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On diregular digraphs with degree two and excess two

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Abstract

An important topic in the design of efficient networks is the construction of \((d, k, +\epsilon)\)-digraphs, i.e. \(k\)-geodetic digraphs with minimum out-degree \(\geq d\) and order \(M(d, k) + \epsilon\), where \(M(d, k)\) represents the Moore bound for degree \(d\) and diameter \(k\) and \(\epsilon > 0\) is the small excess of the digraph. Previous work has shown that there are no \((2, k, +1)\)-digraphs for \(k \geq 2\). In a separate paper, the present author has shown that any \((2, k, +2)\)-digraph must be diregular for \(k \geq 2\). In the present work, this analysis is completed by proving the nonexistence of diregular \((2, k, +2)\)-digraphs for \(k \geq 3\) and classifying diregular \((2, 2, +2)\)-digraphs up to isomorphism.

1. Introduction

An important topic in the design of interconnection networks is the directed degree/diameter problem: what is the largest possible order \(N(d, k)\) of a digraph \(G\) with maximum out-degree \(d\) and diameter \(\leq k\)? A simple inductive argument shows that for \(0 \leq l \leq k\) the number of vertices at distance \(l\) from a fixed vertex \(v\) is bounded above by \(d^l\). Therefore, a natural upper bound for the order of such a digraph is the so-called Moore bound \(M(d, k) = 1 + d + d^2 + \cdots + d^k\). A digraph that attains this upper bound is called a Moore digraph. It is easily seen that a digraph \(G\) is Moore if and only if it is out-regular with degree \(d\), has diameter \(k\) and is \(k\)-geodetric, i.e. for any two vertices \(u, v\) there is at most one \(k\)-path from \(u\) to \(v\).

As it was shown by Bridges and Toueg in [1] that Moore digraphs exist only in the trivial cases \(d = 1\) or \(k = 1\) (the Moore digraphs are directed \((k+1)\)-cycles and complete digraphs \(K_{k+1}\) respectively), much research has been devoted to the study of digraphs that in some sense approximate Moore digraphs. For example, there is an extensive literature on digraphs with maximum out-degree \(d\), diameter \(\leq k\) and order \(M(d, k) - \delta\) for small \(\delta > 0\); this is equivalent to relaxing the \(k\)-geodesity requirement in the conditions for a digraph to be Moore. \(\delta\) is known as the defect of the digraph. The reader is referred to the survey [5] for more information.

In this paper, however, we will consider the following related problem, which is obtained by retaining the \(k\)-geodesity requirement in the above characterisation of Moore digraphs, but allowing the diameter to exceed \(k\): what is the smallest possible order of a \(k\)-geodetic digraph \(G\) with minimum out-degree \(\geq d\)? A \(k\)-geodetic digraph with minimum out-degree \(\geq d\) and order \(M(d, k) + \epsilon\) is said to be a \((d, k, +\epsilon)\)-digraph or to have excess \(\epsilon\). It was shown in [6] that there are no diregular \((2, k, +1)\)-digraphs for \(k \geq 2\). In 2016 it was shown in [3] that digraphs with excess one must be diregular and that there are no \((d, k, +1)\)-digraphs for \(k = 2, 3, 4\) and sufficiently large \(d\). In a separate paper [7], the present author has shown that \((2, k, +2)\)-digraphs must be diregular with degree \(d = 2\) for \(k \geq 2\). In the present paper, we classify the \((2, 2, +2)\)-digraphs up to isomorphism and show that there are no diregular \((2, k, +2)\)-digraphs for \(k \geq 3\), thereby completing the proof of the nonexistence of digraphs with degree \(d = 2\) and excess \(\epsilon = 2\) for \(k \geq 3\). Our reasoning and notation will follow closely that employed in [4] for the corresponding result for defect \(\delta = 2\).
2. Preliminary results

We will let $G$ stand for a $(2, k, +2)$-digraph for arbitrary $k \geq 2$, i.e. $G$ has minimum out-degree $d = 2$, is $k$-geodeic and has order $M(2, k) + 2$. We will denote the vertex set of $G$ by $V(G)$. By the result of [?], $G$ must be diregular with degree $d = 2$ for $k \geq 2$. The distance $d(u, v)$ between vertices $u$ and $v$ is the length of the shortest path from $u$ to $v$. Notice that $d(u, v)$ is not necessarily equal to $d(v, u)$. $u \rightarrow v$ will indicate that there is an arc from $u$ to $v$. We define the in- and out-neighbourhoods of a vertex $u$ by $N^−(u) = \{v \in V(G) : v \rightarrow u\}$ and $N^+(u) = \{v \in V(G) : u \rightarrow v\}$ respectively; more generally, for $0 \leq l \leq k$, the set $\{v \in V(G) : d(u, v) = l\}$ of vertices at distance exactly $l$ from $u$ will be denoted by $N^l(u)$. For $0 \leq l \leq k$ we will also write $T_l(u) = \bigcup_{i=0}^{l} N^i(u)$ for the set of vertices at distance $\leq l$ from $u$. The notation $T_{k−1}(u)$ will be abbreviated by $T(u)$.

It is easily seen that for any vertex $u$ of $G$, there are exactly two distinct vertices that are at distance $\geq k + 1$ from $u$. For any $u \in V(G)$, we will write $O(u)$ for the set of these vertices and call such a set an outlier set and its elements outliers of $u$.

Notice that $O(u) = V(G) − T_{k}(u)$. An elementary counting argument shows that in a diregular $(2, k, +2)$-digraph every vertex is also an outlier of exactly two vertices. We will say that a vertex $u$ can reach a vertex $v$ if $v \notin O(u)$.

Our proof will proceed by an analysis of a pair of vertices with exactly one common out-neighbour. First, we must show that such a pair exists and deduce some elementary properties of pairs of vertices with identical out-neighbourhoods.

Lemma 1. For $k \geq 2$, let $u$ and $v$ be distinct vertices such that $N^+(u) = N^+(v) = \{u_1, u_2\}$. Then $u_1 \in O(u_2)$, $u_2 \in O(u_1)$ and there exists a vertex $x$ such that $O(u) = \{v, x\}$, $O(v) = \{u, x\}$.

Proof. Suppose that $u$ can reach $v$ by a $\leq k$-path. Then $v \in T(u_1) \cup T(u_2)$. As $N^+(v) = N^+(u)$, it follows that there would be a $\leq k$-cycle through $v$, contradicting $k$-geodecity. If $O(u) = \{v, x\}$, then $x \neq v$ and $x \notin T(u_1) \cup T(u_2)$, so that $v$ cannot reach $x$ by a $\leq k$-path. Similarly, if $u_1$ can reach $u_2$ by a $\leq k$-path, then we must have $\{u, v\} \cap T(u_1) \neq \emptyset$, which is impossible. □

Lemma 2. For $k \geq 2$, there exists a pair of vertices $u$, $v$ with $|N^+(u) \cap N^+(v)| = 1$.

Proof. Suppose for a contradiction that there is no such pair of vertices. Define a map $\phi : V(G) \rightarrow V(G)$ as follows. Let $u^+$ be an out-neighbour of a vertex $u$ and let $\phi(u)$ be the in-neighbour of $u^+$ distinct from $u$. By our assumption, it is easily verified that $\phi$ is a well-defined bijection with no fixed points and with square equal to the identity. It follows that $G$ must have even order, whereas $|V(G)| = M(2, k) + 2$ is odd. □

$u$, $v$ will now stand for a pair of vertices with a single common out-neighbour. We will label the vertices of $T_{k}(u)$ according to the scheme $N^+(u) = \{u_1, u_2\}$, $N^+(u_1) = \{u_3, u_4\}$, $N^+(u_2) = \{u_5, u_6\}$, $N^+(u_3) = \{u_7, u_8\}$, $N^+(u_4) = \{u_9, u_{10}\}$ and so on, with the same convention for the vertices of $T_{k}(v)$, where we will assume that $u_2 = v_2$.

3. Classification of $(2, 2, +2)$-digraphs

We begin by classifying the $(2, 2, +2)$-digraphs up to isomorphism. We will prove the following theorem.

Theorem 1. There are exactly two diregular $(2, 2, +2)$-digraphs, which are displayed in Figs. 2 and 5.

Let $G$ be an arbitrary diregular $(2, 2, +2)$-digraph. $G$ has order $M(2, 2) + 2 = 9$. By Lemma 2, $G$ contains a pair of vertices $(u, v)$ such that $|N^+(u) \cap N^+(v)| = 1$; we will assume that $u_2 = v_2$, so that we have the situation shown in Fig. 1. We can immediately deduce some information on the possible positions of $v$ and $v_1$ in $T_{2}(u)$.

Lemma 3. If $v \notin O(u)$, then $v \in N^+(u_1)$. If $v_1 \notin O(u)$, then $v_1 \in N^+(u_1)$.

Proof. $v \notin T(u_2)$ by $2$-geodecity. $v \neq u$ by construction. If we had $v = u_1$, then there would be two distinct $\leq 2$-paths from $u$ to $u_2$. Also $v_1 \notin \{u\} \cup T(u_2)$ by $2$-geodecity and by assumption $u_1 \neq v_1$. □
Since $v$ and $v_1$ cannot both lie in $N^+(u_1)$ by 2-geodecity, we have the following corollary.

**Corollary 1.** $O(u) \cap \{v, v_1\} \neq \varnothing$.

We will call a pair of vertices $(u, v)$ with a single common out-neighbour bad if at least one of $O(u) \cap \{v_1, v_3\} = \varnothing$, $O(v) \cap \{v_1, v_4\} = \varnothing$, $O(v) \cap \{u_1, u_3\} = \varnothing$, $O(u) \cap \{u_1, u_4\} = \varnothing$ holds. Otherwise such a pair will be called good.

**Lemma 4.** There is a unique $(2, 2, +2)$-digraph containing a bad pair.

**Proof.** Assume that there exists a bad pair $(u, v)$. Without loss of generality, $O(u) \cap \{v_1, v_3\} = \varnothing$. By **Lemma 3** we can set $v_1 = u_3$. By 2-geodecity $v_3 = u$. We cannot have $v_4 = v_3 = u$, so $u_4$ must be an outlier of $u$. By **Corollary 1** it follows that $O(u) = \{v, v_4\}$.

Consider the vertex $u_1$. By **Lemma 3**, if $u_1 \notin O(v)$, then $u_1 \in N^+(v_1)$. However, as $v_1 = u_3$, there would be a 2-cycle through $u_1$. Hence $u_1 \in O(v)$. As $O(u) = \{v, v_4\}$, we have $V(G) = \{u, u_1, u_2, u_3 = v_1, u_4, u_5, u_6, v, v_4\}$ and $O(v) = \{u_1, u_4\}$. As neither $u$ nor $v$ lies in $T(u_1)$, we must also have $u_2 \in O(u_1)$. As $u_1$ can reach $u_1, v_1, u_4, u$ and $v_4$, it follows that without loss of generality we either have $O(u_1) = \{u_2, v\}$ and $N^+(u_4) = \{u_5, u_6\} = N^+(u_2)$ or $O(u_1) = \{u_2, u_6\}$ and $N^+(u_4) = \{v, u_5\}$. In either case, $(v, u_1)$ is a good pair.

Suppose firstly that $N^+(u_2) = N^+(u_4)$. Then $u$ is an outlier of $u$ and $u_1$. As each vertex is the outlier of exactly two vertices, $v_1$ must be able to reach $v$ by a $\leq 2$-path. Hence $v_1 \to v$. Likewise $u_2$ can reach $v$, so without loss of generality $u_5 \to u$. Suppose that $O(u_2) \cap \{u, u_1\} = \varnothing$. As $u$ and $v$ have a common out-neighbour, we must have $u_6 \to u$. Since $u \to u_1$, by 2-geodecity we must have $u_5 \to u_1$. However, this is a contradiction, as $v$ and $u_1$ also have a common out-neighbour. Therefore, at least one of $u, u_1$ is an outlier of $u_2$. By **Lemma 1** $u_4$ is an outlier of $u_2$. Therefore either $O(u_2) = \{u, u_4\}$ or $O(u_2) = \{u_1, u_4\}$. If $O(u_2) = \{u, u_4\}$, then $u_2$ must be able to reach $u_1, v_1$ and $v_4, u_5 \to v$ and $v \to v_1$, so $v_1 \in N^+(u_6)$. As $u_1 \to v_1$, we must have $N^+(u_5) = \{v, u_1\}$. As $v$ and $u_1$ have a common out-neighbour, this violates 2-geodecity. Hence $O(u_2) = \{u_1, u_4\}$ and $u_2$ can reach $u, v_1$ and $v_4$. As $v \to v_1, v_1 \in N^+(u_6)$. As $v_1 \to v_4$, it follows that $N^+(u_5) = \{v, v_4\}$.

However, $v_4 \to v$, so this again violates 2-geodecity.

We are left with the case $O(u_1) = \{u_2, u_6\}$ and $N^+(u_4) = \{v, u_5\}$. Then $v_1 \in O(u_2)$, as neither $v$ nor $u_1$ lies in $T(u_2)$. Observe that $u_2$ and $u_4$ have a single common out-neighbour, so by **Corollary 1** $O(u_2) \cap \{u_4, v\} \neq \varnothing$. Therefore either $O(u_2) = \{v, u_4\}$ or $O(u_2) = \{v_1, v\}$. Suppose firstly that $O(u_2) = \{v_1, u_4\}$. Then $N^+(u_2) = \{u, v, u_1, v_4\}$. As $N^+(u_4) = \{v, u_5\}$, $u_5 \not\to v$, so $u_6 \to v$. As $N^+(u) \cap N^+(v) \neq \varnothing$, $u_5 \to u$. If $u \to u_1$, so necessarily $u_6 \in N^+(u_5) = \{v, u_1\}$. However, $v_1 \in N^+(u_1) \cap N^+(v)$, contradicting 2-geodecity.

Hence $O(u_2) = \{v_1, v\}$ and $N^2(u_2) = \{u, u_1, u_4, v_4\}$. As $u_4 \to u_5, u_5 \not\to u_4$. Thus $u_6 \to u_4$. Now $u_1 \to u_4$ and $u \to u_1$ implies that $N^+(u_5) = \{u_1, v_4\}$ and $N^+(u_6) = \{u, u_4\}$. Finally we must have $N^+(v_4) = \{v, u_6\}$. This gives us the $(2, 2, +2)$-digraph shown in **Fig. 2**. \(\square\)

We can now assume that all pairs given by **Lemma 2** are good. Let us fix a pair $(u, v)$ with a single common out-neighbour. It follows from **Corollary 1** and the definition of a good pair that $v_1 \in O(u)$; otherwise $O(u)$ would contain $v, v_3$ and $v_4$, which is impossible. Likewise $u_1 \in O(v)$.
Considering the positions of $v_3$ and $v_4$, we see that there are without loss of generality four possibilities: (1) $u = v_3$, $u_4 = v_4$. (2) $u = v_3$, $O(u) = \{v_1, v_4\}$. (3) $N^+(u_1) = N^+(v_1)$ and (4) $u_3 = v_3$, $O(u) = \{v_1, u_4\}$. A suitable relabelling of vertices shows that case 4 is equivalent to case 1.a below, so we will examine cases 1 to 3 in turn.

**Case 1: $u = v_3$, $u_4 = v_4$**

Depending upon the position of $v$, we must either have $O(u) = \{v_1, v\}$ and $O(v) = \{u_1, u_3\}$ or $v = u_3$ (see Fig. 3).

**Case 1.a: $O(u) = \{v_1, v\}$, $O(v) = \{u_1, u_3\}$**

In this case $V(G) = \{u, u_1, u_2, u_3, u_4, u_5, u_6, v, v_1\}$. $u_1$ and $v_1$ have a single common out-neighbour, namely $u_4$, so, as we are assuming all such pairs to be good, we have $u_3 \in O(v_1)$, $u \in O(u_1)$. By 2-geodecity, $N^+(u_4) \subset \{u_5, u_6, v\}$, so without loss of generality either $N^+(u_4) = \{u_5, u_6\}$ or $N^+(u_4) = \{u_5, v\}$.

Suppose that $N^+(u_4) = \{u_5, u_6\}$. By elimination, $O(v_1) = \{v, u_3\}$. As $G$ is diregular, every vertex is an outlier of exactly two vertices; $v$ is an outlier of $u$ and $v_1$, so both $u_1$ and $u_2$ can reach $v$ by a $\leq 2$-path. Hence $v \in N^+(u_3)$. As $v \to v_1$, we see that $v_1$ is an outlier of $u_1$; as $u$ is also an outlier of $u_1$, we have $O(u_1) = \{u_1, v_1\}$ and $N^+(u_3) = \{v, u_2\}$. As $v \to u_2$, this is impossible.

Now consider $N^+(u_4) = \{u_5, v\}$. We now have $O(v_1) = \{u_3, u_6\}$. Thus $u_3 \in O(v) \cap O(v_1)$, so $u_3 \in T_2(u_4)$, $v$ is not adjacent to $u_3$, so $u_3 \in N^+(u_5)$. $u_2$ and $u_4$ have $u_5$ as a unique common out-neighbour, so $u_6 \in O(u_4)$, $v \in O(u_2)$. As $u_6 \in O(v_1) \cap O(u_4)$, $u_1$ can reach $u_6$. Hence $u_6 \in N^+(u_1)$. Neither $u$ nor $v$ lie in $T(u_1)$, so $u_2 \in O(u_1)$. Therefore either $O(u_1) = \{u, u_2\}$ or $O(u_1) = \{v, u_2\}$. If $O(u_1) = \{u, u_2\}$, then $N^+(u_3) = \{u_6, v, u_2\}$. $u_6$ cannot reach $v_1$, since $u_3 \not\in T(u_2)$, so $O(u_2) = \{v, u_1\}$ and $N^2(u_2) = \{u, u_1, u_2, u_4\}$. As $u_4 \to u_2$, $u_4 \in N^+(u_6)$, $u_1 \to u_4$, so $N^+(u_3) = \{u_1, u_3\}$. As $u_1 \to u_3$, this is a contradiction. Thus $O(u_1) = \{u, u_2\}$, so that $N^+(u_3) = \{u, u_6\}$. $u_1$ must have an in-neighbour apart from $u$, which must be either $u_5$ or $u_6$. As $u_1 \to u_3$, we have $u_1 \in N^+(u_6)$. By elimination, $v$ and $v_1$ must also have in-neighbours in $\{u_5, u_6\}$. As $u_1$ and $v_1$ have a common out-neighbour, we have $N^+(u_5) = \{u_2, v_1\}$, $N^+(u_6) = \{u_1, v\}$. However, both $u_3$ and $v_1$ are adjacent to $u$, violating 2-geodecity.

**Case 1.b: $v = u_3$**

There exists a vertex $x$ such that $V(G) = \{u, u_1, u_2, v, u_4, u_5, u_6, v_1, x\}$, $O(u) = \{v_1, x\}$ and $O(v) = \{u_1, x\}$. As $x \in O(u) \cap O(v)$, $u_1$ and $u_2$ can reach $x$, so without loss of generality $x \in N^+(u_4) \cap N^+(u_5)$. As $u_3$ and $u_4$ have a common out-neighbour, $u_3 \in O(u_1)$. Also, $u_1$ and $v_1$ have $u_4$ as a unique common out-neighbour, so $u \in O(u_1)$ and $O(u_1) = \{u, u_6\}$. Thus $N^+(u_4) = \{x, u_6\}$. Observe that $u_2$ and $u_4$ have the out-neighbour $u_6$ in common. Thus $x \in O(u_2)$, whereas we already have $x \in O(u) \cap O(v)$, a contradiction.

**Case 2: $u = v_3$, $O(u) = \{v_1, u_4\}$**

As $v$ is not equal to $v_1$ or $u_4$, $v$ must lie in $T_2(u)$. Without loss of generality, $v = u_3$. Hence $V(G) = \{u, u_1, u_2, v, u_4, u_5, u_6, v_1, v_4\}$ and $O(v) = \{u_1, u_4\}$. We have the configuration shown in Fig. 4. Hence $u_1$ can reach
u₁, v, u₄, u₂ and v₁, so we have without loss of generality one of the following: a) \( O(u₁) = \{u, v₄\}, N⁺(u₄) = \{u₅, u₆\} \), b) \( O(u₁) = \{u, u₅\}, N⁺(u₄) = \{u₆, v₄\} \), c) \( O(u₁) = \{u₅, u₆\}, N⁺(u₄) = \{u, v₄\} \) or d) \( O(u₁) = \{u₅, v₄\}, N⁺(u₄) = \{u, v₆\} \).

**Case 2.a: O(u₄) = \{u, v₄\}, N⁺(u₄) = \{u₅, u₆\}**

As \( v₄ \in O(u) \cap O(u₁) \), u₂ can reach \( v₄ \) and without loss of generality \( v₄ \in N⁺(u₅), N⁺(u₄) = N⁺(u₄) \), so by Lemma 1 u₂ ∈ O(u₄), u₄ ∈ O(u₅), u₅ ∈ O(u₆) and u₆ ∈ O(u₅). Hence u₄ ∈ O(v) \( \cap \) O(u₂), so v₁ can reach u₄, so u₄ ∈ N⁺(v₄). Neither \( u₅ \) nor u₆ lies in \( N⁺(v₄) \), so \( O(v₁) = \{u₅, u₆\} \) and \( N⁺(u₄) = \{u₄, v₄\} \). Hence \( O(v₄) = \{u, u₁\} \). Observe that \( N⁺(u₁) = N⁺(v₄) \), so that \( v \in O(u₄) \). Therefore \( v \notin N⁺(u₅) \cup N⁺(u₆) \), yielding \( O(u₂) = \{u₄, v\}, N⁺(u₂) = \{u₄, v₁, u, u₁\} \). As v₁ → v₄ and \( N⁺(u₁) = N⁺(v₄) \), we must have \( N⁺(u₄) = \{v₄, u\}, N⁺(u₆) = \{v₁, u₁\} \). This yields the digraph in Fig. 5. Unlike the digraph in Fig. 2, this digraph contains pairs of vertices with identical out-neighbourhoods, so the two are not isomorphic.

**Case 2.b: O(u₄) = \{u, u₅\}, N⁺(u₄) = \{u₆, v₄\}**

As u₄ → v₄, u₄ \( \notin N⁺(v₄) \), so u₄ ∈ O(v₁). Hence u₄ ∈ O(v) \( \cap \) O(v₁), so u₂ can reach u₄. As u₄ → u₆, we must have u₅ → u₄, u₂ and u₄ have u₆ as a common out-neighbour, so \( v₄ \in O(u₂), u₅ \in O(u₄) \). Therefore \( v₄ \in O(u) \cap O(u₂) \), so that u₆ can reach v₄, but \( v₄ \notin T(u₅) \), so \( N⁺(u₄) \) contains an in-neighbour of \( v₄, u₄ \notin N⁺(u₆) \), so we must have u₆ → v₁. We have u₅ ∈ O(u₄) \( \cap \) O(u₁), so v₁ can reach u₅ and hence \( v₄ \rightarrow u₅, v₁ \) cannot reach u₆, as \( u₂, u₄ \notin T(v₁) \), so \( O(v₁) = \{u₄, u₆\} \). Now u₂ and u₄ have u₆ as a unique common out-neighbour, so \( u₆ \in O(v₄), v \in O(u₂) \). Thus \( O(u₂) = \{v, v₄\} \) and \( N²(u₂) = \{u₄, v₁, u, u₁\} \). Taking into account adjacencies between members of \( N²(u₂) \), it follows that \( N⁺(u₄) = \{v₄, u\}, N⁺(u₆) = \{v₁, u₁\} \). However, \( (u₂, u₄) \) now constitutes a bad pair, contradicting our assumption.

**Case 2.c: O(u₁) = \{u, u₅\}, N⁺(u₄) = \{u₆, v₄\}**

As u₄ → v₄, u₄ \( \notin N⁺(v₄) \), hence u₄ ∈ O(v) \( \cap \) O(v₁), implying that u₂ can reach u₄. Without loss of generality, u₅ → u₄. There are three possibilities: (i) \( O(v₁) = \{u₄, u₅\} \), \( N⁺(u₄) = \{v, u₅\} \), (ii) \( O(v₁) = \{u₄, u₅\} \), \( N⁺(u₄) = \{v, u₆\} \), (iii) \( O(v₁) = \{u₄, v\} \), \( N⁺(u₄) = \{v, u₆\} \)

(i) \( O(v₁) = \{u₄, u₅\} \), \( N⁺(u₄) = \{v, u₅\} \)

u₁ and v₄ have v as a unique common out-neighbour, so \( v₄ \in O(u₅) \). However, this contradicts \( v₄ \rightarrow u₅ \rightarrow u₄ \).

(ii) \( O(v₁) = \{u₄, u₅\} \), \( N⁺(u₄) = \{v, u₆\} \)

Neither u₄ nor v₁ lie in \( T(u₂) \), so \( v₄ \in O(u₂) \). Now observe that if u₂ and u₄ have u₆ as unique common out-neighbour, so \( v \in O(u₂) \), yielding \( O(u₂) = \{v, v₄\} \) and \( N²(u₂) = \{u₄, v₁, u, v₁\} \). As u₄ → u and u → u₁, we must have \( N⁺(u₅) = \{u₄, u₁\} \), \( N⁺(u₆) = \{u, v₁\} \), a contradiction, since u₁ → u₄.

(iii) \( O(v₁) = \{u₄, v\} \), \( N⁺(u₄) = \{v, u₅\} \)

We now have \( N⁺(u₂) = N⁺(v₄) \), so u₂ ∈ O(v₄), u₄ ∈ O(u₂), u₅ ∈ O(u₆), u₆ ∈ O(u₅). Also \( N⁺(u₄) = N⁺(v₄) \), so u₄ ∈ O(v₁), v₁ ∈ O(u₄) and u ∈ O(u₄). Since \( u \notin N⁺(u₄) \cup N⁺(u₅) \), we see that u ∈ O(u₂) and hence \( O(u₂) = \{u, v₄\} \) and \( N²(u₂) = \{u₄, u₁, v, u₁\} \). As u₁ → u₄ and u₁ → v, we have \( N⁺(u₅) = \{u₄, v\}, N⁺(u₆) = \{u₁, v₁\} \). It is not difficult to show that this yields a \((2, 2, +2)\)-digraph isomorphic to that in Fig. 5.

**Case 2.d: O(u₁) = \{u₅, v₄\}, N⁺(u₄) = \{u, u₆\}**

In this case \( v₄ \in O(u) \cap O(u₁) \), so u₂ can reach v₄, u₄ and v₁ have unique common out-neighbour u, so \( v₄ \in O(u₄), u₆ \in O(v₁) \). If v₄ → v₄, then we would have u₄ → u₆ → v₄, contradicting \( v₄ \in O(u₄) \), so u₅ → v₄. This also implies that \( u₅ \notin N⁺(v₄) \), so \( u₅ \in O(v₁) \), yielding \( O(v₁) = \{u₅, u₆\} \) and \( N⁺(v₄) = \{v, u₄\} \). Now v₄, u₁ \( \notin T(u₂) \), so \( O(u₂) = \{v, u₄\} \).
4. Main result

We can now complete our analysis by showing that there are no diregular \((2, k, +2)\)-digraphs for \(k \geq 3\). Let \(G\) be such a digraph. By \textbf{Lemma 2}, \(G\) contains vertices \(u\) and \(v\) with a unique common out-neighbour. In accordance with our vertex-labelling convention, we have the situation in \textbf{Fig. 6}. A triangle based at a vertex \(x\) represents the set \(T(x)\).

We now proceed to determine the possible outlier sets of \(u\) and \(v\).

\textbf{Lemma 5.} \(v \in N^{k-1}(u_1) \cup O(u)\) and \(u \in N^{k-1}(v_1) \cup O(v)\). If \(v \in O(u)\), then \(u_2 \in O(u_1)\) and if \(u \in O(v)\), then \(u_2 \in O(v_1)\).

\textbf{Proof.} \(v\) cannot lie in \(T(u)\), or the vertex \(u_2\) would be repeated in \(T_u(u)\). Also, \(v \notin T(u_2)\), or there would be a \(\leq k\)-cycle through \(v\). Therefore, if \(v \notin O(u)\), then \(v \in N^{k-1}(u_1)\). Likewise for the other result. If \(v \in O(u)\), then neither in-neighbour of \(u_2\) lies in \(T(u_1)\), so that \(u_2 \in O(u_1)\).

\textbf{Lemma 6.} Let \(w \in T(v_1)\), with \(d(v_1, w) = l\). Suppose that \(w \in T(u_1)\), with \(d(u_1, w) = m\). Then either \(m \leq l\) or \(w \in N^{k-1}(u_1)\). A similar result holds for \(w \in T(u_1)\).

\textbf{Proof.} Let \(w\) be as described and suppose that \(m > l\). Consider the set \(N^{k-m}(w)\). By construction, \(N^{k-m}(w) \subseteq N^{k}(u_1)\), so by \(k\)-geodecity \(N^{k-m}(w) \cap T(u_1) = \varnothing\). At the same time, we have \(l + k - m \leq k - 1\), so \(N^{k-m}(w) \subseteq T(v_1)\). This implies that \(N^{k-m}(w) \cap T(v_2) = N^{k-m}(w) \cap T(u_2) = \varnothing\). As \(V(G) = \{u\} \cup T(u_1) \cup T(u_2) \cup O(u)\), it follows that \(N^{k-m}(w) \subseteq \{u\} \cup O(u)\). Therefore \(|N^{k-m}(w)| = 2k - m \leq 3\). By assumption \(0 \leq m \leq k - 1\), so it follows that \(m = k - 1\).

\textbf{Corollary 3.} If \(w \in T(v_1)\), then either \(w \in \{u\} \cup O(u)\) or \(w \in T(u_1)\) with \(d(u_1, w) = k - 1\) or \(d(u_1, w) \leq d(v_1, w)\).

\textbf{Proof.} By \(k\)-geodecity and \textbf{Lemma 6}.

\textbf{Corollary 4.} \(v_1 \in N^{k-1}(u_1) \cup O(u)\) and \(u_1 \in N^{k-1}(v_1) \cup O(v)\).
Proof. We prove the first inclusion. By Corollary 3, $v_1 \in \{u\} \cup O(u) \cup \{u_1\} \cup N^{k-1}(u_1)$. By $k$-geodecyy, $v_1 \neq u$ and by construction, $v_1 \neq u_1$. □

We now have enough information to identify one member of $O(u)$ and $O(v)$.

Lemma 7. $v_1 \in O(u)$ and $u_1 \in O(v)$.

Proof. We prove that $v_1 \in O(u)$. Suppose that neither $v_1$ nor $v$ lies in $O(u)$. Then by Lemma 5 and Corollary 4 we have $v, v_1 \in N^{k-1}(u_1)$. As $v_1$ is an out-neighbour of $v$, it follows that $v_1$ appears twice in $T_k(u_1)$, violating $k$-geodecyy. Therefore $O(u) \cap \{v, v_1\} = \emptyset$.

Now assume that $v_1, v_3 \in T_k(u)$. Again by Corollary 4, $v_1 \in N^{k-1}(u_1)$. By $k$-geodecyy we also have $v_3 \in T(u_1)$. However, $v_1 \in N^{+}(v_3)$, so $v_1$ appears twice in $T_k(u_1)$, which is impossible. Hence $O(u) \cap \{v_1, v_3\} = \emptyset$. Similarly, $O(u) \cap \{v_1, v_4\} = \emptyset$. In the terminology of the previous section, $G$ contains no bad pairs. Therefore, if $v_1 \notin O(u)$, then $\{v, v_3, v_4\} \subseteq O(u)$. Since these vertices are distinct, this is a contradiction and the result follows. □

Lemma 7 allows us to conclude that for vertices sufficiently close to $v_1$ one of the potential situations mentioned in Corollary 3 cannot occur.

Lemma 8. $T_{k-3}(v_1) \cap N^{k-1}(u_1) = T_{k-3}(u_1) \cap N^{k-1}(v_1) = \emptyset$.

Proof. Let $w \in T_{k-3}(v_1) \cap N^{k-1}(u_1)$. Consider the position of the vertices of $N^{+}(w)$ in $T_k(u) \cup O(u)$. As $v_1 \notin N^{+}(w)$, it follows from Lemma 7 that at most one of the vertices of $N^{+}(w)$ can be an outlier of $u$, so let us write $w_1 \in N^{+}(w) \setminus O(u)$. By $k$-geodecyy, $w_1 \notin T(u_1) \cup \{u\}$. Hence $w_1 \in T(u_2) = T(v_2)$. However, $u_1$ also lies in $T(v_1)$, so this violates $k$-geodecyy. □

Corollary 5. There is at most one vertex in $T_{k-3}(v_1) \setminus \{v_1\}$ that does not lie in $T(u_1)$; for all other vertices $w \in T_{k-3}(v_1) \setminus \{v_1\}$, $d(u_1, w) = d(v_1, w)$. A similar result for $T_{k-3}(u_1) \setminus \{u_1\}$ also holds.

Lemma 9. For $k = 3$, $N^{+}(u_1) \cap N^{2}(v_1) = N^{+}(v_1) \cap N^{2}(u_1) = \emptyset$.

Proof. Suppose that $v_3 = u_2$. By the reasoning of Lemma 8 we can set $u = v_2$ and $O(u) = \{v_1, v_8\}, v \notin O(u)$ and by 3-geodecyy $v \notin N^{+}(u_3)$, so we can assume that $u = u_9, u_3 \rightarrow v_3$ implies that $u_3 \notin T(v_1)$, so $O(v) = \{u_1, u_1\}$. We must have $\{u_4, u_5, u_{10}\} = \{v_4, v_5, v_{10}\}$. As $u_4 \rightarrow v$, it follows that $v_4 = u_8$ and hence $\{u_4, u_{10}\} = \{v_2, v_{10}\}$, which is impossible. □

As $u_1$ is an outlier of $v$, neither $v_3$ nor $v_4$ can be equal to $u_1$. It follows from Corollary 5 and Lemma 9 that either $N^{+}(u_1) = N^{+}(u_1) = N^{+}(u_1)$ or $u_1$ and $v_1$ have a single common out-neighbour, with one vertex of $N^{+}(v_1)$ being an outlier of $u$.

Lemma 10. $N^{2}(u) \neq N^{2}(v)$

Proof. Let $N^{2}(u) = N^{2}(v)$, with $N^{+}(u_1) = N^{+}(v_1) = \{u_3, u_4\}$. Suppose that $v \notin O(u)$. By Lemma 5, $v \in N^{k-2}(u_2) \cup N^{k-2}(u_4)$. But then there is a $k$-cycle through $v$. It follows that $O(u) = \{v, v_1\}$. $O(v) = \{u_1, u_1\}$. By Lemma 5, $u_2 \in O(u_1) \cap O(v_1)$. Therefore by Lemma 1 $O(u_1) = \{u_2, v_1\}$. $O(v_1) = \{u_2, u_1\}$.

Consider the in-neighbour $v'$ of $u_1$ that is distinct from $u$. We have either $|N^{+}(u') \cap N^{+}(u)| = 1$ or $|N^{+}(u') \cap N^{+}(u)| = 2$. In the first case, it follows from Lemma 7 that $u_2 \in O(u)$. Every vertex of $G$ is an outlier of exactly two vertices, so $u' = u_1$ or $v_1$. In either case, we have a contradiction. Therefore $N^{+}(u') = N^{+}(u)$. It now follows from Lemma 1 that $u' \in O(u) = \{v, v_1\}$, which is impossible. □

Noticing that $u_1$ and $v_1$ also have a unique common out-neighbour, we have the following corollary.

Corollary 6. Without loss of generality, $u_1 = v_3, u_9 = v_9, O(u) = \{v_1, u_4\}, O(v) = \{u_1, u_4\}, O(u_1) = \{v_4, u_{10}\}$ and $O(v_1) = \{u_4, u_{10}\}$.

We are now in a position to complete the proof by deriving a contradiction.

Theorem 2. There are no digereal $(2, k, +2)$-digraphs for $k \geq 3$.

Proof. $u, v \notin \{u_1, u_3, v_1, v_4\}$, so by Lemma 5 $d(u, v) = d(v, u) = k$. In fact, $u_2 = v_2$ implies that $v \in N^{k-2}(u_2)$ and $u \in N^{k-2}(v_2)$. Let $k \geq 4$. Then $u, v \notin \{u_{10}, u_{10}\}$, so $u, v \in T_k(u_1) \setminus T_k(v_1)$. If $u \in T(u_2) = T(v_2)$, then $u$ would appear twice in $T_k(u_1)$, so $u \in N^{k-1}(u_2)$. However, as $u$ and $v$ have a common out-neighbour, this violates $k$-geodecyy.

Finally, suppose that $k = 3$. The above analysis will hold unless $u = u_{10}$ and $v = u_{10}$. Let $N^{+}(u_1) = \{u, u', N^{+}(v_1) = \{v, v'\}$. It is evident that $v' \notin \{v_1, v_4\}$, so that $v' \in T_3(u)$. As $v \in N^{+}(u_4)$, we must have $v' \in N^{2}(u_2)$. Similarly $u' \in N^{2}(v_2)$. Since $u_1$ and $v_1$ have a common out-neighbour, we can assume that $u' \in N^{+}(u_5)$ and $v' \in N^{+}(u_6)$. $v_4$ can be the outlier of only two vertices, namely $u$ and $u_1$, so $v_4 \in N^{3}(u_2)$ and likewise $u_4 \in N^{3}(u_2)$. By 3-geodecyy $v_4 \in N^{2}(u_2)$ and $u_4 \in N^{2}(u_6)$. It follows that $u, v \notin N^{3}(u_2)$, so $u \notin T_3(u_1) \cup T_3(u_2)$. Hence $O(u) = N^{+}(u) = \{v_1, v_4\}$, which again is impossible. □

It is interesting to note that a similar argument can be used to provide an alternative proof of the result of [6].
References