On diregular digraphs with degree two and excess two

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Abstract

An important topic in the design of efficient networks is the construction of \((d, k, +\epsilon)\)-digraphs, i.e. \(k\)-geodetic digraphs with minimum out-degree \(\geq d\) and order \(M(d, k) + \epsilon\), where \(M(d, k)\) represents the Moore bound for degree \(d\) and diameter \(k\) and \(\epsilon > 0\) is the (small) excess of the digraph. Previous work has shown that there are no \((2, k, +1)\)-digraphs for \(k \geq 2\). In a separate paper, the present author has shown that any \((2, k, +2)\)-digraph must be diregular for \(k \geq 2\). In the present work, this analysis is completed by proving the nonexistence of diregular \((2, k, +2)\)-digraphs for \(k \geq 3\) and classifying diregular \((2, 2, +2)\)-digraphs up to isomorphism.

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1. Introduction

An important topic in the design of interconnection networks is the directed degree/diameter problem: what is the largest possible order \(N(d, k)\) of a digraph \(G\) with maximum out-degree \(d\) and diameter \(\leq k\)? A simple inductive argument shows that for \(0 \leq l \leq k\) the number of vertices at distance \(l\) from a fixed vertex \(v\) is bounded above by \(d^l\). Therefore, a natural upper bound for the order of such a digraph is the so-called Moore bound \(M(d, k) = 1 + d + d^2 + \cdots + d^k\). A digraph that attains this upper bound is called a Moore digraph. It is easily seen that a digraph \(G\) is Moore if and only if it is out-regular with degree \(d\), has diameter \(k\) and is \(k\)-geodetic, i.e. for any two vertices \(u, v\) there is at most one \(k\)-path from \(u\) to \(v\).

As it was shown by Bridges and Toueg in [1] that Moore digraphs exist only in the trivial cases \(d = 1\) or \(k = 1\) (the Moore digraphs are directed \((k+1)\)-cycles and complete digraphs \(K_{k+1}\) respectively), much research has been devoted to the study of digraphs that in some sense approximate Moore digraphs. For example, there is an extensive literature on digraphs with maximum out-degree \(d\), diameter \(\leq k\) and order \(M(d, k) - \delta\) for small \(\delta > 0\); this is equivalent to relaxing the \(k\)-geodeticity requirement in the conditions for a digraph to be Moore. \(\delta\) is known as the defect of the digraph. The reader is referred to the survey [5] for more information.

In this paper, however, we will consider the following related problem, which is obtained by retaining the \(k\)-geodeticity requirement in the above characterisation of Moore digraphs, but allowing the diameter to exceed \(k\): what is the smallest possible order of a \(k\)-geodetic digraph \(G\) with minimum out-degree \(\geq d\) and order \(M(d, k) + \epsilon\) is said to be a \((d, k, +\epsilon)\)-digraph or to have excess \(\epsilon\). It was shown in [6] that there are no diregular \((2, k, +1)\)-digraphs for \(k \geq 2\). In 2016 it was shown in [3] that digraphs with excess one must be diregular and that there are no \((d, k, +1)\)-digraphs for \(k = 2, 3, 4\) and sufficiently large \(d\). In a separate paper [7], the present author has shown that \((2, k, +2)\)-digraphs must be diregular with degree \(d = 2\) for \(k \geq 2\). In the present paper, we classify the \((2, 2, +2)\)-digraphs up to isomorphism and show that there are no diregular \((2, k, +2)\)-digraphs for \(k \geq 3\), thereby completing the proof of the nonexistence of digraphs with degree \(d = 2\) and excess \(\epsilon = 2\) for \(k \geq 3\). Our reasoning and notation will follow closely that employed in [4] for the corresponding result for defect \(\delta = 2\).
2. Preliminary results

We will let $G$ stand for a $(2, k, +2)$-digraph for arbitrary $k \geq 2$, i.e. $G$ has minimum out-degree $d = 2$, is $k$-geodetic and has order $M(2, k) + 2$. We will denote the vertex set of $G$ by $V(G)$. By the result of [7], $G$ must be diregular with degree $d = 2$ for $k \geq 2$. The distance $d(u, v)$ between vertices $u$ and $v$ is the length of the shortest path from $u$ to $v$. Notice that $d(u, v)$ is not necessarily equal to $d(v, u)$. $u \rightarrow v$ will indicate that there is an arc from $u$ to $v$. We define the in- and out-neighbourhoods of a vertex $u$ by $N^-(u) = \{ v \in V(G) : v \rightarrow u \}$ and $N^+(u) = \{ v \in V(G) : u \rightarrow v \}$ respectively; more generally, for $0 \leq l \leq k$, the set $\{ v \in V(G) : d(u, v) = l \}$ of vertices at distance exactly $l$ from $u$ will be denoted by $N^l(u)$. For $0 \leq l \leq k$ we will also write $T_l(u) = \bigcup_{0 \leq m \leq l} N^m(u)$ for the set of vertices at distance $\leq l$ from $u$. The notation $T_{k-1}(u)$ will be abbreviated by $T(u)$.

It is easily seen that for any vertex $u$ of $G$, there are exactly two distinct vertices that are at distance $\geq k + 1$ from $u$. For any $u \in V(G)$, we will write $O(u)$ for the set of these vertices and call such a set an outlier set and its elements outliers of $u$. Notice that $O(u) = V(G) - T_k(u)$. An elementary counting argument shows that in a diregular $(2, k, +2)$-digraph every vertex is also an outlier of exactly two vertices. We will say that a vertex $u$ can reach a vertex $v$ if $v \not\in O(u)$.

Our proof will proceed by an analysis of a pair of vertices with exactly one common out-neighbour. First, we must show that such a pair exists and deduce some elementary properties of pairs of vertices with identical out-neighbours.

**Lemma 1.** For $k \geq 2$, let $u$ and $v$ be distinct vertices such that $N^+(u) = N^+(v) = \{ u_1, u_2 \}$. Then $u_1 \in O(u_2)$, $u_2 \in O(u_1)$ and there exists a vertex $x$ such that $O(u) = \{ v, x \}$, $O(v) = \{ u, x \}$.

**Proof.** Suppose that $u$ can reach $v$ by a $\leq k$-path. Then $v \in T(u_1) \cup T(u_2)$. As $N^+(v) = N^+(u)$, it follows that there would be a $\leq k$-cycle through $v$, contradicting $k$-geodeticity. If $O(u) = \{ v, x \}$, then $x \neq v$ and $x \not\in T(u_1) \cup T(u_2)$, so that $v$ cannot reach $x$ by a $\leq k$-path. Similarly, if $u_1$ can reach $u_2$ by a $\leq k$-path, then we must have $\{ u, v \} \cap T(u_1) \neq \emptyset$, which is impossible. □

**Lemma 2.** For $k \geq 2$, there exists a pair of vertices $u$, $v$ with $|N^+(u) \cap N^+(v)| = 1$.

**Proof.** Suppose for a contradiction that there is no such pair of vertices. Define a map $\phi : V(G) \rightarrow V(G)$ as follows. Let $u^+ \in O(u)$ be an out-neighbour of a vertex $u$ and let $\phi(u)$ be the in-neighbour of $u^+$ distinct from $u$. By our assumption, it is easily verified that $\phi$ is a well-defined bijection with no fixed points and with square equal to the identity. It follows that $G$ must have even order, whereas $|V(G)| = M(2, k) + 2$ is odd. □

$u, v$ will now stand for a pair of vertices with a single common out-neighbour. We will label the vertices of $T_k(u)$ according to the scheme $N^+(u) = \{ u_1, u_2 \}$, $N^+(u_1) = \{ u_3, u_4 \}$, $N^+(u_2) = \{ u_5, u_6 \}$, $N^+(u_3) = \{ u_7, u_8 \}$, $N^+(u_4) = \{ u_9, u_{10} \}$ and so on, with the same convention for the vertices of $T_k(v)$, where we will assume that $u_2 = v_2$.

3. Classification of $(2, 2, +2)$-digraphs

We begin by classifying the $(2, 2, +2)$-digraphs up to isomorphism. We will prove the following theorem.

**Theorem 1.** There are exactly two diregular $(2, 2, +2)$-digraphs, which are displayed in Figs. 2 and 5.

Let $G$ be an arbitrary diregular $(2, 2, +2)$-digraph. $G$ has order $M(2, 2) + 2 = 9$. By Lemma 2, $G$ contains a pair of vertices $(u, v)$ such that $|N^+(u) \cap N^+(v)| = 1$; we will assume that $u_2 = v_2$, so that we have the situation shown in Fig. 1.

We can immediately deduce some information on the possible positions of $v$ and $v_1$ in $T_2(u)$.

**Lemma 3.** If $v \not\in O(u)$, then $v \in N^+(u_1)$. If $v_1 \not\in O(u)$, then $v_1 \in N^+(u_1)$.

**Proof.** $v \not\in T(u_2)$ by $2$-geodectity. $v \neq u$ by construction. If we had $v = u_1$, then there would be two distinct $\leq 2$-paths from $u$ to $u_2$. Also $v_1 \not\in \{ u \} \cup T(u_2)$ by $2$-geodectity and by assumption $u_1 \neq v_1$. □
Since \( v \) and \( v_1 \) cannot both lie in \( N^+(u_1) \) by 2-geodecity, we have the following corollary.

**Corollary 1.** \( O(u) \cap \{v, v_1\} \neq \emptyset \).

We will call a pair of vertices \( (u, v) \) with a single common out-neighbour **bad** if at least one of

\[
O(u) \cap \{v_1, v_3\} = \emptyset, O(u) \cap \{v_1, v_4\} = \emptyset, O(v) \cap \{u_1, u_3\} = \emptyset, O(v) \cap \{u_1, u_4\} = \emptyset.
\]

holds. Otherwise such a pair will be called good.

**Lemma 4.** There is a unique \((2, 2, +2)\)-digraph containing a bad pair.

**Proof.** Assume that there exists a bad pair \((u, v)\). Without loss of generality, \( O(u) \cap \{v_1, v_3\} = \emptyset \). By Lemma 3 we can set \( v_1 = u_3 \). By 2-geodecity \( v_3 = u \). We cannot have \( v_4 = v_3 = u \), so \( u_4 \) must be an outlier of \( u \). By Corollary 1 it follows that \( O(u) = \{v, v_4\} \).

Consider the vertex \( u_1 \). By Lemma 3, if \( u_1 \notin O(v) \), then \( u_1 \in N^+(v_1) \). However, \( v_1 = u_3 \), there would be a 2-cycle through \( u_1 \). Hence \( u_1 \in O(v) \). As \( O(u) = \{v, u_4\} \), we have \( V(G) = \{u, u_1, u_2, u_3 = v_1, u_4, u_5, u_6, v, v_4\} \) and \( O(v) = \{u_1, u_4\} \). As neither \( u \) nor \( v \) lies in \( T(u_1) \), we must also have \( v_2 \in O(u_1) \). As \( u_1 \) can reach \( u_1, v_1, u_4, u \) and \( v_4 \), it follows that without loss of generality we either have \( O(u_1) = \{u_2, v\} \) and \( N^+(u_4) = \{u_5, u_6\} = N^+(u_2) \) or \( O(u_1) = \{u_2, u_6\} \) and \( N^+(u_4) = \{v, u_5\} \). In either case, \((v, u_1)\) is a good pair.

Suppose firstly that \( N^+(u_2) = N^+(u_4) \). Then \( v \) is an outlier of \( u \) and \( u_1 \). As each vertex is the outlier of exactly two vertices, \( v_1 \) must be able to reach \( v \) by a \( \leq 2 \)-path. Hence \( v_4 \rightarrow v \). Likewise \( u_2 \) can reach \( v \), so without loss of generality \( u_5 \rightarrow v \). Suppose that \( O(u_2) \cap \{u, u_1\} = \emptyset \). As \( u \) and \( v \) have a common out-neighbour, we must have \( u_6 \rightarrow u \). Since \( u \rightarrow u_1 \), by 2-geodecity we must have \( u_5 \rightarrow u_1 \). However, this is a contradiction, as \( u \) and \( u_1 \) also have a common out-neighbour. Therefore, at least one of \( u, u_1 \) is an outlier of \( u_2 \). By Lemma 1 \( u_4 \) is an outlier of \( u_2 \). Therefore either \( O(u_2) = \{u, u_4\} \) or \( O(u_2) = \{u_1, u_4\} \). If \( O(u_2) = \{u, u_4\} \), then \( u_2 \) must be able to reach \( u_1, v_1 \) and \( u_4, u_5 \rightarrow v \) and \( v \rightarrow v_1 \), so \( v_1 \in N^+(u_6) \). As \( u_1 \rightarrow v_1 \), we must have \( N^+(u_5) = \{v, u_1\} \). As \( v \) and \( u_1 \) have a common out-neighbour, this violates 2-geodecity. Hence \( O(u_2) = \{u_1, u_4\} \) and \( u_2 \) can reach \( u_1, v_1 \) and \( v_4, u_4 \). As \( v \rightarrow v_1, v_1 \in N^+(u_6) \). As \( v_1 \rightarrow v_4 \), it follows that \( N^+(u_5) = \{v, u_4\} \). However, \( v_4 \rightarrow v \), so this again violates 2-geodecity.

We are left with the case \( O(u_1) = \{u_2, u_6\} \) and \( N^+(u_4) = \{v, u_5\} \). Then \( v_1 \in O(u_2) \), as neither \( v \) nor \( u_1 \) lies in \( T(u_2) \). Observe that \( u_2 \) and \( u_4 \) have a single common out-neighbour, so by Corollary 1 \( O(u_2) \cap \{u_4, v\} \neq \emptyset \). Therefore either \( O(u_2) = \{v_1, u_4\} \) or \( O(u_2) = \{v_1, v\} \). Suppose firstly that \( O(u_2) = \{v_1, u_4\} \). Then \( N^+(u_2) = \{u, v, u_1, v_4\} \). As \( N^+(u_4) = \{v, u_5\}, u_5 \not\rightarrow v \), so \( u_6 \rightarrow v \). As \( N^+(u) \cap N^+(v) \neq \emptyset \), \( u_5 \rightarrow u, u \rightarrow u_1 \), so necessarily \( u_6 \in N^+(u_5) = \{v, u_1\} \). However, \( v_1 \in N^+(u_1) \cap N^+(v) \), contradicting 2-geodecity.

Hence \( O(u_2) = \{v_1, v\} \) and \( N^+(u_2) = \{u, u_1, u_4, v_4\} \). As \( u_4 \rightarrow u_5, u_5 \not\rightarrow u_4 \). Thus \( u_6 \rightarrow u_4 \). Now \( u_1 \rightarrow u_4 \) and \( u \rightarrow u_1 \) implies that \( N^+(u_5) = \{u, v_4\} \) and \( N^+(u_6) = \{u, u_4\} \). Finally we must have \( N^+(u_4) = \{v, u_6\} \). This gives us the \((2, 2, +2)\)-digraph shown in Fig. 2. \( \square \)

We can now assume that all pairs given by Lemma 2 are good. Let us fix a pair \((u, v)\) with a single common out-neighbour. It follows from Corollary 1 and the definition of a good pair that \( v_1 \in O(u) \); otherwise \( O(u) \) would contain \( v, v_3 \) and \( v_4 \), which is impossible. Likewise \( u_1 \in O(v) \).
Considering the positions of \(v_3\) and \(v_4\), we see that there are without loss of generality four possibilities: (1) \(u = v_3\), \(u_4 = v_4\). (2) \(u = v_3\), \(O(u) = \{v_1, v_4\}\), (3) \(N^+(u_4) = N^+(v_1)\) and (4) \(u_3 = v_3\), \(O(u) = \{v_1, u_4\}\). A suitable relabelling of vertices shows that case 4 is equivalent to case 1.a below, so we will examine cases 1 to 3 in turn.

**Case 1: \(u = v_3\), \(u_4 = v_4\)**

Depending upon the position of \(v\), we must either have \(O(u) = \{v_1, v\}\) and \(O(v) = \{u_1, u_3\}\) or \(v = u_3\) (see Fig. 3).

**Case 1.a: \(O(u) = \{v_1, v\}\), \(O(v) = \{u_1, u_3\}\)**

In this case \(V(G) = \{u, u_1, u_2, u_3, u_4, u_5, u_6, v, v_1\}\). \(u_1\) and \(v_1\) have a single common out-neighbour, namely \(u_4\), so, as we are assuming all such pairs to be good, we have \(u_3 \in O(v_1)\), \(u \in O(u_1)\). By 2-geodecity, \(N^+(u_4) \subseteq \{u_5, u_6, v\}\), so without loss of generality either \(N^+(u_4) = \{u_5, u_6\}\) or \(N^+(u_4) = \{u_5, v\}\).

Suppose that \(N^+(u_4) = \{u_5, u_6\}\). By elimination, \(O(u_1) = \{v, u_3\}\). As \(G\) is diregular, every vertex is an outlier of exactly two vertices; \(v\) is an outlier of \(u\) and \(v_1\), so both \(u_1\) and \(u_2\) can reach \(v\) by a \(\leq 2\)-path. Hence \(v \in N^+(u_3)\). As \(v \to v_1\), we see that \(v_1\) is an outlier of \(u_1\); as \(u\) is also an outlier of \(u_1\), we have \(O(u_1) = \{u, v_1\}\) and \(N^+(u_3) = \{v, u_2\}\). As \(v \to u_2\), this is impossible.

Now consider \(N^+(u_4) = \{u_5, v\}\). We now have \(O(v_1) = \{u_1, u_6\}\). Thus \(u_3 \in O(v) \cap O(v_1)\), so \(u_3 \in T_2(u_4)\). \(v\) is not adjacent to \(u_3\), so \(u_3 \in N^+(u_5)\). \(u_2\) and \(u_4\) have \(u_5\) as a unique common out-neighbour, so \(u_6 \in O(u_4)\). \(v \in O(u_2)\). As \(u_6 \in O(v_1) \cap O(u_4)\), \(u_1\) can reach \(u_6\). Hence \(u_5 \in N^+(u_3)\). Neither \(u\) nor \(v\) lie in \(T(u_4)\), so \(u_2 \in O(u_1)\). Therefore either \(O(u_1) = \{u, u_2\}\) or \(O(u_1) = \{u_2, v_1\}\). If \(O(u_1) = \{u, u_2\}\), then \(N^+(u_2) = \{u_6, v_1\}\). \(u_2\) cannot reach \(v_1\), since \(u, u_3 \notin T(u_2)\), so \(O(u_2) = \{v, v_1\}\) and \(N^2(u_2) = \{u_1, u_2, u_4\}\). As \(u_4 \to u_5, u_4 \in N^+(u_6)\), \(u_1 \to u_4\), so \(N^+(u) = \{u_1, u_3\}\). As \(u_1 \to u_3\), this is a contradiction.

Thus \(O(u_1) = \{u_2, v_1\}\), so that \(N^+(u_3) = \{u, u_6\}\). \(u_1\) must have an in-neighbour apart from \(u\), which must be either \(u_5\) or \(u_6\). As \(u_1 \to u_3\), we have \(u_1 \in N^+(u_6)\). By elimination, \(v\) and \(v_1\) must also have in-neighbours in \(\{u_5, u_6\}\). As \(u_1\) and \(v_1\) have a common out-neighbour, we have \(N^+(u_5) = \{u_3, v_1\}\), \(N^+(u_6) = \{u_1, v\}\). However, both \(u_3\) and \(v_1\) are adjacent to \(u\), violating 2-geodecity.

**Case 1.b: \(v = u_3\)**

There exists a vertex \(x\) such that \(V(G) = \{u, u_1, u_2, v, u_4, u_5, u_6, v_1, x\}\), \(O(u) = \{v_1, x\}\) and \(O(v) = \{u_1, x\}\). As \(x \in O(u) \cap O(v)\), \(u_1\) and \(u_2\) can reach \(x\), so without loss of generality \(x \in N^+(u_4) \cap N^+(u_5)\). As \(u_5\) and \(u_4\) have a common out-neighbour, \(u_6 \in O(u_4)\). Also, \(u_1\) and \(v_1\) have \(u_4\) as a unique common out-neighbour, so \(u \in O(u_1)\) and \(O(u_1) = \{u, u_6\}\). Thus \(N^+(u_4) = \{x, u_6\}\). Observe that \(u_2\) and \(u_4\) have the out-neighbour \(u_6\) in common. Thus \(x \in O(u_2)\), whereas we already have \(x \in O(u) \cap O(v)\), a contradiction.

**Case 2: \(u = v_3\), \(O(u) = \{v_1, v_4\}\)**

As \(v\) is not equal to \(v_1\) or \(v_4\), \(v\) must lie in \(T_2(u)\). Without loss of generality, \(v = u_3\). Hence \(V(G) = \{u, u_1, u_2, v, u_4, u_5, u_6, v_1, v_4\}\) and \(O(v) = \{u_1, u_4\}\). We have the configuration shown in Fig. 4. Hence \(u_1\) can reach
Taking into account adjacencies between members of difficult to show that this yields a \((2, 2, +2)\)-digraph.

\[ u_1, v, u_4, u_2 \text{ and } v_1, \text{ so we have without loss of generality one of the following: a) } O(u_1) = \{u, v_4\}, N^+(u_4) = \{u_5, u_6\}, \text{ b) } O(u_1) = \{u, u_1\}, N^+(u_4) = \{u_6, v_4\}, \text{ c) } O(u_1) = \{u_5, u_6\}, N^+(u_4) = \{u, v_4\} \text{ or d) } O(u_1) = \{u_5, v_4\}, N^+(u_4) = \{u, u_6\}. \]

**Case 2.a:** \( O(u_4) = \{u, v_4\}, N^+(u_4) = \{u_5, u_6\} \)

As \( v_4 \in O(u) \cap O(u_1), u_2 \) can reach \( v_4 \) and without loss of generality \( v_4 \in N^+(u_5), N^+(u_5) = N^+(u_4), \) so by Lemma 1 \( u_2 \in O(u_4), u_4 \in O(u_2), u_5 \in O(u_6) \) and \( u_6 \in O(u_5). \) Hence \( u_4 \in O(v) \cap O(u_2), \) so \( v_1 \) can reach \( u_4, \) so \( u_4 \in N^+(v_4). \) Neither \( u_5 \) nor \( u_6 \) lies in \( N^+(v_4), \) so \( O(v_1) = \{u_5, u_6\} \) and \( N^+(v_4) = \{u_4, v_4\}. \) Hence \( O(v_4) = \{u, u_1\}. \) Observe that \( N^+(u_1) = N^+(v_4), \) so that \( v \in O(u_4). \) Therefore \( v \notin N^+(u_5) \cup N^+(u_6), \) yielding \( O(u_2) = \{u_4, v\}, N^+(u_2) = \{v_4, v_1, u, u_1\}. \) As \( v_1 \rightarrow v_4 \) and \( N^+(u_1) = N^+(v_4), \) we must have \( N^+(u_5) = \{v_4, u\}, N^+(u_6) = \{v_1, u_1\}. \) This yields the digraph in Fig. 5. Unlike the digraph in Fig. 2, this digraph contains pairs of vertices with identical out-neighbourhoods, so the two are not isomorphic.

**Case 2.b:** \( O(u_1) = \{u, u_5\}, N^+(u_4) = \{u_6, v_4\} \)

As \( u_4 \rightarrow v_4, u_4 \notin N^+(v_4), \) so \( u_4 \in O(v_1). \) Hence \( u_4 \in O(v) \cap O(v_1), \) so \( u_4 \) can reach \( u_4. \) As \( u_4 \rightarrow u_6, \) we must have \( u_5 \rightarrow u_4, u_2 \) and \( u_4 \) have \( u_6 \) as a common out-neighbour, so \( v_4 \in O(u_2), u_5 \in O(u_4). \) Therefore \( u_4 \in O(u) \cap O(u_2), \) so \( u_6 \) can reach \( u_4, \) but \( v_4 \notin T(u_6), \) so \( N^+(u_6) \) contains an in-neighbour of \( v_4, u_4 \notin N^+(u_6), \) so we must have \( u_5 \rightarrow v_4. \) We have \( u_5 \in O(u_4) \cap O(u_1), \) so \( v_1 \) can reach \( u_5 \) and hence \( v_4 \rightarrow u_5, v_1 \) cannot reach \( u_6, \) as \( u_2, u_4 \notin T(v_1), \) so \( O(v_1) = \{u_4, u_6\}, N^+(v_1) = \{u_5, v_4\}. \) Now \( u_2 \) and \( v_4 \) have \( u_5 \) as a unique common out-neighbour, so \( u_6 \in O(v_4), v \in O(u_2). \) Thus \( O(u_2) = \{v, v_4\} \) and \( N^2(u_2) = \{u_4, v_1, u, u_1\}. \) Taking into account adjacencies between members of \( N^2(u_2), \) it follows that \( N^+(u_5) = \{u_4, u\}, N^+(u_6) = \{u_1, v_1\}. \) However, \( (u_2, u_4) \) now constitutes a bad pair, contradicting our assumption.

**Case 2.c:** \( O(u_1) = \{u, u_5, u_6\}, N^+(u_4) = \{u_6, v_4\} \)

As \( u_4 \rightarrow v_4, u_4 \notin N^+(v_4), \) implying that \( u_2 \) can reach \( u_4. \) Without loss of generality, \( u_5 \rightarrow u_4. \) There are three possibilities: (i) \( O(v_1) = \{u_4, u_6\}, N^+(v_4) = \{v, u_5\}, \) (ii) \( O(v_1) = \{u_4, u_5\}, N^+(v_4) = \{v, u_6\} \) and (iii) \( O(v_1) = \{u_4, v\}, N^+(v_4) = \{u_5, u_6\}. \)

(i) \( O(v_1) = \{u_4, u_6\}, N^+(v_4) = \{v, u_5\} \)

\( u_1 \) and \( v_4 \) have \( v \) as a unique common out-neighbour, so \( u_4 \in O(v_4). \) However, this contradicts \( v_4 \rightarrow u_5 \rightarrow u_4. \)

(ii) \( O(v_1) = \{u_4, u_5\}, N^+(v_4) = \{v, u_6\} \)

Neither \( u_4 \) nor \( v_1 \) lie in \( T(u_2), \) so \( v_4 \in O(u_2). \) Now observe that \( u_2 \) and \( v_4 \) have \( u_6 \) as unique common out-neighbour, so \( v \in O(u_2), \) yielding \( O(u_2) = \{v, v_4\} \) and \( N^2(u_2) = \{u_4, u_1, u, v_1\}. \) As \( u_4 \rightarrow u \) and \( u \rightarrow u_1, \) we must have \( N^+(u_5) = \{u_4, u_1\}, N^+(u_6) = \{u, v_1\}, a \) contradiction, since \( u_1 \rightarrow u_4. \)

(iii) \( O(v_1) = \{u_4, v\}, N^+(v_4) = \{u_5, u_6\} \)

We now have \( N^+(u_2) = N^+(v_4), \) so \( u_2 \in O(v_4), v_4 \in O(u_2), u_5 \in O(u_6), u_6 \in O(u_5). \) Also \( N^+(u_4) = N^+(v_1), \) so \( u_4 \in O(v_1), \) \( v_1 \in O(u_4), \) \( u \in O(v_4) \) implies that \( u \notin N^+(u_5) \cup N^+(u_6), \) so we see that \( u \in O(u_2) \) and hence \( O(u_2) = \{u, v_4\} \) and \( N^2(u_2) = \{u_4, u_1, u, v_1\}. \) As \( u_1 \rightarrow u_4 \) and \( u_1 \rightarrow v, \) we have \( N^+(u_5) = \{u_4, v\}, N^+(u_6) = \{u_1, v_1\}. \) It is not difficult to show that this yields a \((2, 2, +2)\)-digraph isomorphic to that in Fig. 5.

**Case 2.d:** \( O(u_1) = \{v_5, v_4\}, N^+(u_4) = \{u, u_6\} \)

In this case \( v_4 \in O(u) \cap O(u_1), u_2 \) can reach \( v_4. \) \( u_4 \) and \( v_1 \) have unique common out-neighbour \( u, \) so \( v_4 \in O(u_4), u_6 \in O(v_1). \) If \( u_6 \rightarrow v_4, \) then we would have \( u_4 \rightarrow u_6 \rightarrow u_4, \) contradicting \( v_4 \in O(u_4), \) so \( u_5 \rightarrow v_4. \) This also implies that \( u_5 \notin N^+(v_4), \) so \( u_5 \in O(v_1), \) yielding \( O(v_1) = \{u_5, v_4\} \) and \( N^+(v_4) = \{v, u_4\} = N^+(u_1). \) Now \( v_4, u_1 \notin T(u_2), \) so \( O(u_2) = \{v, u_4\}. \)
and $N^2(u_2) = \{v_1, v_4, u, u_1\}$. As $v_1 \rightarrow v_4$ and $v_1 \rightarrow u$, it follows that $N^+(u_5) = \{v_4, u\}$, $N^+(u_6) = \{u_1, v_1\}$. However, we now have paths $u_4 \rightarrow u \rightarrow u_1$ and $u_4 \rightarrow u_6 \rightarrow u_1$, which is impossible.

**Case 3:** $N^+(u_1) = N^+(v_1)$

It is easy to see by 2-geodecity that $V(G) = \{u, u_1, u_2, u_3, u_4, u_5, v, v_1\}$, $O(u) = \{v, v_1\}$ and $O(v) = \{u, u_1\}$. As $u_1, v_1 \not\in T(u_2)$, we have $O(u_2) = \{u_3, u_4\}$ and $N^+(u_2) = \{u, u_1, v, v_1\}$. Without loss of generality, $N^+(u_5) = \{u, v_1\}$, $N^+(u_6) = \{v, u_1\}$. $u$ and $v$ have in-neighbours apart from $u_2$ and $u_6$ respectively, so without loss of generality $u_3 \not\in T(u_2)$ and $v_6 \not\in T(v_1)$. Likewise, $u_5$ and $u_6$ have in-neighbours other than $u_2$, so, as $u_3 \rightarrow u$ and $u_6 \rightarrow v$, we must have $N^+(u_3) = \{u, u_6\}$, $N^+(u_4) = \{v, u_5\}$. But now we have paths $u_3 \rightarrow u \rightarrow u_1$ and $u_3 \rightarrow u_6 \rightarrow u_1$, violating 2-geodecity.

**Corollary 2.** There is a unique $(2, 2, +2)$-digraph containing no bad pairs.

This completes our analysis of diregular $(2, 2, +2)$-digraphs. As it was shown in [7] that there are no non-diregular $(2, 2, +2)$-digraphs, $(2, 2, +2)$-digraphs are now classified up to isomorphism. These conclusions have been verified computationally by Erskine [2]. It is interesting to note that neither of the $(2, 2, +2)$-digraphs are vertex-transitive, for in each case there are exactly three vertices contained in two 3-cycles. However, there does exist a Cayley $(2, 2, +5)$-digraph (on the alternating group $A_4$), so it would be interesting to determine the smallest vertex-transitive $(2, 2, +e)$-digraphs.

**4. Main result**

We can now complete our analysis by showing that there are no diregular $(2, k, +e)$-digraphs for $k \geq 3$. Let $G$ be such a digraph. By Lemma 2, $G$ contains vertices $u$ and $v$ with a unique common out-neighbour. In accordance with our vertex-labelling convention, we have the situation in Fig. 6. A triangle based at a vertex $x$ represents the set $T(x)$.

We now proceed to determine the possible outlier sets of $u$ and $v$.

**Lemma 5.** $v \in N^{k-1}(u_1) \cup O(u)$ and $u \in N^{k-1}(v_1) \cup O(v)$. If $v \not\in O(u)$, then $u_2 \in O(u_1)$ and if $u \not\in O(v)$, then $u_2 \in O(v_1)$.

**Proof.** $v$ cannot lie in $T(u)$, or the vertex $u_2$ would be repeated in $T(u)$. Also, $v \not\in T(u_2)$, or there would be a $k$-cycle through $v$. Therefore, if $v \not\in O(u)$, then $v \in N^{k-1}(u_1)$. Likewise for the other result. If $v \in O(u)$, then neither in-neighbour of $u_2$ lies in $T(u_1)$, so that $u_2 \in O(u_1)$. □

**Lemma 6.** Let $w \in T(v_1)$, with $d(v_1, w) = l$. Suppose that $w \in T(u_1)$, with $d(u_1, w) = m$. Then either $m \leq l$ or $w \in N^{k-1}(u_1)$. A similar result holds for $w \in T(u_1)$.

**Proof.** Let $w$ be as described and suppose that $m > l$. Consider the set $N^{k-1}(w)$. By construction, $N^{k-1}(w) \subseteq N^k(u_1)$, so by $k$-geodecity $N^{k-m}(w) \cap T(u_1) = \emptyset$. At the same time, we have $l + k - m \leq k - 1$, so $N^{k-m}(w) \subseteq T(v_1)$. This implies that $N^{k-m}(w) \cap T(v_2) = N^{k-m}(w) \cap T(u_2) = \emptyset$. As $V(G) = \{u\} \cup T(u_1) \cup T(u_2) \cup O(u)$, it follows that $N^{k-m}(w) \subseteq \{u\} \cup O(u)$. Therefore $|N^{k-m}(w)| \leq 3$. By assumption $0 \leq m \leq k - 1$, so it follows that $m = k - 1$. □

**Corollary 3.** If $w \in T(v_1)$, then either $w \in \{u\} \cup O(u)$ or $w \in T(u_1)$ with $d(u_1, w) = k - 1$ or $d(u_1, w) = d(v_1, w)$.

**Proof.** By $k$-geodecity and Lemma 6. □

**Corollary 4.** $v_1 \in N^{k-1}(u_1) \cup O(u)$ and $u_1 \in N^{k-1}(v_1) \cup O(v)$. 

![Fig. 6. Configuration for $k \geq 3$.](image-url)
**Proof.** We prove the first inclusion. By Corollary 3, \(v_1 \in \{u\} \cup O(u) \cup \{u_1\} \cup N^{k-1}(u_1)\). By k-geodecity, \(v_1 \neq u\) and by construction, \(v_1 \neq u_1\). \(\square\)

We now have enough information to identify one member of \(O(u)\) and \(O(v)\).

**Lemma 7.** \(v_1 \in O(u)\) and \(u_1 \in O(v)\).

**Proof.** We prove that \(v_1 \in O(u)\). Suppose that neither \(v_1\) nor \(v\) lies in \(O(u)\). Then by Lemma 5 and Corollary 4 we have \(v, v_1 \in N^{k-1}(u_1)\). As \(v_1\) is an out-neighbour of \(v\), it follows that \(v_1\) appears twice in \(T_k(u_1)\), violating k-geodecity. Therefore \(O(u) \cap \{v, v_1\} \neq \emptyset\).

Now assume that \(v_1, u_1 \in T_k(u)\). Again by Corollary 4, \(v_1 \in N^{k-1}(u_1)\). By k-geodecity we also have \(v_1 \in T(u_1)\). However, \(v_1 \in N^3(v_1)\), so \(v_1\) appears twice in \(T_k(u_1)\), which is impossible. Hence \(O(u) \cap \{v_1, u_1\} \neq \emptyset\). Similarly, \(O(u) \cap \{v_1, v_4\} \neq \emptyset\). In the terminology of the previous section, \(G\) contains no bad pairs. Therefore, if \(v_1 \not\in O(u)\), then \(\{v, v_2, v_4\} \subseteq O(u)\). Since these vertices are distinct, this is a contradiction and the result follows. \(\square\)

Lemma 7 allows us to conclude that for vertices sufficiently close to \(v_1\) one of the potential situations mentioned in Corollary 3 cannot occur.

**Lemma 8.** \(T_{k-3}(v_1) \cap N^{k-1}(u_1) = T_{k-3}(u_1) \cap N^{k-1}(v_1) = \emptyset\).

**Proof.** Let \(w \in T_{k-3}(v_1) \cap N^{k-1}(u_1)\). Consider the position of the vertices of \(N^+(w)\) in \(T_k(u) \cup O(u)\). As \(v_1 \not\in N^+(w)\), it follows from Lemma 7 that at most one of the vertices of \(N^+(w)\) can be an outlier of \(u\), so let us write \(w_1 \in N^+(w)\) \(\not\in O(u)\). By k-geodecity, \(w_1 \not\in T(u_1)\) \(\cup u\). Hence \(w_1 \in T(u_2) = T(v_2)\). However, \(w_1\) also lies in \(T(v_1)\), so this violates k-geodecity. \(\square\)

**Corollary 5.** There is at most one vertex in \(T_{k-3}(v_1) \setminus \{v_1\}\) that does not lie in \(T(u_1)\); for all other vertices \(w \in T_{k-3}(v_1) \setminus \{v_1\}\), \(d(u_1, w) = d(v_1, w)\). A similar result for \(T_{k-3}(u_1) \setminus \{u_1\}\) also holds.

**Lemma 9.** For \(k = 3\), \(N^+(u_1) \cap N^2(v_1) = N^+(v_1) \cap N^2(u_1) = \emptyset\).

**Proof.** Suppose that \(v_3 = u_2\). By the reasoning of Lemma 8 we can set \(u = v_2\) and \(O(u) = \{v_1, v_8\}\) and \(v \not\in O(u)\) and by 3-geodecity \(v \not\in N^+(u_3)\), so we can assume that \(v = u_6\). \(u_3 \rightarrow v_3\) implies that \(u_3 \not\in T(v_1)\), so \(O(v) = \{u_1, u_4\}\). We must have \(\{u_4, u_8, u_{10}\} = \{v_4, v_9, v_{10}\}\). As \(u_4 \rightarrow v\), it follows that \(v_4 \neq u_8\) and hence \(\{u_4, u_{10}\} = \{v_4, v_{10}\}\), which is impossible. \(\square\)

As \(u_1\) is an outlier of \(v\), neither \(v_3\) nor \(v_4\) can be equal to \(u_1\). It follows from Corollary 5 and Lemma 9 that either \(N^+(u_1) = N^+(v_1)\) or \(u_1\) and \(v_1\) have a single common out-neighbour, with one vertex of \(N^+(v_1)\) being an outlier of \(u\).

**Lemma 10.** \(N^2(u) \neq N^2(v)\)

**Proof.** Let \(N^2(u) = N^2(v)\), with \(N^+(u_1) = N^+(v_1) = \{u_3, u_4\}\). Suppose that \(v \not\in O(u)\). By Lemma 5, \(v \in N^{k-2}(u_3) \cup N^{k-2}(u_4)\). But then there is a \(k\)-cycle through \(v\). It follows that \(O(u) = \{v_1, u_1\}\), \(O(v) = \{u_1, u_4\}\). By Lemma 5, \(u_2 \in O(u_1) \cap O(v_1)\). Therefore by Lemma 1 \(O(u_1) = \{u_2, v_1\}\), \(O(v_1) = \{u_2, u_1\}\).

Consider the in-neighbour \(u'\) of \(u_1\) that is distinct from \(u\). We have either \(|N^+(u') \cap N^+(u)| = 1\) or \(|N^+(u') \cap N^+(u)| = 2\). In the first case, it follows from Lemma 7 that \(u_2 \in O(u)\). Every vertex of \(G\) is an outlier of exactly two vertices, so \(u' = u_1\) or \(v_1\). In either case, we have a contradiction. Therefore \(N^+(u') = N^+(u)\). It now follows from Lemma 1 that \(u' \not\in O(u) = \{v_1, u_1\}\), which is impossible. \(\square\)

Noticing that \(u_1\) and \(v_1\) also have a unique common out-neighbour, we have the following corollary.

**Corollary 6.** Without loss of generality, \(u_1 = v_3, u_9 = v_9, O(u) = \{v_1, u_4\}, O(v) = \{u_1, u_4\}, O(u_1) = \{v_4, v_{10}\}\) and \(O(v_1) = \{u_4, u_{10}\}\).

We are now in a position to complete the proof by deriving a contradiction.

**Theorem 2.** There are no digiregular \((2, k, +2)\)-digraphs for \(k \geq 3\).

**Proof.** \(u, v \not\in \{u_3, u_4, v_1, v_4\}\), so by Lemma 5 \(d(u, u) = d(v, u) = k\). In fact, \(u_3 = v_3\) implies that \(v \in N^{k-2}(u_3)\) and \(u \in N^{k-2}(v_3)\). Let \(k \geq 4\). Then \(u, v \not\in \{u_{10}, v_{10}\}\), so \(u, v \in T_k(u_1) \cap T_k(v_1)\). If \(u \in T(u_2) = T(v_2)\), then \(u\) would appear twice in \(T_k(v_1)\), so \(u \in N^{k-1}(u_1)\). However, as \(u\) and \(v\) have a common out-neighbour, this violates k-geodecity.

Finally, suppose that \(k = 3\). The above analysis will hold unless \(u = v_{10}\) and \(v = u_{10}\). Let \(N^-(u_1) = \{u_1, u'\}, N^-(v_1) = \{v, v'\}\). It is evident that \(v' \not\in \{v_1, v_4\}\), so that \(v' \in T_3(u)\). As \(v \in N^2(u_4)\), we must have \(v' \in N^2(u_2)\). Similarly \(u' \in N^2(u_3)\). Since \(u_1\) and \(v_1\) have a common out-neighbour, we can assume that \(u' \not\in N^+(u_3)\) and \(v' \not\in N^+(v_4)\), \(v_4\) can be the outlier of only two vertices, namely \(u\) and \(u_1\), so \(v_4 \in N^3(u_2)\) and likewise \(u_4 \in N^3(u_2)\). By 3-geodecity \(v_4 \in N^2(u_3)\) and \(u_4 \in N^2(u_2)\). It follows that \(u, v \not\in N^2(u_2)\), so \(u \not\in T_3(u_1) \cup T_3(v_2)\). Hence \(O(u) = N^-(u) = \{v_4, v_3\}\), which again is impossible. \(\square\)

It is interesting to note that a similar argument can be used to provide an alternative proof of the result of [6].
References