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Note

On diregular digraphs with degree two and excess two

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Abstract

An important topic in the design of efficient networks is the construction of \((d, k, +\epsilon)\)-digraphs, i.e. \(k\)-geodetic digraphs with minimum out-degree \(\geq d\) and order \(M(d, k) + \epsilon\), where \(M(d, k)\) represents the Moore bound for degree \(d\) and diameter \(k\) and \(\epsilon > 0\) is the (small) excess of the digraph. Previous work has shown that there are no \((2, k, +1)\)-digraphs for \(k \geq 2\). In a separate paper, the present author has shown that any \((2, k, +2)\)-digraph must be diregular for \(k \geq 2\). In the present work, this analysis is completed by proving the nonexistence of diregular \((2, k, +2)\)-digraphs for \(k \geq 2\) and classifying diregular \((2, 2, +2)\)-digraphs up to isomorphism.

1. Introduction

An important topic in the design of interconnection networks is the directed degree/diameter problem: what is the largest possible order \(N(d, k)\) of a digraph \(G\) with maximum out-degree \(d\) and diameter \(\leq k\)? A simple inductive argument shows that for \(0 \leq l \leq k\) the number of vertices at distance \(l\) from a fixed vertex \(v\) is bounded above by \(d^l\). Therefore, a natural upper bound for the order of such a digraph is the so-called Moore bound \(M(d, k) = 1 + d + d^2 + \cdots + d^k\). A digraph that attains this upper bound is called a Moore digraph. It is easily seen that a digraph \(G\) is Moore if and only if it is out-regular with degree \(d\), has diameter \(k\) and is \(k\)-geodetic, i.e. for any two vertices \(u, v\) there is at most one \(k\)-path from \(u\) to \(v\).

As it was shown by Bridges and Toueg in [1] that Moore digraphs exist only in the trivial cases \(d = 1\) or \(k = 1\) (the Moore digraphs are directed \((k + 1)\)-cycles and complete digraphs \(K_{k+1}\) respectively), much research has been devoted to the study of digraphs that in some sense approximate Moore digraphs. For example, there is an extensive literature on digraphs with maximum out-degree \(d\), diameter \(\leq k\) and order \(M(d, k) - \delta\) for small \(\delta > 0\); this is equivalent to relaxing the \(k\)-geodeticity requirement in the conditions for a digraph to be Moore. \(\delta\) is known as the defect of the digraph. The reader is referred to the survey [5] for more information.

In this paper, however, we will consider the following related problem, which is obtained by retaining the \(k\)-geodeticity requirement in the above characterisation of Moore digraphs, but allowing the diameter to exceed \(k\): what is the smallest possible order of a \(k\)-geodetic digraph \(G\) with minimum out-degree \(\geq d\)? A \(k\)-geodetic digraph with minimum out-degree \(\geq d\) and order \(M(d, k) + \epsilon\) is said to be a \((d, k, +\epsilon)\)-digraph or to have excess \(\epsilon\). It was shown in [6] that there are no diregular \((2, k, +1)\)-digraphs for \(k \geq 2\). In 2016 it was shown in [3] that digraphs with excess one must be diregular and that there are no \((d, k, +1)\)-digraphs for \(k = 2, 3, 4\) and sufficiently large \(d\). In a separate paper [7], the present author has shown that \((2, k, +2)\)-digraphs must be diregular with degree \(d = 2\) for \(k \geq 2\). In the present paper, we classify the \((2, 2, +2)\)-digraphs up to isomorphism and show that there are no diregular \((2, k, +2)\)-digraphs for \(k \geq 3\), thereby completing the proof of the nonexistence of digraphs with degree \(d = 2\) and excess \(\epsilon = 2\) for \(k \geq 3\). Our reasoning and notation will follow closely that employed in [4] for the corresponding result for defect \(\delta = 2\).
2. Preliminary results

We will let $G$ stand for a $(2, k, +2)$-digraph for arbitrary $k \geq 2$, i.e. $G$ has minimum out-degree $d = 2$, is $k$-geodetic and has order $M(2, k) + 2$. We will denote the vertex set of $G$ by $V(G)$. By the result of [7], $G$ must be diregular with degree $d = 2$ for $k \geq 2$. The distance $d(u, v)$ between vertices $u$ and $v$ is the length of the shortest path from $u$ to $v$. Notice that $d(u, v)$ is not necessarily equal to $d(v, u)$. $u \rightarrow v$ will indicate that there is an arc from $u$ to $v$. We define the in- and out-neighbourhoods of a vertex $u$ by $N^{-}(u) = \{ v \in V(G) : v \rightarrow u \}$ and $N^{+}(u) = \{ v \in V(G) : u \rightarrow v \}$ respectively; more generally, for $0 \leq l \leq k$, the set $\{ v \in V(G) : d(u, v) = l \}$ of vertices at distance exactly $l$ from $u$ will be denoted by $N^{l}(u)$. For $0 \leq l \leq k$ we will also write $T_{l}(u) = \bigcup_{i=0}^{l} N^{i}(u)$ for the set of vertices at distance $\leq l$ from $u$. The notation $T_{k-1}(u)$ will be abbreviated by $T(u)$.

It is easily seen that for any vertex $u$ of $G$, there are exactly two distinct vertices that are at distance $\geq k + 1$ from $u$. For any $u \in V(G)$, we will write $O(u)$ for the set of these vertices and call such a set an outlier set and its elements outliers of $u$. Notice that $O(u) = V(G) - T_{k}(u)$. An elementary counting argument shows that in a diregular $(2, k, +2)$-digraph every vertex is also an outlier of exactly two vertices. We will say that a vertex $u$ can reach a vertex $v$ if $v \not\in O(u)$.

Our proof will proceed by an analysis of a pair of vertices with exactly one common out-neighbour. First, we must show that such a pair exists and deduce some elementary properties of pairs of vertices with identical out-neighbourhoods.

**Lemma 1.** For $k \geq 2$, let $u$ and $v$ be distinct vertices such that $N^{+}(u) = N^{+}(v) = \{ u_{1}, u_{2} \}$. Then $u_{1} \in O(u_{2})$, $u_{2} \in O(u_{1})$ and there exists a vertex $x$ such that $O(u) = \{ v, x \}$, $O(v) = \{ u, x \}$.

**Proof.** Suppose that $u$ can reach $v$ by a $\leq k$-path. Then $v \in T(u_{1}) \cup T(u_{2})$. As $N^{+}(v) = N^{+}(u)$, it follows that there would be a $\leq k$-cycle through $v$, contradicting $k$-geodeticity. If $O(u) = \{ v, x \}$, then $x \not\in v$ and $x \not\in T(u_{1}) \cup T(u_{2})$, so that $v$ cannot reach $x$ by a $\leq k$-path. Similarly, if $u_{1}$ can reach $u_{2}$ by a $\leq k$-path, then we must have $\{ u, v \} \cap T(u_{1}) \neq \emptyset$, which is impossible. \hfill $\square$

**Lemma 2.** For $k \geq 2$, there exists a pair of vertices $u$, $v$ with $|N^{+}(u) \cap N^{+}(v)| = 1$.

**Proof.** Suppose for a contradiction that there is no such pair of vertices. Define a map $\phi : V(G) \rightarrow V(G)$ as follows. Let $u^{+}$ be an out-neighbour of a vertex $u$ and let $\phi(u^{+})$ be the in-neighbour of $u^{+}$ distinct from $u$. By our assumption, it is easily verified that $\phi$ is a well-defined bijection with no fixed points and with square equal to the identity. It follows that $G$ must have even order, whereas $|V(G)| = M(2, k) + 2$ is odd. \hfill $\square$

$u, v$ will now stand for a pair of vertices with a single common out-neighbour. We will label the vertices of $T_{k}(u)$ according to the scheme $N^{+}(u) = \{ u_{1}, u_{2} \}$, $N^{+}(u_{1}) = \{ u_{3}, u_{4} \}$, $N^{+}(u_{2}) = \{ u_{5}, u_{6} \}$, $N^{+}(u_{3}) = \{ u_{7}, u_{8} \}$, $N^{+}(u_{4}) = \{ u_{9}, u_{10} \}$ and so on, with the same convention for the vertices of $T_{k}(v)$, where we will assume that $u_{2} = v_{2}$.

3. Classification of $(2, 2, +2)$-digraphs

We begin by classifying the $(2, 2, +2)$-digraphs up to isomorphism. We will prove the following theorem.

**Theorem 1.** There are exactly two diregular $(2, 2, +2)$-digraphs, which are displayed in Figs. 2 and 5.

Let $G$ be an arbitrary diregular $(2, 2, +2)$-digraph. $G$ has order $M(2, 2) + 2 = 9$. By Lemma 2, $G$ contains a pair of vertices $(u, v)$ such that $|N^{+}(u) \cap N^{+}(v)| = 1$; we will assume that $u_{2} = v_{2}$, so that we have the situation shown in Fig. 1.

We can immediately deduce some information on the possible positions of $v$ and $v_{1}$ in $T_{2}(u)$.

**Lemma 3.** If $v \not\in O(u)$, then $v \in N^{+}(u_{1})$. If $v_{1} \not\in O(u)$, then $v_{1} \in N^{+}(u_{1})$.

**Proof.** $v \not\in T(u_{2})$ by 2-geodeticity. $v \not\in u$ by construction. If we had $v = u_{1}$, then there would be two distinct $\leq 2$-paths from $u$ to $u_{2}$. Also $v_{1} \not\in \{ u \} \cup T(u_{2})$ by 2-geodeticity and by assumption $u_{1} \not\in v_{1}$. \hfill $\square$
Corollary 1. \( O(u) \cap \{ v, v_1 \} \neq \emptyset \).

We will call a pair of vertices \((u, v)\) with a single common out-neighbour \textit{bad} if at least one of

\[
O(u) \cap \{ v, v_3 \} = \emptyset, \ O(u) \cap \{ v_1, v_4 \} = \emptyset, \ O(v) \cap \{ u_1, u_3 \} = \emptyset, \ O(v) \cap \{ u_1, u_4 \} = \emptyset.
\]

holds. Otherwise such a pair will be called \textit{good}.

Lemma 4. There is a unique \((2, 2, +2)\)-digraph containing a bad pair.

\textbf{Proof.} Assume that there exists a bad pair \((u, v)\). Without loss of generality, \( O(u) \cap \{ v_1, v_3 \} = \emptyset \). By Lemma 3 we can set \( v_1 = u_3 \). By 2-geodecity \( v_3 = u \). We cannot have \( v_4 = v_3 = u \), so \( u_4 \) must be an outlier of \( u \). By Corollary 1 it follows that \( O(u) = \{ v, v_4 \} \).

Consider the vertex \( u_1 \). By Lemma 3, if \( u_1 \notin O(v) \), then \( u_1 \in N^+(v_1) \). However, as \( v_1 = u_3 \), there would be a 2-cycle through \( u_1 \). Hence \( u_1 \in O(v) \). As \( O(u) = \{ v, v_4 \} \), we have \( V(G) = \{ u, u_1, u_2, u_3 = v, u_4, u_5, u_6, v, v_4 \} \) and \( O(v) = \{ u_1, u_4 \} \). As neither \( u \) nor \( v \) lies in \( T(u_1) \), we must also have \( u_2 \in O(u_1) \). As \( u_1 \) can reach \( u_1, u_4, u \) and \( v_4 \), it follows that without loss of generality we either have \( O(u_1) = \{ u_2, v \} \) and \( N^+(u_4) = \{ u_5, u_6 \} = N^+(u_2) \) or \( O(u_1) = \{ u_2, u_6 \} \) and \( N^+(u_4) = \{ v, u_5 \} \). In either case, \( (v, u_1) \) is a good pair.

Suppose firstly that \( N^+(u_2) = N^+(u_4) \). Then \( v \) is an outlier of \( u \) and \( u_1 \). As each vertex is the outlier of exactly two vertices, \( v_1 \) must be able to reach \( v \) by a \( \leq 2 \)-path. Hence \( v_4 \to v \). Likewise \( u_2 \) can reach \( v \), so without loss of generality \( u_5 \to v \). Suppose that \( O(u_2) \cap \{ u, u_1 \} = \emptyset \). As \( u \) and \( v \) have a common out-neighbour, we must have \( u_6 \to u \). Since \( u \to u_1 \), by 2-geodecity we must have \( u_5 \to u_1 \). However, this is a contradiction, as \( u \) and \( u_1 \) also have a common out-neighbour. Therefore, at least one of \( u, u_1 \) is an outlier of \( u_2 \). By Lemma 1 \( u_4 \) is an outlier of \( u_2 \). Therefore either \( O(u_2) = \{ u, u_4 \} \) or \( O(u_2) = \{ u_1, u_4 \} \). If \( O(u_2) = \{ u, u_4 \} \), then \( u_2 \) must be able to reach \( u_1, v_1 \) and \( v_4, u_5 \to v \) and \( v \to v_1 \), so \( v_1 \in N^+(u_6) \). As \( u_1 \to v_1 \), we must have \( N^+(u_5) = \{ v, u_1 \} \). As \( u \) and \( u_1 \) have a common out-neighbour, this violates 2-geodecity. Hence \( O(u_2) = \{ u_1, u_4 \} \) and \( u_2 \) can reach \( u, v_1 \) and \( v_4 \). As \( v \to v_1, v_1 \in N^+(u_6) \). As \( v_1 \to v_4 \), it follows that \( N^+(u_5) = \{ v, v_4 \} \). However, \( u_4 \to v \), so this again violates 2-geodecity.

We are left with the case \( O(u_1) = \{ u_2, u_6 \} \) and \( N^+(u_4) = \{ v, u_5 \} \). Then \( v_1 \in O(u_2) \), as neither \( v \) nor \( u_1 \) lies in \( T(u_2) \). Observe that \( u_2 \) and \( u_4 \) have a single common out-neighbour, so by Corollary 1 \( O(u_2) \cap \{ u, v \} \neq \emptyset \). Therefore either \( O(u_2) = \{ v, u_4 \} \) or \( O(u_2) = \{ v_1, v \} \). Suppose firstly that \( O(u_2) = \{ v_1, u_4 \} \). Then \( N^+(u_2) = \{ u, v, u_1, v_4 \} \). As \( N^+(u_4) = \{ v, u_5 \} \), \( u_5 \not\to v \), so \( u_6 \to v \). As \( N^+(u) \cap N^+(v) \neq \emptyset \), \( u_5 \to u \to u_1 \), so necessarily \( u_6 \to u \). However, \( v_1 \in N^+(u_1) \cap N^+(v) \), contradicting 2-geodecity.

Hence \( O(u_2) = \{ v_1, v \} \) and \( N^+(u_2) = \{ u, u_1, u_4, v_4 \} \). As \( u_4 \to u_5 \) and \( u_5 \not\to u_4 \), thus \( u_6 \to u_4 \). Now \( u_1 \to u_4 \) and \( u \to u_1 \) implies that \( N^+(u_5) = \{ u_1, v_4 \} \) and \( N^+(u_6) = \{ u, u_4 \} \). Finally we must have \( N^+(u_4) = \{ v, u_5 \} \). This gives us the \((2, 2, +2)\)-digraph shown in Fig. 2. £

We can now assume that all pairs given by Lemma 2 are good. Let us fix a pair \((u, v)\) with a single common out-neighbour. It follows from Corollary 1 and the definition of a good pair that \( v_1 \in O(u) \); otherwise \( O(u) \) would contain \( v, v_3 \) and \( v_4 \), which is impossible. Likewise \( u_1 \in O(v) \).

![Fig. 2. The unique (2, 2, +2)-digraph containing a bad pair.](image-url)
Considering the positions of $v_3$ and $v_4$, we see that there are without loss of generality four possibilities: (1) $u = v_3$, $u_4 = v_4$, (2) $u = v_3$, $O(u) = \{v_1, v_4\}$, (3) $N^+(u_1) = N^+(v_1)$ and (4) $u_3 = v_3$, $O(u) = \{v_1, u_4\}$. A suitable relabelling of vertices shows that case 4 is equivalent to case 1a below, so we will examine cases 1 to 3 in turn.

**Case 1: $u = v_3$, $u_4 = v_4$**

Depending upon the position of $v$, we must either have $O(u) = \{v_1, v\}$ and $O(v) = \{u_1, u_3\}$ or $v = u_3$ (see Fig. 3).

**Case 1a: $O(u) = \{v_1, v\}$, $O(v) = \{u_1, u_3\}$**

In this case $V(G) = \{u, u_1, u_2, u_3, u_4, u_5, u_6, v, v_1\}$. $u_1$ and $v_1$ have a single common out-neighbour, namely $u_4$, so, as we are assuming all such pairs to be good, we have $u_3 \in O(v_1)$, $u \in O(u_1)$. By 2-geodecity, $N^+(u_4) \subseteq \{u_5, u_6, v\}$, so without loss of generality either $N^+(u_4) = \{u_5, u_6\}$ or $N^+(u_4) = \{u_5, v\}$.

Suppose that $N^+(u_4) = \{u_5, u_6\}$. By elimination, $O(u_1) = \{v, u_3\}$. As $G$ is diregular, every vertex is an outlier of exactly two vertices; $v$ is an outlier of $u$ and $v_1$, so both $u_1$ and $u_2$ can reach $v$ by a $\leq 2$-path. Hence $v \in N^+(u_3)$. As $v \rightarrow v_1$, we see that $v_1$ is an outlier of $u_1$; as $u$ is also an outlier of $u_1$, we have $O(u_1) = \{u, v_1\}$ and $N^+(u_3) = \{v, u_2\}$. As $v \rightarrow u_2$, this is impossible.

Now consider $N^+(u_4) = \{u_5, v\}$. We now have $O(u_1) = \{u_3, u_6\}$. Thus $u_3 \in O(v) \cap O(u_1)$, so $u_3 \in T_2(u_4)$. $v$ is not adjacent to $u_3$, so $u_3 \in N^+(u_5)$. $u_2$ and $u_4$ have $u_5$ as a unique common out-neighbour, so $u_5 \in O(u_4)$, $v \in O(u_2)$. As $u_5 \in O(v_1) \cap O(u_4)$, $u_1$ can reach $u_6$. Hence $u_6 \in N^+(u_1)$. Neither $u$ nor $v$ lie in $T(u_1)$, so $u_2 \in O(u_1)$. Therefore either $O(u_1) = \{u, u_2\}$ or $O(u_1) = \{u_2, v_1\}$. If $O(u_1) = \{u, u_2\}$, then $N^+(u_2) = \{u_6, v_1\}$. $u_2$ cannot reach $v_1$, since $v, u_3 \notin T(u_2)$, so $O(u_2) = \{v, v_1\}$ and $N^2(u_2) = \{u, u_1, u_2, u_4\}$. As $u_4 \rightarrow u_5, u_4 \in N^+(u_6), u_1 \rightarrow u_4$, so $N^+(u_1) = \{u_1, u_3\}$. As $u_1 \rightarrow u_3$, this is a contradiction. Thus $O(u_1) = \{u_2, v_1\}$, so that $N^+(u_3) = \{u, u_6\}$. $u_1$ must have an in-neighbour apart from $u$, which must be either $u_5$ or $u_6$. As $u_1 \rightarrow u_3$, we have $u_1 \in N^+(u_6)$. By elimination, $v$ and $v_1$ must also have in-neighbours in $\{u_5, u_6\}$. As $u_1$ and $v_1$ have a common out-neighbour, we have $N^+(u_5) = \{u_3, v_1\}, N^+(u_6) = \{u_1, v\}$. However, both $u_3$ and $v_1$ are adjacent to $u$, violating 2-geodecity.

**Case 1b: $v = u_3$**

There exists a vertex $x$ such that $V(G) = \{u, u_1, u_2, v, u_4, u_5, u_6, v_1, x\}, O(u) = \{v_1, x\}$ and $O(v) = \{u_1, x\}$. As $x \in O(u) \cap O(v)$, $u_1$ and $u_2$ can reach $x$, so without loss of generality $x \in N^+(u_4) \cap N^+(u_5)$. As $u_5$ and $u_4$ have a common out-neighbour, $u_6 \in O(u_1)$. Also, $u_1$ and $v_1$ have $u_4$ as a unique common out-neighbour, so $u \in O(u_1)$ and $O(u_1) = \{u, u_5\}$. Thus $N^+(u_4) = \{x, u_6\}$. Observe that $u_2$ and $u_4$ have the out-neighbour $u_6$ in common. Thus $x \in O(u_2)$, whereas we already have $x \in O(u) \cap O(v)$, a contradiction.

**Case 2: $u = v_3$, $O(u) = \{v_1, v_4\}$**

As $v$ is not equal to $v_1$ or $v_4$, $v$ must lie in $T_2(u)$. Without loss of generality, $v = u_3$. Hence $V(G) = \{u, u_1, u_2, v, u_4, u_5, u_6, v_1, v_4\}$ and $O(v) = \{u_1, u_4\}$. We have the configuration shown in Fig. 4. Hence $u_1$ can reach
Taking into account adjacencies between members of Case 2.c: O

\[ u = u \cup v, N^+(u_4) = \{u_6, v_4\}, c) O = \{u_5, u_6, v_4\} \]

\[ N^+(u_4) = \{u, v_4\} \]

or d) O = \{u, v_4\}, N^+(u_4) = \{u, v_4\}.

Case 2.a: O = \{u, v_4\}, N^+(u_4) = \{u_5, u_6\}

As \( v_4 \in O(u) \cup O(u_1) \), \( u_2 \) can reach \( v_4 \) and without loss of generality \( v_4 \in N^+(u_2) \), \( N^+(u_2) = N^+(u_4) \), so by Lemma 1 \( u_2 \in O(u_4), u_4 \in O(u_2), u_5 \in O(u_6) \) and \( u_6 \in O(u_5) \). Hence \( u_4 \in O(v) \cup O(u_2) \), so \( u_1 \) can reach \( u_4 \), so \( u_4 \in N^+(u_4) \) and \( u_5 \in O(u_6) \). Neither \( u_5 \) nor \( u_6 \) lies in \( N^+(u_4) \), so \( O = \{u_5, u_6\} \) and \( N^+(u_4) = \{u, v_4\} \). Hence \( O(v_4) = \{u, u_1\} \). Observe that \( N^+(u_1) = N^+(v_4) \), so that \( v \in O(u_4) \). Therefore \( v \notin N^+(u_5) \cup N^+(u_6) \), yielding \( O = \{u_4, v\}, N^+(u_2) = \{v_4, u, v_1, u_1\} \). As \( v_1 \rightarrow v_4 \) and \( N^+(u_1) = N^+(u_4) \), we must have \( N^+(u_5) = \{v_4, u\}, N^+(u_6) = \{v_1, u_1\} \). This yields the digraph in Fig. 5. Unlike the digraph in Fig. 2, this digraph contains pairs of vertices with identical out-neighbourhoods, so the two are not isomorphic.

Case 2.b: O = \{u, v_4\}, N^+(u_4) = \{u_6, v_4\}

As \( u_4 \rightarrow v, u_4 \notin N^+(u_4) \), so \( u_4 \in O(v_1) \). Hence \( u_4 \in O(v) \cup O(v_1) \), so \( u_2 \) can reach \( u_4 \). As \( u_4 \rightarrow u_6, u_4 \) must have \( u_5 \rightarrow u_4 u_2 \) and \( u_4 \) have \( u_6 \) as a common out-neighbour, so \( v_4 \in O(u_2), u_5 \in O(u_4) \). Therefore \( v_4 \in O(u)^O(u_2) \), so that \( u_6 \) can reach \( u_4, v_4 \notin T(u_6) \), so \( N^+(u_6) \) contains an in-neighbour of \( v_4 \), \( u_4 \notin N^+(u_6) \), so \( u_6 \in O(u_4) \). We have \( u_5 \in O(u_4) \cap O(u_1) \), so \( v_4 \) can reach \( u_5 \), but \( v_4 \notin T(u_5) \), \( u_6 \in O(u_4) \), \( u_6 \notin O(u_4) \). Thus \( O = \{v_4, u_4\} \) and \( N^+(u_2) = \{u_4, v_4\} \). Taking into account adjacencies between members of \( N^+(u_2) \), it follows that \( N^+(u_5) = \{u_4, u\}, N^+(u_6) = \{u_1, v_4\} \). However, \( (u_2, u_4) \) now constitutes a bad pair, contradicting our assumption.

Case 2.c: O = \{u_5, u_6\}, N^+(u_4) = \{u_6, v_4\}

As \( u_4 \rightarrow u_4, u_4 \notin N^+(u_4) \), hence \( u_4 \in O(v) \cup O(v_1) \), implying that \( u_2 \) can reach \( u_4 \). Without loss of generality, \( u_5 \rightarrow u_4 \). There are three possibilities: (i) \( O(u_1) = \{u_4, u_5\}, N^+(u_4) = \{v, u_5\} \), (ii) \( O(v_1) = \{u, u_5\}, N^+(u_4) = \{v, u_5\} \) and (iii) \( O(u_1) = \{u_4, v, u_5\}, N^+(u_4) = \{u_5, u_6\} \).

(i) \( O(v_1) = \{u_4, u_5\}, N^+(u_4) = \{v, u_5\} \)

\( u_1 \) and \( v_4 \) have \( v_4 \) as a unique common out-neighbour, so \( v_4 \in O(v_1) \). However, this contradicts \( v_4 \rightarrow u_5 \rightarrow u_4 \).

(ii) \( O(v_1) = \{u_4, u_5\}, N^+(u_4) = \{v, u_6\} \)

Neither \( u_4 \) nor \( v_4 \) lie in \( T(u_2) \), so \( v_4 \notin O(u_2) \). Now observe that \( u_2 \) and \( v_4 \) have \( u_6 \) as unique common out-neighbour, so \( v \in O(u_2) \), yielding \( O(u_2) = \{v, u_4\} \) and \( N^+(u_2) = \{u_4, v, u_1, v_1\} \). As \( u_4 \rightarrow v, u \rightarrow u_1 \), we must have \( N^+(u_5) = \{u_4, u, v_4\} \), \( N^+(u_6) = \{u, v_1\} \), a contradiction, since \( u_1 \rightarrow u_4 \).

(iii) \( O(v_1) = \{u_4, v\}, N^+(u_4) = \{v, u_5\} \)

We now have \( N^+(u_2) = N^+(u_4) \), so \( u_2 \in O(v_4), v_4 \in O(u_5), u_5 \in O(u_6), u_6 \in O(u_5) \). Also \( N^+(u_4) = N^+(u_1) \), so \( u_4 \in O(u_1), v_4 \in O(u_2) \) and \( u_6 \in O(u_5) \). Hence \( u \notin N^+(u_5) \cup N^+(u_6) \), so we see that \( u \in O(u_2) \) and hence \( O(u_2) = \{u_2, v_4\} \) and \( N^+(u_2) = \{u_4, u_1, v_1\} \). As \( u_1 \rightarrow u_4 \) and \( u_4 \rightarrow u_1 \), we have \( N^+(u_5) = \{u_4, v\}, N^+(u_6) = \{v, u_1\} \). It is not difficult to show that this yields a \((2, 2, +2)\)-digraph isomorphic to that in Fig. 5.

Case 2.d: O = \{u_5, u_6\}, N^+(u_4) = \{u_6, v_4\}

In this case \( v_4 \in O(u) \cup O(u_1) \), so \( u_2 \) can reach \( v_4, u_4 \) and \( v_4 \) have unique common out-neighbour \( v_4 \), so \( v \in O(u_4), u_6 \in O(v_1) \). If \( u_6 \rightarrow v_4 \), then we would have \( u_4 \rightarrow u_6 \rightarrow u_4 \), contradicting \( v_4 \in O(u_4) \), so \( u_5 \rightarrow v_4 \). This also implies that \( u_5 \notin N^+(v_4) \), so \( u_5 \in O(v_1) \), yielding \( O(v_1) = \{u_5, u_6\} \) and \( N^+(u_4) = \{u, v_4\} = N^+(u_1) \). Now \( v_4, u \notin T(u_2) \), so \( O(u_2) = \{v, u_4\} \).

Fig. 5. A second \((2, 2, +2)\)-digraph.
and \( N^2(u_2) = \{v_1, v_4, u, u_1\} \). As \( v_1 \to v_4 \) and \( v_1 \to u \), it follows that \( N^+(u_5) = \{v_4, u\}, N^+(u_6) = \{u_1, v_1\} \). However, we now have paths \( u_4 \to u \to u_1 \) and \( u_4 \to u_6 \to u_1 \), which is impossible.

**Case 3:** \( N^+(u_1) = N^+(v_1) \)

It is easy to see by 2-geodecity that \( V(G) = \{u, u_1, u_2, u_3, u_4, u_5, v, v_1\} \), \( O(u) = \{v, v_1\} \) and \( O(v) = \{u, u_1\} \). As \( u_1, v_1 \not\in T(u_2) \), we have \( O(u_2) = \{u_3, u_4\} \) and \( N^+(u_2) = \{u, u_1, v, v_1\} \). Without loss of generality, \( N^+(u_5) = \{u_1, v_1\}, N^+(u_6) = \{u, u_1\} \). \( u \) and \( v \) have in-neighbours apart from \( u_5 \) and \( u_6 \) respectively, so without loss of generality \( u_3 \to u, u_4 \to v \). Likewise, \( u_5 \) and \( u_6 \) have in-neighbours other than \( u_2 \), so, as \( u_3 \to u \) and \( u_6 \to v \), we must have \( N^+(u_3) = \{u_6\}, N^+(u_4) = \{v, u_5\} \). But now we have paths \( u_3 \to u \to u_1 \) and \( u_3 \to u_6 \to u_1 \), violating 2-geodecity.

**Corollary 2.** There is a unique \((2, 2, +2)\)-digraph containing no bad pairs.

This completes our analysis of diregular \((2, 2, +2)\)-digraphs. As it was shown in [7] that there are no non-diregular \((2, 2, +2)\)-digraphs, \((2, 2, +2)\)-digraphs are now classified up to isomorphism. These conclusions have been verified computationally by Erskine [2]. It is interesting to note that neither of the \((2, 2, +2)\)-digraphs are vertex-transitive, for in each case there are exactly three vertices contained in two 3-cycles. However, there does exist a Cayley \((2, 2, +5)\)-digraph (on the alternating group \(A_4\)), so it would be interesting to determine the smallest vertex-transitive \((2, 2, +e)\)-digraphs.

### 4. Main result

We can now complete our analysis by showing that there are no diregular \((2, k, +2)\)-digraphs for \( k \geq 3 \). Let \( G \) be such a digraph. By **Lemma 2**, \( G \) contains vertices \( u \) and \( v \) with a unique common out-neighbour. In accordance with our vertex-labelling convention, we have the situation in **Fig. 6**. A triangle based at a vertex \( x \) represents the set \( T(x) \).

We now proceed to determine the possible outlier sets of \( u \) and \( v \).

**Lemma 5.** \( v \in N^{k-1}(u_1) \cup O(u) \) and \( u \in N^{k-1}(v_1) \cup O(v) \). If \( v \in O(u) \), then \( u_2 \in O(u_1) \) and if \( u \in O(v) \), then \( u_2 \in O(v_1) \).

**Proof.** \( v \) cannot lie in \( T(u) \), or the vertex \( u_2 \) would be repeated in \( T(v) \). Also, \( v \not\in T(u_2) \), or there would be a \( \leq k \)-cycle through \( v \). Therefore, if \( v \not\in O(u) \), then \( v \in N^{k-1}(u_1) \). Likewise for the other result. If \( v \in O(u) \), then neither in-neighbour of \( u_2 \) lies in \( T(u_1) \), so that \( u_2 \in O(u_1) \). \( \square \)

**Lemma 6.** Let \( w \in T(v_1) \), with \( d(v_1, w) = l \). Suppose that \( w \in T(u_1) \), with \( d(u_1, w) = m \). Then either \( m \leq l \) or \( w \in N^{k-1}(u_1) \). A similar result holds for \( w \in T(u_1) \).

**Proof.** Let \( w \) be as described and suppose that \( m > l \). Consider the set \( N^{k-1-m}(w) \). By construction, \( N^{k-1-m}(w) \subset N^k(u_1) \), so by \( k \)-geodecity \( N^{k-1-m}(w) \cap T(u_1) = \emptyset \). At the same time, we have \( l + k - m \leq k - 1 \), so \( N^{k-1-m}(w) \subset T(v_1) \). This implies that \( N^{k-1-m}(w) \cap T(v_1) = N^{k-1-m}(w) \cap T(u_2) = \emptyset \). As \( V(G) = \{u \} \cup T(u_1) \cup T(u_2) \cup O(u) \), it follows that \( N^{k-1-m}(w) \subset \{u \} \cup O(u) \). Therefore \( |N^{k-1-m}(w)| = 2^{k-m} \leq 3 \). By assumption \( 0 \leq m \leq k - 1 \), so it follows that \( m = k - 1 \). \( \square \)

**Corollary 3.** If \( w \in T(v_1) \), then either \( w \in \{u \} \cup O(u) \) or \( w \in T(u_1) \) with \( d(u_1, w) = k - 1 \) or \( d(u_1, w) \leq d(v_1, w) \).

**Proof.** By \( k \)-geodecity and **Lemma 6**. \( \square \)

**Corollary 4.** \( v_1 \in N^{k-1}(u_1) \cup O(u) \) and \( u_1 \in N^{k-1}(v_1) \cup O(v) \).
Proof. We prove the first inclusion. By Corollary 3, \( v_1 \in \{ u \} \cup O(u) \cup \{ u_1 \} \cup N^{k-1}(u_1) \). By \( k \)-geodecity, \( v_1 \neq u \) and by construction, \( v_1 \neq u_1 \). \( \square \)

We now have enough information to identify one member of \( O(u) \) and \( O(v) \).

Lemma 7. \( v_1 \in O(u) \) and \( u_1 \in O(v) \).

Proof. We prove that \( v_1 \in O(u) \). Suppose that neither \( v_1 \) nor \( v \) lies in \( O(u) \). Then by Lemma 5 and Corollary 4 we have \( v, v_1 \in N^{k-1}(u_1) \). As \( v_1 \) is an out-neighbour of \( v \), it follows that \( v_1 \) appears twice in \( T_k(u_1) \), violating \( k \)-geodecity. Therefore \( O(u) \cap \{ v, v_1 \} \neq \emptyset \).

Now assume that \( v_1, v_3 \in T_k(u_1) \). Again by Corollary 4, \( v_1 \in N^{k-1}(u_1) \). By \( k \)-geodecity we also have \( v_3 \in T(u_1) \). However, \( v_1 \in N^+(v_3) \), so \( v_1 \) appears twice in \( T_3(u_1) \), which is impossible. Hence \( O(u) \cap \{ v_1, v_3 \} \neq \emptyset \). Similarly, \( O(v) \cap \{ v_1, v_4 \} \neq \emptyset \). In the terminology of the previous section, \( G \) contains no bad pairs. Therefore, if \( v_1 \notin O(u) \), then \( \{ v_1, v_3, v_4 \} \subseteq O(u) \). Since these vertices are distinct, this is a contradiction and the result follows. \( \square \)

Lemma 7 allows us to conclude that for vertices sufficiently close to \( v_1 \) one of the potential situations mentioned in Corollary 3 cannot occur.

Lemma 8. \( T_{k-3}(v_1) \cap N^{k-1}(u_1) = T_{k-3}(u_1) \cap N^{k-1}(v_1) = \emptyset \).

Proof. Let \( w \in T_{k-3}(v_1) \cap N^{k-1}(u_1) \). Consider the position of the vertices of \( N^+(w) \) in \( T_k(u) \cup O(u) \). As \( w \notin N^+(w) \), it follows from Lemma 7 that at most one of the vertices of \( N^+(w) \) can be an outlier of \( u \), so let us write \( w_1 \in N^+(w) - O(u) \). By \( k \)-geodecity, \( w_1 \notin T(u_1) \cup \{ u \} \). Hence \( w_1 \in T(t_2) = T(t_2) \). However, \( u_1 \) also lies in \( T(u_1) \), so this violates \( k \)-geodecity. \( \square \)

Corollary 5. There is at most one vertex in \( T_{k-3}(v_1) - \{ v_1 \} \) that does not lie in \( T(u_1) \); for all other vertices \( w \in T_{k-3}(v_1) - \{ v_1 \} \), \( d(u_1, w) = d(v_1, w) \). A similar result for \( T_{k-3}(u_1) - \{ u_1 \} \) also holds.

Lemma 9. For \( k = 3, N^+(u_1) \cap N^2(v_1) = N^+(v_1) \cap N^2(u_1) = \emptyset \).

Proof. Suppose that \( v_3 = u_2 \). By the reasoning of Lemma 8 we can set \( u = v_2 \) and \( O(u) = \{ v_1, v_9 \} \). \( v \notin O(u) \) and by 3-geodecity \( v \notin N^+(u_3) \), so we can assume that \( v = u_0, u_3 \rightarrow v_3 \) implies that \( u_3 \notin T(v_1) \), so \( O(v) = \{ u_1, u_4 \} \). We must have \( \{ u_4, u_5, u_10 \} = \{ u_4, u_5, v_1 \} \). As \( u_4 \rightarrow v_3 \), it follows that \( v_4 = u_5 \) and hence \( \{ u_4, u_10 \} = \{ v_2, v_5 \} \), which is impossible. \( \square \)

As \( u_1 \) is an outlier of \( v_3 \), neither \( v_3 \) nor \( v_4 \) can be equal to \( u_1 \). It follows from Corollary 5 and Lemma 9 that either \( N^+(u_1) = N^+(v_1) \) or \( u_1 \) and \( v_1 \) have a single common out-neighbour, with one vertex of \( N^+(v_1) \) being an outlier of \( u \).

Lemma 10. \( N^2(u) \neq N^2(v) \).

Proof. Let \( N^2(u) = N^2(v) \), with \( N^+(u_1) = N^+(v_1) = \{ u_3, u_4 \} \). Suppose that \( v \notin O(u) \). By Lemma 5, \( v \in N^{k-2}(u_3) \cup N^{k-2}(u_4) \). But then there is a \( k \)-cycle through \( v \). It follows that \( O(u) = \{ v_1, v_4 \}, O(v) = \{ u_1, u_4 \} \). By Lemma 5, \( u_2 \in O(u_1) \cap O(v_1) \). Therefore by Lemma 1 \( O(u_1) = \{ u_2, v_1 \} \). \( O(v_1) = \{ u_2, u_1 \} \).

Consider the in-neighbour \( u' \) of \( u_1 \) that is distinct from \( u \). We have either \( N^+(u') \cap N^+(u_1) = 1 \) or \( N^+(u') \cap N^+(u_1) = 2 \). In the first case, it follows from Lemma 7 that \( u_2 \in O(u) \). Every vertex of \( G \) is an outlier of exactly two vertices, so \( u' = u_1 \) or \( v_1 \). In either case, we have a contradiction. Therefore \( N^+(u') = N^+(u_4) \). It now follows from Lemma 1 that \( u' \notin O(u_1) = \{ v_1, v_4 \} \), which is impossible. \( \square \)

Noticing that \( u_1 \) and \( v_1 \) also have a unique common out-neighbour, we have the following corollary.

Corollary 6. Without loss of generality, \( u_1 = v_3, u_9 = v_9, O(u) = \{ v_1, v_4 \}, O(v) = \{ u_1, u_4 \}, O(u_1) = \{ v_4, v_10 \} \) and \( O(v_1) = \{ u_4, u_10 \} \).

We are now in a position to complete the proof by deriving a contradiction.

Theorem 2. There are no diregular \((2, k, +2)\)-digraphs for \( k \geq 3 \).

Proof. \( u, v \notin \{ u_1, u_3, v_1, v_4 \} \), so by Lemma 5 \( d(u, v) = d(v, u) = k \). In fact, \( u_3 = v_3 \) implies that \( v \in N^{k-2}(u_3) \) and \( u \in N^{k-2}(v_3) \). Let \( k \geq 4 \). Then \( u, v \notin \{ u_10, v_10 \} \), so \( u, v \in T(u_1) \cap T(v_1) \). If \( u \in T(u_3) = T(v_3) \), then \( u \) would appear twice in \( T_k(u_1) \), so \( u \in N^{k-1}(u_1) \). However, as \( u, v \) and \( v \) have a common out-neighbour, this violates \( k \)-geodecity.

Finally, suppose that \( k = 3 \). The above analysis will hold unless \( u = v_10 \) and \( v = u_10 \). Let \( N^+(u_1) = \{ u_1', v_1' \} \). It is evident that \( v' \notin \{ u_1, v_4 \} \), so \( v' \in T(v_3) \). As \( v \in N(u_4) \), we must have \( v' \notin N^2(u_2) \). Similarly \( u' \notin N^2(u_2) \). Since \( u_1 \) and \( v_1 \) have a common out-neighbour, we can assume that \( u' \notin N^+(u_3) \) and \( v' \notin N^+(v_6) \). \( v_4 \) can be the outlier of only two vertices, namely \( u \) and \( u_1 \), so \( v_4 \notin N^2(u_2) \) and likewise \( u_4 \notin N^2(u_2) \). By \( 3 \)-geodecity \( v_4 \notin N^2(u_2) \) and \( u_4 \notin N^2(u_6) \). It follows that \( u, v \notin N^2(u_2) \), so \( u \notin T(u_1) \cap T(v_2) \). Hence \( O(u) = N^+(u) = \{ v_1, v_4 \} \), which again is impossible. \( \square \)

It is interesting to note that a similar argument can be used to provide an alternative proof of the result of [6].
References