On diregular digraphs with degree two and excess two

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Abstract

An important topic in the design of efficient networks is the construction of \((d, k, +\epsilon)-\)digraphs, i.e. \(k\)-geodetic digraphs with minimum out-degree \(\geq d\) and order \(M(d, k)+\epsilon\), where \(M(d, k)\) represents the Moore bound for degree \(d\) and diameter \(k\) and \(\epsilon > 0\) is the (small) excess of the digraph. Previous work has shown that there are no \((2, k, +1)-\)digraphs for \(k \geq 2\). In a separate paper, the present author has shown that any \((2, k, +2)-\)digraph must be diregular for \(k \geq 2\). In the present work, this analysis is completed by proving the nonexistence of diregular \((2, k, +2)-\)digraphs for \(k \geq 3\) and classifying diregular \((2, 2, +2)-\)digraphs up to isomorphism.

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1. Introduction

An important topic in the design of efficient networks is the construction of \((d, k, +\epsilon)-\)digraphs, i.e. \(k\)-geodetic digraphs with minimum out-degree \(\geq d\) and order \(M(d, k)+\epsilon\), where \(M(d, k)\) represents the Moore bound for degree \(d\) and diameter \(k\) and \(\epsilon > 0\). Previous work has shown that there are no \((2, k, +1)-\)digraphs for \(k \geq 2\). In a separate paper, the present author has shown that any \((2, k, +2)-\)digraph must be diregular for \(k \geq 2\). In the present work, this analysis is completed by proving the nonexistence of diregular \((2, k, +2)-\)digraphs for \(k \geq 3\) and classifying diregular \((2, 2, +2)-\)digraphs up to isomorphism.

As it was shown by Bridges and Toueg in [1] that Moore digraphs exist only in the trivial cases \(d = 1\) or \(k = 1\) (the Moore digraphs are directed \((k+1)\)-cycles and complete digraphs \(K_{k+1}\) respectively), much research has been devoted to the study of digraphs that in some sense approximate Moore digraphs. For example, there is an extensive literature on digraphs with maximum out-degree \(d\), diameter \(\leq k\) and order \(M(d, k) - \delta\) for small \(\delta > 0\); this is equivalent to relaxing the \(k\)-geodeticity requirement in the conditions for a digraph to be Moore. \(\delta\) is known as the defect of the digraph. The reader is referred to the survey [5] for more information.

In this paper, however, we will consider the following related problem, which is obtained by retaining the \(k\)-geodeticity requirement in the above characterisation of Moore digraphs, but allowing the diameter to exceed \(k\): what is the smallest possible order of a \(k\)-geodetic digraph \(G\) with minimum out-degree \(\geq d\)? A \(k\)-geodetic digraph with minimum out-degree \(\geq d\) and order \(M(d, k)+\epsilon\) is said to be a \((d, k, +\epsilon)-\)digraph or to have excess \(\epsilon\). It was shown in [6] that there are no diregular \((2, k, +1)-\)digraphs for \(k \geq 2\). In 2016 it was shown in [3] that digraphs with excess one must be diregular and that there are no \((d, k, +1)-\)digraphs for \(k = 2, 3, 4\) and sufficiently large \(d\). In a separate paper [7], the present author has shown that \((2, k, +2)-\)digraphs must be diregular with degree \(d = 2\) for \(k \geq 2\). In the present paper, we classify the \((2, 2, +2)-\)digraphs up to isomorphism and show that there are no diregular \((2, k, +2)-\)digraphs for \(k \geq 3\), thereby completing the proof of the nonexistence of digraphs with degree \(d = 2\) and excess \(\epsilon = 2\) for \(k \geq 3\). Our reasoning and notation will follow closely that employed in [4] for the corresponding result for defect \(\delta = 2\).
Lemma 1. \( u \) is also an outlier of exactly two vertices. We will say that a vertex \( u \) necessarily equal to \( \{1, 2\} \).

Lemma 2. Suppose for a contradiction that there is no such pair of vertices. Define a map \( \phi : V(G) \to V(G) \) as follows. Let \( u^{+} \) be an out-neighbour of a vertex \( u \) and let \( \phi(u) \) be the in-neighbour of \( u^{+} \) distinct from \( u \). By our assumption, it is easily verified that \( \phi \) is a well-defined bijection with no fixed points and with square equal to the identity. It follows that \( G \) must have even order, whereas \( |V(G)| = M(2, k) + 2 \) is odd.

2. Preliminary results

We will let \( G \) stand for a \((2, k, +2)\)-digraph for arbitrary \( k \geq 2 \), i.e. \( G \) has minimum out-degree \( d = 2 \), is \( k \)-geodetic and has order \( M(2, k) + 2 \). We will denote the vertex set of \( G \) by \( V(G) \). By the result of [7], \( G \) must be diregular with degree \( d = 2 \) for \( k \geq 2 \). The distance \( d(u, v) \) between vertices \( u \) and \( v \) is the length of the shortest path from \( u \) to \( v \). Notice that \( d(u, v) \) is not necessarily equal to \( d(v, u) \). \( u \to v \) will indicate that there is an arc from \( u \) to \( v \). We define the in- and out-neighbourhoods of a vertex \( u \) by \( N^{−}(u) = \{ v \in V(G) : v \to u \} \) and \( N^{+}(u) = \{ v \in V(G) : u \to v \} \) respectively; more generally, for \( 0 \leq l \leq k \), the set \( \{ v \in V(G) : d(u, v) = l \} \) of vertices at distance exactly \( l \) from \( u \) will be denoted by \( N^{l}(u) \). For \( 0 \leq l \leq k \) we will also write \( T_{l}(u) = \bigcup_{i=0}^{l} N^{i}(u) \) for the set of vertices at distance \( \leq l \) from \( u \). The notation \( T_{k-1}(u) \) will be abbreviated by \( T(u) \).

It is easily seen that for any vertex \( u \) of \( G \), there are exactly two distinct vertices that are at distance \( \geq k + 1 \) from \( u \). For any \( u \in V(G) \), we will write \( O(u) \) for the set of these vertices and call such a set an outlier set and its elements outliers of \( u \). Notice that \( O(u) = V(G) \setminus T_{k}(u) \). An elementary counting argument shows that in a diregular \((2, k, +2)\)-digraph every vertex is also an outlier of exactly two vertices. We will say that a vertex \( u \) can reach a vertex \( v \) if \( v \not\in O(u) \).

Our proof will proceed by an analysis of a pair of vertices with exactly one common out-neighbour. First, we must show that such a pair exists and deduce some elementary properties of pairs of vertices with identical out-neighbourhoods.

**Lemma 1.** For \( k \geq 2 \), let \( u \) and \( v \) be distinct vertices such that \( N^{+}(u) = N^{+}(v) = \{u_1, u_2\} \). Then \( u_1 \in O(u_2) \), \( u_2 \in O(u_1) \) and there exists a vertex \( x \) such that \( O(u) = \{v, x\} \), \( O(v) = \{u, x\} \).

**Proof.** Suppose that \( u \) can reach \( v \) by a \( \leq k \)-path. Then \( v \in T(u_1) \cup T(u_2) \). As \( N^{+}(v) = N^{+}(u) \), it follows that there would be a \( \leq k \)-cycle through \( v \), contradicting \( k \)-geodeticity. If \( O(u) = \{v, x\} \), then \( x \neq v \) and \( x \not\in T(u_1) \cup T(u_2) \), so that \( v \) cannot reach \( x \) by a \( \leq k \)-path. Similarly, if \( u_1 \) can reach \( u_2 \) by a \( \leq k \)-path, then we must have \( \{u, v\} \cap T(u_1) \neq \emptyset \), which is impossible. \( \square \)

**Lemma 2.** For \( k \geq 2 \), there exists a pair of vertices \( u, v \) with \( |N^{+}(u) \cap N^{+}(v)| = 1 \).

**Proof.** Suppose for a contradiction that there is no such pair of vertices. Define a map \( \phi : V(G) \to V(G) \) as follows. Let \( u^{+} \) be an out-neighbour of a vertex \( u \) and let \( \phi(u) \) be the in-neighbour of \( u^{+} \) distinct from \( u \). By our assumption, it is easily verified that \( \phi \) is a well-defined bijection with no fixed points and with square equal to the identity. It follows that \( G \) must have even order, whereas \( |V(G)| = M(2, k) + 2 \) is odd. \( \square \)

3. Classification of \((2, 2, +2)\)-digraphs

We begin by classifying the \((2, 2, +2)\)-digraphs up to isomorphism. We will prove the following theorem.

**Theorem 1.** There are exactly two diregular \((2, 2, +2)\)-digraphs, which are displayed in Figs. 2 and 5.

Let \( G \) be an arbitrary diregular \((2, 2, +2)\)-digraph. \( G \) has order \( M(2, 2) + 2 = 9 \). By **Lemma 2**, \( G \) contains a pair of vertices \((u, v)\) such that \( |N^{+}(u) \cap N^{+}(v)| = 1 \); we will assume that \( u_2 = v_2 \), so that we have the situation shown in Fig. 1.

We can immediately deduce some information on the possible positions of \( v \) and \( v_1 \) in \( T_{2}(u) \).

**Lemma 3.** If \( v \not\in O(u) \), then \( v \in N^{+}(u_1) \). If \( v_1 \not\in O(u) \), then \( v_1 \in N^{+}(u_1) \).

**Proof.** \( v \notin T(u_2) \) by \( 2 \)-geodeticity. \( v \neq u \) by construction. If we had \( v = u_1 \), then there would be two distinct \( \leq 2 \)-paths from \( u \) to \( u_2 \). Also \( v_1 \notin \{u\} \cup T(u_2) \) by \( 2 \)-geodeticity and by assumption \( u_1 \neq v_1 \). \( \square \)
Since $v$ and $v_1$ cannot both lie in $N^+(u_1)$ by 2-geodecity, we have the following corollary.

**Corollary 1.** $O(u) \cap \{v, v_1\} \neq \emptyset$.

We will call a pair of vertices $(u, v)$ with a single common out-neighbour *bad* if at least one of $O(u) \cap \{v_1, v_3\} = \emptyset$, $O(u) \cap \{v_1, v_4\} = \emptyset$, $O(v) \cap \{u_1, u_3\} = \emptyset$, $O(v) \cap \{u_1, u_4\} = \emptyset$. holds. Otherwise such a pair will be called good.

**Lemma 4.** There is a unique $(2, 2, +2)$-digraph containing a bad pair.

**Proof.** Assume that there exists a bad pair $(u, v)$. Without loss of generality, $O(u) \cap \{v_1, v_3\} = \emptyset$. By **Lemma 3** we can set $v_1 = u_3$. By 2-geodecity $v_3 = u$. We cannot have $v_4 = v_3 = u$, so $v_4$ must be an outlier of $u$. By **Corollary 1** it follows that $O(u) = \{v, v_4\}$.

Consider the vertex $u_1$. By **Lemma 3**, if $u_1 \not\in O(v)$, then $u_1 \notin N^+(v_1)$. However, as $v_1 = u_3$, there would be a 2-cycle through $u_1$. Hence $u_1 \in O(v)$. As $O(u) = \{v, v_4\}$, we have $V(G) = \{u, u_1, u_2, u_3 = v_1, u_4, u_5, u_6, v, v_4\}$ and $O(v) = \{u, u_4\}$. As neither $u$ nor $v$ lies in $T(u_1)$, we must also have $u_2 \notin O(u_1)$. As $u_1$ can reach $u_1, v_1, u_4, u$ and $v_4$, it follows that without loss of generality we either have $O(u_1) = \{u, v\}$ and $N^+(u_1) = \{u_5, u_6\} = N^+(u_2)$ or $O(u_1) = \{u_2, u_6\}$ and $N^+(u_4) = \{v, u_5\}$. In either case, $(v, u_1)$ is a good pair.

Suppose firstly that $N^+(u_2) = N^+(u_4)$. Then $v$ is an outlier of $u$ and $u_1$. As each vertex is the outlier of exactly two vertices, $v_1$ must be able to reach $v$ by a $\leq 2$-path. Hence $v_4 \rightarrow v$. Likewise $u_2$ can reach $u$, so without loss of generality $u_5 \rightarrow v$. Suppose that $O(u_2) \cap \{u, u_1\} = \emptyset$. As $u$ and $v$ have a common out-neighbour, we must have $u_6 \rightarrow u$. Since $u \rightarrow u_1$, by 2-geodecity we must have $u_5 \rightarrow u_1$. However, this is a contradiction, as $v$ and $u_1$ also have a common out-neighbour. Therefore, at least one of $u, u_1$ is an outlier of $u_2$. By **Lemma 1** $u_4$ is an outlier of $u_2$. Therefore either $O(u_2) = \{u, u_4\}$ or $O(u_2) = \{u_1, u_4\}$. If $O(u_2) = \{u, u_4\}$, then $u_2$ must be able to reach $u_1$, $v_1$ and $v_4$. $u_5 \rightarrow u$ and $v \rightarrow v$, so $v_1 \in N^+(u_6)$. As $u_1 \rightarrow v_1$, we must have $N^+(u_5) = \{v, u_1\}$. As $v$ and $u_1$ have a common out-neighbour, this violates 2-geodecity. Hence $O(u_2) = \{u_1, u_4\}$ and $u_2$ can reach $u_1$ and $v_4$. As $v \rightarrow v_1, v_1 \in N^+(u_6)$. As $v_1 \rightarrow v_4$, it follows that $N^+(u_5) = \{v, u_4\}$. However, $v_4 \rightarrow v$, so this again violates 2-geodecity.

We are left with the case $O(u_1) = \{u_2, u_6\}$ and $N^+(u_4) = \{v, u_5\}$. Then $v_1 \in O(u_2)$, as neither $v$ nor $u_1$ lies in $T(u_2)$. Observe that $u_2$ and $u_4$ have a single common out-neighbour, so by **Corollary 1** $O(u_2) \cap \{u_4, v\} \neq \emptyset$. Therefore either $O(u_2) = \{v, u_4\}$ or $O(u_2) = \{v_1, v\}$. Suppose firstly that $O(u_2) = \{v_1, u_4\}$. Then $N^+(u_2) = \{u, v, u_1, v_4\}$. As $u \in N^+(u_4) = \{v, u_5\}$, $u_5 \not\rightarrow v$, so $u_6 \rightarrow v$. As $N^+(u) \cap N^+(v) \neq \emptyset$, $u_5 \rightarrow u \rightarrow u_1$, so necessarily $u_6 \in N^+(u_5) = \{v, u_1\}$. However, $v_1 \in N^+(u_1) \cap N^+(v)$, contradicting 2-geodecity.

Hence $O(u_2) = \{v_1, u_1\}$ and $N^+(u_2) = \{u, u_1, u_4, v_4\}$. As $u_4 \rightarrow u_5, u_5 \not\rightarrow u_4$. Thus $u_6 \rightarrow u_4$. Now $u_1 \rightarrow u_4$ and $u \rightarrow u_1$ implies that $N^+(u_2) = \{v_1, u_4\}$ and $N^+(u_6) = \{u, u_4\}$. Finally we must have $N^+(u_4) = \{v, u_6\}$. This gives us the $(2, 2, +2)$-digraph shown in Fig. 2.

We can now assume that all pairs given by **Lemma 2** are good. Let us fix a pair $(u, v)$ with a single common out-neighbour. It follows from **Corollary 1** and the definition of a good pair that $v_1 \in O(u)$; otherwise $O(u)$ would contain $v, v_3$ and $v_4$, which is impossible. Likewise $u_1 \in O(v)$.
Considering the positions of $v_3$ and $v_4$, we see that there are without loss of generality four possibilities: (1) $u = v_3$, $u_4 = v_4$, (2) $u = v_3$, $O(u) = \{v_1, v_4\}$, (3) $N^+(u_1) = N^+(v_1)$ and (4) $u_3 = v_3$, $O(u) = \{v_1, u_4\}$. A suitable relabelling of vertices shows that case 4 is equivalent to case 1.a below, so we will examine cases 1 to 3 in turn.

**Case 1: $u = v_3$, $u_4 = v_4$**

Depending upon the position of $v$, we must either have $O(u) = \{v_1, v\}$ and $O(v) = \{u_1, u_3\}$ or $v = u_3$ (see Fig. 3).

**Case 1.a: $O(u) = \{v_1, v\}$, $O(v) = \{u_1, u_3\}$**

In this case $V(G) = \{u, u_1, u_2, u_3, u_4, u_5, u_6, v, v_1\}$, $u_1$ and $v_1$ have a single common out-neighbour, namely $u_4$, so, as we are assuming all such pairs to be good, we have $u_3 \in O(v_1)$, $u \in O(u_1)$. By 2-geodecity, $N^+(u_4) \subseteq \{u_5, u_6, v\}$, so without loss of generality either $N^+(u_4) = \{u_5, u_6\}$ or $N^+(u_4) = \{u, v\}$.

Suppose that $N^+(u_4) = \{u_5, u_6\}$. By elimination, $O(u_1) = \{v, u_3\}$. As $G$ is diregular, every vertex is an outlier of exactly two vertices; $v$ is an outlier of $u$ and $v_1$, so both $u_1$ and $u_2$ can reach $v$ by a $\leq 2$-path. Hence $v \in N^+(u_3)$. As $v \rightarrow v_1$, we see that $v_1$ is an outlier of $u_1$; as $u$ is also an outlier of $u_1$, we have $O(u_1) = \{u, v_1\}$ and $N^+(u_3) = \{v, u_2\}$. As $v \rightarrow u_2$, this is impossible.

Now consider $N^+(u_4) = \{u_5, v\}$. We now have $O(u_1) = \{u_3, u_6\}$. Thus $u_3 \in O(v) \cap O(u_1)$, so $u_3 \in T_2(u_4)$. $v$ is not adjacent to $u_3$, so $u_3 \in N^+(u_5)$. $u_2$ and $u_4$ have $u_5$ as a unique common out-neighbour, so $u_6 \in O(u_4)$, $v \in O(u_2)$. As $u_6 \in O(u_1) \cap O(u_4)$, $u_1$ can reach $u_6$. Hence $u_6 \in N^+(u_1)$. Neither $u$ nor $v$ lie in $T(u_1)$, so $u_2 \in O(u_1)$. Therefore either $O(u_1) = \{u, u_2\}$ or $O(u_1) = \{u_2, v_1\}$. If $O(u_1) = \{u, u_2\}$, then $N^+(u_2) = \{u_6, v_1\}$. $u_2$ cannot reach $v_1$, since $v, u_3 \notin T(u_2)$, so $O(u_2) = \{v, v_1\}$ and $N^2(u_2) = \{u_1, u_3, u_4\}$. As $u_4 \rightarrow u_5$, $u_4 \in N^+(u_6)$, $u_1 \rightarrow u_4$, so $N^+(u_1) = \{u_1, u_3\}$. As $u_1 \rightarrow u_3$, this is a contradiction. Thus $O(u_1) = \{u_2, v_1\}$, so that $N^+(u_3) = \{u, u_6\}$. $u_1$ must have an in-neighbour apart from $u$, which must be either $u_5$ or $u_6$. As $u_1 \rightarrow u_3$, we have $u_1 \in N^+(u_6)$. By elimination, $v$ and $v_1$ must also have in-neighbours in $\{u_5, u_6\}$. As $u_1$ and $v_1$ have a common out-neighbour, we have $N^+(u_5) = \{u_3, v_1\}$, $N^+(u_6) = \{u_1, v\}$. However, both $u_3$ and $v_1$ are adjacent to $u$, violating 2-geodecity.

**Case 1.b: $v = u_3$**

There exists a vertex $x$ such that $V(G) = \{u, u_1, u_2, v, u_4, u_5, u_6, v_1, x\}$, $O(u) = \{v_1, x\}$ and $O(v) = \{u_1, x\}$. As $x \in O(u) \cap O(v)$, $u_1$ and $u_2$ can reach $x$, so without loss of generality $x \in N^+(u_4) \cap N^+(u_5)$. As $u_5$ and $u_4$ have a common out-neighbour, $u_6 \in O(u_1)$. Also, $u_1$ and $v_1$ have $u_4$ as a unique common out-neighbour, so $u \in O(u_1)$ and $O(u_1) = \{u, u_6\}$. Thus $N^+(u_4) = \{x, u_6\}$. Observe that $u_2$ and $u_4$ have the out-neighbour $u_6$ in common. Thus $x \in O(u_2)$, whereas we already have $x \in O(u) \cap O(v)$, a contradiction.

**Case 2: $u = v_3$, $O(u) = \{v_1, v_4\}$**

As $v$ is not equal to $v_1$ or $v_4$, $v$ must lie in $T_2(u)$. Without loss of generality, $v = u_3$. Hence $V(G) = \{u, u_1, u_2, v, u_4, u_5, u_6, v_1, v_4\}$ and $O(v) = \{u_1, u_4\}$. We have the configuration shown in Fig. 4. Hence $u_1$ can reach
Taking into account adjacencies between members of \( O_u \), we have

\[ N^+(u_4) = \{u, v_4\}, N^+(u_5) = \{u_5, u_6\} \]

\[ O(u_1) = \{u, u_1\}, N^+(u_1) = \{u_5, v_4\}, c) O(u_1) = \{u_5, u_6\}, N^+(u_1) = \{u, v_4\} \text{ or } d) O(u_1) = \{u_5, v_4\}, N^+(u_1) = \{u, u_6\}. \]

**Case 2.a:** \( O(u_4) = \{u, v_4\}, N^+(u_4) = \{u_5, u_6\} \)

As \( v_4 \in O(u) \cap O(u_1) \), \( u_2 \) can reach \( v_4 \) and without loss of generality \( v_4 \in N^+(u_2) \), \( N^+(u_2) = N^+(u_4) \), so by **Lemma 1** \( u_2 \in O(u_4), u_4 \in O(u_2), u_5 \in O(u_6) \) and \( u_6 \in O(u_5) \). Hence \( u_4 \in O(v) \cap O(u_2) \), so \( v_1 \) can reach \( u_4 \), so \( u_4 \in N^+(v_4) \). Neither \( u_5 \) nor \( u_6 \) lies in \( N^+(v_4) \), so \( O(v_1) = \{u_5, u_6\} \) and \( N^+(v_4) = \{u_4, v_4\} \). Hence \( O(v_4) = \{u, u_1\} \). Observe that \( N^+(u_1) = N^+(u_4) \), so that \( v \in O(u_4) \). Therefore \( v \notin N^+(u_5) \cup N^+(u_6) \), yielding \( O(u_2) = \{u_4, v\} \). \( N^+(u_2) = \{u_4, v, u_1, u_1\} \). As \( v_1 \to v_4 \) and \( N^+(u_1) = N^+(u_4) \), we must have \( N^+(u_1) = \{u_4, u\} \). This yields the digraph in Fig. 5. Unlike the digraph in Fig. 2, this digraph contains pairs of vertices with identical out-neighbourhoods, so the two are not isomorphic.

**Case 2.b:** \( O(u_1) = \{u, u_3\}, N^+(u_4) = \{u_6, v_4\} \)

As \( u_4 \rightarrow v_4, u_4 \notin N^+(u_4), \) so \( u_4 \notin O(v_1) \). Hence \( u_4 \in O(v) \cap O(v_1) \), so \( u_2 \) can reach \( u_4 \). As \( u_4 \rightarrow u_6 \), we must have \( u_5 \rightarrow u_4, u_2 \) and \( u_4 \) have \( u_6 \) as a common out-neighbour, so \( v_4 \in O(u_2), u_5 \in O(u_4) \). Therefore \( v_4 \in O(u) \cap O(u_2) \), so \( u_6 \) can reach \( u_4 \), but \( v_4 \notin T(u_6) \), so \( N^+(u_6) \) contains an in-neighbour of \( v_4, u_4 \notin N^+(u_6) \), so we must have \( u_5 \rightarrow v_1 \). We have \( u_5 \in O(u_4) \cap O(u_1) \), so \( v_1 \) can reach \( u_5 \) and hence \( v_4 \rightarrow u_5, v_1 \) cannot reach \( u_6 \), as \( u_2, u_4 \notin T(v_1) \), so \( O(v_1) = \{u_5, u_6\}, N^+(v_4) = \{v, u_5\} \). Now \( u_2 \) and \( u_4 \) have \( u_6 \) as a unique common out-neighbour, so \( u_6 \in O(v_4), v \in O(u_2) \). Thus \( O(u_2) = \{v, v_4\} \) and \( N^2(u_2) = \{u_4, v, u, u_1, u_1\} \). Taking into account adjacencies between members of \( N^2(u_2) \), it follows that \( N^+(u_1) = \{u_4, u\} \). \( N^+(u_6) = \{u_1, v_1\} \). However, \( u_2, u_4 \) now constitutes a bad pair, contradicting our assumption.

**Case 2.c:** \( O(u_1) = \{u, u_3\}, N^+(u_4) = \{u, v_4\} \)

As \( u_4 \rightarrow v_4, u_4 \notin N^+(v_4), \) so \( u_4 \notin O(v_1) \). Hence \( u_4 \in O(v) \cap O(v_1) \), implying that \( u_2 \) can reach \( u_4 \). Without loss of generality, \( u_5 \rightarrow u_4 \). There are three possibilities: (i) \( O(v_1) = \{u_4, u_6\}, N^+(v_4) = \{v, u_5\} \), (ii) \( O(v_1) = \{u_4, u_5\}, N^+(v_4) = \{v, u_6\} \) and (iii) \( O(v_1) = \{u_4, v\}, N^+(v_4) = \{v, u_5\} \).

(i) \( O(v_1) = \{u_4, u_6\}, N^+(v_4) = \{v, u_5\} \)

\( u_1 \) and \( v_4 \) have \( v \) as a unique common out-neighbour, so \( u_4 \in O(v_4) \). However, this contradicts \( v_4 \rightarrow u_5 \rightarrow u_4 \).

(ii) \( O(v_1) = \{u_4, u_5\}, N^+(v_4) = \{v, u_6\} \)

Neither \( u_4 \) nor \( v_1 \) lie in \( T(u_2) \), so \( v_4 \notin O(u_2) \). Now observe that \( u_2 \) and \( u_4 \) have \( u_6 \) as unique common out-neighbour, so \( v \in O(u_2) \), yielding \( O(u_2) = \{v, v_4\} \) and \( N^2(u_2) = \{u_4, u_1, u_1, v_1\} \). As \( u_4 \rightarrow u_1 \) and \( u \rightarrow u_1 \), we must have \( N^+(u_5) = \{u_4, u_1\}, N^+(u_6) = \{u_1, v_1\} \), a contradiction, since \( u_1 \rightarrow u_4 \).

(iii) \( O(v_1) = \{u_4, v\}, N^+(v_4) = \{v, u_5\} \)

We now have \( N^+(u_2) = N^+(v_4) \), so \( u_2 \in O(v_4), v_4 \in O(u_2), u_5 \in O(u_6), u_6 \in O(u_5) \). Also \( N^+(u_4) = N^+(v_1) \), so \( u_4 \in O(v_1), v_1 \in O(u_4) \) and \( u \in O(u_4) \). If \( u \in O(v_4) \), then \( u \notin N^+(u_5) \cup N^+(u_6) \), so we see that \( u \in O(u_2) \) and hence \( O(u_2) = \{u, v_4\} \) and \( N^2(u_2) = \{u_4, u_1, v, v_1\} \). As \( u_1 \rightarrow u_4 \) and \( u_1 \rightarrow v \), we have \( N^+(u_5) = \{u_4, v\}, N^+(u_6) = \{u_1, v_1\} \). It is not difficult to show that this yields a \((2, 2, +2)\)-digraph isomorphic to that in Fig. 5.

**Case 2.d:** \( O(u_1) = \{u_5, v_4\}, N^+(u_4) = \{u, u_4\} \)

In this case \( v_4 \in O(u) \cap O(u_1) \), so \( u_2 \) can reach \( v_4, u_4 \) and \( v_1 \) have unique common out-neighbour \( u \), so \( v_4 \in O(u_4), u_6 \in O(v_1) \). If \( u_6 \rightarrow v_4 \), then we would have \( u_4 \rightarrow u_6 \rightarrow v_4 \), contradicting \( v_4 \in O(u_4) \), so \( u_5 \rightarrow v_4 \). This also implies that \( u_5 \notin N^+(u_4) \), so \( u_5 \in O(v_1) \), yielding \( O(v_1) = \{u_5, u_6\} \) and \( N^+(v_4) = \{v, u_4\} \). Now \( v_4, u_1 \notin T(u_2) \), so \( O(u_2) = \{v, u_4\} \). No
Main result

4. Main result

We can now complete our analysis by showing that there are no non-diregular \((2, k, +e)-\)digraphs for \(k \geq 3\). Let \(G\) be such a digraph. By Lemma 2, \(G\) contains vertices \(u\) and \(v\) with a unique common out-neighbour. In accordance with our vertex-labelling convention, we have the situation in Fig. 6. A triangle based at a vertex \(x\) represents the set \(T(x)\).

We now proceed to determine the possible outlier sets of \(u\) and \(v\).

Lemma 5. \(v \in N^{k-1}(u_1) \cup O(u)\) and \(u \in N^{k-1}(v_1) \cup O(v)\). If \(v \in O(u)\), then \(u_2 \in O(u_1)\) and if \(u \in O(v)\), then \(u_2 \in O(v_1)\).

Proof. \(v\) cannot lie in \(T(u)\), or the vertex \(u_2\) would be repeated in \(T(u)\). Also, \(v \not\in T(u_2)\), or there would be a \(\leq k\)-cycle through \(v\). Therefore, if \(v \not\in O(u)\), then \(v \in N^{k-1}(u_1)\). Likewise for the other result. If \(v \in O(u)\), then neither in-neighbour of \(u_2\) lies in \(T(u_1)\), so that \(u_2 \in O(u_1)\). □

Lemma 6. Let \(w \in T(v_1)\), with \(d(v_1, w) = l\). Suppose that \(w \in T(u_1)\), with \(d(u_1, w) = m\). Then either \(m \leq l\) or \(w \in N^{k-1}(u_1)\). A similar result holds for \(w \in T(u_1)\).

Proof. Let \(w\) be as described and suppose that \(m > l\). Consider the set \(N^{k-m}(w)\). By construction, \(N^{k-m}(w) \subseteq N^{k}(u_1)\), so by \(k\)-geodecity \(N^{k-m}(w) \cap T(u_1) = \emptyset\). At the same time, we have \(l + k - m \leq k - 1\), so \(N^{k-m}(w) \subseteq T(v_1)\). This implies that \(N^{k-m}(w) \cap T(v_1) = N^{k-m}(w) \cap T(u_2) = \emptyset\). As \(V(G) = \{u\} \cup T(u_1) \cup T(u_2) \cup O(u)\), it follows that \(N^{k-m}(w) \subseteq \{u\} \cup O(u)\). Therefore \(|N^{k-m}(w)| = 2k-m \leq 3\). By assumption \(0 \leq m \leq k - 1\), so it follows that \(m = k - 1\). □

Corollary 3. If \(w \in T(v_1)\), then either \(w \in \{u\} \cup O(u)\) or \(w \in T(u_1)\) with \(d(u_1, w) = k - 1\) or \(d(u_1, w) \leq d(v_1, w)\).

Proof. By \(k\)-geodecity and Lemma 6. □

Corollary 4. \(v_1 \in N^{k-1}(u_1) \cup O(u)\) and \(u_1 \in N^{k-1}(v_1) \cup O(v)\).
Proof. We prove the first inclusion. By Corollary 3, \(v_1 \in \{u\} \cup O(u) \cup \{u_1\} \cup N^{k-1}(u_1)\). By k-geodecity, \(v_1 \neq u\) and by construction, \(v_1 \neq u_1\). □

We now have enough information to identify one member of \(O(u)\) and \(O(v)\).

Lemma 7. \(v_1 \in O(u)\) and \(u_1 \in O(v)\).

Proof. We prove that \(v_1 \in O(u)\). Suppose that neither \(v_1\) nor \(v\) lies in \(O(u)\). Then by Lemma 5 and Corollary 4 we have \(v, v_1 \in N^{k-1}(u_1)\). As \(v_1\) is an out-neighbour of \(v\), it follows that \(v_1\) appears twice in \(T_k(u_1)\), violating k-geodecity. Therefore \(O(u) \cap \{v, v_1\} = \emptyset\).

Now assume that \(v_1, v_3 \in T_k(u_1)\). Again by Corollary 4, \(v_1 \in N^{k-1}(u_1)\). By k-geodecity we also have \(v_3 \in T_k(u_1)\). However, \(v_1 \in N^+(v_3)\), so \(v_3\) appears twice in \(T_k(u_1)\), which is impossible. Hence \(O(u) \cap \{v_1, v_3\} = \emptyset\). Similarly, \(O(u) \cap \{v_1, v_4\} = \emptyset\). In the terminology of the previous section, \(G\) contains no bad pairs. Therefore, if \(v_1 \notin O(u)\), then \(\{v_2, v_3, v_4\} \subseteq O(u)\). Since these vertices are distinct, this is a contradiction and the result follows. □

Lemma 7 allows us to conclude that for vertices sufficiently close to \(v_1\) one of the potential situations mentioned in Corollary 3 cannot occur.

Lemma 8. \(T_{k-3}(v_1) \cap N^{k-1}(u_1) = T_{k-3}(u_1) \cap N^{k-1}(v_1) = \emptyset\).

Proof. Let \(w \in T_{k-3}(v_1) \cap N^{k-1}(u_1)\). Consider the position of the vertices of \(N^+(w)\) in \(T_k(u_1) \cup O(u)\). As \(v_1 \notin N^+(w)\), it follows from Lemma 7 that at most one of the vertices of \(N^+(w)\) can be an outlier of \(u\), so let us write \(w_1 \in N^+(w) \setminus O(u)\). By k-geodecity, \(w_1 \notin T_k(u_1) \cup O(u)\). Hence \(w_1 \in T_k(u_2) \cup T_k(v_2)\). However, \(w_1\) also lies in \(T_k(v_1)\), so this violates k-geodecity. □

Corollary 5. There is at most one vertex in \(T_{k-3}(v_1) \setminus \{v_1\}\) that does not lie in \(T_k(u_1)\); for all other vertices \(w \in T_{k-3}(v_1) \setminus \{v_1\}\), \(d(u_1, w) = d(v_1, w)\). A similar result for \(T_{k-3}(u_1) \setminus \{u_1\}\) also holds.

Lemma 9. For \(k = 3, N^2(u_1) \cap N^2(v_1) = N^+(v_1) \cap N^2(u_1) = \emptyset\).

Proof. Suppose that \(v_3 = u_2\). By the reasoning of Lemma 8 we can set \(u = v\) and \(O(u) = \{v_1, v_3\}, v \notin O(u)\) and by 3-geodecity \(v \notin N^+(u_3)\), so we can assume that \(v = u_0, u_3 \rightarrow v_3\) implies that \(u_1 \notin T_k(v_1)\), so \(O(v) = \{u_1, u_1\}\). We must have \(\{u_4, u_5, u_10\} = \{v_2, v_3, v_10\}\). As \(u_4 \rightarrow v\), it follows that \(v_4 = u_5\) and hence \(\{u_4, u_10\} = \{v_2, v_10\}\), which is impossible. □

As \(u_1\) is an outlier of \(v\), neither \(v_3\) nor \(v_4\) can be equal to \(u_1\). It follows from Corollary 5 and Lemma 9 that either \(N^+(u_1) = N^+(v_1)\) or \(u_1\) and \(v_1\) have a single common out-neighbour, with one vertex of \(N^+(v_1)\) being an outlier of \(u\).

Lemma 10. \(N^2(u) \neq N^2(v)\)

Proof. Let \(N^2(u) = N^2(v)\), with \(N^2(u_1) = N^2(v_1) = \{u_3, u_4\}\). Suppose that \(v \notin O(u)\). By Lemma 5, \(v \in N^{k-2}(u_3) \cup N^{k-2}(u_4)\).

But then there is a \(k\)-cycle through \(u\). It follows that \(O(u) = \{v, u_1\}\), \(O(v) = \{u_1, u_1\}\). By Lemma 5, \(u_2 \in O(u_1) \cap O(v_1)\). Therefore by Lemma 1 \(O(u_1) = \{u_2, v_1\}\), \(O(v_1) = \{u_2, u_1\}\).

Consider the in-neighbour \(u'\) of \(u_1\) that is distinct from \(u\). We have either \(|N^+(u') \cap N^+(u)| = 1\) or \(|N^+(u') \cap N^+(u)| = 2\). In the first case, it follows from Lemma 7 that \(u_2 \in O(u)\). Every vertex of \(G\) is an outlier of exactly two vertices, so \(u' = u_1\) or \(v_1\). In either case, we have a contradiction. Therefore \(N^+(u') \neq N^+(u)\). It now follows from Lemma 1 that \(u' \in O(u) = \{u_1, v_1\}\), which is impossible. □

Noticing that \(u_1\) and \(v_1\) also have a unique common out-neighbour, we have the following corollary.

Corollary 6. Without loss of generality, \(u_1 = v_3, u_9 = v_9, O(u) = \{v_1, v_4\}, O(v) = \{u_1, u_4\}, O(u_1) = \{v_4, v_10\}\) and \(O(v_1) = \{u_4, u_10\}\).

We are now in a position to complete the proof by deriving a contradiction.

Theorem 2. There are no digereular \((2, k, +2)\)-digraphs for \(k \geq 3\).

Proof. \(u, v \notin \{u_1, u_3, v_1, v_4\}\), so by Lemma 5 \(d(u, v) = d(v, u) = k\). In fact, \(u_3 = v_2\) implies that \(v \in N^{k-2}(u_3)\) and \(u \in N^{k-2}(v_2)\). Let \(k \geq 4\). Then \(u, v \notin \{u_10, v_10\}\), so \(u, v \in T_k(u_1) \cap T_k(v_1)\). If \(u \in T_k(u_2) = T(v_2)\), then \(u\) would appear twice in \(T_k(v_1)\), so \(u \in N^{k-1}(u_1)\). However, as \(u\) and \(v\) have a common out-neighbour, this violates k-geodecity.

Finally, suppose that \(k = 3\). The above analysis will hold unless \(u = v_10\) and \(v = u_10\). Let \(N^-(u_1) = \{u, u'\}, N^-(v_1) = \{v, v'\}\). It is evident that \(v' \notin \{v_1, v_4\}\), so that \(v' \in T_k(v)\). As \(v \in N^2(u_4)\), we must have \(v' \notin N^2(u_2)\). Similarly \(u' \notin N^2(u_2)\). Since \(u_1\) and \(v_1\) have a common out-neighbour, we can assume that \(u' \notin N^+\). As \(u_4 \notin N^2(u_6)\), \(v_4\) can be the outlier of only two vertices, namely \(u\) and \(u_1\), so \(v_4 \in N^2(u_2)\) and likewise \(u_4 \in N^2(v_2)\). By 3-geodecity \(v_4 \notin N^2(u_2)\) and \(u_4 \notin N^2(v_2)\). It follows that \(u, v \notin N^2(u_3)\), so \(u \notin T_k(u_1) \cup T_k(v_2)\). Hence \(O(u) = N^+(u) = \{v_1, v_4\}\), which again is impossible. □

It is interesting to note that a similar argument can be used to provide an alternative proof of the result of [6].
References