Note

On diregular digraphs with degree two and excess two

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Abstract

An important topic in the design of efficient networks is the construction of \((d, k, +\epsilon)-\)digraphs, i.e. \(k\)-geodetic digraphs with minimum out-degree \(\geq d\) and order \(M(d, k) + \epsilon\), where \(M(d, k)\) represents the Moore bound for degree \(d\) and diameter \(k\) and \(\epsilon > 0\) is the (small) excess of the digraph. Previous work has shown that there are no \((2, k, +1)-\)digraphs for \(k \geq 2\). In a separate paper, the present author has shown that any \((2, k, +2)-\)digraph must be diregular for \(k \geq 2\). In the present work, this analysis is completed by proving the nonexistence of diregular \((2, k, +2)-\)digraphs for \(k \geq 3\) and classifying diregular \((2, 2, +2)-\)digraphs up to isomorphism.

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1. Introduction

An important topic in the design of interconnection networks is the directed degree/diameter problem: what is the largest possible order \(N(d, k)\) of a digraph \(G\) with maximum out-degree \(d\) and diameter \(\leq k\)? A simple inductive argument shows that for \(0 \leq l \leq k\) the number of vertices at distance \(l\) from a fixed vertex \(v\) is bounded above by \(d^l\). Therefore, a natural upper bound for the order of such a digraph is the so-called Moore bound \(M(d, k) = 1 + d + d^2 + \cdots + d^k\). A digraph that attains this upper bound is called a Moore digraph. It is easily seen that a digraph \(G\) is Moore if and only if it is out-regular with degree \(d\), has diameter \(k\) and is \(k\)-geodetic, i.e. for any two vertices \(u, v\) there is at most one \(k\)-path from \(u\) to \(v\).

As it was shown by Bridges and Toueg in [1] that Moore digraphs exist only in the trivial cases \(d = 1\) or \(k = 1\) (the Moore digraphs are directed \((k + 1)\)-cycles and complete digraphs \(K_{k+1}\) respectively), much research has been devoted to the study of digraphs that in some sense approximate Moore digraphs. For example, there is an extensive literature on digraphs with maximum out-degree \(d\), diameter \(\leq k\) and order \(M(d, k) - \delta\) for small \(\delta > 0\); this is equivalent to relaxing the \(k\)-geodecticy requirement in the conditions for a digraph to be Moore. \(\delta\) is known as the defect of the digraph. The reader is referred to the survey [5] for more information.

In this paper, however, we will consider the following related problem, which is obtained by retaining the \(k\)-geodecticy requirement in the above characterisation of Moore digraphs, but allowing the diameter to exceed \(k\): what is the smallest possible order of a \(k\)-geodetic digraph \(G\) with minimum out-degree \(\geq d\)? A \(k\)-geodetic digraph with minimum out-degree \(\geq d\) and order \(M(d, k) + \epsilon\) is said to be a \((d, k, +\epsilon)-\)digraph or to have excess \(\epsilon\). It was shown in [6] that there are no diregular \((2, k, +1)-\)digraphs for \(k \geq 2\). In 2016 it was shown in [3] that digraphs with excess one must be diregular and that there are no \((d, k, +1)-\)digraphs for \(k = 2, 3, 4\) and sufficiently large \(d\). In a separate paper [7], the present author has shown that \((2, k, +2)-\)digraphs must be diregular with degree \(d = 2\) for \(k \geq 2\). In the present paper, we classify the \((2, 2, +2)-\)digraphs up to isomorphism and show that there are no diregular \((2, k, +2)-\)digraphs for \(k \geq 3\), thereby completing the proof of the nonexistence of digraphs with degree \(d = 2\) and excess \(\epsilon = 2\) for \(k \geq 3\). Our reasoning and notation will follow closely that employed in [4] for the corresponding result for defect \(\delta = 2\).
2. Preliminary results

We will let \( G \) stand for a \((2, k, +2)\)-digraph for arbitrary \( k \geq 2 \), i.e. \( G \) has minimum out-degree \( d = 2 \), is \( k \)-geodetic and has order \( M(2, k) + 2 \). We will denote the vertex set of \( G \) by \( V(G) \). By the result of \([7]\), \( G \) must be digiregular with degree \( d = 2 \) for \( k \geq 2 \). The distance \( d(u, v) \) between vertices \( u \) and \( v \) is the length of the shortest path from \( u \) to \( v \). Notice that \( d(u, v) \) is not necessarily equal to \( d(v, u) \). \( u \rightarrow v \) will indicate that there is an arc from \( u \) to \( v \). We define the in- and out-neighbourhoods of a vertex \( u \) by \( N^-(u) = \{ v \in V(G) : v \rightarrow u \} \) and \( N^+(u) = \{ v \in V(G) : u \rightarrow v \} \) respectively; more generally, for \( 0 \leq l \leq k \), the set \( \{ v \in V(G) : d(u, v) = l \} \) of vertices at distance exactly \( l \) from \( u \) will be denoted by \( N^l(u) \). For \( 0 \leq l \leq k \) we will also write \( T_l(u) = \bigcup_{i=0}^{l} N^i(u) \) for the set of vertices at distance \( \leq l \) from \( u \). The notation \( T_{k-1}(u) \) will be abbreviated by \( T(u) \).

It is easily seen that for any vertex \( u \) of \( G \), there are exactly two distinct vertices that are at distance \( \geq k + 1 \) from \( u \). For any \( u \in V(G) \), we will write \( O(u) \) for the set of these vertices and call such a set an outlier set and its elements outliers of \( u \). Notice that \( O(u) = V(G) - T_k(u) \). An elementary counting argument shows that in a digiregular \((2, k, +2)\)-digraph every vertex is also an outlier of exactly two vertices. We will say that a vertex \( u \) can reach a vertex \( v \) if \( v \notin O(u) \).

Our proof will proceed by an analysis of a pair of vertices with exactly one common out-neighbour. First, we must show that such a pair exists and deduce some elementary properties of pairs of vertices with identical out-neighbourhoods.

**Lemma 1.** For \( k \geq 2 \), let \( u \) and \( v \) be distinct vertices such that \( N^+(u) = N^+(v) = \{ u_1, u_2 \} \). Then \( u_1 \in O(u_2), u_2 \in O(u_1) \) and there exists a vertex \( x \) such that \( O(u) = \{ v, x \} \) and \( O(v) = \{ u, x \} \).

**Proof.** Suppose that \( u \) can reach \( v \) by a \( \leq k \)-path. Then \( v \in T(u_1) \cup T(u_2) \). As \( N^+(v) = N^+(u) \), it follows that there would be a \( \leq k \)-cycle through \( v \), contradicting \( k \)-geodeticity. If \( O(u) = \{ v, x \} \), then \( x \neq v \) and \( x \notin T(u_1) \cup T(u_2) \), so that \( v \) cannot reach \( x \) by a \( \leq k \)-path. Similarly, if \( u_1 \) can reach \( u_2 \) by a \( \leq k \)-path, then we must have \( \{ u, v \} \cap T(u_1) \neq \emptyset \), which is impossible. \( \square \)

**Lemma 2.** For \( k \geq 2 \), there exists a pair of vertices \( u, v \) with \( |N^+(u) \cap N^+(v)| = 1 \).

**Proof.** Suppose for a contradiction that there is no such pair of vertices. Define a map \( \phi : V(G) \rightarrow V(G) \) as follows. Let \( u^* \) be an out-neighbour of a vertex \( u \) and let \( \phi(u) \) be the in-neighbour of \( u^* \) distinct from \( u \). By our assumption, it is easily verified that \( \phi \) is a well-defined bijection with no fixed points and with square equal to the identity. It follows that \( G \) must have even order, whereas \( |V(G)| = M(2, k) + 2 \) is odd. \( \square \)

\( u, v \) will now stand for a pair of vertices with a single common out-neighbour. We will label the vertices of \( T_k(u) \) according to the scheme \( N^+(u) = \{ u_1, u_2 \}, N^+(u_1) = \{ u_3, u_4 \}, N^+(u_2) = \{ u_5, u_6 \}, N^+(u_3) = \{ u_7, u_8 \}, N^+(u_4) = \{ u_9, u_{10} \} \) and so on, with the same convention for the vertices of \( T_k(v) \), where we will assume that \( u_2 = v_2 \).

3. Classification of \((2, 2, +2)\)-digraphs

We begin by classifying the \((2, 2, +2)\)-digraphs up to isomorphism. We will prove the following theorem.

**Theorem 1.** There are exactly two digiregular \((2, 2, +2)\)-digraphs, which are displayed in Figs. 2 and 5.

Let \( G \) be an arbitrary digiregular \((2, 2, +2)\)-digraph. \( G \) has order \( M(2, 2) + 2 = 9 \). By Lemma 2, \( G \) contains a pair of vertices \( (u, v) \) such that \( |N^+(u) \cap N^+(v)| = 1 \); we will assume that \( u_2 = v_2 \), so that we have the situation shown in Fig. 1.

We can immediately deduce some information on the possible positions of \( v \) and \( v_1 \) in \( T_2(u) \).

**Lemma 3.** If \( v \notin O(u) \), then \( v \in N^+(u_1) \). If \( v_1 \notin O(u) \), then \( v_1 \in N^+(u_1) \).

**Proof.** \( v \notin T(u_2) \) by 2-geodicity. \( v \neq u \) by construction. If we had \( v = u_1 \), then there would be two distinct \( \leq 2 \)-paths from \( u \) to \( u_2 \). Also \( v_1 \notin \{ u \} \cup T(u_2) \) by 2-geodcity and by assumption \( u_1 \neq v_1 \). \( \square \)
Since $v$ and $v_1$ cannot both lie in $N^+(u_1)$ by 2-geodecity, we have the following corollary.

**Corollary 1.** $O(u) \cap \{v, v_1\} \neq \emptyset$.

We will call a pair of vertices $(u, v)$ with a single common out-neighbour **bad** if at least one of

$$O(u) \cap \{v_1, v_3\} = \emptyset, O(u) \cap \{v_1, v_4\} = \emptyset, O(u) \cap \{u_1, u_3\} = \emptyset, O(u) \cap \{u_1, u_4\} = \emptyset.$$ 

holds. Otherwise such a pair will be called **good**.

**Lemma 4.** There is a unique $(2, 2, +2)$-digraph containing a bad pair.

**Proof.** Assume that there exists a bad pair $(u, v)$. Without loss of generality, $O(u) \cap \{v_1, v_3\} = \emptyset$. By Lemma 3 we can set $v_1 = u_3$. By 2-geodecity $v_3 = u$. We cannot have $v_4 = v_3 = u$, so $v_4$ must be an outlier of $u$. By Corollary 1 it follows that $O(u) = \{v, v_4\}$.

Consider the vertex $u_1$. By Lemma 3, if $u_1 \not\in O(v)$, then $u_1 \in N^+(v_1)$. However, as $v_1 = u_3$, there would be a 2-cycle through $u_1$. Hence $u_1 \in O(v)$. As $O(u) = \{v, v_4\}$, we have $V(G) = \{u, u_1, u_2, u_3 = v_1, u_4, v, u_4, v_4\}$ and $O(v) = \{u_1, u_4\}$. As neither $u$ nor $v$ lies in $T(u_1)$, we must also have $u_2 \in O(u_1)$. As $u_1$ can reach $u_1, u_4, v$, and $v_4$, it follows that without loss of generality we either have $O(u_1) = \{u_2, v\}$ and $N^+(u_4) = \{u_5, u_6\} = N^+(u_2)$ or $O(u_1) = \{u_2, u_6\}$ and $N^+(u_4) = \{v, u_5\}$. In either case, $(v, u_1)$ is a good pair.

Suppose firstly that $N^+(u_2) = N^+(u_4)$. Then $v$ is an outlier of $u$ and $u_1$. As each vertex is the outlier of exactly two vertices, $v_1$ must be able to reach $v$ by a $\leq 2$-path. Hence $v_4 \rightarrow v$. Likewise $u_2$ can reach $v$, so without loss of generality $u_5 \rightarrow v$. Suppose that $O(u_2) \cap \{u, u_1\} = \emptyset$. As $u$ and $v$ have a common out-neighbour, we must have $u_6 \rightarrow u$. Since $u \rightarrow u_1$, by 2-geodecity we must have $u_5 \rightarrow u_1$. However, this is a contradiction, as $v$ and $u_1$ also have a common out-neighbour. Therefore, at least one of $u, u_1$ is an outlier of $u_2$. By Lemma 1 $u_4$ is an outlier of $u_2$. Therefore either $O(u_2) = \{u, u_4\}$ or $O(u_2) = \{u_1, u_4\}$. If $O(u_2) = \{u, u_4\}$, then $u_2$ must also be able to reach $u_1, v_1$ and $v_4$. $u_5 \rightarrow v$ and $v \rightarrow v_1$, so $v_1 \in N^+(u_6)$. As $u_1 \rightarrow v_1$, we must have $N^+(u_5) = \{v, u_1\}$. As $v$ and $u_1$ have a common out-neighbour, this violates 2-geodecity. Hence $O(u_2) = \{u_1, u_4\}$ and $u_2$ can reach $u, v_1$, and $v$. As $v \rightarrow v_1, v_1 \in N^+(u_6)$. As $v_4 \rightarrow v_4$, it follows that $N^+(u_5) = \{v, v_4\}$. However, $v_4 \rightarrow v_4$, so this again violates 2-geodecity.

We are left with the case $O(u_1) = \{u_2, u_6\}$ and $N^+(u_4) = \{v, u_5\}$. Then $v_1 \in O(u_2)$, as neither $v$ nor $u_1$ lies in $T(u_2)$. Observe that $u_2$ and $u_4$ have a single common out-neighbour, so by Corollary 1 $O(u_2) \cap \{u_4, v\} \neq \emptyset$. Therefore either $O(u_2) = \{v_1, u_4\}$ or $O(u_2) = \{v_1, v\}$. Suppose firstly that $O(u_2) = \{v_1, u_4\}$. Then $N^+(u_2) = \{u, v, u_1, v_4\}$. As $N^+(u_4) = \{v, u_5\}, u_5 \not\rightarrow v$, so $u_6 \rightarrow v$. As $N^+(u) \cap N^+(v) = \emptyset$, $u_5 \rightarrow u \rightarrow v$, so necessarily $u_6 \in N^+(u_1) \cap N^+(v)$, contradicting 2-geodecity.

Hence $O(u_2) = \{v_1, v\}$ and $N^+(u_2) = \{u, u_1, u_4, v_4\}$. As $u_4 \rightarrow u_5, u_5 \not\rightarrow u_4$. Thus $u_6 \rightarrow u_4$. Now $u_1 \rightarrow u_4$ and $u \rightarrow u_1$ implies that $N^+(u_5) = \{u_1, v_4\}$ and $N^+(u_6) = \{u, u_4\}$. Finally we must have $N^+(v_4) = \{v, u_6\}$. This gives us the $(2, 2, +2)$-digraph shown in Fig. 2. 

We can now assume that all pairs given by Lemma 2 are good. Let us fix a pair $(u, v)$ with a single common out-neighbour. It follows from Corollary 1 and the definition of a good pair that $v_1 \in O(u)$; otherwise $O(u)$ would contain $v, v_3$ and $v_4$, which is impossible. Likewise $u_1 \in O(v)$.
Considering the positions of $v_3$ and $v_4$, we see that there are without loss of generality four possibilities: (1) $u = v_3$, $u_4 = v_4$ (2) $u = v_3$, $O(u) = \{v_1, v_4\}$, (3) $N^+(u_1) = N^+(v_1)$ and (4) $u_3 = v_3$, $O(u) = \{v_1, u_4\}$. A suitable relabelling of vertices shows that case 4 is equivalent to case 1.a below, so we will examine cases 1 to 3 in turn.

**Case 1:** $u = v_3$, $u_4 = v_4$

Depending upon the position of $v$, we must either have $O(u) = \{v_1, v\}$ and $O(\nu) = \{u_1, u_3\}$ or $v = u_3$ (see Fig. 3).

**Case 1.a:** $O(u) = \{v_1, v\}$, $O(\nu) = \{u_1, u_3\}$

In this case $V(G) = \{u, u_1, u_2, u_3, u_4, u_5, u_6, v, v_1\}$. $u_1$ and $v_1$ have a single common out-neighbour, namely $u_4$, so, as we are assuming all such pairs to be good, we have $u_3 \in O(v_1)$, $u \in O(u_1)$. By 2-geodecity, $N^+(u_4) \subset \{u_5, u_6, v\}$, so without loss of generality either $N^+(u_4) = \{u_5, u_6\}$ or $N^+(u_4) = \{u_5, v\}$.

Suppose that $N^+(u_4) = \{u_5, u_6\}$. By elimination, $O(u_1) = \{v, u_3\}$. As $G$ is diregular, every vertex is an outlier of exactly two vertices; $v$ is an outlier of $u$ and $v_1$, so both $u_1$ and $u_2$ can reach $v$ by a $\leq 2$-path. Hence $v \in N^+(u_3)$. As $v \to v_1$, we see that $v_1$ is an outlier of $u_1$; as $u$ is also an outlier of $u_1$, we have $O(u_1) = \{u, v_1\}$ and $N^+(u_3) = \{v, u_2\}$. As $u \rightarrow u_2$, this is impossible.

Now consider $N^+(u_4) = \{u_5, v\}$. We now have $O(u_1) = \{u_3, u_6\}$. Thus $u_3 \in O(v) \cap O(v_1)$, so $u_3 \in T_2(u_4)$. $v$ is not adjacent to $u_3$, so $u_3 \in N^+(u_5)$. $u_2$ and $u_4$ have $u_5$ as a unique common out-neighbour, so $u_5 \in O(u_4)$, $v \in O(u_2)$. As $u_6 \in O(u_1) \cap O(u_4)$, $u_1$ can reach $u_6$. Hence $u_6 \in N^+(u_3)$. Neither $u$ nor $v$ lie in $T(u_1)$, so $u_2 \in O(u_1)$. Therefore either $O(u_1) = \{u, u_2\}$ or $O(u_1) = \{u_2, v_1\}$. If $O(u_1) = \{u, u_2\}$, then $N^+(u_2) = \{u_6, v_1\}$. $u_2$ cannot reach $v_1$, since $v, u_3 \notin T(u_2)$, so $O(u_2) = \{v, v_1\}$ and $N^+(u_2) = \{u_1, u_3, v_4\}$. As $u_4 \rightarrow u_5$, $u_4 \in N^+(u_6)$, $u_1 \rightarrow u_4$, so $N^+(u_4) = \{u_1, u_3\}$. As $u_1 \rightarrow u_3$, this is a contradiction. Thus $O(u_1) = \{u_2, v_1\}$, so that $N^+(u_3) = \{u_6, u_5\}$. $u_1$ must have an in-neighbour apart from $u$, which must be either $u_5$ or $u_6$. As $u_1 \rightarrow u_3$, we have $u_1 \in N^+(u_6)$. By elimination, $v$ and $v_1$ must also have in-neighbours in $\{u_5, u_6\}$. As $u_1$ and $v_1$ have a common out-neighbour, we have $N^+(u_5) = \{u_3, v_1\}$, $N^+(u_6) = \{u_1, v\}$. However, both $u_3$ and $v_1$ are adjacent to $u$, violating 2-geodecity.

**Case 1.b:** $v = u_3$

There exists a vertex $x$ such that $V(G) = \{u, u_1, u_2, v, u_4, u_5, u_6, v_1, x\}$, $O(u) = \{v_1, x\}$ and $O(u) = \{u_1, x\}$. As $x \in O(u) \cap O(\nu)$, $u_1$ and $u_2$ can reach $x$, so without loss of generality $x \in N^+(u_4) \cap N^+(u_5)$. As $u_5$ and $u_4$ have a common out-neighbour, $u_5 \in O(u_4)$. Also, $u_1$ and $v_1$ have $u_4$ as a unique common out-neighbour, so $u \in O(u_1)$ and $O(u_1) = \{u, u_6\}$. Thus $N^+(u_4) = \{x, u_6\}$. Observe that $u_2$ and $u_4$ have the out-neighbour $u_6$ in common. Thus $x \in O(u_2)$, whereas we already have $x \in O(u) \cap O(\nu)$, a contradiction.

**Case 2:** $u = v_3$, $O(u) = \{v_1, v_4\}$

As $v$ is not equal to $v_1$ or $v_4$, $v$ must lie in $T_2(u)$. Without loss of generality, $v = u_3$. Hence $V(G) = \{u, u_1, u_2, v, u_4, u_5, u_6, v_1, v_4\}$ and $O(u) = \{u_1, u_4\}$. We have the configuration shown in Fig. 4. Hence $u_1$ can reach
Taking into account adjacencies between members of $N(u)$ and 1, we have without loss of generality one of the following: a) $O(u_1) = \{u, v_4\}$, $N^+(u_4) = \{u_5, u_6\}$, b) $O(u_1) = \{u, v_1\}$, $N^+(u_4) = \{u_6, v_3\}$, c) $O(u_1) = \{u_5, u_6\}$, $N^+(u_4) = \{u, v_4\}$ or d) $O(u_1) = \{u_5, v_4\}$, $N^+(u_4) = \{u, v_6\}$.

**Case 2.a: $O(u_1) = \{u, v_4\}, N^+(u_4) = \{u_5, u_6\}$**

As $v_4 \in O(u) \cap O(u_1)$, $u_2$ can reach $v_4$ and without loss of generality $v_4 \in N^+(u_5), N^+(u_2) = N^+(u_4)$, so by Lemma 1 $u_2 \in O(u_4), u_4 \in O(u_2), u_5 \in O(u_6)$ and $u_6 \in O(u_5)$. Hence $u_4 \in O(v) \cap O(u_2)$, so $v_1$ can reach $u_4$, so $u_4 \in N^+(v_4)$. Neither $u_5$ nor $u_6$ lies in $N^+(v_4)$, so $O(v_1) = \{u_5, u_6\}$ and $N^+(u_4) = \{u_4, v\}$. Hence $O(v_4) = \{u, u_1\}$. Observe that $N^+(u_1) = N^+(v_4)$, so that $v \in O(u_4)$. Therefore $v \notin N^+(u_5) \cup N^+(u_6)$, yielding $O(u_2) = \{u_4, v\}, N^+(u_2) = \{v_4, u_1, u_1\}$. As $v_1 \rightarrow v_4$ and $N^+(u_1) = N^+(v_4)$, we must have $N^+(u_5) = \{v_4, u\}, N^+(u_6) = \{v_1, u_1\}$. This yields the digraph in Fig. 5. Unlike the digraph in Fig. 2, this digraph contains pairs of vertices with identical out-neighbourhoods, so the two are not isomorphic.

**Case 2.b: $O(u_1) = \{u, u_5\}, N^+(u_4) = \{u_6, v_4\}$**

As $u_4 \rightarrow v_4, u_4 \notin N^+(v_4)$, so $u_4 \notin O(v_1)$. Hence $u_4 \notin O(v) \cap O(v_1)$, so $u_2$ can reach $u_4$. As $u_4 \rightarrow u_6$, we must have $u_5 \rightarrow u_4, u_2$ and $u_4, u_6$ as a common out-neighbour, so $v_4 \in O(u_2), u_5 \in O(u_4)$. Therefore $v_4 \in O(u) \cap O(u_2)$, so that $u_6$ can reach $v_4$, but $v_4 \notin T(u_6)$, so $N^+(u_6)$ contains an in-neighbour of $v_3, u_4 \notin N^+(u_6)$. We must have $u_5 \rightarrow v_1$. We have $u_5 \in O(u_4) \cap O(u_5)$, so $v_1$ can reach $u_5$, hence $v_4 \rightarrow u_5, u_1$ cannot reach $u_6$, as $u_2, u_4 \notin T(v_1)$, so $O(v_1) \neq \{u_4, u_6\}, N^+(u_5) = \{v_5, v\}$. Now $u_2$ and $u_3$ have $u_5$ as a unique common out-neighbour, so $u_6 \notin O(v_4), v \in O(u_2)$. Thus $O(u_2) = \{v, u_4\}$ and $N^2(u_2) = \{u_4, v, u_1, u_1\}$. Taking into account adjacencies between members of $N^2(u_2)$, it follows that $N^+(u_5) = \{u_4, u_1\}, N^+(u_6) = \{u_1, v_1\}$. However, $(u_2, u_4)$ now constitutes a bad pair, contradicting our assumption.

**Case 2.c: $O(u_1) = \{u_5, u_6\}, N^+(u_4) = \{u_4, v_4\}$**

As $u_4 \rightarrow v_4, u_4 \notin N^+(v_4)$. Hence $u_4 \in O(v) \cap O(v_1)$, implying that $u_2$ can reach $u_4$. Without loss of generality, $u_5 \rightarrow u_4$. There are three possibilities: (i) $O(v_1) = \{u_4, u_6\}, N^+(u_4) = \{v, u_5\}$, (ii) $O(v_1) = \{u_4, u_5\}, N^+(u_4) = \{u_5, v_6\}$ and (iii) $O(v_1) = \{u_4, v\}, N^+(u_4) = \{u_5, u_6\}$.

- (i) $O(v_1) = \{u_4, u_6\}, N^+(u_4) = \{v, u_5\}$
- $u_1$ and $v_4$ have $v$ as a unique common out-neighbour, so $u_4 \in O(v_4)$. However, this contradicts $v_4 \rightarrow u_5 \rightarrow u_4$.

- (ii) $O(v_1) = \{u_4, u_5\}, N^+(u_4) = \{v, u_6\}$

Neither $u_4$ nor $v_1$ lie in $T(u_2)$, so $v_4 \in O(u_2)$. Now observe that $u_2$ and $v_4$ have $u_6$ as unique common out-neighbour, so $v \in O(u_2)$, yielding $O(u_2) = \{v, u_4\}$ and $N^2(u_2) = \{u_4, u_1, u, v_1\}$. As $u_4 \rightarrow u$ and $u \rightarrow u_1$, we must have $N^+(u_5) = \{u_4, u_1\}, N^+(u_6) = \{u_1, v_1\}$, a contradiction, since $u_1 \rightarrow u_4$.

- (iii) $O(v_1) = \{u_4, v\}, N^+(u_4) = \{u_5, u_6\}$

We now have $N^+(u_2) = N^+(v_4)$, so $u_2 \in O(v_4), v_4 \in O(u_2), u_5 \in O(u_6), u_6 \in O(u_5)$. Also $N^+(u_4) = N^+(v_1)$, so $u_4 \in O(v_1)$, $v_1 \in O(u_4)$ and $u \in O(u_2)$. In $O(u_4)$ implies that $u \notin N^+(u_2) \cup N^+(u_6)$, so we see that $u \in O(u_2)$ and hence $O(u_2) = \{u, v_4\}$ and $N^2(u_2) = \{u_4, u_1, v, v_1\}$. As $u_1 \rightarrow u_4$ and $u_4 \rightarrow v$, we have $N^+(u_5) = \{u_4, v\}, N^+(u_6) = \{u_1, v_1\}$. It is not difficult to show that this yields a $(2, 2, +2)$-digraph isomorphic to that in Fig. 5.

**Case 2.d: $O(u_1) = \{u, v_4\}, N^+(u_4) = \{u, u_6\}$**

In this case $v_4 \in O(u) \cap O(u_1)$, so $u_2$ can reach $v_4, u_4$ and $v_1$ have unique common out-neighbour $u$, so $v_4 \in O(u_4), u_6 \in O(v_1)$. If $u_6 \rightarrow v_4$, then we would have $u_4 \rightarrow u_6 \rightarrow v_4$, contradicting $v_4 \in O(u_4)$, so $u_5 \rightarrow v_4$. This also implies that $u_5 \notin N^+(v_4), u_5 \notin O(v_1)$, yielding $O(v_1) = \{u_5, u_6\}$ and $N^+(u_4) = \{v, u_4\} = N^+(u_1)$. Now $v_4, u_1 \notin T(u_2)$, so $O(u_2) = \{v, u_4\}$.
and $N^2(u_2) = \{v_1, v_4, u, u_1\}$. As $v_1 \to v_4$ and $v_1 \to u$, it follows that $N^+(u_5) = \{v_4, u\}$, $N^+(u_6) = \{u_1, v_1\}$. However, we now have paths $u_4 \to u \to u_1$ and $u_4 \to u_6 \to u_1$, which is impossible.

**Case 3: $N^+(u_1) = N^+(v_1)$**

It is easy to see by 2-geodesity that $V(G) = \{u, u_1, u_2, u_3, u_4, u_5, v, v_1\}$, $O(u) = \{v, v_1\}$ and $O(v) = \{u, u_1\}$. As $u_1, v_1 \not\in T(u_2)$, we have $O(u_2) = \{u_3, u_4\}$ and $N^+(u_2) = \{u, u_1, v, v_1\}$. Without loss of generality, $N^+(u_3) = \{u, v_1\}, N^+(u_6) = \{v, u_1\}, u$ and $v$ have in-neighbours apart from $u_5$ and $u_6$ respectively, so without loss of generality $u_3 \to u$, $u_4 \to v$. Likewise, $u_5$ and $u_6$ have in-neighbours other than $u_2$, so, as $u_3 \to u$ and $u_6 \to v$, we must have $N^+(u_3) = \{u, u_5\}, N^+(u_4) = \{v, u_5\}$. But now we have paths $u_3 \to u \to u_1$ and $u_3 \to u_6 \to u_1$, violating 2-geodesy.

**Corollary 2.** There is a unique $(2, 2, +2)$-digraph containing no bad pairs.

This completes our analysis of diregular $(2, 2, +2)$-digraphs. As it was shown in [7] that there are no non-diregular $(2, 2, +2)$-digraphs, $(2, 2, +2)$-digraphs are now classified up to isomorphism. These conclusions have been verified computationally by Erskine [2]. It is interesting to note that neither of the $(2, 2, +2)$-digraphs are vertex-transitive, for in each case there are exactly three vertices contained in two 3-cycles. However, there does exist a Cayley $(2, 2, +5)$-digraph (on the alternating group $A_4$), so it would be interesting to determine the smallest vertex-transitive $(2, 2, +\epsilon)$-digraphs.

**4. Main result**

We can now complete our analysis by showing that there are no diregular $(2, k, +2)$-digraphs for $k \geq 3$. Let $G$ be such a digraph. By Lemma 2, $G$ contains vertices $u$ and $v$ with a unique common out-neighbour. In accordance with our vertex-labelling convention, we have the situation in Fig. 6. A triangle based at a vertex $x$ represents the set $T(x)$.

We now proceed to determine the possible outlier sets of $u$ and $v$.

**Lemma 5.** $v \in N^{k-1}(u_1) \cup O(u)$ and $u \in N^{k-1}(v_1) \cup O(v)$. If $v \in O(u)$, then $u_2 \in O(u_1)$ and if $u \in O(v)$, then $u_2 \in O(u_1)$.

**Proof.** $v$ cannot lie in $T(u)$, or the vertex $u_2$ would be repeated in $T(u)$. Also, $v \not\in T(u_2)$, or there would be a $\leq k$-cycle through $v$. Therefore, if $v \not\in O(u)$, then $u \in N^{k-1}(u_1)$. Likewise for the other result. If $v \in O(u)$, then neither in-neighbour of $u_2$ lies in $T(u_1)$, so that $u_2 \in O(u_1)$. □

**Lemma 6.** Let $w \in T(v_1)$, with $d(v_1, w) = l$. Suppose that $w \in T(u_1)$, with $d(u_1, w) = m$. Then either $m \leq l$ or $w \in N^{k-1}(u_1)$. A similar result holds for $w \in T(u_1)$.

**Proof.** Let $w$ be as described and suppose that $m > l$. Consider the set $N^{k-m}(w)$. By construction, $N^{k-m}(w) \subseteq N^k(u_1)$, so by $k$-geodesity $N^{k-m}(w) \cap T(u_1) = \emptyset$. At the same time, we have $l + l = k - m \leq k - 1$, so $N^{k-m}(w) \subseteq T(v_1)$. This implies that $N^{k-m}(w) \cap T(v_2) = N^{k-m}(w) \cap T(u_2) = \emptyset$. As $V(G) = \{u\} \cup T(u_1) \cup T(u_2) \cup O(u)$, it follows that $N^{k-m}(w) \subseteq \{u\} \cup O(u)$. Therefore $|N^{k-m}(w)| = 2k-3$. By assumption $0 \leq m \leq k - 1$, so it follows that $m = k - 1$. □

**Corollary 3.** If $w \in T(v_1)$, then either $w \in \{u\} \cup O(u)$ or $w \in T(u_1)$ with $d(u_1, w) = k - 1$ or $d(u_1, w) \leq d(v_1, w)$.

**Proof.** By $k$-geodesity and Lemma 6. □

**Corollary 4.** $v_1 \in N^{k-1}(u_1) \cup O(u)$ and $u_1 \in N^{k-1}(v_1) \cup O(v)$. 

![Fig. 6. Configuration for $k \geq 3$.](image-url)
Proof. We prove the first inclusion. By Corollary 3, \( v_1 \in \{u\} \cup O(u) \cup \{u_1\} \cup N^{k-1}(u_1) \). By \( k \)-geodecity, \( v_1 \neq u \) and by construction, \( v_1 \neq u_1 \). □

We now have enough information to identify one member of \( O(u) \) and \( O(v) \).

Lemma 7. \( v_1 \in O(u) \) and \( u_1 \in O(v) \).

Proof. We prove that \( v_1 \in O(u) \). Suppose that neither \( v_1 \) nor \( v \) lies in \( O(u) \). Then by Lemma 5 and Corollary 4 we have \( v, v_1 \in N^{k-1}(u_1) \). As \( v_1 \) is an out-neighbour of \( v \), it follows that \( v_1 \) appears twice in \( T_k(u_1) \), violating \( k \)-geodecity. Therefore \( O(u) \cap \{v, v_1\} = \emptyset \).

Now assume that \( v_1, v_3 \in T_k(u) \). Again by Corollary 4, \( v_1 \in N^{k-1}(u_1) \). By \( k \)-geodecity we also have \( v_3 \in T(u_1) \). However, \( v_1 \in N_1(v_3) \), so \( v_1 \) appears twice in \( T_k(u_1) \), which is impossible. Hence \( O(u) \cap \{v_1, v_3\} \neq \emptyset \). Similarly, \( O(u) \cap \{v_1, v_4\} \neq \emptyset \). In the terminology of the previous section, \( G \) contains no bad pairs. Therefore, if \( v_1 \notin O(u) \), then \( \{v_1, v_3, v_4\} \subseteq O(u) \). Since these vertices are distinct, this is a contradiction and the result follows. □

Lemma 7 allows us to conclude that for vertices sufficiently close to \( v_1 \) one of the potential situations mentioned in Corollary 3 cannot occur.

Lemma 8. \( T_{k-3}(v_1) \cap N^{k-2}(u_1) = T_{k-3}(u_1) \cap N^{k-2}(v_1) = \emptyset \).

Proof. Let \( w \in T_{k-3}(v_1) \cap N^{k-1}(u_1) \). Consider the position of the vertices of \( N^+(w) \) in \( T_k(u) \cup O(u) \). As \( v_1 \notin N^+(w) \), it follows from Lemma 7 that at most one of the vertices of \( N^+(w) \) can be an outlier of \( u \), so let us write \( w_1 \in N^+(w) - O(u) \). By \( k \)-geodecity, \( u_1 \notin T(u_1) \cup \{u\} \). Hence \( w_1 \in T(u_1) \cup \{u\} \). However, \( w_1 \) also lies in \( T(v_1) \), so this violates \( k \)-geodecity. □

Corollary 5. There is at most one vertex in \( T_{k-3}(v_1) - \{v_1\} \) that does not lie in \( T(u_1) \); for all other vertices \( w \in T_{k-3}(v_1) - \{v_1\} \), \( d(u_1, w) = d(v_1, w) \). A similar result for \( T_{k-3}(u_1) - \{u_1\} \) also holds.

Lemma 9. For \( k = 3 \), \( N^{+}(u_1) \cap N^2(v_1) = N^{+}(v_1) \cap N^2(u_1) = \emptyset \).

Proof. Suppose that \( v_3 = u_2 \). By the reasoning of Lemma 8 we can set \( u = v_2 \) and \( O(u) = \{v_1, v_3\}, v \notin O(u) \) and by 3-geodecity \( v \notin N^+(u_3) \), so we can assume that \( u = u_3, v_3 \rightarrow v_3 \) implies that \( u_1 \notin T(v_1) \), so \( O(v) = \{u_1, v_1\} \). We must have \( \{u_4, u_5, u_{10}\} = \{v_4, v_5, v_{10}\} \). As \( u_4 \rightarrow v_1 \), it follows that \( v_4 = u_5 \) and hence \( \{u_4, u_{10}\} = \{v_2, v_{10}\} \), which is impossible. □

As \( u_1 \) is an outlier of \( v_1 \), neither \( v_3 \) nor \( v_4 \) can be equal to \( u_1 \). It follows from Corollary 5 and Lemma 9 that either \( N^+(u_1) = N^+(v_1) \) or \( u_1 \) and \( v_1 \) have a single common out-neighbour, with one vertex of \( N^+(v_1) \) being an outlier of \( u \).

Lemma 10. \( N^2(u) \neq N^2(v) \)

Proof. Let \( N^2(u) = N^2(v) \), with \( N^+(u_1) = N^+(v_1) = \{u_3, u_4\} \). Suppose that \( v \notin O(u) \). By Lemma 5, \( v \in N^{k-2}(u_3) \cup N^{k-2}(u_4) \). But then there is a \( k \)-cycle through \( v \). It follows that \( O(u) = \{v_1, v_3\}, O(v) = \{u_1, v_1\} \). By Lemma 5, \( u_2 \in O(u) \cap O(v) \). Therefore by Lemma 1 \( O(u) = \{u_2, v_1\} \), \( O(v) = \{u_2, u_1\} \).

Consider the in-neighbour \( u' \) of \( u_1 \) that is distinct from \( u \). We have either \( |N^+(u') \cap N^+(u)| = 1 \) or \( |N^+(u') \cap N^+(u)| = 2 \).

In the first case, it follows from Lemma 7 that \( u_2 \in O(u') \). Every vertex of \( G \) is an outlier of exactly two vertices, so \( u' = u_1 \) or \( v_1 \). In either case, we have a contradiction. Therefore \( N^+(u') = N^+(u) \). It now follows from Lemma 1 that \( u' \in O(u) = \{v_1, v_1\} \), which is impossible. □

Noticing that \( u_1 \) and \( v_1 \) also have a unique common out-neighbour, we have the following corollary.

Corollary 6. Without loss of generality, \( u_1 = v_3, u_9 = v_9, O(u) = \{v_1, v_4\}, O(v) = \{u_1, u_4\}, O(u_1) = \{v_4, v_{10}\} \) and \( O(v_1) = \{u_4, u_{10}\} \).

We are now in a position to complete the proof by deriving a contradiction.

Theorem 2. There are no directrical \((2, k, +2)\)-digraphs for \( k \geq 3 \).

Proof. \( u, v \notin \{u_1, u_3, v_1, v_4\} \), so by Lemma 5 \( \ell(u, u) = d(v, u) = k \). In fact, \( u_3 = v_3 \) implies that \( v \in N^{k-2}(u_3) \) and \( u \in N^{k-2}(v_4) \). Let \( k \geq 4 \). Then \( u, v \notin \{u_{10}, v_{10}\} \), so \( u, v \in T_k(u_1) \cup T_k(v_1) \). If \( u \in T_k(u_2) \), then \( u \) would appear twice in \( T_k(v_1) \), so \( u \notin N^{k-1}(u_4) \). However, as \( u \) and \( v \) have a common out-neighbour, this violates \( k \)-geodecity.

Finally, suppose that \( k = 3 \). The above analysis will hold unless \( u = v_{10} \) and \( v = u_{10} \). Let \( N^+(u_1) = \{u, u'\}, N^+(v_1) = \{v, v'\} \). It is evident that \( u' \notin \{v_1, v_4\} \), so that \( u' \notin T_k(u_3) \). As \( v \in N^+(u_4) \), we must have \( v' \notin N^2(u_2) \). Similarly \( v' \notin N^2(u_2) \).

Since \( u_1 \) and \( v_1 \) have a common out-neighbour, we can assume that \( u' \notin N^+(u_1) \) and \( v' \notin N^+(u_4) \), \( v_4 \) can be the outlier of only two vertices, namely \( u \) and \( u_1 \), so \( v_4 \in N^2(u_2) \). By 3-geodecity \( v_3 \in N^2(u_2) \) and \( u_4 \in N^2(u_6) \). It follows that \( u, v \notin N^2(u_3) \) so \( u \notin T_k(u_1) \cup T_k(u_2) \). Hence \( O(u) = N^+(u) = \{v_1, v_4\} \), which again is impossible. □

It is interesting to note that a similar argument can be used to provide an alternative proof of the result of [6].
References