Note

On diregular digraphs with degree two and excess two

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ABSTRACT

An important topic in the design of efficient networks is the construction of \((d, k, +\epsilon)\)-digraphs, i.e. \(k\)-geodetic digraphs with minimum out-degree \(\geq d\) and order \(M(d, k) + \epsilon\), where \(M(d, k)\) represents the Moore bound for degree \(d\) and diameter \(k\) and \(\epsilon > 0\) is the (small) excess of the digraph. Previous work has shown that there are no \((2, k, +1)\)-digraphs for \(k \geq 2\). In a separate paper, the present author has shown that any \((2, k, +2)\)-digraph must be diregular for \(k \geq 2\). In the present work, this analysis is completed by proving the nonexistence of diregular \((2, k, +2)\)-digraphs for \(k \geq 3\) and classifying diregular \((2, 2, +2)\)-digraphs up to isomorphism.

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1. Introduction

An important topic in the design of efficient networks is the construction of \(d, k\)-digraphs with maximum out-degree \(d\) and diameter \(k\). A simple inductive argument shows that for \(0 \leq l \leq k\) the number of vertices at distance \(l\) from a fixed vertex \(v\) is bounded above by \(d^l\). Therefore, a natural upper bound for the order of such a digraph is the so-called Moore bound \(M(d, k) = 1 + d + d^2 + \cdots + d^k\). A digraph that attains this upper bound is called a Moore digraph. It is easily seen that a digraph \(G\) is Moore if and only if it is out-regular with degree \(d\), has diameter \(k\) and is \(k\)-geodetic, i.e. for any two vertices \(u, v\) there is at most one \(k\)-path from \(u\) to \(v\).

As it was shown by Bridges and Toueg in [1] that Moore digraphs exist only in the trivial cases \(d = 1\) or \(k = 1\) (the Moore digraphs are directed \((k + 1)\)-cycles and complete digraphs \(K^+_{k+1}\) respectively), much research has been devoted to the study of digraphs that in some sense approximate Moore digraphs. For example, there is an extensive literature on digraphs with maximum out-degree \(d\), diameter \(\leq k\) and order \(M(d, k) - \delta\) for small \(\delta > 0\); this is equivalent to relaxing the \(k\)-geodesity requirement in the conditions for a digraph to be Moore. \(\delta\) is known as the defect of the digraph. The reader is referred to the survey [5] for more information.

In this paper, however, we will consider the following related problem, which is obtained by retaining the \(k\)-geodesity requirement in the above characterisation of Moore digraphs, but allowing the diameter to exceed \(k\): what is the smallest possible order of a \(k\)-geodetic digraph \(G\) with minimum out-degree \(\geq d\)? A \(k\)-geodetic digraph with minimum out-degree \(\geq d\) and order \(M(d, k) + \epsilon\) is said to be a \((d, k, +\epsilon)\)-digraph or to have excess \(\epsilon\). It was shown in [6] that there are no \((2, k, +1)\)-digraphs for \(k \geq 2\). In 2016 it was shown in [3] that digraphs with excess one must be diregular and that there are no \((d, k, +1)\)-digraphs for \(k = 3, 4\) and sufficiently large \(d\). In a separate paper [7], the present author has shown that \((2, k, +2)\)-digraphs must be diregular with degree \(d = 2\) for \(k \geq 2\). In the present paper, we classify the \((2, 2, +2)\)-digraphs up to isomorphism and show that there are no diregular \((2, k, +2)\)-digraphs for \(k \geq 3\), thereby completing the proof of the nonexistence of digraphs with degree \(d = 2\) and excess \(\epsilon = 2\) for \(k \geq 3\). Our reasoning and notation will follow closely that employed in [4] for the corresponding result for defect \(\delta = 2\).
2. Preliminary results

We will let $G$ stand for a $(2, k, +2)$-digraph for arbitrary $k \geq 2$, i.e. $G$ has minimum out-degree $d = 2$, is $k$-geodetic and has order $M(2, k) + 2$. We will denote the vertex set of $G$ by $V(G)$. By the result of [7], $G$ must be diregular with degree $d = 2$ for $k \geq 2$. The distance $d(u, v)$ between vertices $u$ and $v$ is the length of the shortest path from $u$ to $v$. Notice that $d(u, v)$ is not necessarily equal to $d(v, u)$. $u \rightarrow v$ will indicate that there is an arc from $u$ to $v$. We define the in- and out-neighbourhoods of a vertex $u$ by $N^-(u) = \{v \in V(G) : v \rightarrow u\}$ and $N^+(u) = \{v \in V(G) : u \rightarrow v\}$ respectively; more generally, for $0 \leq l \leq k$, the set $\{v \in V(G) : d(u, v) = l\}$ of vertices at distance exactly $l$ from $u$ will be denoted by $N_l(u)$. For $0 \leq l \leq k$ we will also write $T_l(u) = \bigcup_{i=0}^{l} N_i(u)$ for the set of vertices at distance $\leq l$ from $u$. The notation $T_{k-1}(u)$ will be abbreviated by $T(u)$.

It is easily seen that for any vertex $u$ of $G$, there are exactly two distinct vertices that are at distance $\geq k + 1$ from $u$. For any $u \in V(G)$, we will write $O(u)$ for the set of these vertices and call such a set an outlier set and its elements outliers of $u$. Notice that $O(u) = V(G) - T_k(u)$. An elementary counting argument shows that in a diregular $(2, k, +2)$-digraph every vertex is also an outlier of exactly two vertices. We will say that a vertex $u$ can reach a vertex $v$ if $v \notin O(u)$.

Our proof will proceed by an analysis of a pair of vertices with exactly one common out-neighbour. First, we must show that such a pair exists and deduce some elementary properties of pairs of vertices with identical out-neighbourhoods.

**Lemma 1.** For $k \geq 2$, let $u$ and $v$ be distinct vertices such that $N^+(u) = N^+(v) = \{u_1, u_2\}$. Then $u_1 \in O(u_2), u_2 \in O(u_1)$ and there exists a vertex $x$ such that $O(u) = \{v, x\}, O(v) = \{u, x\}$.

**Proof.** Suppose that $u$ can reach $v$ by a $\leq k$-path. Then $v \in T(u_1) \cup T(u_2)$. As $N^+(v) = N^+(u)$, it follows that there would be a $\leq k$-cycle through $v$, contradicting $k$-geodeticity. If $O(u) = \{v, x\}$, then $x \neq v$ and $x \notin T(u_1) \cup T(u_2)$, so that $v$ cannot reach $x$ by a $\leq k$-path. Similarly, if $u_1$ can reach $u_2$, by a $\leq k$-path, then we must have $\{u, v\} \cap T(u_1) \neq \emptyset$, which is impossible. □

**Lemma 2.** For $k \geq 2$, there exists a pair of vertices $u, v$ with $|N^+(u) \cap N^+(v)| = 1$.

**Proof.** Suppose for a contradiction that there is no such pair of vertices. Define a map $\phi : V(G) \rightarrow V(G)$ as follows. Let $u^+ = \text{an out-neighbour of a vertex } u$ and let $\phi(u)$ be the in-neighbour of $u^+$ distinct from $u$. By our assumption, it is easily verified that $\phi$ is a well-defined bijection with no fixed points and with square equal to the identity. It follows that $G$ must have even order, whereas $|V(G)| = M(2, k) + 2$ is odd. □

$u, v$ will now stand for a pair of vertices with a single common out-neighbour. We will label the vertices of $T_k(u)$ according to the scheme $N^+(u) = \{u_1, u_2\}, N^+(u_1) = \{u_3, u_4\}, N^+(u_2) = \{u_5, u_6\}, N^+(u_3) = \{u_7, u_8\}, N^+(u_4) = \{u_9, u_{10}\}$ and so on, with the same convention for the vertices of $T_k(v)$, where we will assume that $u_2 = v_2$.

3. Classification of $(2, 2, +2)$-digraphs

We begin by classifying the $(2, 2, +2)$-digraphs up to isomorphism. We will prove the following theorem.

**Theorem 1.** There are exactly two diregular $(2, 2, +2)$-digraphs, which are displayed in Figs. 2 and 5.

Let $G$ be an arbitrary diregular $(2, 2, +2)$-digraph. $G$ has order $M(2, 2) + 2 = 9$. By Lemma 2, $G$ contains a pair of vertices $(u, v)$ such that $|N^+(u) \cap N^+(v)| = 1$; we will assume that $u_2 = v_2$, so that we have the situation shown in Fig. 1.

We can immediately deduce some information on the possible positions of $v$ and $v_1$ in $T_2(u)$.

**Lemma 3.** If $v \notin O(u)$, then $v \in N^+(u_1)$. If $v_1 \notin O(u)$, then $v_1 \in N^+(u_1)$.

**Proof.** $v \notin T(u_2)$ by $2$-geodecticy. $v \neq u$ by construction. If we had $v = u_1$, then there would be two distinct $\leq 2$-paths from $u$ to $u_2$. Also $v_1 \neq \{u\} \cup T(u_2)$ by $2$-geodecticy and by assumption $u_1 \neq v_1$. □
Since $v$ and $v_1$ cannot both lie in $N^+(u_1)$ by 2-geodecity, we have the following corollary.

**Corollary 1.** $O(u) \cap \{v, v_1\} \neq \emptyset$.

We will call a pair of vertices $(u, v)$ with a single common out-neighbour *bad* if at least one of

$$O(u) \cap \{v_1, v_3\} = \emptyset, O(u) \cap \{v_1, v_4\} = \emptyset, O(v) \cap \{u_1, u_3\} = \emptyset, O(v) \cap \{u_1, u_4\} = \emptyset.$$  

holds. Otherwise such a pair will be called good.

**Lemma 4.** There is a unique $(2, 2, +2)$-digraph containing a bad pair.

**Proof.** Assume that there exists a bad pair $(u, v)$. Without loss of generality, $O(u) \cap \{v_1, v_3\} = \emptyset$. By Lemma 3 we can set $v_1 = u_3$. By 2-geodecity $v_3 = u$. We cannot have $v_4 = v_3 = u$, so $u_4$ must be an outlier of $u$. By Corollary 1 it follows that $O(u) = \{v, v_4\}$.

Consider the vertex $u_1$. By Lemma 3, if $u_1 \notin O(v)$, then $u_1 \notin N^+(v_1)$. However, as $v_1 = u_3$, there would be a 2-cycle through $u_1$. Hence $u_1 \in O(v)$. As $O(u) = \{v, v_4\}$, we have $V(G) = \{u, u_1, u_2, u_3 = v_1, u_4, u_5, u_6, v, v_4\}$ and $O(v) = \{u_1, u_4\}$. As neither $u$ nor $v$ lies in $T(u_1)$, we must also have $u_2 \in O(u_1)$. As $u_1$ can reach $u_1, v_1, u_4, u$ and $v_4$, it follows that without loss of generality we either have $O(u_1) = \{u_2, v\}$ and $N^+(u_4) = \{u_5, u_6\} = N^+(u_2)$ or $O(u_1) = \{u_2, u_6\}$ and $N^+(u_4) = \{v, u_5\}$. In either case, $(v, u_1)$ is a good pair.

Suppose firstly that $N^+(u_2) = N^+(u_4)$. Then $v$ is an outlier of $u$ and $u_1$. As each vertex is the outlier of exactly two vertices, $v_1$ must be able to reach $v$ by a $\leq 2$-path. Hence $v_4 \rightarrow v$. Likewise $u_2$ can reach $v$, so without loss of generality $u_5 \rightarrow v$. Suppose that $O(u_2) \cap \{u, u_1\} = \emptyset$. As $u$ and $v$ have a common out-neighbour, we must have $u_6 \rightarrow u$. Since $u \rightarrow u_1$, by 2-geodecity we must have $u_5 \rightarrow u_1$. However, this is a contradiction, as $v$ and $u_1$ also have a common out-neighbour.

Therefore, at least one of $u, u_1$ is an outlier of $u_2$. By Lemma 4 $u_4$ is an outlier of $u_2$. Therefore either $O(u_2) = \{u, u_4\}$ or $O(u_2) = \{u_1, u_4\}$. If $O(u_2) = \{u, u_4\}$, then $u_2$ must be able to reach $u_1, v_1$ and $v_4, u_5 \rightarrow v$ and $v \rightarrow v_1$, so $v_1 \in N^+(u_6)$. As $v_1 \rightarrow v_1$, we must have $N^+(u_5) = \{v_1\}$. As $v$ and $u_1$ have a common out-neighbour, this violates 2-geodecity. Hence $O(u_2) = \{u_1, u_4\}$ and $u_2$ can reach $u, v_1$ and $v_4$. As $v \rightarrow v_1, v_1 \in N^+(u_6)$. As $v_1 \rightarrow v_4$, it follows that $N^+(u_5) = \{v, v_4\}$. However, $v_4 \rightarrow v$, so this again violates 2-geodecity.

We are left with the case $O(u_1) = \{u_2, u_6\}$ and $N^+(u_4) = \{v, u_5\}$. Then $v_1 \in O(u_2)$, as neither $v$ nor $u_1$ lies in $T(u_2)$. Observe that $u_2$ and $u_4$ have a single common out-neighbour, so by Corollary 1 $O(u_2) \cap \{u_4, v\} \neq \emptyset$. Therefore either $O(u_2) = \{v, u_4\}$ or $O(u_2) = \{v_1, v\}$. Suppose firstly that $O(u_2) = \{v_1, v_4\}$. Then $N^+(u_2) = \{u, v, u_1, v_4\}$. As $\exists N^+(u_4) = \{v, u_5\}, u_5 \not\rightarrow v$, so $u_6 \rightarrow v$. As $N^+(u) \cap N^+(v) \neq \emptyset, u_5 \rightarrow u \rightarrow u_1$, so necessarily $u_6 \in N^+(u_1) \cap N^+(v)$, contradicting 2-geodecity.

Hence $O(u_2) = \{v_1, v\}$ and $N^+(u_2) = \{u, u_1, v, v_4\}$. As $u_4 \rightarrow u_5, u_5 \not\rightarrow u_4$. Thus $u_6 \rightarrow u_4$. Now $u_1 \rightarrow u_4$ and $u \rightarrow u_1$ implies that $N^+(u_5) = \{u_4\}$ and $N^+(u_6) = \{u, u_4\}$. Finally we must have $N^+(v_4) = \{u, v_6\}$. This gives us the $(2, 2, +2)$-digraph shown in Fig. 2. $\square$

We can now assume that all pairs given by Lemma 2 are good. Let us fix a pair $(u, v)$ with a single common out-neighbour. It follows from Corollary 1 and the definition of a good pair that $v_1 \in O(u)$; otherwise $O(u)$ would contain $v, v_3$ and $v_4$, which is impossible. Likewise $u_1 \in O(v)$.
Considering the positions of \( v_3 \) and \( v_4 \), we see that there are without loss of generality four possibilities: (1) \( u = v_3, u_4 = v_4 \), (2) \( u = v_3, O(u) = \{v_1, v_4\} \), (3) \( N^+(u) = N^+(v_1) \) and (4) \( u_3 = v_3, O(u) = \{v_1, u_4\} \). A suitable relabelling of vertices shows that case 4 is equivalent to case 1.a below, so we will examine cases 1 to 3 in turn.

**Case 1**: \( u = v_3, u_4 = v_4 \)

Depending upon the position of \( v \), we must either have \( O(u) = \{u_1, v\} \) and \( O(v) = \{u_1, u_3\} \) or \( v = u_3 \) (see Fig. 3).

**Case 1.a**: \( O(u) = \{v_1, v\} \), \( O(v) = \{u_1, u_3\} \)

In this case \( V(G) = \{u, u_1, u_2, u_3, u_4, u_5, u_6, v, v_1\} \), \( u_1 \) and \( v_1 \) have a single common out-neighbour, namely \( u_4 \), so as we are assuming all such pairs to be good, we have \( u_3 \in O(v_1), u \in O(u_1) \). By 2-geodecity, \( N^+(u_4) \subset \{u_5, u_6, v\} \), so without loss of generality either \( N^+(u_4) = \{u_5, u_6\} \) or \( N^+(u_4) = \{u_5, v\} \).

Suppose that \( N^+(u_4) = \{u_5, u_6\} \). By elimination, \( O(u_1) = \{v, u_3\} \). As \( G \) is diregular, every vertex is an outlier of exactly two vertices; \( v \) is an outlier of \( u \) and \( v_1 \), so both \( u_1 \) and \( u_2 \) can reach \( v \) by a \( \leq 2 \)-path. Hence \( v \in N^+(u_3) \). As \( v \rightarrow v_1 \), we see that \( v_1 \) is an outlier of \( u_1 \); as \( u \) is also an outlier of \( u_1 \), we have \( O(u_1) = \{u, v_1\} \) and \( N^+(u_3) = \{v, u_2\} \). As \( v \rightarrow u_2 \), this is impossible.

Now consider \( N^+(u_4) = \{u_5, v\} \). We now have \( O(u_1) = \{u_1, u_6\} \). Thus \( u_3 \in O(v) \cap O(v_1), u_3 \in T_2(u_4) \). \( v \) is not adjacent to \( u_3 \), so \( u_3 \in N^+(u_5) \). \( u_2 \) and \( u_4 \) have \( u_5 \) as a unique common out-neighbour, so \( u_6 \in O(u_4), v \in O(u_2) \). As \( u_6 \in O(v_1) \cap O(u_4) \), \( u_1 \) can reach \( u_6 \). Hence \( u_6 \in N^+(u_1) \).

Neither \( u \) nor \( v \) lie in \( T(u_1) \), so \( u_2 \in O(u_1) \). Therefore either \( O(u_1) = \{u, u_2\} \) or \( O(u_1) = \{v, v_1\} \). If \( O(u_1) = \{u, u_2\} \), then \( N^+(u_2) = \{u_6, v_1\} \). \( u_2 \) cannot reach \( v_1 \), since \( v, u_3 \not\in T(u_2) \), so \( O(u_2) = \{v, v_1\} \) and \( N^2(u_2) = \{u, u_1, u_3, u_4\} \). As \( u_4 \rightarrow u_5 \), \( u_4 \in N^+(u_6) \). \( u_1 \rightarrow u_4 \), so \( N^+(u_1) = \{v_1, u_3\} \). As \( u_3 \rightarrow u_2 \), this is a contradiction. Thus \( O(u_1) = \{u_2, v_1\} \), so that \( N^+(u_2) = \{u, u_6\} \). \( u_1 \) must have an in-neighbour apart from \( u \), which must be either \( u_5 \) or \( u_6 \).

As \( u_1 \rightarrow u_3 \), we have \( u_1 \in N^+(u_6) \). By elimination, \( v \) and \( v_1 \) must also have in-neighbours in \( \{u_5, u_6\} \). As \( u_1 \) and \( v_1 \) have a common out-neighbour, we have \( N^+(u_5) = \{v_3, v_1\}, N^+(u_6) = \{u_1, v\} \). However, both \( u_3 \) and \( v_1 \) are adjacent to \( u \), violating 2-geodecity.

**Case 1.b**: \( v = u_3 \)

There exists a vertex \( x \) such that \( V(G) = \{u, u_1, u_2, v, u_4, u_5, u_6, v_1, x\} \), \( O(u) = \{v_1, x\} \) and \( O(v) = \{u_1, x\} \). As \( x \in O(u) \cap O(v) \), \( u_1 \) and \( u_2 \) can reach \( x \), so without loss of generality \( x \in N^+(u_4) \cap N^+(u_5) \). As \( u_5 \) and \( u_4 \) have a common out-neighbour, \( u_6 \in O(u_1) \). Also, \( u_1 \) and \( v_1 \) have \( u_4 \) as a unique common out-neighbour, so \( u \in O(u_1) \) and \( O(u_1) = \{u, u_6\} \). Thus \( N^+(u_4) = \{x, u_6\} \).

Observe that \( u_2 \) and \( u_4 \) have the out-neighbour \( u_6 \) in common. Thus \( x \in O(u_2) \), whereas we already have \( x \in O(u) \cap O(v) \), a contradiction.

**Case 2**: \( u = v_3 \), \( O(u) = \{v_1, v_4\} \)

As \( v \) is not equal to \( v_1 \) or \( v_4 \), \( v \) must lie in \( T_2(u) \). Without loss of generality, \( v = u_3 \). Hence \( V(G) = \{u, u_1, u_2, v, u_4, u_5, u_6, v_1, v_4\} \) and \( O(v) = \{u_1, u_4\} \). We have the configuration shown in Fig. 4. Hence \( u_1 \) can reach
Taking into account adjacencies between members of $u_1, v, u_4, u_2$ and $v_1$, so we have without loss of generality one of the following: a) $O(u_1) = \{u, v_4\}, N^+(u_4) = \{u_5, u_6\}$, b) $O(u_1) = \{u, u_5\}, N^+(u_4) = \{u_6, v_4\}$, c) $O(u_1) = \{u_5, u_6\}, N^+(u_4) = \{u, v_4\}$ or d) $O(u_1) = \{u_5, v_4\}, N^+(u_4) = \{u, v_4\}$.

**Case 2.a: $O(u_4) = \{u, v_4\}, N^+(u_4) = \{u_5, u_6\}$**

As $v_4 \in O(u) \cap O(u_1)$, $u_2$ can reach $v_4$ and without loss of generality $v_6 \in N^+(u_5), N^+(u_2) = N^+(u_4)$, so by Lemma 1 $u_2 \in O(u_4), u_4 \in O(u_2), u_5 \in O(u_6)$ and $u_6 \in O(u_5)$. Hence $u_4 \in O(v) \cap O(u_2)$, so $v_1$ can reach $u_4$, so $u_4 \in N^+(v_4)$. Neither $u_5$ nor $u_6$ lies in $N^+(v_4)$, so $O(v_1) = \{u_5, u_6\}$ and $N^+(v_4) = \{u_4, v\}$. Hence $O(v_4) = \{u, u_1\}$. Observe that $N^+(u_1) = N^+(v_4)$, so that $v \in O(u_1)$. Therefore $v \not\in N^+(u_5) \cup N^+(u_6)$, yielding $O(u_2) = \{u_4, v\}, N^+(u_2) = \{u_4, v_1, u, u_1\}$. As $v_1 \rightarrow v_4$ and $N^+(u_1) = N^+(v_4)$, we must have $N^+(u_5) = \{v_4, u\}, N^+(u_6) = \{v_1, u_1\}$. This yields the digraph in Fig. 5. Unlike the digraph in Fig. 2, this digraph contains pairs of vertices with identical out-neighbours, so the two are not isomorphic.

**Case 2.b: $O(u_1) = \{u, u_5\}, N^+(u_4) = \{u_6, v_4\}$**

As $u_4 \rightarrow v_4, u_4 \not\in N^+(v_4)$, so $u_4 \in O(v_1)$. Hence $u_4 \in O(v) \cap O(v_1)$, so $u_2$ can reach $u_4$. As $u_4 \rightarrow u_6$, we must have $u_5 \rightarrow u_4, u_2$ and $u_4$ have $u_6$ as a common out-neighbour, so $v_4 \in O(u_5), u_5 \in O(u_4)$. Therefore $u_4 \in O(u) \cap O(u_2)$, so that $u_6$ can reach $v_4$, but $v_4 \not\in T(u_6)$, so $N^+(u_6)$ contains an in-neighbour of $v_4, u_4 \not\in N^+(u_6)$, so we must have $u_6 \rightarrow v_1$. We have $u_5 \in O(u_4) \cap O(u_1)$, so $v_1$ can reach $u_5$ and hence $v_4 \rightarrow u_5, v_1$ cannot reach $u_6$, as $u_2, u_4 \not\in T(v_1)$, so $O(v_1) = \{u_4, u_6\}, N^+(u_5) = \{v_5, v\}$. Now $u_2$ and $u_4$ have $u_6$ as a unique common out-neighbour, so $u_6 \in O(v_4), v \in O(u_2)$. Thus $O(u_2) = \{v, v_4\}$ and $N^2(u_2) = \{u_4, v_1, u, u_1\}$. Taking into account adjacencies between members of $N^2(u_4)$, it follows that $N^+(u_5) = \{u_4, u\}, N^+(u_6) = \{v_1, u_1\}$. However, $(u_2, u_4)$ now constitutes a bad pair, contradicting our assumption.

**Case 2.c: $O(u_1) = \{u, u_5\}, N^+(u_4) = \{u_6, v_4\}$**

As $u_4 \rightarrow v_4, u_4 \not\in N^+(v_4)$, so $u_4 \in O(v_1)$. Hence $u_4 \in O(v) \cap O(v_1)$, implying that $u_2$ can reach $u_4$. Without loss of generality, $u_5 \rightarrow u_4$.

There are three possibilities: (i) $O(v_1) = \{u_4, u_5\}, N^+(v_4) = \{v, u_5\}$, (ii) $O(v_1) = \{u_4, u_5\}, N^+(v_4) = \{v, u_6\}$ and (iii) $O(v_1) = \{u_4, v\}, N^+(v_4) = \{u_5, u_6\}$.

(i) $O(v_1) = \{u_4, u_5\}, N^+(v_4) = \{v, u_5\}$

$u_1$ and $v_4$ have $v$ as a unique common out-neighbour, so $u_4 \in O(v_4)$. However, this contradicts $v_4 \rightarrow u_5 \rightarrow u_4$.

(ii) $O(v_1) = \{u_4, u_5\}, N^+(v_4) = \{v, u_6\}$

Neither $u_4$ nor $v_1$ lie in $T(u_2)$, so $v_4 \in O(u_2)$. Now observe that $u_2$ and $u_4$ have $u_6$ as unique common out-neighbour, so $v \in O(u_2)$, yielding $O(u_2) = \{v, v_4\}$ and $N^2(u_2) = \{u_4, u_1, u, v_1\}$. As $u_4 \rightarrow u$ and $u \rightarrow u_1$, we must have $N^+(u_5) = \{u_4, u_1\}, N^+(u_6) = \{u, v_1\}$, a contradiction, since $u_1 \rightarrow u_4$.

(iii) $O(v_1) = \{u_4, v\}, N^+(v_4) = \{u_5, u_6\}$

We now have $N^+(u_2) = N^+(v_4)$, so $u_2 \in O(v_4), v_4 \in O(u_2), u_5 \in O(u_6), u_6 \in O(u_5)$. Also $N^+(u_4) = N^+(v_1)$, so $u_4 \in O(v_1), v_1 \in O(u_4)$ and $u \in O(u_4)$. $u \in O(u_4)$ implies that $u \not\in N^+(u_5) \cup N^+(u_6)$, so we see that $u \in O(u_2)$ and hence $O(u_2) = \{u, v_4\}$ and $N^2(u_2) = \{u_4, u_1, u, v_1\}$. As $u_1 \rightarrow u_4$ and $u_4 \rightarrow v$, we have $N^+(u_5) = \{u_4, v\}, N^+(u_6) = \{u_1, v_1\}$. It is not difficult to show that this yields a $(2, 2, +2)$-digraph isomorphic to that in Fig. 5.

**Case 2.d: $O(u_1) = \{u_5, v_4\}, N^+(u_4) = \{u, u_6\}$**

In this case $v_4 \in O(u) \cap O(u_1)$, so $u_2$ can reach $v_4, u_4$ and $v_1$ have unique common out-neighbour $u$, so $v_4 \in O(u_4), u_6 \in O(v_1)$. If $u_6 \rightarrow v_4$, then we would have $u_4 \rightarrow u_6 \rightarrow v_4$, contradicting $v_4 \in O(u_4)$, so $u_5 \rightarrow v_4$. This also implies that $u_5 \not\in N^+(u_4)$, so $u_5 \in O(v_1)$, yielding $O(v_1) = \{u_5, u_6\}$ and $N^+(v_4) = \{v, u_4\} = N^+(u_1)$. Now $v_4, u_1 \not\in T(u_2)$, so $O(u_2) = \{v, u_4\}$.
A. Each case there are exactly three vertices contained in two 3-cycles. However, there does exist a Cayley \( (2, k) \).

**Corollary 2.** There is a unique \((2, 2, +2)\)-digraph containing no bad pairs.

This completes our analysis of diregular \((2, 2, +2)\)-digraphs. As it was shown in [7] that there are no non-diregular \((2, 2, +2)\)-digraphs, \((2, 2, +2)\)-digraphs are now classified up to isomorphism. These conclusions have been verified computationally by Erskine [2]. It is interesting to note that neither of the \((2, 2, +2)\)-digraphs are vertex-transitive, for in each case there are exactly three vertices contained in two 3-cycles. However, there does exist a Cayley \((2, 2, +5)\)-digraph (on the alternating group \(A_4\)), so it would be interesting to determine the smallest vertex-transitive \((2, 2, +e)\)-digraphs.

**4. Main result**

We can now complete our analysis by showing that there are no diregular \((2, k, +2)\)-digraphs for \(k \geq 3\). Let \(G\) be such a digraph. By **Lemma 2**, \(G\) contains vertices \(u\) and \(v\) with a unique common out-neighbour. In accordance with our vertex-labelling convention, we have the situation in **Fig. 6**. A triangle based at a vertex \(x\) represents the set \(T(x)\).

We now proceed to determine the possible outlier sets of \(u\) and \(v\).

**Lemma 5.** \(v \in N^{k-1}(u_1) \cup O(u)\) and \(u \in N^{k-1}(v_1) \cup O(v)\). If \(v \in O(u)\), then \(u_2 \in O(u_1)\) and if \(u \in O(v)\), then \(u_2 \in O(v_1)\).

**Proof.** \(v\) cannot lie in \(T(u)\), or the vertex \(u_2\) would be repeated in \(T(u)\). Also, \(v \notin T(u_2)\), or there would be a \(a \leq k\)-cycle through \(v\). Therefore, if \(v \notin O(u)\), then \(v \in N^{k-1}(u_1)\). Likewise for the other result. If \(v \in O(u)\), then neither in-neighbour of \(u_2\) lies in \(T(u_1)\), so that \(u_2 \notin O(u_1)\). \(\Box\)

**Lemma 6.** Let \(w \in T(v_1)\), with \(d(v_1, w) = l\). Suppose that \(w \in T(u_1)\), with \(d(u_1, w) = m\). Then either \(m \leq l\) or \(w \in N^{k-1}(u_1)\). A similar result holds for \(w \in T(u_1)\).

**Proof.** Let \(w\) be as described and suppose that \(m > l\). Consider the set \(N^{k-m}(w)\). By construction, \(N^{k-m}(w) \subseteq N^{k}(u_1)\), so by \(k\)-geodecity \(N^{k-m}(w) \cap T(u_1) = \emptyset\). At the same time, we have \(I + k - m \leq k - 1\), so \(N^{k-m}(w) \subseteq T(v_1)\). This implies that \(N^{k-m}(w) \cap T(v_2) = N^{k-m}(w) \cap T(u_2) = \emptyset\). As \(V(G) = \{u\} \cup T(u_1) \cup T(u_2) \cup O(u)\), it follows that \(N^{k-m}(w) \subseteq \{u\} \cup O(u)\). Therefore \(|N^{k-m}(w)| = 2k - m \leq 3\). By assumption \(0 \leq m \leq k - 1\), so it follows that \(m = k - 1\). \(\Box\)

**Corollary 3.** If \(w \in T(v_1)\), then either \(w \in \{u\} \cup O(u)\) or \(w \in T(u_1)\) with \(d(u_1, w) = k - 1\) or \(d(u_1, w) \leq d(v_1, w)\).

**Proof.** By \(k\)-geodecity and **Lemma 6**. \(\Box\)

**Corollary 4.** \(v_1 \in N^{k-1}(u_1) \cup O(u)\) and \(u_1 \in N^{k-1}(v_1) \cup O(v)\).
Proof. We prove the first inclusion. By Corollary 3, \( v_1 \in \{ u \} \cup O(u) \cup \{ u_1 \} \cup N^{k-1}(u_1) \). By \( k \)-geodecity, \( v_1 \neq u \) and by construction, \( v_1 \neq u_1 \). □

We now have enough information to identify one member of \( O(u) \) and \( O(v) \).

Lemma 7. \( v_1 \in O(u) \) and \( u_1 \in O(v) \).

Proof. We prove that \( v_1 \in O(u) \). Suppose that neither \( v_1 \) nor \( v \) lies in \( O(u) \). Then by Lemma 5 and Corollary 4 we have \( v, v_1 \in N^{k-1}(u_1) \). As \( v_1 \) is an out-neighbour of \( v \), it follows that \( v_1 \) appears twice in \( T_k(u_1) \), violating \( k \)-geodecity. Therefore \( O(u) \cap \{ v, v_1 \} \neq \emptyset \).

Now assume that \( v_1, u_1 \in T_k(u) \). Again by Corollary 4, \( v_1 \in N^{k-1}(u_1) \). By \( k \)-geodecity we also have \( v_3 \in T(u_1) \). However, \( v_1 \in N^+(v_3) \), so \( v_1 \) appears twice in \( T_k(u_1) \), which is impossible. Hence \( O(u) \cap \{ v_1, v_3 \} \neq \emptyset \). Similarly, \( O(u) \cap \{ v_1, v_4 \} \neq \emptyset \). In the terminology of the previous section, \( G \) contains no bad pairs. Therefore, if \( v_1 \notin O(u) \), then \( \{ v, v_3, v_4 \} \subseteq O(u) \). Since these vertices are distinct, this is a contradiction and the result follows. □

Lemma 7 allows us to conclude that for vertices sufficiently close to \( v_1 \) one of the potential situations mentioned in Corollary 3 cannot occur.

Lemma 8. \( T_{k-3}(v_1) \cap N^{k-1}(u_1) = T_{k-3}(u_1) \cap N^{k-1}(v_1) = \emptyset \).

Proof. Let \( w \in T_{k-3}(v_1) \cap N^{k-1}(u_1) \). Consider the position of the vertices of \( N^+(w) \) in \( T_k(u) \cup O(u) \). As \( v_1 \notin N^+(w) \), it follows from Lemma 7 that at most one of the vertices of \( N^+(w) \) can be an outlier of \( u \), so let us write \( w_1 \in N^+(w) \cap O(u) \). By \( k \)-geodecity, \( u_1 \notin T(u_1) \cap u_1 \). Hence \( w_1 \in T(u_2) = T(v_2) \). However, \( w_1 \) also lies in \( T(v_1) \), so this violates \( k \)-geodecity. □

Corollary 5. There is at most one vertex in \( T_{k-3}(v_1) - \{ v_1 \} \) that does not lie in \( T(u_1) \); for all other vertices \( w \) in \( T_{k-3}(v_1) - \{ v_1 \} \), \( d(u_1, w) = d(v_1, w) \). A similar result for \( T_{k-3}(u_1) - \{ u_1 \} \) also holds.

Lemma 9. For \( k = 3 \), \( N^+(u_1) \cap N^2(v_1) = N^+(v_1) \cap N^2(u_1) = \emptyset \).

Proof. Suppose that \( v_3 = u_2 \). By the reasoning of Lemma 8 we can set \( u = v_2 \) and \( O(u) = \{ v_1, v_8 \} \), \( v \notin O(u) \) and by \( 3 \)-geodecity \( v \notin N^+(u_3) \), so we can assume that \( u = u_0, u_3 \to v_3 \) implies that \( v_3 \notin T(v_1) \), so \( O(v) = \{ u_1, u_1 \} \). We must have \( \{ u_4, u_5, u_10 \} = \{ v_4, v_5, v_10 \} \). As \( u_4 \to v_2 \), it follows that \( v_4 = u_8 \) and hence \( \{ u_4, u_10 \} = \{ v_2, v_10 \} \), which is impossible. □

As \( u_1 \) is an outlier of \( v \), neither \( v_3 \) nor \( v_4 \) can be equal to \( u_1 \). It follows from Corollary 5 and Lemma 9 that either \( N^+(u_1) = N^+(v_1) \) or \( u_1 \) and \( v_1 \) have a single common out-neighbour, with one vertex of \( N^+(v_1) \) being an outlier of \( u \).

Lemma 10. \( N^2(u) \neq N^2(v) \)

Proof. Let \( N^2(u) = N^2(v) \), with \( N^+(u_1) = N^+(v_1) = \{ u_3, u_4 \} \). Suppose that \( v \notin O(u) \). By Lemma 5, \( v \in N^{k-2}(u_3) \cup N^{k-2}(u_4) \). But then there is a \( k \)-cycle through \( v \). It follows that \( O(u) = \{ v, v_1 \} \). Then \( O(v) = \{ u_1, u_1 \} \). By Lemma 5, \( u_2 \in O(u_1) \cap O(v_1) \). Therefore by Lemma 1 \( O(u_1) = \{ u_2, v_1 \} \), \( O(v_1) = \{ u_2, u_1 \} \).

Consider the in-neighbour \( u' \) of \( u_1 \) that is distinct from \( u \). We have either \( N^+(u') \cap N^+(u) = \{ v \} \) or \( N^+(u') \cap N^+(u) = \{ v, v' \} \). In the first case, it follows from Lemma 7 that \( u_2 \in O(u) \). Every vertex of \( G \) is an outlier of exactly two vertices, so \( u' = u_1 \) or \( v_1 \). In either case, we have a contradiction. Therefore \( N^+(u') \cap N^+(u) = \{ v \} \). It now follows from Lemma 1 that \( u' \in O(u) = \{ v, v_1 \} \), which is impossible. □

Noticing that \( u_1 \) and \( v_1 \) also have a unique common out-neighbour, we have the following corollary.

Corollary 6. Without loss of generality, \( u_3 = v_3, u_9 = v_9, O(u) = \{ v_1, v_4 \} \), \( O(v) = \{ u_1, u_4 \} \), \( O(u_1) = \{ v_4, v_10 \} \) and \( O(v_1) = \{ u_4, u_10 \} \).

We are now in a position to complete the proof by deriving a contradiction.

Theorem 2. There are no digereular \((2, k, +2)\)-digraphs for \( k \geq 3 \).

Proof. \( u, v \notin \{ u_1, u_3, v_1, v_4 \} \), so by Lemma 5 \( d(u, v) = d(v, u) = k \). In fact, \( u_3 = v_3 \) implies that \( v \in N^{k-2}(u_3) \) and \( u \in N^{k-2}(v_4) \). Let \( k \geq 4 \). Then \( u, v \notin \{ u_10, v_10 \} \), so \( u, v \in T_3(u_1) \cap T_3(v_1) \). If \( u \in T_2(u_2) = T(v_2) \), then \( u \) would appear twice in \( T_2(v_1) \), so \( u \in N^{k-1}(u_1) \). However, as \( u \) and \( v \) have a common out-neighbour, this violates \( k \)-geodecity.

Finally, suppose that \( k = 3 \). The above analysis will hold unless \( u = v_10 \) and \( v = u_10 \). Let \( N^-(u_1) = \{ u_1, u_1 \} \). It is evident that \( u' \notin \{ v_1, v_4 \} \), so \( v' \notin T_3(u) \). As \( v \in N^2(u_4) \), we must have \( v' \in N^2(u_2) \). Similarly \( u' \in N_2(u_2) \).

Since \( u_1 \) and \( v_1 \) have a common out-neighbour, we can assume that \( u' \in N^+(u_3) \) and \( v' \in N^+(u_5) \). \( u_3 \) and \( v_5 \) will be the outlier of only two vertices, namely \( u \) and \( v_1 \). As \( u_3 = v_3 \), we must have \( u' \in N^2(u_2) \). Similarly \( u' \in N^2(u_2) \).

It is interesting to note that a similar argument can be used to provide an alternative proof of the result of \([6]\).
References