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On diregular digraphs with degree two and excess two

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Abstract

An important topic in the design of efficient networks is the construction of \((d, k, +\epsilon)\)-digraphs, i.e. \(k\)-geodetic digraphs with minimum out-degree \(\geq d\) and order \(M(d, k) + \epsilon\), where \(M(d, k)\) represents the Moore bound for degree \(d\) and diameter \(k\) and \(\epsilon > 0\) is the (small) excess of the digraph. Previous work has shown that there are no \((2, k, +1)\)-digraphs for \(k \geq 2\). In a separate paper, the present author has shown that any \((2, k, +2)\)-digraph must be regular for \(k \geq 2\). In the present work, this analysis is completed by proving the nonexistence of regular \((2, k, +2)\)-digraphs for \(k \geq 3\) and classifying regular \((2, 2, +2)\)-digraphs up to isomorphism.

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1. Introduction

An important topic in the design of interconnection networks is the directed degree diameter problem: what is the largest possible order \(N(d, k)\) of a digraph \(G\) with maximum out-degree \(d\) and diameter \(\leq k\)? A simple inductive argument shows that for \(0 \leq l \leq k\) the number of vertices at distance \(l\) from a fixed vertex \(v\) is bounded above by \(d^l\). Therefore, a natural upper bound for the order of such a digraph is the so-called Moore bound \(M(d, k) = 1 + d + d^2 + \cdots + d^k\). A digraph that attains this upper bound is called a Moore digraph. It is easily seen that a digraph \(G\) is Moore if and only if it is out-regular with degree \(d\), has diameter \(k\) and is \(k\)-geodetic, i.e. for any two vertices \(u, v\) there is at most one \(k\)-path from \(u\) to \(v\).

As it was shown by Bridges and Toueg in [1] that Moore digraphs exist only in the trivial cases \(d = 1\) or \(k = 1\) (the Moore digraphs are directed \((k + 1)\)-cycles and complete digraphs \(K_{k+1}\) respectively), much research has been devoted to the study of digraphs that in some sense approximate Moore digraphs. For example, there is an extensive literature on digraphs with maximum out-degree \(d\), diameter \(\leq k\) and order \(M(d, k) - \delta\) for small \(\delta > 0\); this is equivalent to relaxing the \(k\)-geodetic requirement in the conditions for a digraph to be Moore. \(\delta\) is known as the defect of the digraph. The reader is referred to the survey [5] for more information.

In this paper, however, we will consider the following related problem, which is obtained by retaining the \(k\)-geodetic requirement in the above characterisation of Moore digraphs, but allowing the diameter to exceed \(k\): what is the smallest possible order of a \(k\)-geodetic digraph \(G\) with minimum out-degree \(\geq d\)? A \(k\)-geodetic digraph with minimum out-degree \(\geq d\) and order \(M(d, k) + \epsilon\) is said to be a \((d, k, +\epsilon)\)-digraph or to have excess \(\epsilon\). It was shown in [6] that there are no regular \((2, k, +1)\)-digraphs for \(k \geq 2\). In 2016 it was shown in [3] that digraphs with excess one must be diregular and that there are no \((d, k, +1)\)-digraphs for \(k = 2, 3, 4\) and sufficiently large \(d\). In a separate paper [7], the present author has shown that \((2, k, +2)\)-digraphs must be diregular with degree \(d = 2\) for \(k \geq 2\). In the present paper, we classify the \((2, 2, +2)\)-digraphs up to isomorphism and show that there are no regular \((2, k, +2)\)-digraphs for \(k \geq 3\), thereby completing the proof of the nonexistence of digraphs with degree \(d = 2\) and excess \(\epsilon = 2\) for \(k \geq 3\). Our reasoning and notation will follow closely that employed in [4] for the corresponding result for defect \(\delta = 2\).
2. Preliminary results

We will let $G$ stand for a $(2, k, +2)$-digraph for arbitrary $k \geq 2$, i.e. $G$ has minimum out-degree $d = 2$, is $k$-geodetic and has order $M(2, k) + 2$. We will denote the vertex set of $G$ by $V(G)$. By the result of [7], $G$ must be diregular with degree $d = 2$ for $k \geq 2$. The distance $d(u, v)$ between vertices $u$ and $v$ is the length of the shortest path from $u$ to $v$. Notice that $d(u, v)$ is not necessarily equal to $d(v, u)$. $u \rightarrow v$ will indicate that there is an arc from $u$ to $v$. We define the in- and out-neighbourhoods of a vertex $u$ by $N^-(u) = \{v \in V(G) : v \rightarrow u\}$ and $N^+(u) = \{v \in V(G) : u \rightarrow v\}$ respectively; more generally, for $0 \leq l \leq k$, the set $\{v \in V(G) : d(u, v) = l\}$ of vertices at distance exactly $l$ from $u$ will be denoted by $N^l(u)$. For $0 \leq l \leq k$ we will also write $T_l(u) = \bigcup_{l' = 0}^{l'} N^l(u)$ for the set of vertices at distance $\leq l$ from $u$. The notation $T_{k-1}(u)$ will be abbreviated by $T(u)$.

It is easily seen that for any vertex $u$ of $G$, there are exactly two distinct vertices that are at distance $\geq k + 1$ from $u$. For any $u \in V(G)$, we will write $O(u)$ for the set of these vertices and call such a set an outlier set and its elements outliers of $u$. Notice that $O(u) = V(G) − T_k(u)$. An elementary counting argument shows that in a diregular $(2, k, +2)$-digraph every vertex is also an outlier of exactly two vertices. We will say that a vertex $u$ can reach a vertex $v$ if $v \notin O(u)$.

Our proof will proceed by an analysis of a pair of vertices with exactly one common out-neighbour. First, we must show that such a pair exists and deduce some elementary properties of pairs of vertices with identical out-neighbourhoods.

**Lemma 1.** For $k \geq 2$, let $u$ and $v$ be distinct vertices such that $N^+(u) = N^+(v) = \{u_1, u_2\}$. Then $u_1 \in O(u_2)$, $u_2 \in O(u_1)$ and there exists a vertex $x$ such that $O(u) = \{v, x\}$, $O(v) = \{u, x\}$.

**Proof.** Suppose that $u$ can reach $v$ by a $\leq k$-path. Then $v \in T(u_1) \cup T(u_2)$. As $N^+(v) = N^+(u)$, it follows that there would be a $\leq k$-cycle through $v$, contradicting $k$-geodeticity. If $O(u) = \{v, x\}$, then $x \neq v$ and $x \notin T(u_1) \cup T(u_2)$, so that $v$ cannot reach $x$ by a $\leq k$-path. Similarly, if $u_1$ can reach $u_2$ by a $\leq k$-path, then we must have $\{u, v\} \cap T(u_1) \neq \emptyset$, which is impossible. □

**Lemma 2.** For $k \geq 2$, there exists a pair of vertices $u, v$ with $|N^+(u) \cap N^+(v)| = 1$.

**Proof.** Suppose for a contradiction that there is no such pair of vertices. Define a map $\phi : V(G) \rightarrow V(G)$ as follows. Let $u^+$ be an out-neighbour of a vertex $u$ and let $\phi(u)$ be the in-neighbour of $u^+$ distinct from $u$. By our assumption, it is easily verified that $\phi$ is a well-defined bijection with no fixed points and with square equal to the identity. It follows that $G$ must have even order, whereas $|V(G)| = M(2, k) + 2$ is odd. □

$u, v$ will now stand for a pair of vertices with a single common out-neighbour. We will label the vertices of $T_k(u)$ according to the scheme $N^+(u) = \{u_1, u_2\}$, $N^+(u_1) = \{u_3, u_4\}$, $N^+(u_2) = \{u_5, u_6\}$, $N^+(u_3) = \{u_7, u_8\}$, $N^+(u_4) = \{u_9, u_{10}\}$ and so on, with the same convention for the vertices of $T_k(v)$, where we will assume that $u_2 = v_2$.

3. Classification of $(2, 2, +2)$-digraphs

We begin by classifying the $(2, 2, +2)$-digraphs up to isomorphism. We will prove the following theorem.

**Theorem 1.** There are exactly two diregular $(2, 2, +2)$-digraphs, which are displayed in Figs. 2 and 5.

Let $G$ be an arbitrary diregular $(2, 2, +2)$-digraph. $G$ has order $M(2, 2) + 2 = 9$. By Lemma 2, $G$ contains a pair of vertices $(u, v)$ such that $|N^+(u) \cap N^+(v)| = 1$; we will assume that $u_2 = v_2$, so that we have the situation shown in Fig. 1.

We can immediately deduce some information on the possible positions of $v$ and $v_1$ in $T_2(u)$.

**Lemma 3.** If $v \notin O(u)$, then $v \in N^+(u_1)$. If $v_1 \notin O(u)$, then $v_1 \in N^+(u_1)$.

**Proof.** $v \notin T(u_2)$ by $2$-geodecty. $v \neq u$ by construction. If we had $v = u_1$, then there would be two distinct $\leq 2$-paths from $u$ to $u_2$. Also $v_1 \notin \{u\} \cup T(u_2)$ by $2$-geodecty and by assumption $u_1 \neq v_1$. □
Since $v$ and $v_1$ cannot both lie in $N^+(u_1)$ by 2-geodecity, we have the following corollary.

**Corollary 1.** $O(u) \cap \{v, v_1\} \neq \emptyset$.

We will call a pair of vertices $(u, v)$ with a single common out-neighbour *bad* if at least one of

$$O(u) \cap \{v_1, v_3\} = \emptyset, O(u) \cap \{v_1, v_4\} = \emptyset, O(u) \cap \{u_1, u_3\} = \emptyset, O(u) \cap \{u_1, u_4\} = \emptyset.$$  

holds. Otherwise such a pair will be called *good*.

**Lemma 4.** There is a unique $(2, 2, +2)$-digraph containing a bad pair.

**Proof.** Assume that there exists a bad pair $(u, v)$. Without loss of generality, $O(u) \cap \{v_1, v_3\} = \emptyset$. By Lemma 3 we can set $v_1 = u_3$. By 2-geodecity $v_3 = u$. We cannot have $v_4 = v_3 = u$, so $v_4$ must be an outlier of $u$. By Corollary 1 it follows that $O(u) = \{v, v_4\}$.

Consider the vertex $u_1$. By Lemma 3, if $u_1 \not\in O(v)$, then $u_1 \in N^+(v_1)$. However, as $v_1 = u_3$, there would be a 2-cycle through $u_1$. Hence $u_1 \in O(v)$. As $O(u) = \{v, v_4\}$, we have $V(G) = \{u, u_1, u_2, u_3 = v_1, u_4, u_5, u_6, v, v_4\}$ and $O(v) = \{u_1, u_4\}$. As neither $u$ nor $v$ lies in $T(u_1)$, we must also have $u_2 \in O(u_1)$. As $u_1$ can reach $u_1, v_1, u_4, u$ and $v_4$, it follows that without loss of generality we either have $O(u_1) = \{u_2, v\}$ and $N^+(u_4) = \{u_5, u_6\} = N^+(u_2)$ or $O(u_1) = \{u_2, u_6\}$ and $N^+(u_4) = \{v, u_5\}$. In either case, $(v, u_1)$ is a good pair.

Suppose firstly that $N^+(u_2) = N^+(u_4)$. Then $v$ is an outlier of $u$ and $u_1$. As each vertex is the outlier of exactly two vertices, $v_1$ must be able to reach $v$ by a $\leq 2$-path. Hence $v_4 \rightarrow v$. Likewise $u_2$ can reach $v$, so without loss of generality $u_5 \rightarrow v$. Suppose that $O(u_2) \cap \{u, u_1\} = \emptyset$. As $u$ and $v$ have a common out-neighbour, we must have $u_6 \rightarrow v$. Since $u \rightarrow u_1$, by 2-geodecity we must have $u_5 \rightarrow u_1$. However, this is a contradiction, as $v$ and $u_1$ also have a common out-neighbour.

Therefore, at least one of $u, u_1$ is an outlier of $u_2$. By Lemma 1 $u_4$ is an outlier of $u_2$. Therefore either $O(u_2) = \{u, u_4\}$ or $O(u_2) = \{u_1, u_4\}$. If $O(u_2) = \{u, u_4\}$, then $u_2$ must be able to reach $v$, $v_1$ and $v_4$. $u_5 \rightarrow v$ and $v \rightarrow v_1$, so $v_1 \in N^+(u_5)$. As $u_1 \rightarrow v_1$, we must have $N^+(u_5) = \{v_1, v_1\}$. As $v$ and $u_1$ have a common out-neighbour, this violates 2-geodecity. Hence $O(u_2) = \{u_1, u_4\}$ and $u_2$ can reach $u$, $v_1$ and $v_4$. As $v \rightarrow v_1, v_1 \in N^+(u_5)$. As $v_4 \rightarrow v$, it follows that $N^+(u_5) = \{v, v_4\}$. However, $v_4 \rightarrow v$, so this again violates 2-geodecity.

We are left with the case $O(u_1) = \{u_2, u_6\}$ and $N^+(u_4) = \{v, u_5\}$. Then $v_1 \in O(u_2)$, as neither $v$ nor $u_1$ lies in $T(u_2)$. Observe that $u_2$ and $u_4$ have a single common out-neighbour, so by Corollary 1 $O(u_2) \cap \{u_4\} \neq \emptyset$. Therefore either $O(u_2) = \{v_1, u_4\}$ or $O(u_2) = \{v_1, v\}$. Suppose firstly that $O(u_2) = \{v_1, u_4\}$. Then $N^+(u_2) = \{u, v, u_1, v_4\}$. As $N^+(u_4) = \{v, u_5\}$, $u_5 \not\rightarrow v$, so $u_6 \rightarrow v$. As $N^+(u) \cap N^+(v) \neq \emptyset$, $u_5 \rightarrow u \rightarrow u_1$, so necessarily $u_6 \in N^+(u_1) \cap N^+(v)$, contradicting 2-geodecity.

Hence $O(u_2) = \{v_1, v\}$ and $N^+(u_2) = \{u, u_1, u_4, v_4\}$. As $u_4 \rightarrow u_5, u_5 \not\rightarrow u_4$. Thus $u_6 \rightarrow u_4$. Now $u_1 \rightarrow u_4$ and $u \rightarrow u_1$ implies that $N^+(u_5) = \{u_1, v_4\}$ and $N^+(u_6) = \{u, u_4\}$. Finally we must have $N^+(u_4) = \{v, u_6\}$. This gives us the $(2, 2, +2)$-digraph shown in Fig. 2. 

![Fig. 2. The unique $(2, 2, +2)$-digraph containing a bad pair.](image-url)
Considering the positions of $v_3$ and $v_4$, we see that there are without loss of generality four possibilities: (1) $u = v_3$, $u_4 = v_4$, (2) $u = v_3$, $O(u) = \{v_1, v_4\}$, (3) $N^+(u_4) = N^+(v_1)$ and (4) $u_3 = v_3$, $O(u) = \{v_1, u_4\}$. A suitable relabelling of vertices shows that case 4 is equivalent to case 1.a below, so we will examine cases 1 to 3 in turn.

**Case 1:** $u = v_3$, $u_4 = v_4$

Depending upon the position of $v$, we must either have $O(u) = \{v_1, v\}$ and $O(v) = \{u_1, u_3\}$ or $u = v_3$ (see Fig. 3).

**Case 1.a:** $O(u) = \{v_1, v\}$, $O(v) = \{u_1, u_3\}$

In this case $V(G) = \{u, u_1, u_2, u_3, u_4, u_5, u_6, v, v_1\}$. $u_1$ and $v_1$ have a single common out-neighbour, namely $u_4$, so, as we are assuming all such pairs to be good, we have $u_3 = O(v_1)$, $u \in O(u_1)$. By 2-geodecacy, $N^+(u_4) \subseteq \{u_5, u_6, v\}$, so without loss of generality either $N^+(u_4) = \{u_5, u_6\}$ or $N^+(u_4) = \{u_5, v\}$.

Suppose that $N^+(u_4) = \{u_5, u_6\}$. By elimination, $O(u_1) = \{v, u_3\}$. As $G$ is diregular, every vertex is an outlier of exactly two vertices; $v$ is an outlier of $u$ and $v_1$, so both $u_1$ and $u_2$ can reach $v$ by a $\leq 2$-path. Hence $v \in N^+(u_3)$. As $v \rightarrow v_1$, we see that $v_1$ is an outlier of $u_1$; as $u$ is also an outlier of $u_1$, we have $O(u_1) = \{u, v_1\}$ and $N^+(u_3) = \{v, u_2\}$. As $v \rightarrow u_2$, this is impossible.

Now consider $N^+(u_4) = \{u_5, v\}$. We now have $O(v_1) = \{u_3, u_6\}$. Thus $u_3 \in O(v) \cap O(v_1)$, so $u_3 \in T_2(u_4)$. $v$ is not adjacent to $u_3$, so $u_3 \in N^+(u_5)$. $u_2$ and $u_4$ have $u_5$ as a unique common out-neighbour, so $u_6 \in O(u_4)$, $v \in O(u_2)$. As $u_6 \in O(v_1) \cap O(u_4)$, $u_1$ can reach $u_6$. Hence $u_6 \in N^+(u_1)$. Neither $u$ nor $v$ lie in $T(u_1)$, so $u_2 \in O(u_1)$. Therefore either $O(u_1) = \{u, u_2\}$ or $O(u_1) = \{u_2, v_1\}$. If $O(u_1) = \{u, u_2\}$, then $N^+(u_2) = \{u_6, v_1\}$. $u_2$ cannot reach $v_1$, since $v, u_3 \notin T(u_2)$, so $O(u_2) = \{v, v_1\}$ and $N^+(u_2) = \{u, u_1, u_3, u_4\}$. As $u_4 \rightarrow u_5, u_4 \in N^+(u_6), u_1 \rightarrow u_4$, so $N^+(u_1) = \{u_1, u_3\}$. As $u_1 \rightarrow u_3$, this is a contradiction. Thus $O(u_1) = \{u_2, v_1\}$, so that $N^+(u_3) = \{u, u_6\}$. $u_1$ must have an in-neighbour apart from $u$, which must be either $u_5$ or $u_6$. As $u_1 \rightarrow u_3$, we have $u_1 \in N^+(u_6)$. By elimination, $v$ and $v_1$ must also have in-neighbours in $\{u_5, u_6\}$. As $u_1$ and $v_1$ have a common out-neighbour, we have $N^+(u_5) = \{u_2, v_1\}, N^+(u_6) = \{u_1, v\}$. However, both $u_3$ and $v_1$ are adjacent to $u$, violating 2-geodecacy.

**Case 1.b:** $v = u_3$

There exists a vertex $x$ such that $V(G) = \{u, u_1, u_2, v, u_4, u_5, u_6, v_1, x\}$, $O(u) = \{v_1, x\}$ and $O(v) = \{u_1, x\}$. As $x \in O(u) \cap O(v)$, $u_1$ and $u_2$ can reach $x$, so without loss of generality $x \in N^+(u_4) \cap N^+(u_5)$. As $u_3$ and $u_4$ have a common out-neighbour, $u_5 \in O(u_1)$. Also, $u_1$ and $v_1$ have $u_4$ as a unique common out-neighbour, so $u \in O(u_1)$ and $O(u_1) = \{u_1, u_6\}$. Thus $N^+(u_4) = \{x, u_6\}$. Observe that $u_2$ and $u_4$ have the out-neighbour $u_6$ in common. Thus $x \in O(u_2)$, whereas we already have $x \in O(u) \cap O(v)$, a contradiction.

**Case 2:** $u = v_3$, $O(u) = \{v_1, v_4\}$

As $v$ is not equal to $v_1$ or $v_4$, $v$ must lie in $T_2(u)$. Without loss of generality, $v = u_3$. Hence $V(G) = \{u, u_1, u_2, v, u_4, u_5, u_6, v_1, v_4\}$ and $O(v) = \{u_1, u_4\}$. We have the configuration shown in Fig. 4. Hence $u_1$ can reach
Taking into account adjacencies between members of $O$, it is difficult to show that this yields a $(2, 2, +2)$-digraph.

**Case 2.a:** $O(u_4) = \{u, v_4\}$, $N^+(u_4) = \{u_5, u_6\}$

As $v_4 \in O(u) \cap O(u_1)$, $u_2$ can reach $v_4$ and without loss of generality $u_4 \in N^+(u_5)$. $N^+(u_2) = N^+(u_4)$, so by Lemma 1 $u_2 \in O(u_4)$, $u_4 \in O(u_2)$, $u_5 \in O(u_6)$ and $u_6 \in O(u_5)$. Hence $u_4 \in O(v) \cap O(u_2)$, so $v_1$ can reach $u_4$, so $u_4 \in N^+(v_1)$. Neither $u_5$ nor $u_6$ lies in $N^+(v_1)$, so $O(v_1) = \{u_5, u_6\}$ and $N^+(v_1) = \{u_4, v_1\}$. Hence $O(v_4) = \{u, u_1\}$. Observe that $N^+(u_1) = N^+(v_4)$, so that $v_4 \in O(u_4)$. Therefore $v \notin N^+(u_5) \cup N^+(u_6)$, yielding $O(u_2) = \{u_4, v_4\}$. $N^+(u_2) = \{u_5, v_1, u, u_1\}$. As $v_1 \to v_4$, and $N^+(u_1) = N^+(v_4)$, we must have $N^+(u_5) = \{u_4, u\}$, $N^+(v_6) = \{v_1, u_1\}$. This yields the digraph in Fig. 5. Unlike the digraph in Fig. 2, this digraph contains pairs of vertices with identical out-neighbourhoods, so the two are not isomorphic.

**Case 2.b:** $O(u_4) = \{u, v_4\}$, $N^+(u_4) = \{u_6, v_4\}$

As $u_4 \to v_4$, $u_4 \notin N^+(v_4)$, so $u_4 \in O(v_1)$. Hence $u_4 \in O(v) \cap O(v_1), u_2$ can reach $u_4$. As $u_4 \to u_6$, we must have $u_5 \to u_4, u_2$ and $u_4$ have $u_6$ as a common out-neighbour, so $u_4 \in O(u_2)$, $u_5 \in O(u_4)$. Therefore $u_4 \in O(u) \cap O(u_2)$, so that $u_6$ can reach $u_4$, but $u_4 \notin T(u_6)$, so $N^+(u_6)$ contains an in-neighbour of $v_4, u_4 \notin N^+(u_6)$, so we must have $u_6 \to v_1$. We have $u_5 \in O(u_4) \cap O(u_1)$, so $v_1$ can reach $u_4$, and hence $v_4 \to u_5, v_1$ cannot reach $u_6$, as $u_2, u_4 \notin T(v_1)$, so $O(v_1) = \{u_5, u_6\}$, $N^+(v_5) = \{v_5, v\}$. Now $u_2$ and $u_4$ have $u_5$ as a unique common out-neighbour, so $u_6 \in O(v_4), v \in O(u_2)$. Thus $O(u_2) = \{v_4, u\}$ and $N^+(u_2) = \{u_4, v_1, u, u_1\}$. Taking into account adjacencies between members of $N^+(u_2)$, it follows that $N^+(u_5) = \{u_4, u\}, N^+(u_6) = \{v_1, u_1\}$. However, ($u_2, u_4$) now constitutes a bad pair, contradicting our assumption.

**Case 2.c:** $O(u_4) = \{u, v_4\}$, $N^+(u_4) = \{u_5, v_4\}$

As $u_4 \to v_4$, $u_4 \notin N^+(v_4)$. Hence $u_4 \in O(v) \cap O(v_1)$, implying that $u_2$ can reach $u_4$. Without loss of generality, $u_5 \to u_4$. There are three possibilities: (i) $O(v_1) = \{u_4, u_6\}, N^+(v_4) = \{v, u_5\}$, (ii) $O(v_1) = \{u_4, u_5\}, N^+(v_4) = \{v, u_6\}$ and (iii) $O(v_1) = \{u_4, v\}, N^+(v_4) = \{u_5, u_6\}$.

(i) $O(v_1) = \{u_4, u_6\}, N^+(v_4) = \{v, u_5\}$

$u_1$ and $v_4$ have $v$ as a unique common out-neighbour, so $u_4 \in O(v_4)$. However, this contradicts $v_4 \to u_5 \to u_4$.

(ii) $O(v_1) = \{u_4, u_6\}, N^+(v_4) = \{v, u_6\}$

Neither $u_4$ nor $v_1$ lie in $T(u_2)$, so $v_4 \in O(u_2)$. Now observe that $u_2$ and $u_4$ have $u_6$ as unique common out-neighbour, so $v \in O(u_2)$, yielding $O(u_2) = \{v_4, u_4\}$ and $N^+(u_2) = \{u_4, u_1, u, u_1\}$. As $u_4 \to u$ and $u \to u_1$, we must have $N^+(u_5) = \{u_4, u_1\}, N^+(u_6) = \{u, v_1\}$, a contradiction, since $u_1 \to u_4$.

(iii) $O(v_1) = \{u_4, v\}, N^+(v_4) = \{u_5, u_6\}$

We now have $N^+(u_5) = N^+(v_4)$, so $u_2 \in O(v_4), v_4 \in O(u_2), u_5 \in O(u_6), u_6 \in O(u_5)$. Also $N^+(u_4) = N^+(v_4)$, so $u_4 \in O(v_1), v_1 \in O(u_4)$ and $u \in O(u_4)$. If $u \in O(u_4)$, $u_4 \in O(u_1)$ implies that $u \notin N^+(u_5) \cup N^+(u_6)$, so we see that $u \in O(u_2)$ and hence $O(u_2) = \{u_4, v\}$ and $N^+(u_2) = \{u_4, u_1, v, v_1\}$. As $u_1 \to u_4$ and $u_4 \to v$, we have $N^+(u_5) = \{u_4, v\}, N^+(u_6) = \{u_1, v_1\}$. It is not difficult to show that this yields a $(2, 2, +2)$-digraph isomorphic to that in Fig. 5.

**Case 2.d:** $O(u_4) = \{v, u_4\}$, $N^+(u_4) = \{u_5, u_6\}$

In this case $v_4 \in O(u) \cap O(u_1)$, so $u_2$ can reach $v_4, u_4$ and $v_1$ have unique common out-neighbour $u$, so $v_4 \in O(u_4), u_6 \in O(v_1)$. If $u_5 \to v_4$, then we would have $u_4 \to u_6 \to v_4$, contradicting $v_4 \in O(u_4)$, so $u_5 \to v_4$. This also implies that $u_5 \notin N^+(v_4), so u_5 \notin O(v_1)$, yielding $O(v_1) = \{u_5, u_6\}$ and $N^+(u_4) = \{v, u_4\} = N^+(u_1)$. Now $v_4, u_1 \notin T(u_2)$, so $O(u_2) = \{v, u_4\}$.

Fig. 5. A second (2, 2, +2)-digraph.
and \( N^2(u_2) = \{v_1, v_4, u, u_1\} \). As \( v_1 \to v_4 \) and \( v_1 \to u \), it follows that \( N^+(u_5) = \{v_4, u\}, N^+(u_6) = \{u_1, v_1\} \). However, we now have paths \( u_4 \to u \to u_1 \) and \( u_4 \to u_6 \to u_1 \), which is impossible.

**Case 3:** \( N^+(u_1) = N^+(v_1) \)

It is easy to see by 2-geodesity that \( V(G) = \{u, u_1, u_2, u_3, u_4, u_5, v, v_1\} \), \( O(u) = \{v, v_1\} \) and \( O(v) = \{u, u_1\} \). As \( u, v_1 \notin T(u_2) \), we have \( O(u_2) = \{u_1, u_4\} \) and \( N^+(u_2) = \{u, u_1, v, v_1\} \). Without loss of generality, \( N^+(u_3) = \{u, v_1\}, N^+(u_6) = \{v, u_1\} \). \( u \) and \( v \) have in-neighbours apart from \( u_5 \) and \( u_6 \) respectively, so without loss of generality \( u_3 \to u, u_4 \to v \). Likewise, \( u_5 \) and \( u_6 \) have in-neighbours other than \( u_2 \), so, as \( u_3 \to u \) and \( u_6 \to v \), we must have \( N^+(u_3) = \{u, u_6\}, N^+(u_4) = \{v, u_5\} \). But now we have paths \( u_3 \to u \to u_1 \) and \( u_3 \to u_6 \to u_1 \), violating 2-geodesity.

**Corollary 2.** There is a unique \((2, 2, +2)\)-digraph containing no bad pairs.

This completes our analysis of diregular \((2, 2, +2)\)-digraphs. As it was shown in [7] that there are no non-diregular \((2, 2, +2)\)-digraphs, \((2, 2, +2)\)-digraphs are now classified up to isomorphism. These conclusions have been verified computationally by Erskine [2]. It is interesting to note that neither of the \((2, 2, +2)\)-digraphs are vertex-transitive, for in each case there are exactly three vertices contained in two 3-cycles. However, there does exist a Cayley \((2, 2, +5)\)-digraph (on the alternating group \( A_4 \)), so it would be interesting to determine the smallest vertex-transitive \((2, 2, +e)\)-digraphs.

### 4. Main result

We can now complete our analysis by showing that there are no diregular \((2, k, +2)\)-digraphs for \( k \geq 3 \). Let \( G \) be such a digraph. By **Lemma 2**, \( G \) contains vertices \( u \) and \( v \) with a unique common out-neighbour. In accordance with our vertex-labelling convention, we have the situation in **Fig. 6**. A triangle based at a vertex \( x \) represents the set \( T(x) \).

We now proceed to determine the possible outlier sets of \( u \) and \( v \).

**Lemma 5.** \( v \in N^{k-1}(u_1) \cup O(u) \) and \( u \in N^{k-1}(v_1) \cup O(v) \). If \( v \notin O(u) \), then \( u_2 \in O(u_1) \) and if \( u \notin O(v) \), then \( u_2 \in O(v_1) \).

**Proof.** \( v \) cannot lie in \( T(u) \), or the vertex \( u_2 \) would be repeated in \( T_0(u) \). Also, \( v \notin T(u_2) \), or there would be a \( \leq k \)-cycle through \( v \). Therefore, if \( v \notin O(u) \), then \( v \in N^{k-1}(u_1) \). Likewise for the other result. If \( v \notin O(u) \), then neither in-neighbour of \( u_2 \) lies in \( T(u_1) \), so that \( u_2 \in O(u_1) \). \( \Box \)

**Lemma 6.** Let \( w \in T(v_1) \), with \( d(v_1, w) = l \). Suppose that \( w \in T(u_1) \), with \( d(u_1, w) = m \). Then either \( m \leq l \) or \( w \in N^{k-1}(u_1) \). A similar result holds for \( w \in T(u_1) \).

**Proof.** Let \( w \) be as described and suppose that \( m > l \). Consider the set \( N^{k-m}(w) \). By construction, \( N^{k-m}(w) \subseteq N^k(u_1) \), so by \( k \)-geodesity \( N^{k-m}(w) \cap T(u_1) = \emptyset \). At the same time, we have \( l + k - m \leq k - 1 \), so \( N^{k-m}(w) \subseteq T(v_1) \). This implies that \( N^{k-m}(w) \cap T(v_2) = N^{k-m}(w) \cap T(u_2) = \emptyset \). As \( V(G) = \{u\} \cup T(u_1) \cup T(u_2) \cup O(u) \), it follows that \( N^{k-m}(w) \subseteq \{u\} \cup O(u) \). Therefore \( |N^{k-m}(w)| \leq 2^{k-m} \leq 3 \). By assumption \( 0 \leq m \leq k - 1 \), so it follows that \( m = k - 1 \). \( \Box \)

**Corollary 3.** If \( w \in T(v_1) \), then either \( w \in \{u\} \cup O(u) \) or \( w \in T(u_1) \) with \( d(u_1, w) = k - 1 \) or \( d(u_1, w) \leq d(v_1, w) \).

**Proof.** By \( k \)-geodesity and **Lemma 6**. \( \Box \)

**Corollary 4.** \( v_1 \in N^{k-1}(u_1) \cup O(u) \) and \( u_1 \in N^{k-1}(v_1) \cup O(v) \).
Proof. We prove the first inclusion. By Corollary 3, \( v_1 \in \{u\} \cup O(u) \cup \{u_1\} \cup N^{k-1}(u_1) \). By \( k \)-geodecity, \( v_1 \neq u \) and by construction, \( v_1 \neq u_1 \).

We now have enough information to identify one member of \( O(u) \) and \( O(v) \).

**Lemma 7.** \( v_1 \in O(u) \) and \( u_1 \in O(v) \).

Proof. We prove that \( v_1 \in O(u) \). Suppose that neither \( v_1 \) nor \( v \) lies in \( O(u) \). Then by Lemma 5 and Corollary 4 we have \( v, v_1 \in N^{k-2}(u_1) \). As \( v_1 \) is an out-neighbour of \( v \), it follows that \( v_1 \) appears twice in \( T_k(u_1) \), violating \( k \)-geodecity. Therefore \( O(u) \cap \{v, v_1\} = \emptyset \).

Now assume that \( v_1, v_2 \in T_k(u_1) \). Again by Corollary 4, \( v_1, v_2 \in N^{k-2}(u_1) \). By \( k \)-geodecity we also have \( v_3 \in T(u_1) \). However, \( v_3 \in N^+(v_1) \), so \( v_3 \) appears twice in \( T_k(u_1) \), which is impossible. Hence \( O(u) \cap \{v_1, v_2\} = \emptyset \). Similarly, \( O(u) \cap \{v_1, v_4\} = \emptyset \). In the terminology of the previous section, \( G \) contains no bad pairs. Therefore, if \( v_1 \notin O(u) \), then \( \{v_2, v_3, v_4\} \subseteq O(u) \). Since these vertices are distinct, this is a contradiction and the result follows. \( \square \)

**Corollary 5.** There is at most one vertex in \( T_k-3(v_1) - \{v_1\} \) that does not lie in \( T(u_1) \); for all other vertices \( w \in T_k-3(v_1) - \{v_1\} \), \( d(u_1, w) = d(v, w) \). A similar result for \( T_k-3(u_1) - \{u_1\} \) also holds.

**Lemma 8.** \( T_k-3(v_1) \cap N^k-1(u_1) = T_k-3(u_1) \cap N^k-1(v_1) = \emptyset \).

Proof. Let \( w \in T_k-3(v_1) \cap N^k-1(u_1) \). Consider the position of the vertices of \( N^+(w) \) in \( T_k(u_1) \cup O(u) \). As \( v \notin N^+(w) \), it follows from Lemma 7 that at most one of the vertices of \( N^+(w) \) can be an outlier of \( w \), so let us write \( w_1 \in N^+(w) - O(u) \). By \( k \)-geodecity, \( w_1 \notin T(u_1) \cup O(u) \). Hence \( w_1 \in T(v_2) = T(v_2) \). However, \( w_1 \) also lies in \( T(u_1) \), so this violates \( k \)-geodecity. \( \square \)

**Corollary 5.** There is at most one vertex in \( T_k-3(v_1) - \{v_1\} \) that does not lie in \( T(u_1) \); for all other vertices \( w \in T_k-3(v_1) - \{v_1\} \), \( d(u_1, w) = d(v, w) \). A similar result for \( T_k-3(u_1) - \{u_1\} \) also holds.

**Lemma 9.** For \( k = 3 \), \( N^+(u_1) \cap N^2(v_1) = N^+(v_1) \cap N^2(u_1) = \emptyset \).

Proof. Suppose that \( v_3 = u_2 \). By the reasoning of Lemma 8 we can set \( u_2 = v = u_2 \) and \( O(u) = \{v_1, v_2\}, v \notin O(u) \) and by \( 3 \)-geodecity \( v \notin N^3(v_2) \), so we can assume that \( v = u_9, u_3 \rightarrow v_3 \) implies that \( u_3 \notin T(v) \), so \( O(v) = \{u_1, u_1\} \). We must have \( \{u_4, u_5, u_10\} = \{v_4, v_5, v_10\} \). As \( u_4 \rightarrow v \), it follows that \( v_4 = u_8 \) and hence \( \{u_4, u_10\} = \{v_2, v_10\} \), which is impossible. \( \square \)

As \( u_1 \) is an outlier of \( v \), neither \( v_3 \) nor \( v_4 \) can be equal to \( u_1 \). It follows from Corollary 5 and Lemma 9 that either \( N^+(u_1) = N^+(v_1) \) or \( u_1 \) and \( v_1 \) have a single common out-neighbour, with one vertex of \( N^+(v_1) \) being an outlier of \( u \).

**Lemma 10.** \( N^2(u) \neq N^2(v) \)

Proof. Let \( N^2(u) = N^2(v) \), with \( N^+(u_1) = N^+(v_1) \). Suppose that \( v \notin O(u) \). By Lemma 5, \( v \in N^{k-2}(u_2) \cup N^{k-2}(u_4) \). But then there is a \( k \)-cycle through \( v \). It follows that \( O(u) = \{v_1, v_1\} \). \( O(v) = \{u_1, u_1\} \). By Lemma 5, \( u_2 \in O(u_1) \cap O(v_1) \). Therefore by Lemma 1 \( O(u_1) = \{u_2, v_1\} \), \( O(v_1) = \{v_2, u_1\} \).

Consider the in-neighbour \( u' \) of \( u_1 \) that is distinct from \( u \). We have either \( |N^+(u') \cap N^+(u)| = 1 \) or \( |N^+(u') \cap N^+(u)| = 2 \). In the first case, it follows from Lemma 7 that \( u_2 \in O(u) \). Every vertex of \( G \) is an outlier of exactly two vertices, so \( u' = u_1 \) or \( u_1 \). In either case, we have a contradiction. Therefore \( N^+(u') = N^+(u) \). It now follows from Lemma 1 that \( u' \notin O(u) = \{v_1, v_1\} \), which is impossible. \( \square \)

Noticing that \( u_1 \) and \( v_1 \) also have a unique common out-neighbour, we have the following corollary.

**Corollary 6.** Without loss of generality, \( u_1 = v_1 = v_3, u_4 = v_9 \), \( O(u) = \{v_1, v_4\} \), \( O(v) = \{u_1, u_4\} \), \( O(u_1) = \{v_4, v_10\} \) and \( O(v_1) = \{u_4, u_10\} \).

We are now in a position to complete the proof by deriving a contradiction.

**Theorem 2.** There are no digereular \((2, k, +2)\)-digraphs for \( k \geq 3 \).

**Proof.** \( u, v \notin \{u_1, u_2, v_1, v_4\} \), so by Lemma 5 \( d(u, v) = d(v, u) = k \). In fact, \( u_2 = v_2 \) implies that \( v \in N^{k-2}(u_2) \) and \( u \in N^{k-2}(v_2) \). Let \( k \geq 4 \). Then \( u, v \notin \{u_10, v_10\} \), so \( u, v \in T_k(u_1) \cap T_k(v_1) \) if \( u \in T_k(u_2) = T(v_2) \), then \( u \) would appear twice in \( T_k(u_1) \), so \( u \in N^{k-1}(u_4) \). However, as \( u \) and \( v \) have a common out-neighbour, this violates \( k \)-geodecity.

Finally, suppose that \( k = 3 \). The above analysis will hold unless \( u = v_10 \) and \( v = u_10 \). Let \( N^-(u_1) = \{u, u'\}, N^-(v_1) = \{v, v'\} \). It is evident that \( u' \notin \{v_1, v_4\} \), so that \( v' \notin T_3(u) \). As \( v \in N^+(u_4) \), we must have \( v' \notin N^2(u_2) \). Similarly \( u' \notin N^2(u_2) \). Since \( u_1 \) and \( v_1 \) have a common out-neighbour, we can assume that \( u' \notin N^+(u_5) \) and \( v' \notin N^+(u_6) \). \( u_4 \) and \( v_4 \) can be the outlier of only two vertices, namely \( u_1 \) and \( v_1 \), so \( u_4 \in N^3(u_2) \) and likewise \( u_4 \in N^3(u_2) \). By \( 3 \)-geodecity \( v_4 \in N^2(u_2) \) and \( u_4 \in N^2(u_6) \). It follows that \( u, v \notin N^2(u_2) \), so \( u \notin T_3(u_1) \cup T_3(v_2) \). Hence \( O(u) = N^-(u) \neq \{v_1, v_4\} \), which again is impossible. \( \square \)

It is interesting to note that a similar argument can be used to provide an alternative proof of the result of [6].
References