Ondiregulardigraphswithdegree two and excess two

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Abstract

An important topic in the design of efficient networks is the construction of \((d, k, +\epsilon)\)-digraphs, i.e. \(k\)-geodetic digraphs with minimum out-degree \(\geq d\) and order \(M(d, k) + \epsilon\), where \(M(d, k)\) represents the Moore bound for degree \(d\) and diameter \(k\) and \(\epsilon > 0\) is the (small) excess of the digraph. Previous work has shown that there are no \((2, k, +1)\)-digraphs for \(k \geq 2\). In a separate paper, the present author has shown that any \((2, k, +2)\)-digraph must be regular for \(k \geq 2\). In the present work, this analysis is completed by proving the nonexistence of diregular \((2, k, +2)\)-digraphs for \(k \geq 3\) and classifying \((2, 2, +2)\)-digraphs up to isomorphism.

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1. Introduction

An important topic in the design of interconnection networks is the directed degree/diameter problem: what is the largest possible order \(N(d, k)\) of a digraph \(G\) with maximum out-degree \(d\) and diameter \(\leq k\)? A simple inductive argument shows that for \(0 \leq l \leq k\) the number of vertices at distance \(l\) from a fixed vertex \(u\) is bounded above by \(d^l\). Therefore, a natural upper bound for the order of such a digraph is the so-called Moore bound \(M(d, k) = 1 + d + d^2 + \cdots + d^k\). A digraph that attains this upper bound is called a Moore digraph. It is easily seen that a digraph \(G\) is Moore if and only if it is out-regular with degree \(d\), has diameter \(k\) and is \(k\)-geometric, i.e. for any two vertices \(u, v\) there is at most one \(k\)-path from \(u\) to \(v\).

As it was shown by Bridges and Toueg in [1] that Moore digraphs exist only in the trivial cases \(d = 1\) or \(k = 1\) (the Moore digraphs are directed \((k + 1)\)-cycles and complete digraphs \(K_{k+1}\) respectively), much research has been devoted to the study of digraphs that in some sense approximate Moore digraphs. For example, there is an extensive literature on digraphs with maximum out-degree \(d\), diameter \(\leq k\) and order \(M(d, k) - \delta\) for small \(\delta > 0\); this is equivalent to relaxing the \(k\)-geometricity requirement in the conditions for a digraph to be Moore. \(\delta\) is known as the defect of the digraph. The reader is referred to the survey [5] for more information.

In this paper, however, we will consider the following related problem, which is obtained by retaining the \(k\)-geometricity requirement in the above characterisation of Moore digraphs, but allowing the diameter to exceed \(k\): what is the smallest possible order of a \(k\)-geometric digraph \(G\) with minimum out-degree \(\geq d\)? A \(k\)-geometric digraph with minimum out-degree \(\geq d\) and order \(M(d, k) + \epsilon\) is said to be a \((d, k, +\epsilon)\)-digraph or to have excess \(\epsilon\). It was shown in [6] that there are no \((2, k, +1)\)-digraphs for \(k \geq 2\). In 2016 it was shown in [3] that digraphs with excess one must be diregular and that there are no \((d, k, +1)\)-digraphs for \(k = 2, 3, 4\) and sufficiently large \(d\). In a separate paper [7], the present author has shown that \((2, k, +2)\)-digraphs must be regular with degree \(d = 2\) for \(k \geq 2\). In the present paper, we classify the \((2, 2, +2)\)-digraphs up to isomorphism and show that there are no diregular \((2, k, +2)\)-digraphs for \(k \geq 3\), thereby completing the proof of the nonexistence of digraphs with degree \(d = 2\) and excess \(\epsilon = 2\) for \(k \geq 3\). Our reasoning and notation will follow closely that employed in [4] for the corresponding result for defect \(\delta = 2\).
2. Preliminary results

We will let $G$ stand for a $(2, k, +2)$-digraph for arbitrary $k \geq 2$, i.e. $G$ has minimum out-degree $d = 2$, is $k$-geodetic and has order $M(2, k) + 2$. We will denote the vertex set of $G$ by $V(G)$. By the result of [7], $G$ must be digiregular with degree $d = 2$ for $k \geq 2$. The distance $d(u, v)$ between vertices $u$ and $v$ is the length of the shortest path from $u$ to $v$. Notice that $d(u, v)$ is not necessarily equal to $d(v, u)$. $u \rightarrow v$ will indicate that there is an arc from $u$ to $v$. We define the in- and out-neighbourhoods of a vertex $u$ by $N^-(u) = \{v \in V(G) : v \rightarrow u\}$ and $N^+(u) = \{v \in V(G) : u \rightarrow v\}$ respectively; more generally, for $0 \leq l \leq k$, the set $\{v \in V(G) : d(u, v) = l\}$ of vertices at distance exactly $l$ from $u$ will be denoted by $N^l(u)$. For $0 \leq l \leq k$ we will also write $T_l(u) = \bigcup_{i=0}^{l} N^i(u)$ for the set of vertices at distance $\leq l$ from $u$. The notation $T_{k-1}(u)$ will be abbreviated by $T(u)$.

It is easily seen that for any vertex $u$ of $G$, there are exactly two distinct vertices that are at distance $\geq k + 1$ from $u$. For any $u \in V(G)$, we will write $O(u)$ for the set of these vertices and call such a set an outlier set and its elements outliers of $u$. Notice that $O(u) = V(G) - T_k(u)$. An elementary counting argument shows that in a digiregular $(2, k, +2)$-digraph every vertex is also an outlier of exactly two vertices. We will say that a vertex $u$ can reach a vertex $v$ if $v \notin O(u)$.

Our proof will proceed by an analysis of a pair of vertices with exactly one common out-neighbour. First, we must show that such a pair exists and deduce some elementary properties of pairs of vertices with identical out-neighbourhoods.

**Lemma 1.** For $k \geq 2$, let $u$ and $v$ be distinct vertices such that $N^+(u) = N^+(v) = \{u_1, u_2\}$. Then $u_1 \in O(u_2)$, $u_2 \in O(u_1)$ and there exists a vertex $x$ such that $O(u) = \{v, x\}$, $O(v) = \{u, x\}$.

**Proof.** Suppose that $u$ can reach $v$ by a $\leq k$-path. Then $v \in T(u_1) \cup T(u_2)$. As $N^+(v) = N^+(u)$, it follows that there would be a $\leq k$-cycle through $v$, contradicting $k$-geodeticity. If $O(u) = \{v, x\}$, then $x \neq v$ and $x \notin T(u_1) \cup T(u_2)$, so that $v$ cannot reach $x$ by a $\leq k$-path. Similarly, if $u_1$ can reach $u_2$ by a $\leq k$-path, then we must have $\{u, v\} \cap T(u_1) \neq \emptyset$, which is impossible. □

**Lemma 2.** For $k \geq 2$, there exists a pair of vertices $u, v$ with $|N^+(u) \cap N^+(v)| = 1$.

**Proof.** Suppose for a contradiction that there is no such pair of vertices. Define a map $\phi : V(G) \rightarrow V(G)$ as follows. Let $u^+ \in$ an out-neighbour of a vertex $u$ and let $\phi(u)$ be the in-neighbour of $u^+$ distinct from $u$. By our assumption, it is easily verified that $\phi$ is a well-defined bijection with no fixed points and with square equal to the identity. It follows that $G$ must have even order, whereas $|V(G)| = M(2, k) + 2$ is odd. □

$u, v$ will now stand for a pair of vertices with a single common out-neighbour. We will label the vertices of $T_k(u)$ according to the scheme $N^+(u) = \{u_1, u_2\}$, $N^+(u_1) = \{u_3, u_4\}$, $N^+(u_2) = \{u_5, u_6\}$, $N^+(u_3) = \{u_7, u_8\}$, $N^+(u_4) = \{u_9, u_{10}\}$ and so on, with the same convention for the vertices of $T_k(v)$, where we will assume that $u_2 = v_2$.

3. Classification of $(2, 2, +2)$-digraphs

We begin by classifying the $(2, 2, +2)$-digraphs up to isomorphism. We will prove the following theorem.

**Theorem 1.** There are exactly two digiregular $(2, 2, +2)$-digraphs, which are displayed in Figs. 2 and 5.

Let $G$ be an arbitrary digiregular $(2, 2, +2)$-digraph. $G$ has order $M(2, 2) + 2 = 9$. By Lemma 2, $G$ contains a pair of vertices $(u, v)$ such that $|N^+(u) \cap N^+(v)| = 1$; we will assume that $u_2 = v_2$, so that we have the situation shown in Fig. 1.

We can immediately deduce some information on the possible positions of $v$ and $v_1$ in $T_2(u)$.

**Lemma 3.** If $v \notin O(u)$, then $v \in N^+(u_1)$. If $v_1 \notin O(u)$, then $v_1 \in N^+(u_1)$.

**Proof.** $v \notin T(u_2)$ by 2-geodecty. $v \neq u$ by construction. If we had $v = u_1$, then there would be two distinct $\leq 2$-paths from $u$ to $u_2$. Also $v_1 \notin \{u\} \cup T(u_2)$ by 2-geodecty and by assumption $u_1 \neq v_1$. □
Proof. Assume that there exists a bad pair \((v, u)\) by 2-geodecity, we have the following corollary.

**Corollary 1.** \(O(u) \cap \{v, v_1\} \neq \emptyset\).

We will call a pair of vertices \((u, v)\) with a single common out-neighbour bad if at least one of
\[
O(u) \cap \{v_1, v_3\} = \emptyset, O(u) \cap \{v_1, v_4\} = \emptyset, O(v) \cap \{u_1, u_5\} = \emptyset, O(v) \cap \{u_1, u_4\} = \emptyset.
\]
holds. Otherwise such a pair will be called good.

**Lemma 4.** There is a unique \((2, 2, +2)\)-digraph containing a bad pair.

**Proof.** Assume that there exists a bad pair \((u, v)\). Without loss of generality, \(O(u) \cap \{v_1, v_3\} = \emptyset\). By Lemma 3 we can set \(v_1 = u_3\). By 2-geodecity \(v_3 = u\). We cannot have \(v_4 = v_3 = u\), so \(v_4\) must be an outlier of \(u\). By Corollary 1 it follows that \(O(u) = \{v, v_4\}\).

Consider the vertex \(u_1\). By Lemma 3, if \(u_1 \notin O(v)\), then \(u_1 \in N^+(v_1)\). However, as \(v_1 = u_3\), there would be a 2-cycle through \(u_1\). Hence \(u_1 \in O(v)\). As \(O(u) = \{v, v_4\}\), we have \(V(G) = \{u, u_1, u_2, u_3 = v_1, u_4, u_5, u_6, v, v_4\}\) and \(O(v) = \{u_1, u_4\}\). As neither \(u\) nor \(v\) lies in \(T(u_1)\), we must also have \(u_2 \in O(u_1)\). As \(u_1\) can reach \(u_1, u_4, u\) and \(v_4\), it follows that without loss of generality we either have \(O(u_1) = \{u_2, v\}\) and \(N^+(u_4) = \{u_5, u_6\} = N^+(u_2)\) or \(O(u_1) = \{u_2, u_6\}\) and \(N^+(u_4) = \{v, u_5\}\). In either case, \((v, u_1)\) is a good pair.

Suppose firstly that \(N^+(u_2) = N^+(u_4)\). Then \(v\) is an outlier of \(u\) and \(u_1\). As each vertex is the outlier of exactly two vertices, \(v_1\) must be able to reach \(v\) by a \(\leq 2\)-path. Hence \(v_1 \rightarrow v\). Likewise \(u_2\) can reach \(v\), so without loss of generality \(u_5 \rightarrow v\). Suppose that \(O(u_2) \cap \{u, u_1\} = \emptyset\). As \(v\) and \(u\) have a common out-neighbour, we must have \(u_6 \rightarrow u\). Since \(v_1 \rightarrow u_1\) by 2-geodecity we must have \(u_5 \rightarrow u_1\). However, this is a contradiction, as \(v\) and \(u_1\) also have a common out-neighbour. Therefore, at least one of \(u, u_1\) is an outlier of \(u_2\). By Lemma 1 \(u_4\) is an outlier of \(u_2\). Therefore either \(O(u_2) = \{u, u_4\}\) or \(O(u_2) = \{u_1, u_4\}\). If \(O(u_2) = \{u, u_4\}\), then \(u_2\) must be able to reach \(u_1, v_1\) and \(v_4, u_5 \rightarrow v\) and \(v \rightarrow v_1\), so \(v_1 \in N^+(u_6)\). As \(v_1 \rightarrow v_1\), we must have \(N^+(u_5) = \{v, u_1\}\). As \(v\) and \(u_1\) have a common out-neighbour, this violates 2-geodecity. Hence \((v, u_1)\) is a good pair.

We are left with the case \(O(u_1) = \{u_2, u_6\}\) and \(N^+(u_4) = \{v, u_5\}\). Then \(v_1 \in O(u_2)\), as neither \(v\) nor \(u_1\) lies in \(T(u_2)\). Observe that \(u_2\) and \(u_4\) have a single common out-neighbour, so by Corollary 1 \(O(u_2) \cap \{u_4, v\} \neq \emptyset\). Therefore either \(O(u_2) = \{v_1, u_4\}\) or \(O(u_2) = \{v_1, v\}\). Suppose firstly that \(O(u_2) = \{v_1, u_4\}\). Then \(N^+(u_2) = \{u, v, u_1, v_4\}\). As \(N^+(u_4) = \{v, u_5\}\), \(u_5 \nrightarrow v\), so \(u_6 \rightarrow v\). As \(N^+(u) \cap N^+(v) \neq \emptyset\), \(u_5 \rightarrow u\), so necessarily \(u_5 \in N^+(u_6)\). However, \(v_1 \in N^+(u_1) \cap N^+(v)\), contradicting 2-geodecity.

Hence \(O(u_2) = \{v_1, v\}\) and \(N^2(u_2) = \{u, u_1, v_4, v_4\}\). As \(u_4 \rightarrow u_5, u_5 \nrightarrow u_4\). Thus \(u_6 \rightarrow u_4\). Now \(u_1 \rightarrow u_4\) and \(u \rightarrow u_1\) implies that \(N^+(u_5) = \{u_1, v_4\}\) and \(N^+(u_6) = \{u, u_4\}\). Finally we must have \(N^+(u_4) = \{u, u_6\}\). This gives us the \((2, 2, +2)\)-digraph shown in Fig. 2. □

We can now assume that all pairs given by Lemma 2 are good. Let us fix a pair \((u, v)\) with a single common out-neighbour. It follows from Corollary 1 and the definition of a good pair that \(v_1 \in O(u)\); otherwise \(O(u)\) would contain \(v, v_3\) and \(v_4\), which is impossible. Likewise \(u_1 \in O(v)\).
Considering the positions of \(v_3\) and \(v_4\), we see that there are without loss of generality four possibilities: (1) \(u = v_3, u_4 = v_4\), (2) \(u = v_3, O(u) = \{v_1, v_4\}\), (3) \(N^+(u_1) = N^+(v_1)\) and (4) \(u_3 = v_3, O(u) = \{v_1, u_4\}\). A suitable relabelling of vertices shows that case 4 is equivalent to case 1.a below, so we will examine cases 1 to 3 in turn.

**Case 1:** \(u = v_3, u_4 = v_4\)

Depending upon the position of \(v\), we must either have \(O(u) = \{v_1, v\}\) and \(O(v) = \{u_1, u_3\}\) or \(v = u_3\) (see Fig. 3).

**Case 1.a:** \(O(u) = \{v_1, v\}, O(v) = \{u_1, u_3\}\)

In this case \(V(G) = \{u, u_1, u_2, u_3, u_4, u_5, u_6, v, v_1\}\). \(u_1\) and \(v_1\) have a single common out-neighbour, namely \(u_4\), so, as we are assuming all such pairs to be good, we have \(u_3 \in O(v_1), u \in O(u_1)\). By 2-geodecity, \(N^+(u_4) \subset \{u_5, u_6, v\}\), so without loss of generality either \(N^+(u_4) = \{u_5, u_6\}\) or \(N^+(u_4) = \{u_5, v\}\).

Suppose that \(N^+(u_4) = \{u_5, u_6\}\). By elimination, \(O(u_1) = \{v, u_3\}\). As \(G\) is digraph, every vertex is an outlier of exactly two vertices; \(v\) is an outlier of \(u_1\) and \(v_1\), so both \(u_1\) and \(u_2\) can reach \(v\) by a \(\leq 2\)-path. Hence \(v \in N^+(u_3)\). As \(v \rightarrow v_1\), we see that \(v_1\) is an outlier of \(u_1\); as \(u\) is also an outlier of \(u_1\), we have \(O(u_1) = \{u, v_1\}\) and \(N^+(u_3) = \{v, u_2\}\). As \(v \rightarrow u_2\), this is impossible.

Now consider \(N^+(u_4) = \{u_5, v\}\). We now have \(O(u_1) = \{u_3, u_6\}\). Thus \(u_3 \in O(v) \cap O(v_1), u_3 \in T_2(u_4), v\) is not adjacent to \(u_3\), so \(u_3 \in N^+(u_5)\). \(u_2\) and \(u_4\) have \(u_5\) as a unique common out-neighbour, so \(u_6 \in O(u_4), v \in O(u_2)\). As \(u_6 \in O(u_1) \cap O(u_4)\), \(u_1\) can reach \(u_6\). Hence \(u_6 \in N^+(u_3)\). Neither \(u\) nor \(v\) lie in \(T(u_1)\), thus \(O(u_1) = \{u_2, v_1\}\). Therefore either \(O(u_1) = \{u, u_2\}\) or \(O(v_1) = \{u_2, v_1\}\).

If \(O(u_1) = \{u, u_2\}\), then \(N^+(u_2) = \{u_6, v_1\}\). As \(u_2 \rightarrow u_3\), we have \(u_3 \in O(v_1)\). As \(u_4 \rightarrow u_5\), \(u_4 \in N^+(u_6)\), \(u_1 \rightarrow u_4\), so \(N^+(u_3) = \{u_1, u_3\}\). As \(u_1 \rightarrow u_3\), this is a contradiction.

Thus \(O(u_1) = \{u_2, v_1\}\), so that \(N^+(u_3) = \{u, u_6\}\). \(u_1\) must have an in-neighbour apart from \(u\), which must be either \(u_5\) or \(u_6\).

As \(u_1 \rightarrow u_3\), we have \(u_3 \in N^+(u_6)\). By elimination, \(v\) and \(v_1\) must also have in-neighbours in \(\{u_5, u_6\}\). As \(u_1\) and \(v_1\) have a common out-neighbour, we have \(N^+(u_5) = \{u_3, v_1\}\), \(N^+(u_6) = \{u_1, v\}\). However, both \(u_3\) and \(v\) are adjacent to \(u\), violating 2-geodecity.

**Case 1.b:** \(v = u_3\)

There exists a vertex \(x\) such that \(V(G) = \{u, u_1, u_2, v, u_4, u_5, u_6, v_1, x\}\), \(O(u) = \{v_1, x\}\) and \(O(v) = \{u_1, x\}\). As \(x \in O(u) \cap O(v)\), \(u_1\) and \(u_2\) can reach \(x\), so without loss of generality \(x \in N^+(u_4) \cap N^+(u_5)\). As \(u_3\) and \(u_4\) have a common out-neighbour, \(u_5 \in O(u_4)\). Also, \(u_1\) and \(v_1\) have \(u_4\) as a unique common out-neighbour, so \(u \in O(u_1)\) and \(O(u_1) = \{u, u_5\}\). Thus \(N^+(u_4) = \{x, u_6\}\). Observe that \(u_2\) and \(u_4\) have the out-neighbour \(u_6\) in common. Thus \(x \in O(u_2)\), whereas we already have \(x \in O(u) \cap O(v)\), a contradiction.

**Case 2:** \(u = v_3, O(u) = \{v_1, v_4\}\)

As \(v\) is not equal to \(v_1\) or \(v_4\), \(v\) must lie in \(T_2(u)\). Without loss of generality, \(v = u_3\). Hence \(V(G) = \{u, u_1, u_2, v, u_4, u_5, u_6, v_1, v_4\}\) and \(O(v) = \{u_1, u_4\}\). We have the configuration shown in Fig. 4. Hence \(u_1\) can reach
Taking into account adjacencies between members of $u_1, v, u_4, u_2$ and $v_1$, so we have without loss of generality one of the following: a) $O(u_1) = \{u, v_4\}, N^+(u_4) = \{u_2, u_5\}$, b) $O(u_1) = \{u, u_1\}, N^+(u_4) = \{u_6, v_4\}$, c) $O(u_1) = \{u_5, u_6\}, N^+(u_4) = \{u, v_4\}$ or d) $O(u_1) = \{u_5, v_4\}, N^+(u_4) = \{u, u_6\}$.

**Case 2.a:** $O(u_1) = \{u, v_4\}, N^+(u_4) = \{u_5, u_6\}$

As $v_4 \in O(u) \cap O(u_1)$, $u_2$ can reach $v_4$ and without loss of generality $v_4 \in N^+(u_4), N^+(u_2) = N^+(u_4)$, so by Lemma 1 $u_2 \in O(u_4), u_4 \in O(u_2), u_5 \in O(u_6)$ and $u_6 \in O(u_5)$. Hence $u_4 \in O(v) \cap O(u_2)$, so $v_1$ can reach $u_4$, so $u_4 \in N^+(v_4)$. Neither $u_5$ nor $u_6$ lies in $N^+(v_4)$, so $O(v_1) = \{u_5, u_6\}$ and $N^+(u_4) = \{u_4, v_4\}$. Hence $O(v_4) = \{u, u_1\}$. Observe that $N^+(u_1) = N^+(v_4)$, so that $v \in O(u_1)$. Therefore $v \not\in N^+(u_5) \cup N^+(u_6)$, yielding $O(u_2) = \{u_4, v\}, N^+(u_2) = \{u_4, v_1, u, u_1\}$. As $v_1 \to v_4$ and $N^+(u_1) = N^+(v_4)$, we must have $N^+(u_5) = \{v, u\}, N^+(u_6) = \{v_1, u_1\}$. This yields the digraph in Fig. 5. Unlike the digraph in Fig. 2, this digraph contains pairs of vertices with identical out-neighborhoods, so the two are not isomorphic.

**Case 2.b:** $O(u_1) = \{u, u_1\}, N^+(u_4) = \{u_6, v_4\}$

As $u_4 \to v_4, u_4 \not\in N^+(v_4)$, so $u_4 \in O(v_1)$. Hence $u_4 \in O(v) \cap O(v_1)$, so $u_2$ can reach $u_4$. As $u_4 \to u_6$, we must have $u_5 \to u_4, u_2$ and $u_4$ have $u_6$ as a common out-neighbor, so $v_4 \in O(u_2), u_5 \in O(u_4)$. Therefore $v_4 \in O(u) \cap O(u_2)$, so that $u_6$ can reach $v$, but $v_4 \not\in T(u_6)$, so $N^+(u_6)$ contains an in-neighbor of $v_4, u_4 \not\in N^+(u_6)$, so we must have $u_6 \to v_3$. We have $u_5 \in O(u_4) \cap O(u_1)$, so $v_1$ can reach $u_5$ and hence $v_4 \to u_5, v_3$ cannot reach $u_6$, as $u_2, u_4 \not\in T(v_1)$, so $O(v_1) = \{u_4, u_6\}, N^+(u_2) = \{u_4, v_4\}$. Now $u_2$ and $u_4$ have $u_5$ as a unique common out-neighbor, so $u_6 \in O(u_4), v \in O(u_2)$. Thus $O(u_2) = \{v, u_4\}$ and $N^+(u_2) = \{u_4, v_1, u, u_1\}$. Taking into account adjacencies between members of $N^+(u_4)$, it follows that $N^+(u_5) = \{u_4, u\}, N^+(u_6) = \{v_1, u_1\}$. However, $(u_2, u_4)$ now constitutes a bad pair, contradicting our assumption.

**Case 2.c:** $O(u_1) = \{u_5, u_6\}, N^+(u_4) = \{u_6, v_4\}$

As $u_4 \to u_4, u_4 \not\in N^+(v_4)$. Hence $u_4 \in O(v) \cap O(v_1)$, implying that $u_2$ can reach $u_4$. Without loss of generality, $u_5 \to u_4$.

There are three possibilities: (i) $O(v_1) = \{u_4, u_6\}, N^+(u_4) = \{v, u_5\}, N^+(v_4) = \{v, u_6\}$ and (iii) $O(v_1) = \{u_4, v\}, N^+(v_4) = \{u_5, u_6\}$.

(i) $O(v_1) = \{u_4, u_6\}, N^+(v_4) = \{v, u_5\}$

$u_1$ and $v_4$ have $v$ as a unique common out-neighbor, so $u_4 \in O(v_4)$, however, this contradicts $v_4 \to u_5 \to u_4$.

(ii) $O(v_1) = \{u_4, u_5\}, N^+(v_4) = \{v, u_6\}$

Neither $u_4 \not\in v_1 \in T(u_2)$, so $v_4 \in O(u_2)$. Now observe that $u_2$ and $u_4$ have $u_6$ as unique common out-neighbor, so $v \in O(u_2)$, yielding $O(u_2) = \{v, u_4\}$ and $N^+(u_2) = \{u_4, u_1, u, v_1\}$. As $u_4 \to u$ and $u \to u_1$, we must have $N^+(u_5) = \{u_4, u_1\}, N^+(u_6) = \{u, v_1\}$, a contradiction, since $u_1 \to u_4$.

(iii) $O(v_1) = \{u_4, v\}, N^+(v_4) = \{u_5, u_6\}$

We now have $N^+(u_2) = N^+(v_4)$, so $u_2 \in O(u_4), v_4 \in O(u_2), u_5 \in O(u_6), v_6 \in O(u_5)$. Also $N^+(u_4) = N^+(v_4)$, so $u_4 \in O(v_1), v_1 \in O(u_4)$ and $u \in O(u_4)$ implies that $u \not\in N^+(u_5) \cup N^+(u_6)$, so we see that $u \in O(u_2)$ and hence $O(u_2) = \{u_4, v_4\}$ and $N^+(u_2) = \{u_4, u_1, v, v_1\}$. As $u_1 \to u_4$ and $u_1 \to v$, we have $N^+(u_5) = \{u_4, v\}, N^+(u_6) = \{u_1, v_1\}$. It is not difficult to show that this yields a $(2, 2, +2)$-digraph isomorphic to that in Fig. 5.

**Case 2.d:** $O(u_1) = \{u_5, v_4\}, N^+(u_4) = \{u_6, v_4\}$

In this case $v_4 \in O(u) \cap O(u_1)$, so $u_2$ can reach $v_4, u_4$ and $v_4$ have unique common out-neighbor $u$, so $v_4 \in O(u_4), u_6 \in O(v_1)$. If $u_6 \to v_4$, then we would have $u_4 \to u_6 \to v_4$, contradicting $v_4 \in O(u_4)$, so $u_5 \to v_4$. This also implies that $u_5 \not\in N^+(v_4), u_5 \in O(v_1)$, yielding $O(v_1) = \{u_5, u_6\}$ and $N^+(v_4) = \{v, u_4\} = N^+(u_1)$. Now $v_4, u_1 \not\in T(u_2)$, so $O(u_2) = \{v, u_4\}$.
and $N^2(u_2) = \{v_1, v_4, u, u_1\}$. As $v_1 \to v_4$ and $v_1 \to u$, it follows that $N^+(u_5) = \{v_4, u\}$, $N^+(u_6) = \{u_1, v_1\}$. However, we now have paths $u_4 \to u \to u_1$ and $u_4 \to u_6 \to u_1$, which is impossible.

**Case 3:** $N^+(u_1) = N^+(v_1)$

It is easy to see by 2-geodesity that $V(G) = \{u, u_1, u_2, u_3, u_4, u_5, v, v_1\}$, $O(u) = \{v, v_1\}$ and $O(v) = \{u, u_1\}$. As $u_1, v_1 \notin T(u_2)$, we have $O(u_2) = \{u_3, u_4\}$ and $N^+(u_2) = \{u, u_1, v, v_1\}$. Without loss of generality, $N^+(u_5) = \{u, v_1\}$, $N^+(u_6) = \{v, u_1\}$. $u$ and $v$ have in-neighbours apart from $u_5$ and $u_6$ respectively, so without loss of generality $u_3 \to u$, $u_4 \to v$. Likewise, $u_5$ and $u_6$ have in-neighbours other than $u_2$, so, as $u_3 \to u$ and $u_6 \to v$, we must have $N^+(u_3) = \{u, u_6\}$, $N^+(u_4) = \{v, u_5\}$. But now we have paths $u_3 \to u \to u_1$ and $u_3 \to u_6 \to u_1$, violating 2-geodesity.

**Corollary 2.** There is a unique $(2, 2, +2)$-digraph containing no bad pairs.

This completes our analysis of diregular $(2, 2, +2)$-digraphs. As it was shown in [7] that there are no non-diregular $(2, 2, +2)$-digraphs, $(2, 2, +2)$-digraphs are now classified up to isomorphism. These conclusions have been verified computationally by Erskine [2]. It is interesting to note that neither of the $(2, 2, +2)$-digraphs are vertex-transitive, for in each case there are exactly three vertices contained in two 3-cycles. However, there does exist a Cayley $(2, 2, +5)$-digraph (on the alternating group $A_4$), so it would be interesting to determine the smallest vertex-transitive $(2, 2, +\epsilon)$-digraphs.

**4. Main result**

We can now complete our analysis by showing that there are no diregular $(2, k, +2)$-digraphs for $k \geq 3$. Let $G$ be such a digraph. By Lemma 2, $G$ contains vertices $u$ and $v$ with a unique common out-neighbour. In accordance with our vertex-labelling convention, we have the situation in Fig. 6. A triangle based at a vertex $x$ represents the set $T(x)$.

We now proceed to determine the possible outlier sets of $u$ and $v$.

**Lemma 5.** $v \in N^{k-1}(u_1) \cup O(u)$ and $u \in N^{k-1}(v_1) \cup O(v)$. If $v \in O(u)$, then $u_2 \in O(u_1)$ and if $u \in O(v)$, then $u_2 \in O(v_1)$.

**Proof.** $v$ cannot lie in $T(u)$, or the vertex $u_2$ would be repeated in $T(u)$. Also, $v \notin T(u_2)$, or there would be a $< k$-cycle through $v$. Therefore, if $v \notin O(u)$, then $v \in N^{k-1}(u_1)$. Likewise for the other result. If $v \in O(u)$, then neither in-neighbour of $u_2$ lies in $T(u_1)$, so that $u_2 \in O(u_1)$. \hfill $\square$

**Lemma 6.** Let $w \in T(v_1)$, with $d(v_1, w) = l$. Suppose that $w \in T(u_1)$, with $d(u_1, w) = m$. Then either $m \leq l$ or $w \in N^{k-1}(u_1)$. A similar result holds for $w \in T(v_1)$.

**Proof.** Let $w$ be as described and suppose that $m > l$. Consider the set $N^{k-m}(w)$. By construction, $N^{k-m}(w) \subseteq N^k(u_1)$, so by $k$-geodesity $N^{k-m}(w) \cap T(u_1) = \emptyset$. At the same time, we have $l + k - m \leq k - 1$, so $N^{k-m}(w) \subseteq T(v_1)$. This implies that $N^{k-m}(w) \cap T(v_2) = N^{k-m}(w) \cap T(u_2) = \emptyset$. As $V(G) = \{u\} \cup T(u_1) \cup T(u_2) \cup O(u)$, it follows that $N^{k-m}(w) \subseteq \{u\} \cup O(u)$. Therefore $|N^{k-m}(w)| = 2^{k-m} \leq 3$. By assumption $0 \leq m \leq k - 1$, so it follows that $m = k - 1$. \hfill $\square$

**Corollary 3.** If $w \in T(v_1)$, then either $w \in \{u\} \cup O(u)$ or $w \in T(u_1)$ with $d(u_1, w) = k - 1$ or $d(u_1, w) \leq d(v_1, w)$.

**Proof.** By $k$-geodesity and Lemma 6. \hfill $\square$

**Corollary 4.** $v_1 \in N^{k-1}(u_1) \cup O(u)$ and $u_1 \in N^{k-1}(v_1) \cup O(v)$.
Proof. We prove the first inclusion. By Corollary 3, \( v_1 \in \{u\} \cup O(u) \cup \{u_1\} \cup N^{k-1}(u_1) \). By \( k \)-geodecity, \( v_1 \neq u \) and by construction, \( v_1 \neq u_1 \). \( \square \)

We now have enough information to identify one member of \( O(u) \) and \( O(v) \).

Lemma 7. \( v_1 \in O(u) \) and \( u_1 \in O(v) \).

Proof. We prove that \( v_1 \in O(u) \). Suppose that neither \( v_1 \) nor \( v \) lies in \( O(u) \). Then by Lemma 5 and Corollary 4 we have \( v, v_1 \in N^{k-1}(u_1) \). As \( v_1 \) is an out-neighbour of \( v \), it follows that \( v_1 \) appears twice in \( T_k(u_1) \), violating \( k \)-geodecity. Therefore \( O(u) \cap \{v_1, v\} \neq \emptyset \).

Now assume that \( v_1, v_3 \in T_k(u) \). Again by Corollary 4, \( v_1 \in N^{k-1}(u_1) \). By \( k \)-geodecity we also have \( v_3 \in T(u_1) \). However, \( v_1 \in N^+(v_3) \), so \( v_1 \) appears twice in \( T_k(u_1) \), which is impossible. Hence \( O(u) \cap \{v_1, v_3\} \neq \emptyset \). Similarly, \( O(u) \cap \{v_1, v_4\} \neq \emptyset \). In the terminology of the previous section, \( G \) contains no bad pairs. Therefore, if \( v_1 \notin O(u) \), then \( \{v_3, v_4\} \subseteq O(u) \). Since these vertices are distinct, this is a contradiction and the result follows. \( \square \)

Lemma 7 allows us to conclude that for vertices sufficiently close to \( v_1 \) one of the potential situations mentioned in Corollary 3 cannot occur.

Lemma 8. \( T_{k-3}(v_1) \cap N^{k-1}(u_1) = T_{k-3}(u_1) \cap N^{k-1}(v_1) = \emptyset \).

Proof. Let \( w \in T_{k-3}(v_1) \cap N^{k-1}(u_1) \). Consider the position of the vertices of \( N^+(w) \) in \( T_k(u) \cup O(u) \). As \( v \notin N^+(w) \), it follows from Lemma 7 that at most one of the vertices of \( N^+(w) \) can be an outlier of \( u \), so let us write \( w_1 \in N^+(w) - O(u) \). By \( k \)-geodecity, \( u_1 \notin T(u_1) \cup \{u_1\} \). Hence \( w_1 \in T(u_2) = T(v_2) \). However, \( w_1 \) also lies in \( T(v_1) \), so this violates \( k \)-geodecity. \( \square \)

Corollary 5. There is at most one vertex in \( T_{k-3}(v_1) \cap \{v_1\} \) that does not lie in \( T(u_1) \): for all other vertices \( w \in T_{k-3}(v_1) \setminus \{v_1\} \), \( d(u_1, w) = d(v_1, w) \). A similar result for \( T_{k-3}(u_1) \setminus \{u_1\} \) also holds.

Lemma 9. For \( k = 3, N^+(u_1) \cap N^2(v_1) = N^+(v_1) \cap N^2(u_1) = \emptyset \).

Proof. Suppose that \( v_3 = u_2 \). By the reasoning of Lemma 8 we can set \( u = v_2 \) and \( O(u) = \{v_1, v_8\} \). As \( 3 \)-geodecity \( v \notin N^+(u_3) \), so we can assume that \( v = u_9, u_3 \rightarrow v_3 \) implies that \( u_3 \notin T(v_1) \), so \( O(v) = \{u_1, u_4\} \). We must have \( \{u_4, u_9, u_10\} = \{v_4, v_9, v_10\} \). As \( u_4 \rightarrow v \), it follows that \( v_4 = u_9 \) and hence \( \{u_4, u_10\} = \{v_4, v_10\} \), which is impossible. \( \square \)

As \( u_1 \) is an outlier of \( v \), neither \( v_3 \) nor \( v_4 \) can be equal to \( u_1 \). It follows from Corollary 5 and Lemma 9 that either \( N^+(u_1) = N^+(v_1) \) or \( u_1 \) and \( v_1 \) have a single common out-neighbour, with one vertex of \( N^+(v_1) \) being an outlier of \( u \).

Lemma 10. \( N^2(u) \neq N^2(v) \).

Proof. Let \( N^2(u) = N^2(v) \), with \( N^+(u_1) = N^+(v_1) = \{u_3, u_4\} \). Suppose that \( v \notin O(u) \). By Lemma 5, \( v \in N^{k-2}(u_2) \cup N^{k-2}(u_4) \). But then there is a \( k \)-cycle through \( v \). It follows that \( O(u) = \{v, v_1\} \), \( O(v) = \{u_1, u_4\} \). By Lemma 5, \( u_2 \in O(u_1) \cap O(v_1) \). Therefore by Lemma 1 \( O(1) = \{u_2, v_1\} \), \( O(v_1) = \{u_2, u_1\} \).

Consider the in-neighbour \( u' \) of \( v_1 \) that is distinct from \( u \). We have either \( |N^+(u') \cap N^+(u)| = 1 \) or \( |N^+(u') \cap N^+(u)| = 2 \). In the first case, it follows from Lemma 7 that \( u_2 \in O(u') \). Every vertex of \( G \) is an outlier of exactly two vertices, so \( u' = u_1 \) or \( u_1 \). In either case, we have a contradiction. Therefore \( N^+(u') \neq N^+(u) \). It now follows from Lemma 1 that \( u' \notin O(u) = \{v, v_1\} \), which is impossible. \( \square \)

Noticing that \( u_1 \) and \( v_1 \) also have a unique common out-neighbour, we have the following corollary.

Corollary 6. Without loss of generality, \( u_1 = v_3, u_9 = v_9, O(u) = \{v_1, v_4\} \), \( O(v) = \{u_1, u_4\} \), \( O(u_1) = \{v_4, v_10\} \) and \( O(v_1) = \{u_4, u_10\} \).

We are now in a position to complete the proof by deriving a contradiction.

Theorem 2. There are no digereular \((2, k, +2)\)-digraphs for \( k \geq 3 \).

Proof. \( u, v \notin \{u_1, u_3, v_1, v_4\} \), so by Lemma 5 \( d(u, v) = d(v, u) = k \). In fact, \( u_2 = v_2 \) implies that \( v \in N^{k-2}(u_2) \) and \( u \in N^{k-2}(v_2) \). Let \( k \geq 4 \). Then \( u, v \notin \{u_1, u_9, v_1, v_4\} \), so \( u, v \in T_k(u) \cap T_k(v) \). If \( u \in T(u_2) = T(v_2) \), then \( u \) would appear twice in \( T_k(u) \), so \( u \in N^{k-1}(u_2) \). However, as \( u \) and \( v \) have a common out-neighbour, this violates \( k \)-geodecity.

Finally, suppose that \( k = 3 \). The above analysis will hold unless \( u = v_10 \) and \( v = u_10 \). Let \( N^-(u_1) = \{u, u', N^-(v_1) = \{v, v'\} \). It is evident that \( v' \notin \{v_1, v_4\} \), so that \( v' \in T_3(u) \). As \( v \in N^+(u_4) \), we must have \( v' \in N^2(u_2) \). Similarly \( u' \in N^2(u_2) \). Since \( u_1 \) and \( v_1 \) have a common out-neighbour, we can assume that \( u' \in N^+(u_3) \) and \( v' \in N^+(v_6) \). \( v_4 \) can be the outlier of only two vertices, namely \( u \) and \( u_1 \), so \( v_4 \in N^3(u_2) \) and likewise \( u_4 \in N^3(v_2) \). By \( 3 \)-geodecity \( v_4 \in N^2(u_6) \) and \( u_4 \in N^2(v_6) \). It follows that \( u, v \notin N^2(u_2) \), so \( u \notin T_3(u_1) \cup T_3(u_2) \). Hence \( O(u) = N^-(u) = \{v_1, v_4\} \), which again is impossible. \( \square \)

It is interesting to note that a similar argument can be used to provide an alternative proof of the result of [6].
References