On diregular digraphs with degree two and excess two

Journal Item

How to cite:

© 2017 Elsevier

Version: Version of Record

Link(s) to article on publisher’s website:
http://dx.doi.org/doi:10.1016/j.dam.2017.10.034

Copyright and Moral Rights for the articles on this site are retained by the individual authors and/or other copyright owners. For more information on Open Research Online’s data policy on reuse of materials please consult the policies page.
On diregular digraphs with degree two and excess two

James Tuite

Department of Mathematics and Statistics, Open University, Walton Hall, Milton Keynes, United Kingdom

Abstract

An important topic in the design of efficient networks is the construction of \((d, k, +\epsilon)\)-digraphs, i.e. \(k\)-geodetic digraphs with minimum out-degree \(\geq d\) and order \(M(d, k) + \epsilon\), where \(M(d, k)\) represents the Moore bound for degree \(d\) and diameter \(k\) and \(\epsilon > 0\) is the (small) excess of the digraph. Previous work has shown that there are no \((2, k, +1)\)-digraphs for \(k \geq 2\). In a separate paper, the present author has shown that any \((2, k, +2)\)-digraph must be diregular for \(k \geq 2\). In the present work, this analysis is completed by proving the nonexistence of diregular \((2, k, +2)\)-digraphs for \(k \geq 3\) and classifying diregular \((2, 2, +2)\)-digraphs up to isomorphism.

© 2017 Elsevier B.V. All rights reserved.

1. Introduction

An important topic in the design of interconnection networks is the directed degree/diameter problem: what is the largest possible order \(N(d, k)\) of a digraph \(G\) with maximum out-degree \(d\) and diameter \(\leq k\)? A simple inductive argument shows that for \(0 \leq l \leq k\) the number of vertices at distance \(l\) from a fixed vertex \(v\) is bounded above by \(d^l\). Therefore, a natural upper bound for the order of such a digraph is the so-called Moore bound \(M(d, k) = 1 + d + d^2 + \cdots + d^k\). A digraph that attains this upper bound is called a Moore digraph. It is easily seen that a digraph \(G\) is Moore if and only if it is out-regular with degree \(d\), has diameter \(k\) and is \(k\)-geodetic, i.e. for any two vertices \(u, v\) there is at most one \(k\)-path from \(u\) to \(v\).

As it was shown by Bridges and Toueg in [1] that Moore digraphs exist only in the trivial cases \(d = 1\) or \(k = 1\) (the Moore digraphs are directed \((k+1)\)-cycles and complete digraphs \(K_{k+1}\) respectively), much research has been devoted to the study of digraphs that in some sense approximate Moore digraphs. For example, there is an extensive literature on digraphs with maximum out-degree \(d\), diameter \(\leq k\) and order \(M(d, k) - \delta\) for small \(\delta > 0\); this is equivalent to relaxing the \(k\)-geodeticity requirement in the conditions for a digraph to be Moore. \(\delta\) is known as the defect of the digraph. The reader is referred to the survey [5] for more information.

In this paper, however, we will consider the following related problem, which is obtained by retaining the \(k\)-geodeticity requirement in the above characterisation of Moore digraphs, but allowing the diameter to exceed \(k\): what is the smallest possible order of a \(k\)-geodetic digraph \(G\) with minimum out-degree \(\geq d\)? A \(k\)-geodetic digraph with minimum out-degree \(\geq d\) and order \(M(d, k) + \epsilon\) is said to be a \((d, k, +\epsilon)\)-digraph or to have excess \(\epsilon\). It was shown in [6] that there are no diregular \((2, k, +1)\)-digraphs for \(k \geq 2\). In 2016 it was shown in [3] that digraphs with excess one must be diregular and that there are no \((d, k, +1)\)-digraphs for \(k = 2, 3, 4\) and sufficiently large \(d\). In a separate paper [7], the present author has shown that \((2, k, +2)\)-digraphs must be diregular with degree \(d = 2\) for \(k \geq 2\). In the present paper, we classify the \((2, 2, +2)\)-digraphs up to isomorphism and show that there are no diregular \((2, k, +2)\)-digraphs for \(k \geq 3\), thereby completing the proof of the nonexistence of digraphs with degree \(d = 2\) and excess \(\epsilon = 2\) for \(k \geq 3\). Our reasoning and notation will follow closely that employed in [4] for the corresponding result for defect \(\delta = 2\).
2. Preliminary results

We will let $G$ stand for a $(2, k, +2)$-digraph for arbitrary $k \geq 2$, i.e. $G$ has minimum out-degree $d = 2$, is $k$-geodetic and has order $M(2, k) + 2$. We will denote the vertex set of $G$ by $V(G)$. By the result of [7], $G$ must be diregular with degree $d = 2$ for $k \geq 2$. The distance $d(u, v)$ between vertices $u$ and $v$ is the length of the shortest path from $u$ to $v$. Notice that $d(u, v)$ is not necessarily equal to $d(v, u)$. $u \rightarrow v$ will indicate that there is an arc from $u$ to $v$. We define the in- and out-neighbourhoods of a vertex $u$ by $N^-(u) = \{v \in V(G) : v \rightarrow u\}$ and $N^+(u) = \{v \in V(G) : u \rightarrow v\}$ respectively; more generally, for $0 \leq l \leq k$, the set $\{v \in V(G) : d(u, v) = l\}$ of vertices at distance exactly $l$ from $u$ will be denoted by $N^l(u)$. For $0 \leq l \leq k$ we will also write $T_l(u) = \bigcup_{0 \leq l \leq k}N^l(u)$ for the set of vertices at distance $\leq l$ from $u$. The notation $T_{k-1}(u)$ will be abbreviated by $T(u)$.

It is easily seen that for any vertex $u$ of $G$, there are exactly two distinct vertices that are at distance $\geq k + 1$ from $u$. For any $u \in V(G)$, we will write $O(u)$ for the set of these vertices and call such a set an outlier set and its elements outliers of $u$. Notice that $O(u) = V(G) - T_k(u)$. An elementary counting argument shows that in a diregular $(2, k, +2)$-digraph every vertex is also an outlier of exactly two vertices. We will say that a vertex $u$ can reach a vertex $v$ if $v \notin O(u)$.

Our proof will proceed by an analysis of a pair of vertices with exactly one common out-neighbour. First, we must show that such a pair exists and deduce some elementary properties of pairs of vertices with identical out-neighbourhoods.

**Lemma 1.** For $k \geq 2$, let $u$ and $v$ be distinct vertices such that $N^+(u) = N^+(v) = \{u_1, u_2\}$. Then $u_1 \in O(u_2)$, $u_2 \in O(u_1)$ and there exists a vertex $x$ such that $O(u) = \{v, x\}$, $O(v) = \{u, x\}$.

**Proof.** Suppose that $u$ can reach $v$ by a $\leq k$-path. Then $v \in T(u_1) \cup T(u_2)$. As $N^+(v) = N^+(u)$, it follows that there would be a $\leq k$-cycle through $v$, contradicting $k$-geodeticity. If $O(u) = \{v, x\}$, then $x \neq v$ and $x \notin T(u_1) \cup T(u_2)$, so that $v$ cannot reach $x$ by a $\leq k$-path. Similarly, if $u_1$ can reach $u_2$ by a $\leq k$-path, then we must have $\{u, v\} \cap T(u_1) \neq \emptyset$, which is impossible. \qed

**Lemma 2.** For $k \geq 2$, there exists a pair of vertices $u$, $v$ with $|N^+(u) \cap N^+(v)| = 1$.

**Proof.** Suppose for a contradiction that there is no such pair of vertices. Define a map $\phi : V(G) \rightarrow V(G)$ as follows. Let $u^+$ be an out-neighbour of a vertex $u$ and let $\phi(u)$ be the in-neighbour of $u^+$ distinct from $u$. By our assumption, it is easily verified that $\phi$ is a well-defined bijection with no fixed points and with square equal to the identity. It follows that $G$ must have even order, whereas $|V(G)| = M(2, k) + 2$ is odd. \qed

$u$, $v$ will now stand for a pair of vertices with a single common out-neighbour. We will label the vertices of $T_{k}(u)$ according to the scheme $N^+(u) = \{u_1, u_2\}$, $N^+(u_1) = \{u_3, u_4\}$, $N^+(u_2) = \{u_5, u_6\}$, $N^+(u_3) = \{u_7, u_8\}$, $N^+(u_4) = \{u_9, u_{10}\}$ and so on, with the same convention for the vertices of $T_{k}(v)$, where we will assume that $u_2 = v_2$.

3. Classification of $(2, 2, +2)$-digraphs

We begin by classifying the $(2, 2, +2)$-digraphs up to isomorphism. We will prove the following theorem.

**Theorem 1.** There are exactly two diregular $(2, 2, +2)$-digraphs, which are displayed in Figs. 2 and 5.

Let $G$ be an arbitrary diregular $(2, 2, +2)$-digraph. $G$ has order $M(2, 2) + 2 = 9$. By Lemma 2, $G$ contains a pair of vertices $(u, v)$ such that $|N^+(u) \cap N^+(v)| = 1$; we will assume that $u_2 = v_2$, so that we have the situation shown in Fig. 1.

We can immediately deduce some information on the possible positions of $v$ and $v_1$ in $T_{2}(u)$.

**Lemma 3.** If $v \notin O(u)$, then $v \in N^+(u_1)$. If $v_1 \notin O(u)$, then $v_1 \in N^+(u_1)$.

**Proof.** If $v \notin T(u_2)$ by 2-geodeticity. $v \neq u$ by construction. If we had $v = u_1$, then there would be two distinct $\leq 2$-paths from $u$ to $u_2$. Also $v_1 \notin \{u\} \cup T(u_2)$ by 2-geodeticity and by assumption $u_1 \neq v_1$. \qed

![Fig. 1. The vertices $u$ and $v$.](image-url)
Since \(v\) and \(v_1\) cannot both lie in \(N^+(u_1)\) by 2-geodecity, we have the following corollary.

**Corollary 1.** \(O(u) \cap \{v, v_1\} \neq \emptyset\).

We will call a pair of vertices \((u, v)\) with a single common out-neighbour **bad** if at least one of

\[
O(u) \cap \{v_1, v_3\} = \emptyset, O(u) \cap \{v_1, v_4\} = \emptyset, O(v) \cap \{u_1, u_3\} = \emptyset, O(v) \cap \{u_1, u_4\} = \emptyset.
\]

holds. Otherwise such a pair will be called good.

**Lemma 4.** There is a unique \((2, 2, +2)\)-digraph containing a bad pair.

**Proof.** Assume that there exists a bad pair \((u, v)\). Without loss of generality, \(O(u) \cap \{v_1, v_3\} = \emptyset\). By Lemma 3 we can set \(v_1 = u_3\). By 2-geodecity \(v_3 = u\). We cannot have \(v_4 = v_3 = u\), so \(v_4\) must be an outlier of \(u\). By Corollary 1 it follows that \(O(u) = \{v, v_4\}\).

Consider the vertex \(u_1\). By Lemma 3, if \(u_1 \notin O(v)\), then \(u_1 \in N^+(v_1)\). However, as \(v_1 = u_3\), there would be a 2-cycle through \(u_1\). Hence \(u_1 \in O(v)\). As \(O(u) = \{v, v_4\}\), we have \(V(G) = \{u, u_1, u_2, u_3 = v_1, u_4, u_5, u_6, v, v_4\}\) and \(O(v) = \{u_1, u_4\}\). As neither \(u\) nor \(v\) lies in \(T(u_1)\), we must also have \(u_2 \in O(u_1)\). As \(u_1\) can reach \(u_1, u_4, u\) and \(v_4\), it follows that without loss of generality we either have \(O(u_1) = \{u_2, v\}\) and \(N^+(u_4) = \{u_5, u_6\} = N^+(u_2)\) or \(O(u_1) = \{u_2, u_6\}\) and \(N^+(u_4) = \{v, u_5\}\). In either case, \((u, u_1)\) is a good pair.

Suppose firstly that \(N^+(u_2) = N^+(u_4)\). Then \(v\) is an outlier of \(u\) and \(u_1\). As each vertex is the outlier of exactly two vertices, \(v_1\) must be able to reach \(v\) by a \(\leq 2\)-path. Hence \(v_4 \rightarrow v\). Likewise \(u_2\) can reach \(v\), so without loss of generality \(u_5 \rightarrow v\). Suppose that \(O(u_2) \cap \{u, u_1\} = \emptyset\). As \(u\) and \(v\) have a common out-neighbour, we must have \(u_6 \rightarrow u\). Since \(u \rightarrow u_1\), by 2-geodecity we must have \(u_6 \rightarrow u_1\). However, this is a contradiction, as \(u\) and \(u_1\) also have a common out-neighbour.

Therefore, at least one of \(u, u_1\) is an outlier of \(u_2\). By Lemma 1 \(u_4\) is an outlier of \(u_2\). Therefore either \(O(u_2) = \{u, u_4\}\) or \(O(u_2) = \{u_1, u_4\}\). If \(O(u_2) = \{u, u_4\}\), then \(u_2\) must be able to reach \(u_1, v_1\) and \(v_4\). As \(u_5 \rightarrow v\), so \(v_1 \in N^+(u_6)\). As \(u_1 \rightarrow v_1\), we must have \(N^+(u_5) = \{v, u_1\}\). As \(v\) and \(u_1\) have a common out-neighbour, this violates 2-geodecity. Hence \(O(u_2) = \{u_1, u_4\}\) and \(u_2\) can reach \(u_1, v_1\) and \(v_4\). As \(v \rightarrow v_1, v_1 \in N^+(u_6)\). As \(v_1 \rightarrow v_4\), it follows that \(N^+(u_5) = \{v, v_4\}\). However, \(u_4 \rightarrow v\), so this again violates 2-geodecity.

We are left with the case \(O(u_1) = \{u_2, u_6\}\) and \(N^+(u_4) = \{v, u_5\}\). Then \(v_1 \in O(u_2)\), as neither \(v\) nor \(u_1\) lies in \(T(u_1)\). Observe that \(u_2\) and \(u_4\) have a single common out-neighbour, so by Corollary 1 \(O(u_2) \cap \{u_4, v\} \neq \emptyset\). Therefore either \(O(u_2) = \{v_1, u_4\}\) or \(O(u_2) = \{v_1, v\}\). Suppose firstly that \(O(u_2) = \{v_1, u_4\}\). Then \(N^+(u_2) = \{u, v, u_1, v_4\}\). As \(N^+(u_4) = \{v, u_5\}\), \(u_5 \not\rightarrow v\), so \(u_6 \rightarrow v\). As \(N^+(u) \cap N^+(v) \neq \emptyset\), \(u_5 \rightarrow u\), so necessarily \(u_6 \in N^+(u_1) \cap N^+(v)\), contradicting 2-geodecity.

Hence \(O(u_2) = \{v_1, v\}\) and \(N^+(u_2) = \{u, u_1, u_4, v_4\}\). As \(u_4 \rightarrow u_5, u_5 \not\rightarrow u_4\). Thus \(u_6 \rightarrow u_4\). Now \(u_1 \rightarrow u_4\) and \(u \rightarrow u_1\) implies that \(N^+(u_5) = \{u_1, v_4\}\) and \(N^+(u_6) = \{u, u_4\}\). Finally we must have \(N^+(v_4) = \{v, u_5\}\). This gives us the \((2, 2, +2)\)-digraph shown in Fig. 2. \(\square\)

We can now assume that all pairs given by Lemma 2 are good. Let us fix a pair \((u, v)\) with a single common out-neighbour. It follows from Corollary 1 and the definition of a good pair that \(v_1 \in O(u)\); otherwise \(O(u)\) would contain \(v, v_3\) and \(v_4\), which is impossible. Likewise \(u_1 \in O(v)\).**
Considering the positions of \(v_3\) and \(v_4\), we see that there are without loss of generality four possibilities: (1) \(u = v_3, u_4 = v_4\), (2) \(u = v_4, u_3 = v_3\), \(O(u) = \{v_1, v_4\}\), \(3) N^+(u_4) = N^+(v_1)\text{ and } (4) u_3 = v_3, O(u) = \{v_1, u_4\}\). A suitable relabelling of vertices shows that case 4 is equivalent to case 1.a below, so we will examine cases 1 to 3 in turn.

**Case 1: \(u = v_3, u_4 = v_4\)**

Depending upon the position of \(v\), we must either have \(O(u) = \{v_1, v\}\) and \(O(v) = \{u_1, u_3\}\) or \(v = u_3\) (see Fig. 3).

**Case 1.a: \(O(u) = \{v_1, v\}\), \(O(v) = \{u_1, u_3\}\)**

In this case \(V(G) = \{u, u_1, u_2, u_3, u_4, u_5, u_6, v_1, v_2\}\), \(u_1\) and \(v_1\) have a single common out-neighbour, namely \(u_4\), so, as we are assuming all such pairs to be good, we have \(u_3 \in O(v_1), u \in O(u_1)\). By 2-geodecity, \(N^+(u_4) \subseteq \{u_5, u_6, v\}\), so without loss of generality either \(N^+(u_4) = \{u_5, u_6\}\) or \(N^+(u_4) = \{u_5, v\}\).

Suppose that \(N^+(u_4) = \{u_5, u_6\}\). By elimination, \(O(u_1) = \{v, u_3\}\). As \(G\) is directly oriented, every vertex is an outlier of exactly two vertices; \(v\) is an outlier of \(u\) and \(v_1\), so both \(u_1\) and \(u_2\) can reach \(v\) by a \(\leq 2\)-path. Hence \(v \in N^+(u_3)\). As \(v \rightarrow v_1\), we see that \(v_1\) is an outlier of \(u_1\); as \(u\) is also an outlier of \(u_1\), we have \(O(u_1) = \{u, v_1\}\) and \(N^+(u_3) = \{v, u_2\}\). As \(v \rightarrow u_2\), this is impossible.

Now consider \(N^+(u_4) = \{u_5, v\}\). We now have \(O(v_1) = \{u_3, u_6\}\). Thus \(u_3 \in O(v) \cap O(v_1)\), so \(u_3 \in T_2(u_4)\). \(v\) is not adjacent to \(u_3\), so \(u_3 \in N^+(u_5)\). \(u_2\) and \(u_4\) have \(u_5\) as a unique common out-neighbour, so \(u_6 \in O(u_4)\), \(v \in O(u_2)\). As \(u_6 \in O(v_1) \cap O(u_4)\), \(u_1\) can reach \(u_6\). Hence \(u_6 \in N^+(u_1)\). Neither \(u\) nor \(v\) lie in \(T(u_1)\), so \(u_2 \in O(u_1)\). Therefore either \(O(u_1) = \{u, u_2\}\) or \(O(u_1) = \{u_2, v_1\}\). If \(O(u_1) = \{u, u_2\}\), then \(N^+(u_2) = \{u_6, v_1\}\). \(u_2\) cannot reach \(v_1\), since \(v, u_3 \notin T(u_2)\), so \(O(u_2) = \{v, v_1\}\) and \(N^2(u_2) = \{u, u_1, u_2, u_3\}\). As \(u_4 \rightarrow u_5, u_4 \in N^+(u_6), u_1 \rightarrow u_4, \text{ so } N^+(u_1) = \{u_1, u_3\}\). As \(u_1 \rightarrow u_3\), this is a contradiction. Thus \(O(u_1) = \{u_2, v_1\}\), so that \(N^+(u_3) = \{u, u_6\}\). \(u_1\) must have an in-neighbour apart from \(u\), which must be either \(u_5\) or \(u_6\). As \(u_1 \rightarrow u_5\), we have \(u_1 \in N^+(u_6)\). By elimination, \(v\) and \(v_1\) must also have in-neighbours in \(\{u_5, u_6\}\). As \(u_1\) and \(v_1\) have a common out-neighbour, we have \(N^+(u_5) = \{u_3, v_1\}, N^+(u_6) = \{u_1, v\}\). However, both \(u_3\) and \(v_1\) are adjacent to \(u\), violating 2-geodecity.

**Case 1.b: \(v = u_3\)**

There exists a vertex \(x\) such that \(V(G) = \{u, u_1, u_2, v, u_4, u_5, u_6, v_1, x\}\), \(O(u) = \{v_1, x\}\) and \(O(v) = \{u_1, x\}\). As \(x \in O(u) \cap O(v)\), \(u_1\) and \(u_2\) can reach \(x\), so without loss of generality \(x \in N^+(u_4) \cap N^+(u_5)\). As \(u_5\) and \(u_4\) have a common out-neighbour, \(u_6 \in O(u_1)\). Also, \(u_1\) and \(v_1\) have \(u_4\) as a unique common out-neighbour, so \(u \in O(u_1)\) and \(O(u_1) = \{u, u_6\}\). Thus \(N^+(u_4) = \{x, u_6\}\). Observe that \(u_2\) and \(u_4\) have the out-neighbour \(u_6\) in common. Thus \(x \in O(u_2)\), whereas we already have \(x \in O(u) \cap O(v)\), a contradiction.

**Case 2: \(u = v_3, O(u) = \{v_1, v_4\}\)**

As \(v\) is not equal to \(v_1\) or \(v_4\), \(v\) must lie in \(T_2(u)\). Without loss of generality, \(v = u_3\). Hence \(V(G) = \{u, u_1, u_2, v, u_4, u_5, u_6, v_1, v_4\}\) and \(O(v) = \{u_1, u_4\}\). We have the configuration shown in Fig. 4. Hence \(u_1\) can reach
There are three possibilities: (i) $O(u_1) = \{u, v_4\}$, $N^+(u_4) = \{u_5, u_6\}$, (ii) $O(u_1) = \{u, u_4\}$, $N^+(u_4) = \{u_6, v_4\}$, (iii) $O(u_1) = \{u_5, u_6\}$, $N^+(u_4) = \{u, v_4\}$ or (iv) $O(u_1) = \{u_5, v_4\}$, $N^+(u_4) = \{u, u_6\}$.

**Case 2.a:** $O(u_1) = \{u, v_4\}, N^+(u_4) = \{u_5, u_6\}$

As $v_4 \in O(u) \cap O(u_1)$, $u_2$ can reach $v_4$ and without loss of generality $v_4 \in N^+(u_5)$. $N^+(u_2) = N^+(u_4)$, so by Lemma 1 $u_2 \in O(u_4)$, $u_4 \in O(u_2)$, $u_5 \in O(u_5)$ and $u_6 \in O(u_5)$. Hence $u_4 \in O(v) \cap O(u_2)$, so $v_1$ can reach $u_4$, so $u_4 \in N^+(v_4)$. Neither $u_5$ nor $u_6$ lies in $N^+(v_4)$, so $O(v_1) = \{u_5, u_6\}$ and $N^+(v_4) = \{u_4, v\}$. Hence $O(v_4) = \{u, u_1\}$. Observe that $N^+(u_1) = N^+(v_4)$, so that $v \in O(u_4)$. Therefore $v \notin N^+(u_5) \cup N^+(u_6)$, yielding $O(u_2) = \{u_4, v\}$, $N^+(u_2) = \{v_4, v_1, u, u_1\}$. As $v_1 \to v_4$ and $N^+(u_1) = N^+(v_4)$, we must have $N^+(v_5) = \{v_4, u\}$, $N^+(u_6) = \{v_1, u_1\}$. This yields the digraph in Fig. 5. Unlike the digraph in Fig. 2, this digraph contains pairs of vertices with identical out-neighbourhoods, so the two are not isomorphic.

**Case 2.b:** $O(u_1) = \{u_5, u_6\}, N^+(u_4) = \{u_6, v_4\}$

As $u_4 \to v_4$, $u_4 \notin N^+(u_4)$, so $u_4 \in O(v_1)$. Hence $u_4 \in O(v) \cap O(v_1)$, so $u_2$ can reach $u_4$. As $u_4 \to u_6$, we must have $u_5 \to u_4$, $u_2$ and $u_4$ have $u_6$ as a common out-neighbour, so $v_4 \in O(u_5)$, $u_5 \in O(u_4)$. Therefore $v_4 \in O(u) \cap O(u_2)$, so that $u_6$ can reach $u_4$, but $v_4 \notin T(u_6)$, so $N^+(u_6)$ contains an in-neighbour of $v_4$, $u_4 \notin N^+(u_6)$, so we must have $u_6 \to v_1$. We have $u_5 \in O(u_4) \cap O(u_1)$, so $v_1$ can reach $u_5$ and hence $v_4 \to u_5$, $v_1$ cannot reach $u_6$, as $u_2, u_4 \notin T(v_1)$, so $O(v_1) = \{u_4, u_6\}$, $N^+(v_5) = \{u_5, v\}$. Now $u_2$ and $v_4$ have $u_5$ as a unique common out-neighbour, so $u_6 \in O(v_4)$, $v \in O(u_2)$. Thus $O(u_2) = \{v, v_4\}$ and $N^+(u_2) = \{u_4, v, u, u_1\}$. Taking into account adjacencies between members of $N^+(v_2)$, it follows that $N^+(v_5) = \{u_4, u\}$, $N^+(u_6) = \{u_1, v_1\}$. However, $(u_2, u_4)$ now constitutes a bad pair, contradicting our assumption.

**Case 2.c:** $O(u_1) = \{u_5, u_6\}, N^+(u_4) = \{u_4, v_4\}$

As $u_4 \to v_4$, $u_4 \notin N^+(v_4)$. Hence $u_4 \in O(v) \cap O(v_1)$, implying that $u_2$ can reach $u_4$. Without loss of generality, $u_5 \to u_4$. There are three possibilities: (i) $O(v_1) = \{u_4, u_6\}, N^+(v_4) = \{v, u_5\}$, (ii) $O(v_1) = \{u_4, u_5\}, N^+(v_4) = \{v, u_6\}$ and (iii) $O(v_1) = \{u_4, v\}, N^+(v_4) = \{u_5, u_6\}$.

(i) $O(v_1) = \{u_4, u_6\}, N^+(v_4) = \{v, u_5\}$

$u_1$ and $v_4$ have $v$ as a unique common out-neighbour, so $u_4 \in O(v_4)$. However, this contradicts $v_4 \to u_5 \to u_4$.

(ii) $O(v_1) = \{u_4, u_5\}, N^+(v_4) = \{v, u_6\}$

Neither $u_4$ nor $v_1$ lie in $T(u_2)$, so $v_4 \in O(u_2)$. Now observe that $u_2$ and $u_4$ have $u_6$ as unique common out-neighbour, so $v \in O(u_2)$, yielding $O(u_2) = \{v, v_4\}$ and $N^+(u_2) = \{u_4, u_1, u, v_1\}$. As $u_4 \to u$ and $u \to u_1$, we must have $N^+(u_5) = \{u_4, u_1\}, N^+(u_6) = \{u, v_1\}$, a contradiction, since $u_1 \to u_4$.

(iii) $O(v_1) = \{u_4, v\}, N^+(v_4) = \{u_5, u_6\}$

We now have $N^+(u_2) = N^+(v_4)$, so $u_2 \in O(v_4), v_4 \in O(u_2), u_5 \in O(u_6), u_6 \in O(u_5)$. Also $N^+(u_4) = N^+(v_4)$, so $u_4 \in O(v_1), v_1 \in O(u_4)$ and $u \in O(u_4)$. $u \in O(v_1)$ implies that $u \notin N^+(u_5) \cup N^+(u_6)$, so we see that $u \in O(u_2)$ and hence $O(u_2) = \{u, v_4\}$ and $N^+(u_2) = \{u_4, v_1, u, v_1\}$. As $u_1 \to u_4$ and $u_4 \to v$, we have $N^+(u_5) = \{u_4, v\}, N^+(u_6) = \{u_1, v_1\}$. It is not difficult to show that this yields a $(2, 2, +2)$-digraph isomorphic to that in Fig. 5.

**Case 2.d:** $O(u_1) = \{u_5, v_4\}, N^+(u_4) = \{u, u_6\}$

In this case $v_4 \in O(u) \cap O(u_1)$, so $u_2$ can reach $v_4$, $u_4$ and $v_1$ have unique common out-neighbour $u$, so $v_4 \in O(u_4), u_6 \in O(v_1)$. If $u_6 \to v_4$, then we would have $u_4 \to u_6 \to v_4$, contradicting $v_4 \in O(u_4)$, so $u_5 \to v_4$. This also implies that $u_5 \notin N^+(v_4)$, so $u_5 \in O(v_1)$, yielding $O(v_1) = \{u_5, u_6\}$ and $N^+(v_4) = \{v, u_4\} = N^+(u_1)$. Now $v_4, u_1 \notin T(u_2)$, so $O(u_2) = \{v, u_4\}$.
and \(N^2(u_2) = \{v_1, v_4, u, u_1\}\). As \(v_1 \to v_4\) and \(v_1 \to u\), it follows that \(N^+ (u_5) = \{v_4, u\}\), \(N^+ (u_6) = \{u_1, v_1\}\). However, we now have paths \(u_4 \to u \to u_1\) and \(u_4 \to u_6 \to u_1\), which is impossible.

**Case 3:** \(N^+(u_1) = N^+(v_1)\)

It is easy to see by 2-geodecity that \(V(G) = \{u, u_1, u_2, u_3, u_4, u_5, u_6, v, v_1\}\), \(O(u) = \{v, v_1\}\) and \(O(v) = \{u, u_1\}\). As \(u_1, v_1 \notin T(u_2)\), we have \(O(u_2) = \{u_3, u_4\}\) and \(N^2(u_2) = \{u, u_1, v, v_1\}\). Without loss of generality, \(N^+(u_3) = \{u, v_1\}\), \(N^+(u_6) = \{v, u_1\}\). \(u\) and \(v\) have in-neighbours apart from \(u_5\) and \(u_6\) respectively, so without loss of generality \(u_3 \to u\), \(u_4 \to v\). Likewise, \(u_5\) and \(u_6\) have in-neighbours other than \(u_2\), so, as \(u_3 \to u\) and \(u_6 \to v\), we must have \(N^+(u_3) = \{u, u_6\}\), \(N^+(u_4) = \{v, u_5\}\). But now we have paths \(u_3 \to u \to u_1\) and \(u_3 \to u_6 \to u_1\), violating 2-geodecity.

**Corollary 2.** There is a unique \((2, 2, +2)\)-digraph containing no bad pairs.

This completes our analysis of diregular \((2, 2, +2)\)-digraphs. As it was shown in [7] that there are no non-diregular \((2, 2, +2)\)-digraphs, \((2, 2, +2)\)-digraphs are now classified up to isomorphism. These conclusions have been verified computationally by Erskine [2]. It is interesting to note that neither of the \((2, 2, +2)\)-digraphs are vertex-transitive, for in each case there are exactly three vertices contained in two 3-cycles. However, there does exist a Cayley \((2, 2, +5)\)-digraph (on the alternating group \(A_4\)), so it would be interesting to determine the smallest vertex-transitive \((2, 2, +e)\)-digraphs.

### 4. Main result

We can now complete our analysis by showing that there are no diregular \((2, k, +2)\)-digraphs for \(k \geq 3\). Let \(G\) be such a digraph. By **Lemma 2**, \(G\) contains vertices \(u\) and \(v\) with a unique common out-neighbour. In accordance with our vertex-labelling convention, we have the situation in Fig. 6. A triangle based at a vertex \(x\) represents the set \(T(x)\).

We now proceed to determine the possible outlier sets of \(u\) and \(v\).

**Lemma 5.** \(v \in N^{k-1}(u_1) \cup O(u)\) and \(u \in N^{k-1}(v_1) \cup O(v)\). If \(v \in O(u)\), then \(u_2 \in O(u_1)\) and if \(u \in O(v)\), then \(u_2 \in O(v_1)\).

**Proof.** \(v\) cannot lie in \(T(u)\), or the vertex \(u_2\) would be repeated in \(T(u)\). Also, \(v \notin T(u_2)\). Therefore, if \(v \notin O(u)\), then \(u \in N^{k-1}(u_1)\). Likewise for the other result. If \(v \in O(u)\), then neither in-neighbour of \(u_2\) lies in \(T(u_1)\), so that \(u_2 \in O(u_1)\). \(□\)

**Lemma 6.** Let \(w \in T(v_1)\), with \(d(u_1, w) = l\). Suppose that \(w \in T(u_1)\), with \(d(u_1, w) = m\). Then either \(m \leq l\) or \(w \in N^{k-1}(u_1)\) and \(A\) similar result holds for \(w \in T(u_1)\).

**Proof.** Let \(w\) be as described and suppose that \(m > l\). Consider the set \(N^{k-m}(w)\). By construction, \(N^{k-m}(w) \subseteq N^{k}(u_1)\), so by \(k\)-geodecity \(N^{k-m}(w) \cap T(u_1) = \emptyset\). At the same time, we have \(l + k - m \leq k - 1\), so \(N^{k-m}(w) \subseteq T(v_1)\). This implies that \(N^{k-m}(w) \cap T(v_2) = N^{k-m}(w) \cap T(u_2) = \emptyset\). As \(V(G) = \{u\} \cup T(u_1) \cup T(u_2) \cup O(u)\), it follows that \(N^{k-m}(w) \subseteq \{u\} \cup O(u)\). Therefore \(|N^{k-m}(w)| = 2^{k-m} \leq 3\). By assumption \(0 \leq m \leq k - 1\), so it follows that \(m = k - 1\). \(□\)

**Corollary 3.** If \(w \in T(v_1)\), then either \(w \in \{u\} \cup O(u)\) or \(w \in T(u_1)\) with \(d(u_1, w) = k - 1\) or \(d(u_1, w) \leq d(v_1, w)\).

**Proof.** By \(k\)-geodecity and **Lemma 6**. \(□\)

**Corollary 4.** \(v_1 \in N^{k-1}(u_1) \cup O(u)\) and \(u_1 \in N^{k-1}(v_1) \cup O(v)\).
Proof. We prove the first inclusion. By Corollary 3, $v_1 \in \{u\} \cup O(u) \cup \{u_1\} \cup N^{k-1}(u_1)$. By $k$-geodecity, $v_1 \neq u$ and by construction, $v_1 \neq u_1$. □

We now have enough information to identify one member of $O(u)$ and $O(v)$.

Lemma 7. $v_1 \in O(u)$ and $u_1 \in O(v)$.

Proof. We prove that $v_1 \in O(u)$. Suppose that neither $v_1$ nor $v$ lies in $O(u)$. Then by Lemma 5 and Corollary 4 we have $v, v_1 \in N^{k-1}(u_1)$. As $v_1$ is an out-neighbour of $v$, it follows that $v_1$ appears twice in $T_k(u_1)$, violating $k$-geodecity. Therefore $O(u) \cap \{v, v_1\} \neq \emptyset$.

Now assume that $v_1, v \in T_k(u)$. Again by Corollary 4, $v_1 \in N^{k-1}(u_1)$. By $k$-geodecity we also have $v_1 \in T(u_1)$. However, $v_1 \in N^+(v_1)$, so $v_1$ appears twice in $T_k(u_1)$, which is impossible. Hence $O(u) \cap \{v_1, v_3\} \neq \emptyset$. Similarly, $O(u) \cap \{v_1, v_4\} \neq \emptyset$. In the terminology of the previous section, $G$ contains no bad pairs. Therefore, if $v_1 \notin O(u)$, then $\{v, v_3, v_4\} \subseteq O(u)$. Since these vertices are distinct, this is a contradiction and the result follows. □

Lemma 7 allows us to conclude that for vertices sufficiently close to $v_1$ one of the potential situations mentioned in Corollary 3 cannot occur.

Lemma 8. $T_{k-3}(v_1) \cap N^{k-1}(u_1) = T_{k-3}(u_1) \cap N^{k-1}(v_1) = \emptyset$.

Proof. Let $w \in T_{k-3}(v_1) \cap N^{k-1}(u_1)$. Consider the position of the vertices of $N^+(w)$ in $T_k(u) \cup O(u)$. As $v \notin N^+(w)$, it follows from Lemma 7 that at most one of the vertices of $N^+(w)$ can be an outlier of $u$, so let us write $w_1 \in N^+(w) - O(u)$. By $k$-geodecity, $w_1 \notin T(u_1) \cup \{u\}$. Hence $w_1 \in T(u_2) = T(v_2)$. However, $u_1$ also lies in $T(v_1)$, so this violates $k$-geodecity. □

Corollary 5. There is at most one vertex in $T_{k-3}(v_1) - \{v_1\}$ that does not lie in $T(u_1)$: for all other vertices $w \in T_{k-3}(v_1) - \{v_1\}$, $d(u_1, w) = d(v_1, w)$. A similar result for $T_{k-3}(u_1) - \{u_1\}$ also holds.

Lemma 9. For $k = 3, N^+(u_1) \cap N^2(v_1) = N^+(v_1) \cap N^2(u_1) = \emptyset$.

Proof. Suppose that $v_3 = u_2$. By the reasoning of Lemma 8 we can set $w = v_2$ and $O(u) = \{v_1, v_9\}$, $v \notin O(u)$ and by $3$-geodecity $v \notin N^+(u_3)$, so we can assume that $v = u_9, u_3 \rightarrow v_3$ implies that $u_3 \notin T(v_1)$. So $O(v) = \{u_1, u_4\}$. We must have $\{u_4, u_9, u_{10}\} = \{v_4, v_9, v_{10}\}$. As $u_4 \rightarrow v$, it follows that $v_4 = u_8$ and hence $\{u_4, u_{10}\} = \{v_2, v_{10}\}$, which is impossible. □

As $u_1$ is an outlier of $v$, neither $v_3$ nor $v_4$ can be equal to $u_1$. It follows from Corollary 5 and Lemma 9 that either $N^+(u_1) = N^+(v_1)$ or $u_1$ and $v_1$ have a single common out-neighbour, with one vertex of $N^+(v_1)$ being an outlier of $u$.

Lemma 10. $N^2(u) \neq N^2(v)$

Proof. Let $N^2(u) = N^2(v)$, with $N^+(u_1) = N^+(v_1) = \{u_3, u_4\}$. Suppose that $v \notin O(u)$. By Lemma 5, $v \in N^{k-2}(u_3) \cup N^{k-2}(u_4)$. But then there is a $k$-cycle through $v$. It follows that $O(u) = \{v, v_1\}$, $O(v) = \{u, u_1\}$. By Lemma 5, $u_2 \in O(u_1) \cap O(v_1)$. Therefore by Lemma 1 $O(u_1) = \{u_2, v_1\}$, $O(v_1) = \{u_2, u_1\}$.

Consider the in-neighbour $u'$ of $u_1$ that is distinct from $u$. We have either $|N^+(u') \cap N^+(u)| = 1$ or $|N^+(u') \cap N^+(u)| = 2$. In the first case, it follows from Lemma 7 that $u_2 \in O(u)$. Every vertex of $G$ is an outlier of exactly two vertices, so $u' = u_1$ or $v_1$. In either case, we have a contradiction. Therefore $N^+(u') = N^+(u)$. It now follows from Lemma 1 that $u' \in O(u) = \{v, v_1\}$, which is impossible. □

Noticing that $u_1$ and $v_1$ also have a unique common out-neighbour, we have the following corollary.

Corollary 6. Without loss of generality, $u_1 = v_3, u_9 = v_9, O(u) = \{v_1, v_4\}, O(v) = \{u_1, u_4\}, O(u_1) = \{v_4, v_{10}\}$ and $O(v_1) = \{u_4, u_{10}\}$.

We are now in a position to complete the proof by deriving a contradiction.

Theorem 2. There are no digereular $(2, k, +2)$-digraphs for $k \geq 3$.

Proof. Let $u, v \notin \{u_1, u_4, v_1, v_4\}$, so by Lemma 5 $d(u, v) = d(v, u) = k$. In fact, $u_2 = v_3$ implies that $v \in N^{k-2}(u_3)$ and $u \in N^{k-2}(v_4)$. Let $k \geq 4$. Then $u, v \notin \{u_{10}, v_{10}\}$, so $u, v \in T_k(u_1) \cap T_k(v_1)$. If $u \in T(u_2) = T(v_2)$, then $u$ would appear twice in $T_k(u_1)$, so $u \in N^{k-1}(u_3)$. However, as $u$ and $v$ have a common out-neighbour, this violates $k$-geodecity.

Finally, suppose that $k = 3$. The above analysis will hold unless $u = v_{10}$ and $v = u_{10}$. Let $N^+(u_1) = \{u, u', N^-(v_1) = \{v, v'\}$. It is evident that $u' \notin \{v_1, v_4\}$, so that $v' \in T_3(u)$. As $v \in N^+(u_4)$, we must have $v' \in N^2(u_2)$. Similarly $u' \in N^2(u_2)$. Since $u_1$ and $v_1$ have a common out-neighbour, we can assume that $u' \in N^+(u_5)$ and $v' \in N^+(u_6)$, $v_3$ can be the outlier of only two vertices, namely $u$ and $u_1$, so $v_3 \in N^2(u_2)$ and likewise $u_4 \in N^2(u_3)$. By $3$-geodecity $v_4 \in N^2(u_5)$ and $u_4 \in N^2(u_6)$. It follows that $u, v \notin N^2(u_3)$, so $u \notin T_3(u_1) \cup T_3(v_1)$. Hence $O(u) = N^+(u) = \{v_4, v_3\}$, which again is impossible. □

It is interesting to note that a similar argument can be used to provide an alternative proof of the result of [6].
References