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On the number of transversals in a class of Latin squares

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Abstract
Denote by $A_{pk}^n$ the Latin square of order $n = p^k$ formed by the Cayley table of the additive group $(\mathbb{Z}_{pk}^n, +)$, where $p$ is an odd prime and $k$ is a positive integer. It is shown that for each $p$ there exists $Q > 0$ such that for all sufficiently large $k$, the number of transversals in $A_{pk}^n$ exceeds $(nQ)^{\frac{n}{p(n-1)}}$.

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1 Introduction

Several recent papers have addressed the issue of bounds on the numbers of transversals in Latin squares. So, suppose that $S$ is a Latin square. Denote by $T(S)$ the number of transversals in $S$, and put

$$T(n) = \max\{T(S) : S \text{ is a Latin square of order } n\}.$$

It was shown by McKay, McLeod and Wanless [4] that for $n \geq 5$, $15^{n/5} \leq T(n) \leq 0.6135^n n! \sqrt{n}$.

The Cayley table of any finite group forms a Latin square, and such squares are called group-based. Let $A_n$ denote the cyclic Latin square of order $n$, that is the square formed by the Cayley table of the cyclic group $(\mathbb{Z}_n, +)$. If $n$ is even then $T(A_n) = 0$, but for odd $n$ it was conjectured by Vardi [6] that there exist positive constants $c$ and $d$ such that $c^n n! \leq T(A_n) \leq d^n n!$. Subsequently Cavenagh and Wanless [1] proved that for all sufficiently large $n$, $T(A_n) > (3.246)^n n!$, and this appears to remain the best lower bound for any class of group-based Latin squares obtained to date.

More recently, Taranenko [5] proved that $T(n) \leq \left(1 + o(1)\right) e^n n!$, while Glebov and Luria [3] have shown that $T(n) \geq \left(1 - o(1)\right) e^n n!$. The latter result is based on a probabilistic argument employing random Latin squares. These more recent results lend credence to Vardi’s conjecture but do not address group-based squares directly.

In the current paper we take $p$ to be an odd prime and $k$ to be a positive integer. Then the Cayley table of the additive group $(\mathbb{Z}_p^k, +)$ forms a Latin square of order $n = p^k$ which we denote by $A_{p^k}$. We will assume that this square has its rows and columns labelled in the natural way by elements of $\mathbb{Z}_p^k$ represented as $k$-vectors over $\mathbb{Z}_p$, and when $k = 1$ we write $A_p$ rather than $A_{p^1}$. We prove that, for all sufficiently large $k$, $A_{p^k}$ has more than $(nQ)^{p^k-1}$ transversals, where $Q > 0$ depends only on $p$ and is independent of $k$.

Note added in proof: Since drafting our current paper, our attention has been drawn to the arXiv paper [2] which claims a proof of Vardi’s conjecture.

2 Results

We start with the observation that $A_{p^k}$ has a transversal $T$ formed from its leading diagonal. We will construct a large number of transversals by carrying out transversal trades on $T$. These trades are based on the square $A_p$ and involve transversals within this square that do not contain the (row, column, entry) triple $(0, 0, 0)$. So let $T^*$ denote the number of transversals of $A_p$ that do not contain this triple. By transitivity, the number of transversals in $A_{p^k}$ that contain the triple $(0, 0, 0)$ is $T(A_{p^k})/p^k$, so the number of transversals not containing this triple is $T(A_{p^k})(1 - \frac{1}{p^k})$. In particular, $T^* = T(A_p)(1 - \frac{1}{p})$, and note rather trivially that $T(A_p) \geq p$. 


For $k \geq 2$, the square $A^k_p$ can be partitioned into $p^2$ subarrays by writing the row labels, the column labels and the entries in the form $(z, i)$ where $z \in \mathbb{Z}^{k-1}_p$ and $i \in \mathbb{Z}_p$. This is shown schematically in Figure 1 with the row and column labels omitted.

$$A^k_p = \begin{pmatrix}
(A^k_p, 0) & (A^k_p, 0, 1) & \cdots & (A^k_p, 0, p - 1) \\
(A^k_p, 1) & (A^k_p, 1, 2) & \cdots & (A^k_p, 0, 0) \\
\vdots & \vdots & \ddots & \vdots \\
(A^k_p, p - 1) & (A^k_p, 0, 1) & \cdots & (A^k_p, 0, p - 2)
\end{pmatrix}$$

$$= \begin{pmatrix}
A_{0,0} & A_{0,1} & \cdots & A_{0,p-1} \\
A_{1,0} & A_{1,1} & \cdots & A_{1,p-1} \\
\vdots & \vdots & \ddots & \vdots \\
A_{p-1,0} & A_{p-1,1} & \cdots & A_{p-1,p-1}
\end{pmatrix}$$

Figure 1: Partitioning $A^k_p$.

Taken without the row and column labels inherited from $A^k_p$, the subarrays $A_{i,j}$ and $A_{i',j'}$ are identical when $i + j = i' + j'$ in $\mathbb{Z}_p$. However, we will associate each of these subarrays with their original row and column labels.

Our transversal trades will be based on copies of $A^k_p$, each having precisely one entry from each $A_{i,j}$. Specifically, one (row, column, entry) triple is selected from the leading diagonal of $A_{0,0}$, say $((a, 0), (a, 0), (2a, 0))$, and one triple is selected from $A_{0,1}$ having the same row entry, say $((a, 0), (b, 1), (a+b, 1))$. These two choices are sufficient to determine a copy of $A^k_p$, denoted by $A(a, b)$, as shown in Figure 2, which also shows the inherited row and column labels.

$$\begin{pmatrix}
(a, 0) & (b, 1) & (2b - a, 2) & \cdots & (2a - b, p - 1) \\
(2a, 0) & (a + b, 1) & (2b, 2) & \cdots & (3a - b, p - 1) \\
(2a + b, 1) & (2b, 2) & (3b - a, 3) & \cdots & (2a, 0) \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
(3a - b, p - 1) & (2a, 0) & (a + b, 1) & \cdots & (4a - 2b, p - 2)
\end{pmatrix}$$

Figure 2: The array $A(a, b)$.

Note that the row and column labels of $A(a, b)$, inherited from $A^k_p$, have the form $(rb - (r - 1)a, r)$ and the entries have the form $(rb - (r - 2)a, r)$, both for $r = 0, 1, \ldots, p - 1$.

The leading diagonal of $A(a, b)$ lies in the leading diagonal of $A^k_p$ and therefore this diagonal of $A(a, b)$ forms a part of the transversal $T$. There are $T^*$ transversals in $A(a, b)$ that do not contain the triple $((a, 0), (a, 0), (2a, 0))$. If the diagonal transversal of $A(a, b)$ in $T$ is traded for any one of these $T^*$ transversals, then a new transversal in $A^k_p$ is obtained that does not contain the triple $((a, 0), (a, 0), (2a, 0))$. Hence, for each given $a \in \mathbb{Z}^{k-1}_p$, $T^*$ distinct transversals of $A^k_p$ may be obtained for each $b \in \mathbb{Z}^{k-1}_p$. Furthermore, for two different
values $b, b' \in \mathbb{Z}_p^{k-1}$, the arrays $A(a, b)$ and $A(a, b')$ only intersect in the cell $((a, 0), (a, 0))$, and so by varying $b$, a total of $p^{k-1} T^*$ distinct transversals of $A_\rho^k$ may be obtained that do not contain the triple $((a, 0), (a, 0), (2a, 0))$.

In principle, we wish to carry out these trades sequentially for as many values of $a$ as is possible. The obstacle is that having carried out a trade using $A(a, b)$, and having chosen $a' \neq a$, the choice of $b'$ is constrained by the need to ensure that $A(a', b')$ avoids the rows, columns and entries of $A(a, b)$. So suppose that trades have already been made using $c - 1$ choices of $(a, b)$ and that a $c^{th}$ choice is to be made. If $(a, b)$ defines one of the previous choices and $(a', b')$ is the proposed $c^{th}$ choice, with $a' \neq a$, then to ensure that rows and columns do not clash it is necessary and sufficient that $(r' b' - (r' - 1) a', r')$ and $(r b - (r - 1) a, r)$ are unequal for all $r, r' = 0, 1, \ldots, p - 1$. But these two quantities can only be equal if $r' = r$, and then only if $r b' - (r - 1) a' = r b - (r - 1) a$. Hence the rows and columns of $A(a, b)$ and $A(a', b')$ are distinct provided that $b' \neq b + \frac{1}{p - 1} (a' - a)$ for $r = 1, 2, \ldots, p - 1$. As $r$ varies from 1 to $p - 1$, $\frac{1}{p - 1}$ takes all values in $\mathbb{Z}_p$, apart from the value 1. Hence in selecting $b'$ it is necessary to avoid the $p - 1$ values $b + p(a' - a)$ for $p = 0, 2, 3, \ldots, p - 1$ for each previous choice of $(a, b)$.

By arguing in a similar fashion regarding entries, we obtain exactly the same condition to avoid entry clashes between $A(a, b)$ and $A(a', b')$. It follows that at the $c^{th}$ choice, there are at least $p^{k-1} - (c - 1)(p - 1)$ choices for $b'$ (rather more if there is multiple counting of excluded rows, columns and entries).

Now put $C = \lfloor \frac{k - 1}{p - 1} \rfloor = \frac{k - 1}{p - 1}$ and let $c \leq C$ be a positive integer. Then it is possible to choose $c$ subarrays of the form $A(a, b)$ that are pairwise disjoint as regards rows, columns and entries. Suppose that the subarrays chosen are $A(a_i, b_i)$ for $i = 1, 2, \ldots, c$. Then the number of transversals in $A_\rho^k$ that do not contain any of the triples $((a_i, 0), (a_i, 0), (2a_i, 0))$ for $i = 1, 2, \ldots, c$, and which can be constructed by trades on these arrays is at least

$$(T^*)^c (p^{k-1} - (p - 1))(p^{k-1} - 2(p - 1)) \ldots (p^{k-1} - (c - 1)(p - 1)) \ldots > (T^*(p - 1))^c \frac{c^c}{(C - c)!}$$

To see that these transversals are all distinct, consider any one of them, say $T^*$. Each $a_i$, for $i = 1, 2, \ldots, c$ can be identified from those diagonal entries of $A_0, 0$ that do not form part of $T^*$. Having identified an $a_i$, there will be a triple of $T^*$ of the form $((a_i, 0), (r b_i - (r - 1) a_i, r), (r b_i - (r - 2) a_i, r))$ where $r \neq 0$. From this triple, $r$ can be identified and hence also $b_i$. Thus the subarrays $A(a_i, b_i)$ can be recovered from $T^*$, and the distinctness of the transversals follows. In fact any distinct choices of up to $C$ values for $a_i$ will yield distinct transversals. Hence we obtain the following theorem.

**Theorem 2.1** If $p$ is an odd prime and $k$ is a positive integer, then the number of transversals in the Latin square $A_\rho^k$, denoted by $T(A_\rho^k)$, satisfies the inequality

$$T(A_\rho^k) > \sum_{c=0}^{C} \left( \begin{array}{c} p^{k-1} \end{array} \right)(T^*(p - 1))^c \frac{C!}{(C - c)!} \quad (1)$$

4
where \( C = \frac{p^{k-1}-1}{p-1} \) and \( T^* = T(A_p)(1 - \frac{1}{p}) \).

The final term in the summation (1) gives

\[
T(A^k_p) > \left( \frac{p^{k-1}}{C} \right) (T^*(p-1))^C!
= \frac{(p^{k-1})!}{(p^{k-1} - C)!} (T^*(p-1))^C
\]

Applying Stirling’s Theorem in the form \( r! = r^{r+\frac{1}{2}} e^{-r} \sqrt{2\pi} e^{o(1)} \) (as \( r \to \infty \)) to this expression for large \( k \) gives

\[
T(A^k_p) > [p^{k-1} T^*(p-1)e^{-1}]^C \left( 1 - \frac{C}{p^{k-1}} \right)^{-(p^{k-1} - C + \frac{1}{2})} \cdot e^{o(1)}. \tag{2}
\]

For \( p \geq 3 \) and \( k \geq 2 \) we have \((1 - \frac{C}{p^{k-1}}) \leq (1 - \frac{1}{p})\) and \( p^{k-1} - C + \frac{1}{2} > (p - 2)C\). Hence

\[
T(A^k_p) > \left[ p^k \left( \frac{p}{p-1} \right)^{p-4} T(A_p)e^{-1} \right]^C \cdot e^{o(1)}.
\]

The square \( A^k_p \) has order \( n = p^k \) and \( C = \frac{p^k}{p^{k-1}} - \frac{1}{p-1} \), so taking \( Q \) to be slightly less than \((\frac{p^k}{p^{k-1}})^{p-4} T(A_p)e^{-1}\) gives the following corollary.

**Corollary 2.1** If \( p \) is an odd prime, there exists \( Q > 0 \) such that for all sufficiently large \( k \),

\[
T(A^k_p) > (nQ)^{\frac{n}{p^{k-1}}},
\]

where \( n = p^k \).

In fact if \( p \) is also sufficiently large, then using the result of [1], we may take \( Q = (3.246)^p \). However, the bound is clearly best when \( p \) is small. In the case \( p = 3 \), inequality (2) simplifies as follows. Firstly \( T(A_3) = 3 \), so \( T^* = 2 \). Also \( C = (3^{k-1} - 1)/2 \) and \( 3^{k-1} - C + \frac{1}{2} = 3^{k-1}/2 + 1 \). Hence

\[
T(A^k_3) > (4 \cdot 3^{k-1} \cdot e^{-1})^C \left( \frac{1}{2} + \frac{1}{3^{k-1}} \right)^{-(3^{k-1} - 1)} \cdot e^{o(1)}
= \left( \frac{4n}{3e} \right)^C \cdot 2^{C+\frac{1}{2}} \left( 1 + \frac{1}{3^{k-1}} \right)^{-(3^{k-1} - 1)} \cdot e^{o(1)}
= \left( \frac{8n}{3e} \right)^C \cdot 2^{\sqrt{2}} \cdot \frac{1}{\sqrt{e}} \cdot e^{o(1)},
\]

since \( (1 + \frac{1}{p})^{-r} \to e^{-1} \) as \( r \to \infty \). Noting that \( 8/3e \) is 0.981 and that \( C = \frac{n}{6} - \frac{1}{7} \), we obtain
Corollary 2.2 For all sufficiently large $k$, $T(A^k_3) > (0.981n)^{\frac{5}{6}}$, where $n = 3^k$.

Finally we remark that, by transitivity, the number of orthogonal mates of the Latin square $A^k_p$ is $T(A^k_p)/n$ (where $n = p^k$) and so Theorem 2.1 and its corollaries also provide lower bounds for this quantity.

References


