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Note

On diregular digraphs with degree two and excess two

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1. Introduction

An important topic in the design of efficient networks is the construction of \((d, k, +\epsilon)\)-digraphs, i.e. \(k\)-geodetic digraphs with minimum out-degree \(\geq d\) and order \(M(d, k) + \epsilon\), where \(M(d, k)\) represents the Moore bound for degree \(d\) and diameter \(k\) and \(\epsilon > 0\) is the (small) excess of the digraph. Previous work has shown that there are no \((2, k, +1)\)-digraphs for \(k \geq 2\). In a separate paper, the present author has shown that any \((2, k, +2)\)-digraph must be diregular for \(k \geq 2\). In the present work, this analysis is completed by proving the nonexistence of diregular \((2, k, +2)\)-digraphs for \(k \geq 3\) and classifying diregular \((2, 2, +2)\)-digraphs up to isomorphism.

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2. Preliminary results

We will let $G$ stand for a $(2, k, +2)$-digraph for arbitrary $k \geq 2$, i.e. $G$ has minimum out-degree $d = 2$, is $k$-geodetic and has order $M(2, k) + 2$. We will denote the vertex set of $G$ by $V(G)$. By the result of [7], $G$ must be diregular with degree $d = 2$ for $k \geq 2$. The distance $d(u, v)$ between vertices $u$ and $v$ is the length of the shortest path from $u$ to $v$. Notice that $d(u, v)$ is not necessarily equal to $d(v, u)$. $u \rightarrow v$ will indicate that there is an arc from $u$ to $v$. We define the in- and out-neighbourhoods of a vertex $u$ by $N^-(u) = \{v \in V(G) : v \rightarrow u\}$ and $N^+(u) = \{v \in V(G) : u \rightarrow v\}$ respectively; more generally, for $0 \leq l \leq k$, the set $\{v \in V(G) : d(u, v) = l\}$ of vertices at distance exactly $l$ from $u$ will be denoted by $N^l(u)$. For $0 \leq l \leq k$ we will also write $T_l(u) = \bigcup_{v \in N^l(u)} N^l(u)$ for the set of vertices at distance $\leq l$ from $u$. The notation $T_{k-1}(u)$ will be abbreviated by $T(u)$.

It is easily seen that for any vertex $u$ of $G$, there are exactly two distinct vertices that are at distance $\geq k + 1$ from $u$. For any $u \in V(G)$, we will write $O(u)$ for the set of these vertices and call such a set an outlier set and its elements outliers of $u$. Notice that $O(u) = V(G) - T_k(u)$. An elementary counting argument shows that in a diregular $(2, k, +2)$-digraph every vertex is also an outlier of exactly two vertices. We will say that a vertex $u$ can reach a vertex $v$ if $v \notin O(u)$.

Our proof will proceed by an analysis of a pair of vertices with exactly one common out-neighbour. First, we must show that such a pair exists and deduce some elementary properties of pairs of vertices with identical out-neighbourhoods.

**Lemma 1.** For $k \geq 2$, let $u$ and $v$ be distinct vertices such that $N^+(u) = N^+(v) = \{u_1, u_2\}$. Then $u_1 \in O(u_2)$, $u_2 \in O(u_1)$ and there exists a vertex $x$ such that $O(u) = \{v, x\}$, $O(v) = \{u, x\}$.

**Proof.** Suppose that $u$ can reach $v$ by a $\leq k$-path. Then $v \in T(u_1) \cup T(u_2)$. As $N^+(v) = N^+(u)$, it follows that there would be a $\leq k$-cycle through $v$, contradicting $k$-geodeticity. If $O(u) = \{v, x\}$, then $x \neq v$ and $x \notin T(u_1) \cup T(u_2)$, so that $v$ cannot reach $x$ by a $\leq k$-path. Similarly, if $u_1$ can reach $u_2$ by a $\leq k$-path, then we must have $\{u, v\} \cap T(u_1) \neq \emptyset$, which is impossible. □

**Lemma 2.** For $k \geq 2$, there exists a pair of vertices $u, v$ with $|N^+(u) \cap N^+(v)| = 1$.

**Proof.** Suppose for a contradiction that there is no such pair of vertices. Define a map $\phi : V(G) \rightarrow V(G)$ as follows. Let $u^+\in$ an out-neighbour of a vertex $u$ and let $\phi(u)$ be the in-neighbour of $u^+$ distinct from $u$. By our assumption, it is easily verified that $\phi$ is a well-defined bijection with no fixed points and with square equal to the identity. It follows that $G$ must have even order, whereas $|V(G)| = M(2, k) + 2$ is odd. □

$u, v$ will now stand for a pair of vertices with a single common out-neighbour. We will label the vertices of $T_k(u)$ according to the scheme $N^+(u) = \{u_1, u_2\}$, $N^+(u_1) = \{u_3, u_4\}$, $N^+(u_2) = \{u_5, u_6\}$, $N^+(u_3) = \{u_7, u_8\}$, $N^+(u_4) = \{u_9, u_{10}\}$ and so on, with the same convention for the vertices of $T_k(v)$, where we will assume that $u_2 = v_2$.

3. Classification of $(2, 2, +2)$-digraphs

We begin by classifying the $(2, 2, +2)$-digraphs up to isomorphism. We will prove the following theorem.

**Theorem 1.** There are exactly two diregular $(2, 2, +2)$-digraphs, which are displayed in Figs. 2 and 5.

Let $G$ be an arbitrary diregular $(2, 2, +2)$-digraph. $G$ has order $M(2, 2) + 2 = 9$. By Lemma 2, $G$ contains a pair of vertices $(u, v)$ such that $|N^+(u) \cap N^+(v)| = 1$; we will assume that $u_2 = v_2$, so that we have the situation shown in Fig. 1.

We can immediately deduce some information on the possible positions of $v$ and $v_1$ in $T_2(u)$.

**Lemma 3.** If $v \notin O(u)$, then $v \in N^+(u_1)$. If $v_1 \notin O(u)$, then $v_1 \in N^+(u_1)$.

**Proof.** $v \notin T(u_2)$ by $2$-geodecty. $v \neq u$ by construction. If we had $v = u_1$, then there would be two distinct $\leq 2$-paths from $u$ to $u_2$. Also $v_1 \notin \{u\} \cup T(u_2)$ by $2$-geodecty and by assumption $u_1 \neq v_1$. □
Since \( v \) and \( v_1 \) cannot both lie in \( N^+(u_1) \) by 2-geodecity, we have the following corollary.

**Corollary 1.** \( O(u) \cap \{ v, v_1 \} \neq \emptyset \).

We will call a pair of vertices \((u, v)\) with a single common out-neighbour **bad** if at least one of \( O(u) \cap \{ v_1, v_3 \} = \emptyset, O(u) \cap \{ v_1, v_4 \} = \emptyset, O(v) \cap \{ u_1, u_3 \} = \emptyset, O(v) \cap \{ u_1, u_4 \} = \emptyset \). holds. Otherwise such a pair will be called **good**.

**Lemma 4.** There is a unique \((2, 2, +2)\)-digraph containing a bad pair.

**Proof.** Assume that there exists a bad pair \((u, v)\). Without loss of generality, \( O(u) \cap \{ v_1, v_3 \} = \emptyset \). By Lemma 3 we can set \( v_1 = u_3 \). By 2-geodecity \( v_3 = u \). We cannot have \( v_4 = v_3 = u \), so \( u_4 \) must be an outlier of \( u \). By Corollary 1 it follows that \( O(u) = \{ v, v_4 \} \).

Consider the vertex \( u_1 \). By Lemma 3, if \( u_1 \notin O(v) \), then \( u_1 \in N^+(v_1) \). However, as \( v_1 = u_3 \), there would be a 2-cycle through \( u_1 \), hence \( u_1 \notin O(v) \). As \( O(u) = \{ v, v_4 \} \), we have \( V(G) = \{ u, u_1, u_2, u_3 = v_1, u_4, u_5, u_6, v, v_4 \} \) and \( O(v) = \{ u_1, u_4 \} \). As neither \( u \) nor \( v \) lies in \( T(u_1) \), we must also have \( u_2 \in O(u_1) \). As \( u_1 \) can reach \( u_1, v_1, v_4, u_4 \) and \( v_4 \), it follows that without loss of generality we either have \( O(u_1) = \{ u_2, v \} \) and \( N^+(u_4) = \{ u_5, u_6 \} = N^+(u_2) \) or \( O(u_1) = \{ u_2, u_6 \} \) and \( N^+(u_4) = \{ v, u_5 \} \). In either case, \((v, u_1)\) is a good pair.

Suppose firstly that \( N^+(u_2) = N^+(u_4) \). Then \( v \) is an outlier of \( u \) and \( u_1 \). As each vertex is the outlier of exactly two vertices, \( v_1 \) must be able to reach \( v \) by a \( \leq 2 \)-path. Hence \( v_4 \rightarrow v \). Likewise \( u_2 \) can reach \( v \), so without loss of generality \( u_5 \rightarrow u_1 \). However, this is a contradiction, as \( v \) and \( u_1 \) also have a common out-neighbour. Therefore, at least one of \( u, u_1 \) is an outlier of \( u_2 \). By Lemma 1 \( u_4 \) is an outlier of \( u_2 \). Therefore either \( O(u_2) = \{ u, u_4 \} \) or \( O(u_2) = \{ u_1, u_4 \} \). If \( O(u_2) = \{ u, u_4 \} \), then \( v_2 \) must be able to reach \( u_1, v_1 \) and \( v_4, u_5 \rightarrow v \) and \( v \rightarrow v_1 \), so \( v_1 \in N^+(u_6) \). As \( u_1 \rightarrow v_1 \), we must have \( N^+(u_5) = \{ v, u_1 \} \). As \( v \) and \( u_1 \) have a common out-neighbour, this violates 2-geodecity. Hence \( O(u_2) = \{ u_1, u_4 \} \) and \( u_2 \) can reach \( u, v_1 \) and \( v_4 \). As \( v \rightarrow v_1, v_1 \in N^+(u_6) \). As \( v_1 \rightarrow v_4 \), it follows that \( N^+(u_5) = \{ v, v_4 \} \). However, \( v_4 \rightarrow v \), so this again violates 2-geodecity.

We are left with the case \( O(u_1) = \{ u_2, u_6 \} \) and \( N^+(u_4) = \{ v, u_5 \} \). Then \( v_1 \in O(u_2) \), as neither \( v \) nor \( u_1 \) lies in \( T(u_2) \). Observe that \( u_2 \) and \( u_4 \) have a single common out-neighbour, so by Corollary 1 \( O(u_2) \cap \{ u_4, v \} \neq \emptyset \). Therefore either \( O(u_2) = \{ v_1, u_4 \} \) or \( O(u_2) = \{ v_1, v \} \). Suppose firstly that \( O(u_2) = \{ v_1, u_4 \} \). Then \( N^+(u_2) = \{ u, v, u_1, v_4 \} \). As \( N^+(u_4) = \{ v_5, u_5 \} \) \( u_5 \nrightarrow v \), so \( u_6 \rightarrow v \). As \( N^+(u) \cap N^+(v) \neq \emptyset, u_5 \rightarrow u_1 \rightarrow u_1 \), so necessarily \( N^+(u_6) = \{ u_1 \} \). However, \( v_1 \in N^+(u_1) \cap N^+(v) \), contradicting 2-geodecity.

Hence \( O(u_2) = \{ v_1, v \} \) and \( N^+(u_2) = \{ u, u_1, u_4, v_4 \} \). As \( u_4 \rightarrow u_5, u_5 \nrightarrow u_4 \). Thus \( u_6 \rightarrow u_4 \). Now \( u_1 \rightarrow u_4 \) and \( u \rightarrow u_1 \) implies that \( N^+(u_5) = \{ u_1, v_4 \} \) and \( N^+(u_6) = \{ u, u_4 \} \). Finally we must have \( N^+(v_4) = \{ v, u_6 \} \). This gives us the \((2, 2, +2)\)-digraph shown in Fig. 2.

![Fig. 2. The unique \((2, 2, +2)\)-digraph containing a bad pair.](image-url)
Considering the positions of \( v_3 \) and \( v_4 \), we see that there are without loss of generality four possibilities: (1) \( u = v_3, \ u_4 = v_4 \), (2) \( u = v_3, \ u_4 \in V(G) \), (3) \( u \in V(G) \), (4) \( u \in V(G) \). A suitable relabelling of vertices shows that case 4 is equivalent to case 1a below, so we will examine cases 1 to 3 in turn.

**Case 1:** \( u = v_3, \ u_4 = v_4 \)

Depending upon the position of \( v \), we must either have \( O(u) = \{ v_1, v \} \) and \( O(v) = \{ u_1, u_3 \} \), or \( v = u_3 \). (see Fig. 3).

**Case 1a:** \( O(u) = \{ v_1, v \} \), \( O(v) = \{ u_1, u_3 \} \)

In this case \( V(G) = \{ u, u_1, u_2, u_3, u_4, u_5, u_6, v, v_1 \} \). If \( u_1 \) and \( v_1 \) have a single common out-neighbour, namely \( u_4 \), so, as we are assuming all such pairs to be good, we have \( u_3 \in O(v_1), \ u \in O(u_1) \). By 2-geodecity, \( N^+(u_4) \), \( N^+(u_3) \) and \( N^+(u_5) \), \( N^+(u_6) \), \( v \), and \( v_1 \), without loss of generality either \( N^+(u_4) = \{ u_5, u_6 \} \) or \( N^+(u_4) = \{ u, v \} \).

Suppose that \( N^+(u_4) = \{ u_5, u_6 \} \). By elimination, \( O(u_1) = \{ v, u_3 \} \). As \( G \) is diregular, every vertex is an outlier of exactly two vertices; \( v \) in the set of \( u_1 \), and \( u_3 \), so both \( u_1 \) and \( u_3 \) can reach \( v \) by a \( 2 \)-path. Hence \( v \in N^+(u_3) \). As \( v \rightarrow v_1 \), we see that \( v_1 \) is an outlier of \( u_3 \); as \( u \) is also an outlier of \( u_1 \), we have \( O(u_1) = \{ u, v_1 \} \). As \( v \rightarrow u_2 \), this is impossible. Now consider \( N^+(u_4) = \{ u_5, v \} \). We now have \( O(u_1) = \{ u_3, u_5 \} \). Thus \( u_3 \in O(v) \cap O(v_1) \). This \( u_3 \in T_2(u_4) \). As \( u_4 \in O(v_1) \), \( u_2 \) and \( u_3 \) have a unique common out-neighbour, namely \( u_5 \). Without loss of generality either \( u \in O(u_1) \) or \( u \in O(u_2) \). As \( u_5 \in O(u_1) \), \( u_2 \) cannot reach \( v \), since \( v, u_3 \notin T_2(u_2) \). Hence \( u_5 \in O(u_1) \) and \( N^2(u_2) = \{ u_1, u_2, u_3, u_4 \} \). As \( u_4 \rightarrow u_2, u_4 \in O(u_1) \), \( u_1 \rightarrow u_2 \), so \( N^+(u_4) = \{ u_1, u_2 \} \). As \( u_1 \rightarrow u_2 \), this is impossible. Thus \( O(u_1) = \{ u_2, v_1 \} \), so that \( N^+(u_3) = \{ u, u_6 \} \). \( u_1 \) must have an in-neighbour apart from \( u \), which must be either \( u_5 \) or \( u_6 \). As \( u_1 \rightarrow u_3 \), we have \( u_1 \in N^+(u_6) \). By elimination, \( v \) and \( v_1 \) must also have in-neighbours in \( \{ u_5, u_6 \} \). As \( u_1 \) and \( v_1 \) have a common out-neighbour, we have \( N^+(u_5) = \{ u_2, v_1 \} \), \( N^+(u_6) = \{ u_1, v \} \). However, both \( u_3 \) and \( v_1 \) are adjacent to \( u \), violating 2-geodecity.

**Case 1b:** \( v = u_3 \)

There exists a vertex \( x \) such that \( V(G) = \{ u, u_1, u_2, v, u_4, u_5, v_1, x \} \). As \( x \in O(u) \cap O(v) \), \( u_1 \) and \( u_2 \) can reach \( x \), so without loss of generality \( x \in N^+(u_4) \cap N^+(u_5) \). As \( u_5 \) and \( u_4 \) have a common out-neighbour, \( u_5 \in O(u_1) \). Also, \( u_1 \) and \( v_1 \) have \( u_4 \) as a unique common out-neighbour, so \( u \in O(u_1) \). As \( u \in O(u_1) \) and \( O(u_1) = \{ u_4, u_5 \} \). Hence \( N^+(u_4) = \{ x, u_5 \} \). Observe that \( u_2 \) and \( u_4 \) have the out-neighbour \( u_5 \) in common. Thus \( x \in O(u_2) \), whereas we already have \( x \in O(u) \cap O(v) \), a contradiction.

**Case 2:** \( u = v_3, \ O(u) = \{ v_1, v_4 \} \)

As \( v \) is not equal to \( v_1 \) or \( v_4 \), \( v \) must lie in \( T_2(u) \). Without loss of generality, \( v = u_3 \). Hence \( V(G) = \{ u, u_1, u_2, v, u_4, u_5, v_1, v_4 \} \) and \( O(v) = \{ u_1, v_4 \} \). We have the configuration shown in Fig. 4. Hence \( u_1 \) can reach

![Fig. 3. Case 1 configuration.](image_url)

![Fig. 4. Case 2 configuration.](image_url)
Taking into account adjacencies between members of $N^v_1$, we have without loss of generality one of the following: a) $O(u_1) = \{u, v_4\}$, $N^+(u_4) = \{u_5, u_6\}$, b) $O(u_1) = \{u, u_1\}$, $N^+(u_4) = \{u_6, v_4\}$, c) $O(u_1) = \{u_5, u_6\}$, $N^+(u_4) = \{u, v_4\}$ or d) $O(u_1) = \{u, v_4\}$, $N^+(u_4) = \{u, u_6\}$.

**Case 2.a: $O(u_4) = \{u, v_4\}$, $N^+(u_4) = \{u_6, u_0\}$**

As $v_4 \in O(u) \cap O(u_1)$, $u_2$ can reach $v_4$ and without loss of generality $v_4 \in N^+(u)$, $N^+(u_2) = N^+(u_4)$, so by Lemma 1 $u_2 \in O(u_2)$, $u_4 \in O(u_2)$, $u_5 \in O(u_5)$ and $u_6 \in O(u_5)$. Hence $u_4 \in O(v) \cap O(u_2)$, so $v_1$ can reach $u_4$, so $u_4 \in N^+(v_4)$. Neither $u_5$ nor $u_6$ lies in $N^+(u)$, so $O(v_1) = \{u_5, u_6\}$ and $N^+(u_4) = \{u_4, v\}$. Hence $O(v_4) = \{u, u_1\}$. Observe that $N^+(u_1) = N^+(u_4)$, so that $v \in O(u_4)$. Therefore $v \notin N^+(u_5) \cup N^+(u_6)$, yielding $O(u_2) = \{u, v, u_4\}$, $N^+(u_2) = \{u_4, v_1, u, u_1\}$. As $v_1 \to v_4$ and $N^+(u_1) = N^+(u_4)$, we must have $N^+(u_5) = \{v_4, u\}$, $N^+(u_6) = \{v_1, u_1\}$. This yields the digraph in Fig. 5. Unlike the digraph in Fig. 2, this digraph contains pairs of vertices with identical out-neighbourhoods, so the two are not isomorphic.

**Case 2.b: $O(u_1) = \{u, u_3\}$, $N^+(u_4) = \{u_6, v_4\}$**

As $u_4 \to v_4$, $u_4 \notin N^+(v_4)$, so $u_4 \in O(v_1)$. Hence $u_4 \in O(v) \cap O(v_1)$, so $u_2$ can reach $u_4$. As $u_4 \to u_6$, we must have $u_5 \to u_4$, $u_2$ and $u_4$ have $u_6$ as a common out-neighbour, so $v_4 \in O(u_2)$, $u_5 \in O(u_4)$. Therefore $v_4 \in O(u) \cap O(u_2)$, so that $u_6$ can reach $v_4$, but $v_4 \notin T(u_6)$, so $N^+(u_6)$ contains an in-neighbour of $v_4$, $u_4 \notin N^+(u_6)$, so we must have $u_6 \to v_1$. We have $u_5 \in O(u_4) \cap O(u_1)$, so $v_1$ can reach $u_5$ and hence $v_4 \to u_5, u_1$ cannot reach $u_6$ as $u_2, u_4 \notin T(v_1)$, so $O(v_1) = \{u_4, u_6\}$, $N^+(v_5) = \{u_5, v\}$. Now $u_2$ and $v_4$ have $u_5$ as a unique common out-neighbour, so $u_6 \in O(v_4), v \in O(u_2)$. Thus $O(u_2) = \{v_4, u\}$ and $N^+(u_2) = \{u_4, v_1, u, u_1\}$. Taking into account adjacencies between members of $N^2(u_2)$, it follows that $N^+(u_6) = \{u_4, v\}, N^+(u_6) = \{u_1, v_1\}$. However, $(u_2, u_4)$ now constitutes a bad pair, contradicting our assumption.

**Case 2.c: $O(u_1) = \{u, u_3\}$, $N^+(u_4) = \{u_6, v_4\}$**

As $u_4 \to v_4$, $u_4 \notin N^+(v_4)$, so $u_4 \in O(v)$, implying that $u_2$ can reach $u_4$. Without loss of generality, $u_5 \to u_4$.

There are three possibilities: (i) $O(v_1) = \{u_4, v_6\}, N^+(v_4) = \{v, u_5\}$, (ii) $O(v_1) = \{u_4, u_5\}, N^+(v_4) = \{v, u_6\}$ and (iii) $O(v_1) = \{u_4, v\}, N^+(v_4) = \{u_5, u_6\}$.

(i) $O(v_1) = \{u_4, v_6\}, N^+(v_4) = \{v, u_5\}$

$u_1$ and $v_4$ have $v$ as a unique common out-neighbour, so $u_4 \in O(v_4)$. However, this contradicts $v_4 \to u_5 \to u_4$.

(ii) $O(v_1) = \{u_4, u_5\}, N^+(v_4) = \{v, u_6\}$

Neither $u_4$ nor $v_1$ lies in $T(u_2)$, so $v_4 \in O(u_2)$. Now observe that $u_2$ and $u_4$ have $u_6$ as unique common out-neighbour, so $v \in O(u_2)$, yielding $O(u_2) = \{v, u_4\}$ and $N^2(u_2) = \{u_4, u_1, v, u_1\}$. As $u_4 \to u$ and $u \to u_1$, we must have $N^+(u_5) = \{u_4, u_1\}, N^+(u_6) = \{u_1, v_1\}$, a contradiction, since $u_1 \to u_4$.

(iii) $O(v_1) = \{u_4, v\}, N^+(v_4) = \{u_5, u_6\}$

We now have $N^+(u_2) = N^+(v_4)$, so $u_2 \in O(v_4), v_4 \in O(u_2), u_5 \in O(u_6), u_6 \in O(u_5)$. Also $N^+(u_4) = N^+(v_1)$, so $u_4 \in O(v_1), v_1 \in O(u_4)$ and $u \in O(u_2)$. $u \in O(u_2)$ implies that $u \notin N^+(u_5) \cup N^+(u_6)$, so we see that $u \in O(u_2)$ and hence $O(u_2) = \{u, v_4\}$ and $N^2(u_2) = \{u_4, u_1, v, u_1\}$. As $u_1 \to u_4$ and $u_4 \to v$, we have $N^+(u_5) = \{u_4, v\}, N^+(u_6) = \{u_1, v_1\}$. It is not difficult to show that this yields a $(2, 2, +2)$-digraph isomorphic to that in Fig. 5.

**Case 2.d: $O(u_1) = \{u_5, v_4\}$, $N^+(u_4) = \{u, u_6\}$**

In this case $v_4 \in O(u) \cap O(u_1)$, so $u_2$ can reach $v_4$, $u_4$ and $v_1$ have unique common out-neighbour $u$, so $v_4 \in O(u_4), u_6 \in O(v_1)$. If $u_6 \to v_4$, then we would have $u_4 \to u_5 \to u_4$, contradicting $v_4 \in O(u_4)$, so $u_5 \to v_4$. This also implies that $u_5 \notin N^+(u_4)$, so $u_5 \in O(v_1)$, yielding $O(v_1) = \{u_5, u_6\}$ and $N^+(v_4) = \{v, u_4\} = N^+(u_1)$. Now $v_4, u_1 \notin T(u_2)$, so $O(u_2) = \{v, u_4\}$.
and $N^2(u_2) = \{v_1, v_4, u, u_1\}$. As $v_1 \rightarrow v_4$ and $v_1 \rightarrow u$, it follows that $N^+(u_5) = \{v_4, u\}$, $N^+(u_6) = \{u_1, v_1\}$. However, we now have paths $u_4 \rightarrow u \rightarrow u_1$ and $u_4 \rightarrow u_6 \rightarrow u_1$, which is impossible.

**Case 3:** $N^+(u_1) = N^+(v_1)$

It is easy to see by 2-geodecity that $V(G) = \{u, u_1, u_2, u_3, u_4, u_5, v, v_1\}$, $O(u) = \{v, v_1\}$ and $O(v) = \{u, u_1\}$. As $u_1, v_1 \notin T(u_2)$, we have $O(u_2) = \{u_3, u_4\}$ and $N^+(u_2) = \{u_1, v_1\}$. Without loss of generality, $N^+(u_5) = \{u_1, v\}$, $N^+(u_6) = \{v, u_1\}$. $u$ and $v$ have in-neighbours apart from $u_5$ and $u_6$ respectively, so without loss of generality $u_3 \rightarrow u$, $u_4 \rightarrow v$. Likewise, $u_5$ and $u_6$ have in-neighbours other than $u_2$, so, as $u_2 \rightarrow u$ and $u_6 \rightarrow v$, we must have $N^+(u_3) = \{u, u_6\}$, $N^+(u_4) = \{v, u_5\}$. But now we have paths $u_3 \rightarrow u \rightarrow u_1$ and $u_3 \rightarrow u_6 \rightarrow u_1$, violating 2-geodecity.

**Corollary 2.** There is a unique $(2, 2, +2)$-digraph containing no bad pairs.

This completes our analysis of diregular $(2, 2, +2)$-digraphs. As it was shown in [7] that there are no non-diregular $(2, 2, +2)$-digraphs, $(2, 2, +2)$-digraphs are now classified up to isomorphism. These conclusions have been verified computationally by Erskine [2]. It is interesting to note that neither of the $(2, 2, +2)$-digraphs are vertex-transitive, for in each case there are exactly three vertices contained in two 3-cycles. However, there does exist a Cayley $(2, 2, +5)$-digraph (on the alternating group $A_4$), so it would be interesting to determine the smallest vertex-transitive $(2, 2, +\epsilon)$-digraphs.

### 4. Main result

We can now complete our analysis by showing that there are no diregular $(2, k, +\epsilon)$-digraphs for $k \geq 3$. Let $G$ be such a digraph. By Lemma 2, $G$ contains vertices $u$ and $v$ with a unique common out-neighbour. In accordance with our vertex-labelling convention, we have the situation in Fig. 6. A triangle based at a vertex $x$ represents the set $T(x)$.

We now proceed to determine the possible outlier sets of $u$ and $v$.

**Lemma 5.** $v \in N^{k-1}(u_1) \cup O(u)$ and $u \in N^{k-1}(v_1) \cup O(v)$. If $v \in O(u)$, then $u_2 \in O(u_1)$ and if $u \in O(v)$, then $u_2 \in O(v_1)$.  

**Proof.** $v$ cannot lie in $T(u)$, or the vertex $u_2$ would be repeated in $T(u)$. Also, $v \notin T(u_2)$, or there would be a $\leq k$-cycle through $v$. Therefore, if $v \notin O(u)$, then $v \in N^{k-1}(u_1)$. Likewise for the other result. If $v \in O(u)$, then neither in-neighbour of $u_2$ lies in $T(u_1)$, so that $u_2 \in O(u_1)$. □

**Lemma 6.** Let $w \in T(v_1)$, with $d(v_1, w) = l$. Suppose that $w \in T(u_1)$, with $d(u_1, w) = m$. Then either $m \leq l$ or $w \in N^{k-1}(u_1)$. A similar result holds for $w \in T(u_1)$.

**Proof.** Let $w$ be as described and suppose that $m > l$. Consider the set $N^{k-m}(w)$. By construction, $N^{k-m}(w) \subseteq N^k(u_1)$, so by $k$-geodecity $N^{k-m}(w) \cap T(u_1) = \emptyset$. At the same time, we have $l + k - m \leq k - 1$, so $N^{k-m}(w) \subseteq T(v_1)$. This implies that $N^{k-m}(w) \cap T(u_2) = N^{k-m}(w) \cap T(u_2) = \emptyset$. As $V(G) = \{u\} \cup T(u_1) \cup T(u_2) \cup O(u)$, it follows that $N^{k-m}(w) \subseteq \{u\} \cup O(u)$. Therefore $|N^{k-m}(w)| = 2^{k-m} \leq 3$. By assumption $0 \leq m \leq k - 1$, so it follows that $m = k - 1$. □

**Corollary 3.** If $w \in T(v_1)$, then either $w \in \{u\} \cup O(u)$ or $w \in T(u_1)$ with $d(u_1, w) = k - 1$ or $d(u_1, w) \leq d(v_1, w)$.

**Proof.** By $k$-geodecity and Lemma 6. □

**Corollary 4.** $v_1 \in N^{k-1}(u_1) \cup O(u)$ and $u_1 \in N^{k-1}(v_1) \cup O(v)$.  

Fig. 6. Configuration for $k \geq 3$. 

172

Proof. We prove the first inclusion. By Corollary 3, \( v_1 \in \{u\} \cup O(u) \cup \{u_1\} \cup N_k^{-1}(u_1) \). By \( k \)-geodecity, \( v_1 \neq u \) and by construction, \( v_1 \neq u_1 \). □

We now have enough information to identify one member of \( O(u) \) and \( O(v) \).

**Lemma 7.** \( v_1 \in O(u) \) and \( u_1 \in O(v) \).

**Proof.** We prove that \( v_1 \in O(u) \). Suppose that neither \( v_1 \) nor \( v \) lies in \( O(u) \). Then by Lemma 5 and Corollary 4 we have \( v, v_1 \in N_k^{-1}(u_1) \). As \( v_1 \) is an out-neighbour of \( v \), it follows that \( v_1 \) appears twice in \( T_k(u) \), violating \( k \)-geodecity. Therefore \( O(u) \cap \{v, v_1\} = \emptyset \).

Now assume that \( v_1, v_3 \in T_k(u) \). Again by Corollary 4, \( v_1 \in N_k^{-1}(u_1) \). By \( k \)-geodecity we also have \( v_3 \in T(u_1) \). However, \( v_1, v_3 \in N_k^{-1}(v_1) \), so \( v_1 \) appears twice in \( T_k(u_1) \), which is impossible. Hence \( O(u) \cap \{v_1, v_3\} = \emptyset \). Similarly, \( O(u) \cap \{v_1, v_4\} = \emptyset \). In the terminology of the previous section, \( G \) contains no bad pairs. Therefore, if \( v_1 \not\in O(u) \), then \( \{v, v_3, v_4\} \subseteq O(u) \). Since these vertices are distinct, this is a contradiction and the result follows. □

Lemma 7 allows us to conclude that for vertices sufficiently close to \( v_1 \) one of the potential situations mentioned in Corollary 3 cannot occur.

**Lemma 8.** \( T_{k-3}(v_1) \cap N_k^{-1}(u_1) = T_{k-3}(u_1) \cap N_k^{-1}(v_1) = \emptyset \).

**Proof.** Let \( w \in T_{k-3}(v_1) \cap N_k^{-1}(u_1) \). Consider the position of the vertices of \( N_k^+(w) \) in \( T_k(u) \cup O(u) \). As \( v_1 \not\in N_k^+(w) \), it follows from Lemma 7 that at most one of the vertices of \( N_k^+(w) \) can be an outlier of \( u \), so let us write \( w_1 \in N_k^+(w) \setminus O(u) \). By \( k \)-geodecity, \( u_1 \not\in T(u_1) \cup O(u) \). Hence \( w_1 \in T_k(u) \cap O(u) \). However, \( u_1 \) also lies in \( T_k(v_1) \), so this violates \( k \)-geodecity. □

**Corollary 5.** There is at most one vertex in \( T_{k-3}(v_1) \setminus \{v_1\} \) that does not lie in \( T(u_1) \); for all other vertices \( w \in T_{k-3}(v_1) \setminus \{v_1\} \), \( d(u_1, w) = d(v_1, w) \). A similar result for \( T_{k-3}(u_1) \setminus \{u_1\} \) also holds.

**Lemma 9.** \( k = 3, N^+(u_1) \cap N^2(v_1) = N^+(v_1) \cap N^2(u_1) = \emptyset \).

**Proof.** Suppose that \( v_3 = u_2 \). By the reasoning of Lemma 8 we can set \( u = v_2 \) and \( O(u) = \{v_1, v_8\} \). \( v \not\in O(u) \) and by \( 3 \)-geodecity \( v \not\in N_3^+(u_3) \), so we can assume that \( u = v_9 \). \( u_3 \rightarrow v_3 \) implies that \( u_1 \not\in T(v_1) \), so \( O(v) = \{u_1, v_1\} \). We must have \( \{v_4, v_5, v_10\} = \{v_4, v_3, v_10\} \). As \( u_4 \rightarrow v \), it follows that \( v_4 = u_8 \) and hence \( \{u_4, v_10\} = \{v_2, v_10\} \), which is impossible. □

As \( u_1 \) is an outlier of \( v, v_3 \) nor \( v_4 \) can be equal to \( u_1 \). It follows from Corollary 5 and Lemma 9 that either \( N^+(u_1) = N^+(v_1) \) or \( u_1 \) and \( v_1 \) have a single common out-neighbour, with one vertex of \( N^+(v_1) \) being an outlier of \( u \).

**Lemma 10.** \( N^2(u) \neq N^2(v) \)

**Proof.** Let \( N_2^2(u) = N_2^2(v), \) with \( N_2^2(u_1) = N_2^2(v_1) = \{u_3, u_4\} \). Suppose that \( v \not\in O(u) \). By Lemma 5, \( v \in N_k^{-2}(u_3) \cup N_k^{-2}(u_4) \). But then there is a \( k \)-cycle through \( v \). It follows that \( O(u) = \{v, v_1\} \), \( O(v) = \{u_1, u_1\} \). By Lemma 5, \( u_2 \in O(u_1) \cap O(v_1) \). Therefore by Lemma 1 \( O(u_1) = \{u_2, v_1\} \), \( O(v_1) = \{u_2, u_1\} \).

Consider the in-neighbour \( u' \) of \( u_1 \) that is distinct from \( u \). We have either \( |N^+(u') \cap N^+(u)| = 1 \) or \( |N^+(u') \cap N^+(u)| = 2 \). In the first case, it follows from Lemma 7 that \( u_2 \in O(u) \). Every vertex of \( G \) is an outlier of exactly two vertices, so \( u' \neq u_1 \) or \( v_1 \). In either case, we have a contradiction. Therefore \( N^+(u') = N^+(u) \). It now follows from Lemma 1 that \( u' \in O(u) = \{v, v_1\} \), which is impossible. □

Noticing that \( u_1 \) and \( v_1 \) also have a unique common out-neighbour, we have the following corollary.

**Corollary 6.** Without loss of generality, \( u_1 = v_3, u_9 = v_9, O(u) = \{v_1, v_4\}, O(v) = \{u_1, u_4\}, O(u_1) = \{v_4, v_10\} \) and \( O(v_1) = \{u_4, u_10\} \).

We are now in a position to complete the proof by deriving a contradiction.

**Theorem 2.** There are no digereular \((2, k, +2)\)-digraphs for \( k \geq 3 \).

**Proof.** Suppose \( u, v \not\in \{u_1, u_3, v_1, v_4\} \), so by Lemma 5 \( d(u, v) = d(v, u) = k \). In fact, \( u_2 = v_2 \) implies that \( v \in N_k^{-2}(u_4) \) and \( u \in N_k^{-2}(v_4) \). Let \( k \geq 4 \). Then \( u, v \not\in \{u_10, v_10\} \), so \( u, v \in T_k(u_1) \cup T_k(v_1) \). If \( u \in T_k(u_2) = T(v_2) \), then \( u \) would appear twice in \( T_k(v_1) \), so \( u \in N_k^{-2}(u_4) \). However, as \( u \) and \( v \) have a common out-neighbour, this violates \( k \)-geodecity.

Finally, suppose that \( k = 3 \). The above analysis will hold unless \( u = v_10 \) and \( v = u_10 \). Let \( N_k^{-1}(u_1) = \{u, u'\}, N_k^{-1}(v_1) = \{v, v'\} \). It is evident that \( u' \neq \{v_1, v_4\} \), so that \( u' \in T(v_1) \). As \( v \in N_k^{-1}(u_4) \), we must have \( v' \in N_k^2(u_2) \). Similarly \( u' \in N_k^2(u_2) \). Since \( u_1 \) and \( v_1 \) have a common out-neighbour, we can assume that \( u' \in N_k^2(u_5) \) and \( v' \in N_k^2(v_6) \). As \( u_1 \) and \( v_1 \) are the outlier of only two vertices, namely \( u \) and \( v_1 \), so \( u_4 \in N_k^2(u_2) \) and likewise \( u_4 \in N_k^2(v_2) \). By \( 3 \)-geodecity \( v_4 \in N_k^2(u_2) \) and \( u_4 \in N_k^2(v_2) \). It follows that \( u, v \not\in N_k^2(u_2) \), so \( u \not\in T(v_5) \cup T(v_3) \). Hence \( O(u) = N_k^{-1}(u) = \{v_1, v_4\} \), which again is impossible. □

It is interesting to note that a similar argument can be used to provide an alternative proof of the result of [6].
References