a paper by Kaski and Östergård [13]; there are 11 084 874 829 such systems and this result was obtained by a computer search. It seems unlikely that the next case, STS(21), will be enumerated any time soon since it is estimated that the number of these is several orders of magnitude greater than in the STS(19) case. It is known that the number of STS( $v$ )s is  $v^{v^2(\frac{1}{6}-o(1))}$  as  $v \rightarrow \infty$  [20] (see also [3] Section 5.2.), a result that extends to MMPTS( $v$ )s.

In the case when  $v \equiv 5 \pmod{6}$ , the MMPTS(5) is unique up to isomorphism, but there are two MMPTS(11)s [2] and these are given in Table 1. There are generally two types of MMPTS( $v$ ) designs for  $v \equiv 5 \pmod{6}$ . The first type is when the leave  $\{a, b\}$ ,  $\{b, c\}$ ,  $\{c, d\}$ ,  $\{d, a\}$  is accompanied by a pair of intersecting triples of the design,  $\{a, c, e\}$  and  $\{b, d, e\}$ . Putting the 6 pairs from these two triples together with the leave gives a copy of the complete graph  $K_5$ , and we will refer to this as the *quintuple* case, and such designs as being of Type-Q. The second type is when the leave is accompanied by a pair of disjoint triples of the design,  $\{a, c, e\}$  and  $\{b, d, f\}$  ( $e \neq f$ ), and we will refer to this as the *non-quintuple* case, and such designs as being of Type-N. In Table 1, the design  $Q$  is of Type-Q, and the design  $N$  is of Type-N. In both cases the point set is  $\{0, 1, \dots, 10\}$  (with 10 denoted by  $A$ ) and the leave is the 4-cycle  $(0, 1, 2, 3)$ . Here and elsewhere, when listing pairs and triples, set brackets and/or commas may be omitted, so that  $\{a, b, c\}$  may be written as  $abc$ . A reader with some patience can construct these two designs by hand and verify that there is precisely one of each type up to isomorphism.

Given the correspondence between MMPTS( $v$ )s and MMPTS( $v+1$ )s, we be-

| System | triples  |
|--------|--|
| $Q$    | 024, 056, 078, 09A, 134, 159, 168, 17A, 257, 269, 28A, 35A, 367, 389, 458, 46A, 479. |
| $N$    | 024, 057, 068, 09A, 135, 148, 169, 17A, 259, 26A, 278, 34A, 367, 389, 456, 479, 58A. |

Table 1: The two maximum partial triple systems of order 11.

gin by enumerating MMPTS(17)s. Our results then enable us to enumerate the MMPTS(16)s. We first show that there are precisely 35 810 097 MMPTS(17)s. We find that there are 2 350 733 MMPTS(17)s of Type-Q, and 33 459 364 MMPTS(17)s of Type-N. These results were obtained by a computer search described in Section 2. The next value of  $v \equiv 5 \pmod{6}$  is  $v = 23$ . The number of MMPTS(23)s is likely to be further orders of magnitude greater than the number of STS(21)s and so will probably remain out of reach for a considerable period of time.

There is a general construction of MMPTS( $v$ )s for  $v \equiv 5 \pmod{6}$  from PBD( $v, \{3, 5\}$ )s having precisely one block of size 5; such designs (denoted as PBD( $v, \{3, 5^*\}$ )s) exist for all  $v \equiv 5 \pmod{6}$  (see [3] Section 6). If the block of size 5 is  $\{a, b, c, d, e\}$ , this can be replaced with four pairs  $ab, bc, cd$  and  $da$  and two triples  $ace$  and  $bde$ . There are fifteen distinct ways to do this, and the resulting systems may or may not be isomorphic. Conversely, an MMPTS( $v$ ) of Type-Q with leave  $(a, b, c, d)$ , and triples  $ace$  and  $bde$  can be converted to a PBD( $v, \{3, 5^*\}$ ) having precisely one block of size 5. We take advantage of this correspondence in Section 4 to enumerate PBD(17,  $\{3, 5^*\}$ )s.

In the analysis of MMPTS(17)s (and other designs), the numbers of Pasch and mitre configurations are determined. A *Pasch* configuration in a PTS( $v$ ), also known as a *quadrilateral*, comprises a set of four triples of the system on six points, having the form  $abc, ade, bdf, cef$ . A *mitre* configuration has five triples on seven points, having the form  $abc, ade, afg, bdf, ceg$ ; the point  $a$  that appears in three triples will be called the *root* of the mitre and the other six points will be called the *leaves*. An  $(r, s)$ -configuration in a PTS( $v$ ) is a configuration having  $r$  triples and  $s$  points. So a Pasch configuration is a  $(4, 6)$ -configuration, and in fact the only one. A mitre is a  $(5, 7)$ -configuration and there is one other, obtained from a Pasch configuration by adding a triple  $afg$ . It is a longstanding conjecture regarding Steiner triple systems that, given  $r \geq 4$ , there exists  $v_r$  such that for all admissible  $v \geq v_r$ , there exists an STS( $v$ ) having no  $(r', r' + 2)$ -configurations for all  $r'$  with  $4 \leq r' \leq r$  [6]. Such an STS( $v$ ) is said to be *r-sparse*. A 4-sparse system is therefore one without Pasch configurations, and is also known as an *anti-Pasch* or *quadrilateral-free* system. An *anti-mitre* system is one without mitres, and a 5-sparse system is one without both mitres and Pasch configurations, equivalently it is both anti-Pasch and anti-mitre. It is

known that there are anti-Pasch STS( $v$ )s for every admissible  $v \neq 7, 13$  [16, 10], and an anti-mitre STS( $v$ ) for every admissible  $v \neq 9$  [22]. It was shown in [21] that 5-sparse STS( $v$ ) exist for almost all admissible  $v$ , and their existence for orders  $v \equiv 3 \pmod{6}$  ( $v \neq 9, 15$ ) was established in [23]. Further results are given in [7], but the exact set of values  $v$  for which 5-sparse STS( $v$ )s exist remains undetermined.

Given an MMPTS(17) and one of the points of its leave, by deleting this point and the 7 triples that contain it, an MMPTS(16) is formed. Since all MMPTS(16)s may be formed in this way, it is possible to determine all such systems from the MMPTS(17)s. After applying an isomorphism check, we establish that there are 47 744 568 MMPTS(16)s. Details of these systems are given in Section 5.

## 2 Searching for MMPTS(17)s

### 2.1 Seeding the search

The point set of each design was taken to be  $V = \{0, 1, \dots, 16\}$  and the leave was taken to be  $\{01, 03, 12, 23\}$ . With equipment available to us at the present time it was not feasible to search all possible triples on  $V$  exhaustively, and this would anyway have been a colossal waste of resources. Instead, a small number of partial designs (*seeds*) were constructed that, without loss of generality, could potentially be extended to provide collectively at least one representative of each isomorphism class of MMPTS(17)s. These seeds were formed as follows.

Consider first the non-quintuple case. Without loss of generality, the leave and some of the triples of a non-quintuple MMPTS(17) may be taken as:

$$\{0, 1\}, \{0, 3\}, \{1, 2\}, \{2, 3\}, \{0, 2, 4\}, \{1, 3, 5\}, \{0, 5, 6\}, \\ \{0, 7, 8\}, \{0, 9, 10\}, \{0, 11, 12\}, \{0, 13, 14\}, \{0, 15, 16\}, \{2, 5, 7\}.$$

Again without loss of generality, the design may be taken to contain either the triple  $\{2, 6, 8\}$  or  $\{2, 6, 9\}$ . For the first of these possibilities, the next triple may be taken to be  $\{2, 9, 11\}$ , while for the second possibility the next triple may be taken as either  $\{2, 8, 10\}$  or  $\{2, 8, 11\}$ , all without loss of generality. The design requires three further blocks containing the point 2. By considering the possibilities it will be seen that, without loss of generality, the design must be an extension of one of the five possibilities  $N_1, N_2, N_3, N_4$  and  $N_5$  shown in Table 2.

Next consider the quintuple case. The argument is almost identical to the non-quintuple case. The only difference is that the triple  $\{1, 3, 5\}$  is replaced by the triple  $\{1, 3, 4\}$ . Without loss of generality, the design must be an extension of one of the five possibilities  $Q_1, Q_2, Q_3, Q_{2a}$  and  $Q_4$  shown in Table 3. The reason that the fourth of these alternatives is denoted  $Q_{2a}$  is that there exists an isomorphism of this design and  $Q_2$ . An isomorphism taking  $Q_2$  to  $Q_{2a}$  is given by the permutation  $(5, 13)(6, 14)(7, 15, 10)(8, 16, 9)$ . Consequently, design  $Q_{2a}$  is not required as a seed in an exhaustive search.

| $N_1$     | $N_2$     | $N_3$     | $N_4$     | $N_5$     |
|-----------|-----------|-----------|-----------|-----------|
| 0, 2, 4   | 0, 2, 4   | 0, 2, 4   | 0, 2, 4   | 0, 2, 4   |
| 1, 3, 5   | 1, 3, 5   | 1, 3, 5   | 1, 3, 5   | 1, 3, 5   |
| 0, 5, 6   | 0, 5, 6   | 0, 5, 6   | 0, 5, 6   | 0, 5, 6   |
| 0, 7, 8   | 0, 7, 8   | 0, 7, 8   | 0, 7, 8   | 0, 7, 8   |
| 0, 9, 10  | 0, 9, 10  | 0, 9, 10  | 0, 9, 10  | 0, 9, 10  |
| 0, 11, 12 | 0, 11, 12 | 0, 11, 12 | 0, 11, 12 | 0, 11, 12 |
| 0, 13, 14 | 0, 13, 14 | 0, 13, 14 | 0, 13, 14 | 0, 13, 14 |
| 0, 15, 16 | 0, 15, 16 | 0, 15, 16 | 0, 15, 16 | 0, 15, 16 |
| 2, 5, 7   | 2, 5, 7   | 2, 5, 7   | 2, 5, 7   | 2, 5, 7   |
| 2, 6, 8   | 2, 6, 8   | 2, 6, 9   | 2, 6, 9   | 2, 6, 9   |
| 2, 9, 11  | 2, 9, 11  | 2, 8, 10  | 2, 8, 11  | 2, 8, 11  |
| 2, 10, 12 | 2, 10, 13 | 2, 11, 13 | 2, 10, 12 | 2, 10, 13 |
| 2, 13, 15 | 2, 12, 15 | 2, 12, 15 | 2, 13, 15 | 2, 12, 15 |
| 2, 14, 16 | 2, 14, 16 | 2, 14, 16 | 2, 14, 16 | 2, 14, 16 |

Table 2: Seeds for the computer search in the non-quintuple case.

| $Q_1$     | $Q_2$     | $Q_3$     | $Q_{2a}$  | $Q_4$     |
|-----------|-----------|-----------|-----------|-----------|
| 0, 2, 4   | 0, 2, 4   | 0, 2, 4   | 0, 2, 4   | 0, 2, 4   |
| 1, 3, 4   | 1, 3, 4   | 1, 3, 4   | 1, 3, 4   | 1, 3, 4   |
| 0, 5, 6   | 0, 5, 6   | 0, 5, 6   | 0, 5, 6   | 0, 5, 6   |
| 0, 7, 8   | 0, 7, 8   | 0, 7, 8   | 0, 7, 8   | 0, 7, 8   |
| 0, 9, 10  | 0, 9, 10  | 0, 9, 10  | 0, 9, 10  | 0, 9, 10  |
| 0, 11, 12 | 0, 11, 12 | 0, 11, 12 | 0, 11, 12 | 0, 11, 12 |
| 0, 13, 14 | 0, 13, 14 | 0, 13, 14 | 0, 13, 14 | 0, 13, 14 |
| 0, 15, 16 | 0, 15, 16 | 0, 15, 16 | 0, 15, 16 | 0, 15, 16 |
| 2, 5, 7   | 2, 5, 7   | 2, 5, 7   | 2, 5, 7   | 2, 5, 7   |
| 2, 6, 8   | 2, 6, 8   | 2, 6, 9   | 2, 6, 9   | 2, 6, 9   |
| 2, 9, 11  | 2, 9, 11  | 2, 8, 10  | 2, 8, 11  | 2, 8, 11  |
| 2, 10, 12 | 2, 10, 13 | 2, 11, 13 | 2, 10, 12 | 2, 10, 13 |
| 2, 13, 15 | 2, 12, 15 | 2, 12, 15 | 2, 13, 15 | 2, 12, 15 |
| 2, 14, 16 | 2, 14, 16 | 2, 14, 16 | 2, 14, 16 | 2, 14, 16 |

Table 3: Seeds for the computer search in the quintuple case.

## 2.2 The searching programs

Given one of the nine seeds described in the previous section, the problem is to cover all uncovered pairs (apart from those in the leave) by triples. This can be represented as a graph covering problem by taking a graph  $G$  on the points  $0, 1, \dots, 16$  whose edges are the pairs that are neither in the leave nor covered by triples in the seed. A solution corresponds to covering each edge of  $G$  precisely once with triangles of edges from  $G$ , and the program must find all possible solutions. A program written by the current authors was used for this purpose. If a point is reached where an uncovered pair has no remaining triples available to cover it, the program backtracks. If it reaches a point where all pairs are covered, it records the solution and then backtracks. The principal difficulty is adjusting the records for backtracks. In each of the nine cases, the program took about 90 minutes on a desktop computer to enumerate the solutions. It was then adapted to test solutions for isomorphisms and to record solutions as described below. It is prudent to verify computational results independently and so a minor variant of Knuth's *dancing links* graph covering program [14, 15] was used to check the number of solutions from each of the 9 runs; each of these checks took about 8 minutes.

The outcomes of the nine runs are shown in Table 4. The total number

| Seed  | Number of designs | Seed  | Number of designs |
|-------|-------------------|-------|-------------------|
| $N_1$ | 161 885 696       | $Q_1$ | 134 263 296       |
| $N_2$ | 165 881 472       | $Q_2$ | 140 978 304       |
| $N_3$ | 166 118 112       | $Q_3$ | 142 761 312       |
| $N_4$ | 170 428 416       | $Q_4$ | 144 371 376       |
| $N_5$ | 171 571 376       | Total | 562 374 288       |
| Total | 835 885 072       |       |                   |

Table 4: Search results for the numbers of MMPTS(17)s.

of systems obtained in the nine runs was 1 398 259 360. No solution for the quintuple case can be isomorphic with any solution from the non-quintuple case, but within each class there are many isomorphisms.

## 2.3 Isomorphism testing

The software package *nauty* [18] was used to analyse the search outputs for isomorphisms. This treats each solution as a bipartite graph obtained from the design with an edge joining point  $x$  to triple  $B$  if and only if  $x \in B$ . Two designs are isomorphic if and only if the two bipartite graphs are isomorphic. Nauty assigns a canonical labelling to each graph so that two graphs are isomorphic if and only if they have the same canonical labelling. The canonical labellings



were stored, as they were produced, in a sorted list. The time taken to insert a new labelling into such a list is proportional to the logarithm of the number of objects in the list. If the labelling was already present in the list, the system was immediately recognised as an isomorphic copy and so could be discarded. Nauty is a long-established package with an excellent record for reliability, so there is a high degree of confidence in the results. Nevertheless, steps were taken to check these results.

Given a solution produced by the searching program, three numbers were obtained for each point  $x$  of the design: the number  $n_p(x)$  of Pasch configurations incident with  $x$ , the number of mitres  $n_r(x)$  with  $x$  as a root, and the number of mitres  $n_l(x)$  with  $x$  as a leaf. In the quintuple case, any isomorphism between two solutions must preserve the point sets  $V_1 = \{0, 1, 2, 3\}$ ,  $V_2 = \{4\}$ ,  $V_3 = \{5, 6 \dots, 16\}$ , and in the non-quintuple case any isomorphism must preserve the point sets  $V_1 = \{0, 1, 2, 3\}$ ,  $V_2 = \{4, 5\}$ ,  $V_3 = \{6, 7 \dots, 16\}$ . Consequently the three sets of ordered triples  $\{(n_p(x), n_r(x), n_l(x)) : x \in V_i\}$  for  $i = 1, 2, 3$  form an invariant in each case. Using this invariant, it was easy to determine whether or not two solutions with the same invariant were isomorphic since the number of possible isomorphisms was small in every case. Samples of the search output were taken and the results produced by consideration of this invariant were found to agree with those produced by nauty. It had been hoped that this invariant might be complete, but this is not the case: there are non-isomorphic solutions sharing the same invariant.

Table 5 shows the number of isomorphism classes in the two cases. Altogether, up to isomorphism, there are precisely 35 810 097 MMPTS(17)s.

|                     |            |
|---------------------|------------|
| Non-quintuple case: | 33 459 364 |
| Quintuple case:     | 2 350 733  |
| Total number:       | 35 810 097 |

Table 5: The numbers of isomorphism classes.

## 2.4 Storing the solutions

Storing a representative of each isomorphism class in the form of 44 triples requires about 14 GB of storage space. However this can be brought down to a modest 1.5 GB as follows. Given one of these designs, each triple  $\{i, j, k\}$  is ordered so that  $i < j < k$ . These triples are taken in ascending order by their first entry, and then by their second. The list of 44 third entries is recorded with numbers represented by characters. The numbers 0-9 are represented as characters 0-9 and the numbers 10-16 are represented as characters a-g. In this way a design can be represented as a character string of length 44. In principle, not all 44 triples are needed because some triples are fixed in all 9 runs, but incorporating this saving isn't worth the extra effort. Compressing

the resulting files in .rar format reduces the required storage space to 13 MB for the quintuple case, and 179 MB for the non-quintuple case. The files are available as MMPTS(17)\_Type-Q.rar and MMPTS(17)\_Type-N.rar from [5].

### 3 Properties of the MMPTS(17)s

The number of Pasch configurations, mitres, and STS(7) subsystems (Fano planes) in each design was determined by two independently written programs.

In the quintuple case, 78 designs have no Pasch configurations, and 2 designs have the maximum number of 44 Pasch configurations. In the non-quintuple case, 843 designs have no Pasch configurations, and 1 design has the maximum number of 47 Pasch configurations. Hence there are anti-Pasch (4-sparse) MMPTS(17)s of both types. Table 6 gives the numbers of Type-Q and Type-N MMPTS(17)s having  $P$  Pasch configurations for  $0 \leq P \leq 47$ .

| $P$ | Type-Q  | Type-N    | $P$ | Type-Q  | Type-N    | $P$ | Type-Q | Type-N |
|-----|---------|-----------|-----|---------|-----------|-----|--------|--------|
| 0   | 78      | 843       | 16  | 106 560 | 1 487 105 | 32  | 105    | 868    |
| 1   | 487     | 7 531     | 17  | 76 724  | 1 072 999 | 33  | 35     | 578    |
| 2   | 2 415   | 37 136    | 18  | 52 071  | 745 422   | 34  | 39     | 303    |
| 3   | 8 161   | 124 920   | 19  | 34 839  | 501 176   | 35  | 42     | 177    |
| 4   | 21 834  | 326 465   | 20  | 21 773  | 325 369   | 36  | 9      | 133    |
| 5   | 46 990  | 696 112   | 21  | 13 707  | 206 993   | 37  | 4      | 87     |
| 6   | 85 332  | 1 258 629 | 22  | 8 038   | 128 418   | 38  | 11     | 32     |
| 7   | 133 629 | 1 962 982 | 23  | 4 957   | 78 709    | 39  | 9      | 34     |
| 8   | 185 730 | 2 697 978 | 24  | 2 782   | 48 182    | 40  | 6      | 6      |
| 9   | 227 426 | 3 292 317 | 25  | 1 919   | 29 380    | 41  | 1      | 17     |
| 10  | 255 310 | 3 642 666 | 26  | 1 088   | 17 815    | 42  | 0      | 14     |
| 11  | 261 623 | 3 697 358 | 27  | 676     | 11 220    | 43  | 4      | 1      |
| 12  | 248 943 | 3 477 394 | 28  | 423     | 6 664     | 44  | 2      | 1      |
| 13  | 220 388 | 3 053 390 | 29  | 271     | 4 032     | 45  | 0      | 4      |
| 14  | 182 791 | 2 528 980 | 30  | 208     | 2 456     | 46  | 0      | 1      |
| 15  | 143 166 | 1 984 888 | 31  | 127     | 1 578     | 47  | 0      | 1      |

Table 6: Pasch configuration frequencies in MMPTS(17)s

In the quintuple case, 4 designs have the minimum number of 5 mitres, and 3 designs have the maximum number of 48 mitres. In the non-quintuple case, 1 design has the minimum number of 3 mitres, and 1 design has the maximum number of 47 mitres. Note there are no anti-mitre and, consequently, no 5-sparse MMPTS(17)s. Table 7 gives the numbers of Type-Q and Type-N MMPTS(17)s having  $M$  mitres for  $0 \leq M \leq 48$ .

Table 8 gives the numbers of Type-Q and Type-N MMPTS(17)s having  $S$  STS(7) subsystems (also known as Fano planes) for  $0 \leq S \leq 4$  (the maximum number in any MMPTS(17)).

The automorphism groups of the designs were also determined. The numbers of each order were checked by two independently written programs. Table 9 gives full details. The results in Table 9 can be also used to determine the total number of MMPTS(17)s on a specified point set from the orbit-stabiliser

| $M$ | Type-Q | Type-N |
|-----|--------|--------|
| 0   | 0      | 0      |
| 1   | 0      | 0      |
| 2   | 0      | 0      |
| 3   | 0      | 1      |
| 4   | 0      | 1      |
| 5   | 4      | 13     |
| 6   | 13     | 55     |
| 7   | 21     | 195    |
| 8   | 69     | 695    |
| 9   | 224    | 2368   |
| 10  | 533    | 6855   |
| 11  | 1367   | 18010  |
| 12  | 3406   | 43678  |
| 13  | 7139   | 98058  |
| 14  | 14637  | 199350 |
| 15  | 26544  | 373500 |

| $M$ | Type-Q | Type-N  |
|-----|--------|---------|
| 16  | 44626  | 643122  |
| 17  | 69604  | 1020033 |
| 18  | 100089 | 1493054 |
| 19  | 134564 | 2026513 |
| 20  | 167649 | 2560458 |
| 21  | 197183 | 3014058 |
| 22  | 217341 | 3317425 |
| 23  | 225116 | 3422818 |
| 24  | 220267 | 3300754 |
| 25  | 205106 | 2992985 |
| 26  | 180070 | 2547896 |
| 27  | 150370 | 2042250 |
| 28  | 119906 | 1536133 |
| 29  | 89588  | 1086488 |
| 30  | 64205  | 720441  |
| 31  | 43397  | 449377  |

| $M$ | Type-Q | Type-N |
|-----|--------|--------|
| 32  | 27912  | 263661 |
| 33  | 17221  | 143731 |
| 34  | 10441  | 74012  |
| 35  | 5749   | 35097  |
| 36  | 3131   | 15668  |
| 37  | 1640   | 6630   |
| 38  | 852    | 2498   |
| 39  | 376    | 985    |
| 40  | 201    | 320    |
| 41  | 73     | 101    |
| 42  | 56     | 46     |
| 43  | 12     | 19     |
| 44  | 24     | 10     |
| 45  | 3      | 1      |
| 46  | 1      | 0      |
| 47  | 0      | 1      |
| 48  | 3      | 0      |

Table 7: Mitre frequencies in MMPTS(17)s

| $S$ | Type-Q  | Type-N   |
|-----|---------|----------|
| 0   | 2335059 | 33198472 |
| 1   | 15499   | 258750   |
| 2   | 125     | 2104     |
| 3   | 46      | 38       |
| 4   | 4       | 0        |

Table 8: STS(7) frequencies in MMPTS(17)s.

theorem as

$$\sum_D \frac{17!}{|Aut(D)|},$$

where  $Aut(D)$  denotes the automorphism group of the design  $D$ , and  $D$  runs through the representatives of the isomorphism classes of the MMPTS(17)s. The number of Type-Q is 834 485 388 804 587 520 000 and the number of Type-N is 11 900 365 174 781 411 328 000. Hence the total number of MMPTS(17)s is 12 734 850 563 585 998 848 000.

| Automorphisms |                                    | Type-Q            | Type-N            |
|---------------|------------------------------------|-------------------|-------------------|
| order         | group                              | number of systems | number of systems |
| 12            | dihedral $D_{12}$                  | 3                 | 0                 |
| 8             | dihedral $D_8$                     | 20                | 0                 |
| 8             | $\mathbb{Z}_4 \times \mathbb{Z}_2$ | 4                 | 0                 |
| 6             | dihedral $D_6 (= S_3)$             | 27                | 0                 |
| 6             | $\mathbb{Z}_6$                     | 3                 | 0                 |
| 4             | Klein                              | 269               | 0                 |
| 4             | $\mathbb{Z}_4$                     | 74                | 0                 |
| 3             | $\mathbb{Z}_3$                     | 108               | 0                 |
| 2             | $\mathbf{Z}_2$                     | 8 470             | 3 992             |
| 1             | trivial                            | 2 341 755         | 33 455 372        |

Table 9: Automorphisms of MMPTS(17)s.

## 4 The PBD(17, {3, 5\*})s

The PBD(17, {3, 5\*})s were enumerated by two methods. The first was to take the 2 350 733 non-isomorphic MMPTS(17)s of Type-Q, delete the blocks 024 and 134 (giving a leave forming the complete graph  $K_5$  on the set  $\{0, 1, 2, 3, 4\}$ ), and then test the resulting systems for isomorphisms. The second was to start again from Table 3, delete the blocks 024, 134 from each seed, generate systems from the seeds, and then test for isomorphisms. Isomorphisms were identified using nauty and checked with our own independent program. Both approaches gave the number of non-isomorphic PBD(17, {3, 5\*})s as 157 151. Thus the vast majority of these designs must each generate 15 non-isomorphic MMPTS(17)s when two intersecting triples are formed from the leave  $K_5$  in all possible ways. The non-isomorphic PBD(17, {3, 5\*})s are available in compact form as strings of length 42 from [5] in the file PBD(17,{3,5}).rar (1 MB).

Analysis of these systems shows that there are 39 systems with no Pasch configurations, and 1 system with the maximum number of 34 Pasch configurations. Table 10 gives the number  $n_p$  of systems having  $P$  Pasch configurations for  $0 \leq P \leq 34$ .

| $P$ | $n_p$ |
|-----|-------|
| 0   | 39    |
| 1   | 259   |
| 2   | 911   |
| 3   | 2701  |
| 4   | 5440  |
| 5   | 9324  |
| 6   | 13619 |
| 7   | 17149 |
| 8   | 19363 |

| $P$ | $n_p$ |
|-----|-------|
| 9   | 19285 |
| 10  | 17904 |
| 11  | 14844 |
| 12  | 11935 |
| 13  | 8479  |
| 14  | 5958  |
| 15  | 3982  |
| 16  | 2468  |
| 17  | 1467  |

| $P$ | $n_p$ |
|-----|-------|
| 18  | 855   |
| 19  | 472   |
| 20  | 270   |
| 21  | 176   |
| 22  | 88    |
| 23  | 61    |
| 24  | 35    |
| 25  | 23    |
| 26  | 9     |

| $P$ | $n_p$ |
|-----|-------|
| 27  | 8     |
| 28  | 12    |
| 29  | 1     |
| 30  | 5     |
| 31  | 5     |
| 32  | 1     |
| 33  | 2     |
| 34  | 1     |

Table 10: Pasch configuration frequencies in PBD(17, {3, 5\*})s.

The minimum number of mitres in a PBD(17, {3, 5\*}) is 5 (10 systems have this number), and 2 systems have the maximum number of 36 mitres. Table 11 gives the number  $n_m$  of systems having  $M$  mitres for  $5 \leq M \leq 36$ .

| $M$ | $n_m$ |
|-----|-------|
| 5   | 10    |
| 6   | 19    |
| 7   | 52    |
| 8   | 153   |
| 9   | 430   |
| 10  | 1001  |
| 11  | 2211  |
| 12  | 4026  |

| $M$ | $n_m$ |
|-----|-------|
| 13  | 6640  |
| 14  | 10146 |
| 15  | 13561 |
| 16  | 16615 |
| 17  | 18614 |
| 18  | 18821 |
| 19  | 17540 |
| 20  | 15142 |

| $M$ | $n_m$ |
|-----|-------|
| 21  | 11618 |
| 22  | 8449  |
| 23  | 5448  |
| 24  | 3306  |
| 25  | 1809  |
| 26  | 892   |
| 27  | 386   |
| 28  | 171   |

| $M$ | $n_m$ |
|-----|-------|
| 29  | 54    |
| 30  | 22    |
| 31  | 2     |
| 32  | 9     |
| 34  | 1     |
| 35  | 1     |
| 36  | 2     |

Table 11: Mitre frequencies in PBD(17, {3, 5\*})s.

Each PBD(17, {3, 5\*}) has either  $S = 0$  or  $S = 1$  STS(7) subsystems (Fano planes). Table 12 gives the number  $n_s$  of systems having  $S$  STS(7) subsystems.

| $S$ | $n_s$  |
|-----|--------|
| 0   | 156713 |
| 1   | 438    |

Table 12: STS(7) frequencies in PBD(17, {3, 5\*})s.

Table 13 tabulates the automorphism groups of the PBD(17, {3, 5\*})s. Again, the numbers of each order were checked by two independently written programs. The groups of order 24 were identified using the GAP system [8]. The symbol  $\times$  denotes a semidirect product. The total number of PBD(17, {3, 5\*})s on a specified point set can be obtained from the orbit-stabiliser theorem, and is 55 632 359 253 639 168 000.

| Automorphisms |   | number of PBD(17, {3, 5*})s |
|---------------|---|-----------------------------|
| order         | group   |                             |
| 24            | $S_4$   | 2                           |
| 24            | $\mathbb{Z}_4 \times S_3$                                 | 1                           |
| 24            | $(\mathbb{Z}_6 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2$ | 1                           |
| 12            | $\mathbb{Z}_{12}$   | 2                           |
| 12            | dihedral $D_{12}$   | 2                           |
| 12            | $\mathbb{Z}_3 \rtimes \mathbb{Z}_4$                       | 2                           |
| 8             | dihedral $D_8$  | 5                           |
| 8             | $\mathbb{Z}_4 \times \mathbb{Z}_2$                        | 3                           |
| 6             | $\mathbb{Z}_6$  | 15                          |
| 6             | dihedral $D_6 (= S_3)$                                    | 18                          |
| 4             | $\mathbb{Z}_4$  | 19                          |
| 4             | Klein   | 63                          |
| 3             | $\mathbf{Z}_3$  | 64                          |
| 2             | $\mathbf{Z}_2$  | 1 190                       |
| 1             | trivial   | 155 764                     |

Table 13: Automorphisms of PBD(17, {3, 5\*})s.

## 5 The MMPTS(16)s

As explained in the Introduction, the MMPTS(16)s can be obtained by deleting one of the points 0, 1, 2 or 3 of each of the MMPTS(17)s described in Section 2. This results in  $4 \times 35\,810\,097 = 143\,240\,388$  systems. But amongst these there are isomorphisms, and nauty was used to eliminate isomorphic copies. It was found that there are 47 744 568 non-isomorphic solutions. The dancing links program was used to check this number with an independent search. Of these solutions, 3 133 686 originate from MMPTS(17)s of Type-Q, and the remaining 44 610 882 are from systems of Type-N. If an MMPTS(16) originates from an MMPTS(17) of Type-Q, then adding a new point to each of the seven non-intersecting pairs and to one of the three intersecting pairs of its leave gives a Type-Q MMPTS(17). Hence no MMPTS(16) originating from a quintuple type MMPTS(17) can be isomorphic to one originating from a non-quintuple type MMPTS(17). A MMPTS(16) derived from a quintuple type MMPTS(17) will be described as Type-Q, and a MMPTS(16) derived from a non-quintuple type MMPTS(17) will be described as Type-N. The designs are available in files MMPTS(16)\_Type-Q.rar (17 MB) and MMPTS(16)\_Type-N.rar (208 MB) from [5]. The designs are mapped so that the leave is  $\{01, 02, 03, 45, 67, \dots, ef\}$ , with characters a to f representing points 10 to 15, and the designs are specified in compact form as strings of length 37.

For MMPTS(16)s of Type-Q, 1 346 designs have no Pasch configurations, and 1 design has the maximum number of 43 Pasch configurations. For MMPTS(16)s of type-N, 15 877 designs have no Pasch configurations, and 1 design has the

maximum number of 44 Pasch configurations. Hence there are anti-Pasch (4-sparse) MMPTS(16)s of both types. Table 14 gives the numbers of Type-Q and Type-N MMPTS(16)s having  $P$  Pasch configurations for  $0 \leq P \leq 44$ .

| $P$ | Type-Q  | Type-N    | $P$ | Type-Q | Type-N  | $P$ | Type-Q | Type-N |
|-----|---------|-----------|-----|--------|---------|-----|--------|--------|
| 0   | 1 346   | 15 877    | 15  | 36 534 | 565 757 | 30  | 29     | 314    |
| 1   | 9 571   | 125 270   | 16  | 22 010 | 344 867 | 31  | 18     | 163    |
| 2   | 38 020  | 502 936   | 17  | 12 897 | 207 237 | 32  | 14     | 125    |
| 3   | 100 154 | 1 347 901 | 18  | 7 562  | 123 640 | 33  | 10     | 76     |
| 4   | 198 137 | 2 704 815 | 19  | 4 448  | 73 743  | 34  | 2      | 38     |
| 5   | 309 441 | 4 278 201 | 20  | 2 606  | 43 954  | 35  | 5      | 36     |
| 6   | 395 447 | 5 549 182 | 21  | 1 661  | 26 351  | 36  | 1      | 17     |
| 7   | 432 141 | 6 121 166 | 22  | 1 013  | 15 917  | 37  | 1      | 12     |
| 8   | 414 833 | 5 905 615 | 23  | 653    | 9 497   | 38  | 2      | 10     |
| 9   | 358 284 | 5 122 025 | 24  | 355    | 5 761   | 39  | 2      | 3      |
| 10  | 282 328 | 4 069 661 | 25  | 264    | 3 480   | 40  | 1      | 3      |
| 11  | 206 951 | 3 016 887 | 26  | 148    | 2 096   | 41  | 1      | 5      |
| 12  | 142 368 | 2 113 200 | 27  | 120    | 1 327   | 42  | 0      | 1      |
| 13  | 94 556  | 1 407 703 | 28  | 85     | 755     | 43  | 1      | 0      |
| 14  | 59 617  | 904 747   | 29  | 49     | 510     | 44  | 0      | 1      |

Table 14: Pasch configuration frequencies in MMPTS(16)s

For MMPTS(16)s of Type-Q, 2 designs have no mitres, and 1 design has the maximum number of 48 mitres. For MMPTS(16)s of type-N, 5 designs have no mitres, and 1 design has the maximum number of 45 mitres. Hence there are anti-mitre MMPTS(16)s of both types. Table 15 gives the numbers of Type-Q and Type-N MMPTS(16)s having  $M$  mitres for  $0 \leq M \leq 48$ . The 7 designs with no mitres all have Pasch configurations, so there are no 5-sparse MMPTS(16)s.

| $M$ | Type-Q  | Type-N    | $M$ | Type-Q  | Type-N    | $M$ | Type-Q | Type-N |
|-----|---------|-----------|-----|---------|-----------|-----|--------|--------|
| 0   | 2       | 5         | 16  | 335 963 | 4 967 389 | 32  | 153    | 469    |
| 1   | 3       | 14        | 17  | 289 977 | 4 289 496 | 33  | 41     | 310    |
| 2   | 26      | 119       | 18  | 231 425 | 3 427 280 | 34  | 77     | 153    |
| 3   | 116     | 879       | 19  | 172 907 | 2 532 692 | 35  | 4      | 89     |
| 4   | 526     | 5 005     | 20  | 121 074 | 1 746 307 | 36  | 35     | 87     |
| 5   | 2 165   | 21 590    | 21  | 79 603  | 1 122 077 | 37  | 8      | 16     |
| 6   | 6 559   | 73 898    | 22  | 50 082  | 677 356   | 38  | 10     | 23     |
| 7   | 17 563  | 206 848   | 23  | 29 615  | 385 028   | 39  | 1      | 15     |
| 8   | 39 376  | 488 364   | 24  | 16 792  | 205 365   | 40  | 9      | 3      |
| 9   | 77 123  | 988 266   | 25  | 9 284   | 103 465   | 41  | 0      | 6      |
| 10  | 131 409 | 1 741 921 | 26  | 4 974   | 50 430    | 42  | 3      | 14     |
| 11  | 198 113 | 2 700 157 | 27  | 2 591   | 23 424    | 43  | 0      | 1      |
| 12  | 267 765 | 3 725 024 | 28  | 1 548   | 10 649    | 44  | 1      | 0      |
| 13  | 324 803 | 4 620 691 | 29  | 695     | 4 811     | 45  | 0      | 1      |
| 14  | 359 205 | 5 184 315 | 30  | 441     | 2 148     | 46  | 0      | 0      |
| 15  | 361 423 | 5 303 746 | 31  | 195     | 936       | 47  | 0      | 0      |
|     |         |           |     |         |           | 48  | 1      | 0      |

Table 15: Mitre frequencies in MMPTS(16)s

Table 16 gives the numbers of Type-Q and Type-N MMPTS(16)s having  $S$

STS(7) subsystems (Fano planes) for  $0 \leq S \leq 4$  (the maximum number in any MMPTS(16)).

| $S$ | Type-Q    | Type-N     |
|-----|-----------|------------|
| 0   | 3 120 879 | 44 379 670 |
| 1   | 12 703    | 229 796    |
| 2   | 80        | 1 392      |
| 3   | 22        | 24         |
| 4   | 2         | 0          |

Table 16: STS(7) frequencies in MMPTS(16)s.

Table 17 tabulates the automorphism groups of the MMPTS(16)s. Again, the numbers of each order were checked by two independently written programs. The total number of MMPTS(16)s on a specified point set can be obtained from the orbit-stabiliser theorem, and this gives the number of Type-Q systems as 65 449 834 416 046 080 000 and the number of Type-N systems as 933 361 974 492 659 712 000. Thus the overall total number of MMPTS(16)s is 998 811 808 908 705 792 000.

| Automorphisms |                                    | Type-Q            | Type-N            |
|---------------|------------------------------------|-------------------|-------------------|
| order         | group                              | number of systems | number of systems |
| 12            | dihedral $D_{12}$                  | 9                 | 0                 |
| 12            | alternating $A_4$                  | 2                 | 0                 |
| 12            | $\mathbb{Z}_6 \times \mathbb{Z}_2$ | 1                 | 0                 |
| 6             | dihedral $D_6 (= S_3)$             | 33                | 6                 |
| 6             | $\mathbb{Z}_6$                     | 28                | 0                 |
| 4             | Klein                              | 470               | 0                 |
| 4             | $\mathbb{Z}_4$                     | 32                | 0                 |
| 3             | $\mathbb{Z}_3$                     | 280               | 504               |
| 2             | $\mathbf{Z}_2$                     | 9 802             | 1 434             |
| 1             | trivial                            | 3 123 029         | 44 608 938        |

Table 17: Automorphisms of MMPTS(16)s.

## 6 Concluding remarks

Details of MMPTS( $v$ )s for  $v \leq 11$  are given in [2]. In this Section we give some details of the systems of orders 12 and 14 that have been enumerated elsewhere. As recorded in [9], there are precisely five non-isomorphic MMPTS(12)s arising from the two non-isomorphic STS(13)s, but these are not explicitly listed. Table 18 gives them here in compact form with the numbers of Pasch configurations and mitres that they contain, and their automorphism groups. In each case the



leave is  $\{01, 23, \dots, ab\}$  with  $a, b$  representing points 10, 11. Since no STS(13) has an STS(7) subsystem, neither do these MMPTS(12)s. The total number of MMPTS(12)s on a specified point set is 1 197 504 000.

| MMPTS(12)            | #Pasch | #mitre | Automorphism group     |
|----------------------|--------|--------|------------------------|
| 468ab798abb8ab7aa99b | 4      | 6      | $\mathbb{Z}_2$         |
| 468ab79a6b9baba889ba | 5      | 3      | $\mathbb{Z}_2$         |
| 468ab78ab96ba9abb89a | 5      | 3      | dihedral $D_6 (= S_3)$ |
| 468ab94ba87baa9b8ba9 | 4      | 5      | trivial                |
| 468aba498b79bb9ab8aa | 7      | 0      | $\mathbb{Z}_3$         |

Table 18: The non-isomorphic MMPTS(12)s.

There are 787 non-isomorphic MMPTS(14)s, and these are enumerated in [11] where they are recorded as  $(14_6, 28_3)$  configurations. Here we give counts for Pasch configurations, mitres and STS(7) subsystems, and details of the automorphism groups. The designs are available in the file MMPTS(14).txt (23KB) from [5]. They are mapped so that the leave is  $\{01, 23, \dots, cd\}$  with characters  $a$  to  $d$  representing points 10 to 13, and they are specified in compact form as strings of length 28.

There are 6 MMPTS(14)s with no Pasch configurations, and 1 MMPTS(14) has the maximum number of 63 Pasch configurations. Table 19 gives the number  $n_p$  of MMPTS(14)s having  $P$  Pasch configurations for  $0 \leq P \leq 63$ .

There are 15 MMPTS(14)s with no mitres, and 1 MMPTS(14) has the maximum number of 22 mitres. Table 20 gives the number  $n_m$  of MMPTS(14)s having  $M$  mitres for  $0 \leq M \leq 22$ . The 15 designs with no mitres all have Pasch configurations, so there are no 5-sparse MMPTS(14)s.

Table 21 gives the number  $n_s$  of MMPTS(14)s having  $S$  STS(7) subsystems (Fano planes) for  $0 \leq S \leq 8$  (the maximum number in any MMPTS(14)).

Table 22 tabulates the automorphism groups of the MMPTS(14)s. Again, the numbers of each order were checked by two independently written programs.

| $P$ | $n_p$ | $P$ | $n_p$ | $P$ | $n_p$ | $P$ | $n_p$ |
|-----|-------|-----|-------|-----|-------|-----|-------|
| 0   | 6     | 9   | 36    | 19  | 12    | 29  | 2     |
| 1   | 7     | 10  | 26    | 20  | 7     | 31  | 7     |
| 2   | 34    | 11  | 30    | 21  | 5     | 33  | 1     |
| 3   | 94    | 12  | 13    | 22  | 4     | 35  | 1     |
| 4   | 94    | 13  | 29    | 23  | 10    | 39  | 1     |
| 5   | 117   | 14  | 23    | 25  | 2     | 43  | 1     |
| 6   | 72    | 15  | 9     | 26  | 1     | 47  | 1     |
| 7   | 62    | 16  | 3     | 27  | 2     | 63  | 1     |
| 8   | 56    | 17  | 17    | 28  | 1     |     |       |

Table 19: Pasch configuration frequencies in MMPTS(14)s

| $M$ | $n_m$ |
|-----|-------|
| 0   | 15    |
| 2   | 5     |
| 3   | 8     |
| 4   | 24    |
| 5   | 12    |

| $M$ | $n_m$ |
|-----|-------|
| 6   | 44    |
| 7   | 35    |
| 8   | 73    |
| 9   | 87    |
| 10  | 90    |

| $M$ | $n_m$ |
|-----|-------|
| 11  | 96    |
| 12  | 100   |
| 13  | 71    |
| 14  | 51    |
| 15  | 40    |

| $M$ | $n_m$ |
|-----|-------|
| 16  | 23    |
| 17  | 7     |
| 18  | 1     |
| 19  | 3     |
| 20  | 1     |
| 22  | 1     |

Table 20: Mitre frequencies in MMPTS(14)s

| $S$ | $n_s$ |
|-----|-------|
| 0   | 730   |
| 1   | 43    |
| 2   | 11    |
| 4   | 2     |
| 8   | 1     |

Table 21: STS(7) frequencies in MMPTS(14)s.

The groups of orders 24 and above were identified using the GAP system [8]. The symbol  $\rtimes$  denotes a semidirect product. The system with the automorphism group of order 1344 is the system obtained from the projective STS(15) that has an automorphism group  $A_8$ , the alternating group of order 20160, and whose automorphism partition consists of a single part containing all 15 points; this is STS(15) number 1 in the listing of [17]. The total number of MMPTS(14)s on a specified point set can be obtained from the orbit-stabiliser theorem, and is 60 281 712 691 200.

Collecting together previous results and the results of this paper, Table 23 gives the numbers of non-isomorphic MMPTS( $v$ )s for  $v \leq 19$ . The entry for  $v = 18$  comes from a personal communication [12] cited in [11] and given also in [1, page 354].

| Automorphisms |  | number of MMPTS(14)s |
|---------------|--|----------------------|
| order         | group  |                      |
| 1344          | $\mathbb{Z}_2^3 \rtimes PSL(3, 2)$   | 1                    |
| 192           | $((\mathbb{Z}_2 \times D_8) \rtimes \mathbb{Z}_2) \rtimes \mathbb{Z}_3 \rtimes \mathbb{Z}_2$ | 1                    |
| 96            | $(\mathbb{Z}_2^2 \times A_4) \rtimes \mathbb{Z}_2$   | 1                    |
| 32            | $(\mathbb{Z}_2^4) \rtimes \mathbb{Z}_2$  | 3                    |
| 24            | symmetric $S_4$  | 3                    |
| 24            | $A_4 \times \mathbb{Z}_2$  | 2                    |
| 21            | $\mathbb{Z}_7 \rtimes \mathbb{Z}_3$  | 2                    |
| 16            | $\mathbb{Z}_2^4$   | 1                    |
| 12            | alternating $A_4$  | 4                    |
| 8             | dihedral $D_8$   | 10                   |
| 6             | dihedral $D_6 (= S_3)$   | 1                    |
| 6             | $\mathbb{Z}_6$   | 1                    |
| 4             | Klein  | 18                   |
| 4             | $\mathbb{Z}_4$   | 13                   |
| 3             | $\mathbb{Z}_3$   | 37                   |
| 2             | $\mathbb{Z}_2$   | 40                   |
| 1             | trivial  | 649                  |

Table 22: Automorphisms of MMPTS(14)s.

| $v$      | The number of non-isomorphic MMPTS( $v$ )s |
|----------|--|
| $\leq 9$ | 1  |
| 10       | 2  |
| 11       | 2  |
| 12       | 5  |
| 13       | 2  |
| 14       | 787  |
| 15       | 80   |
| 16       | 47 744 568                                 |
| 17       | 35 810 097                                 |
| 18       | 210 611 385 743                            |
| 19       | 11 084 874 829                             |

Table 23: The number of non-isomorphic MMPTS( $v$ )s.

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