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On the number of transversals in a class of Latin squares

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Abstract

Denote by $A^p_k$ the Latin square of order $n = p^k$ formed by the Cayley table of the additive group $(\mathbb{Z}_p^k, +)$, where $p$ is an odd prime and $k$ is a positive integer. It is shown that for each $p$ there exists $Q > 0$ such that for all sufficiently large $k$, the number of transversals in $A^p_k$ exceeds $(nQ)^{n/3p^k/4}$. 

Running head: Transversals.
AMS classification: 05B15.
Keywords: Latin square; Transversal.

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1 Introduction

Several recent papers have addressed the issue of bounds on the numbers of transversals in Latin squares. So, suppose that $S$ is a Latin square. Denote by $T(S)$ the number of transversals in $S$, and put

$$T(n) = \max\{T(S) : S \text{ is a Latin square of order } n\}.$$ 

It was shown by McKay, McLeod and Wanless [4] that for $n \geq 5$, $15^{n/5} \leq T(n) \leq 0.6135n!/\sqrt{n}$.

The Cayley table of any finite group forms a Latin square, and such squares are called group-based. Let $A_n$ denote the cyclic Latin square of order $n$, that is the square formed by the Cayley table of the cyclic group $(\mathbb{Z}_n, +)$. If $n$ is even then $T(A_n) = 0$, but for odd $n$ it was conjectured by Vardi [6] that there exist positive constants $c$ and $d$ such that $c^n n! \leq T(A_n) \leq d^n n!$. Subsequently Cavenagh and Wanless [1] proved that for all sufficiently large $n$, $T(A_n) > (3.246)^n$, and this appears to remain the best lower bound for any class of group-based Latin squares obtained to date.

More recently, Taranenko [5] proved that $T(n) \leq [(1 + o(1)) \frac{n}{2\sqrt{2}}]n$, while Glebov and Luria [3] have shown that $T(n) \geq [(1 - o(1)) \frac{\pi}{2}]n$. The latter result is based on a probabilistic argument employing random Latin squares. These more recent results lend credence to Vardi’s conjecture but do not address group-based squares directly.

In the current paper we take $p$ to be an odd prime and $k$ to be a positive integer. Then the Cayley table of the additive group $(\mathbb{Z}_p^k, +)$ forms a Latin square of order $n = p^k$ which we denote by $A_{kp}$. We will assume that this square has its rows and columns labelled in the natural way by elements of $\mathbb{Z}_p^k$ represented as $k$-vectors over $\mathbb{Z}_p$, and when $k = 1$ we write $A_p$ rather than $A_{1p}$. We prove that, for all sufficiently large $k$, $A_{kp}$ has more than $(nQ)\frac{p}{(p-1)^{3Q}}$ transversals, where $Q > 0$ depends only on $p$ and is independent of $k$.

Note added in proof: Since drafting our current paper, our attention has been drawn to the arXiv paper [2] which claims a proof of Vardi’s conjecture.

2 Results

We start with the observation that $A_{kp}$ has a transversal $T$ formed from its leading diagonal. We will construct a large number of transversals by carrying out transversal trades on $T$. These trades are based on the square $A_p$ and involve transversals within this square that do not contain the (row, column, entry) triple $(0,0,0)$. So let $T^*$ denote the number of transversals of $A_p$ that do not contain this triple. By transitivity, the number of transversals in $A_{kp}$ that contain the triple $(0,0,0)$ is $T(A_{kp})/p^k$, so the number of transversals not containing this triple is $T(A_{kp})(1 - \frac{1}{p^k})$. In particular, $T^* = T(A_p)(1 - \frac{1}{p})$, and note rather trivially that $T(A_p) \geq p$. 

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For \( k \geq 2 \), the square \( A_p^k \) can be partitioned into \( p^2 \) subarrays by writing the row labels, the column labels and the entries in the form \((z,i)\) where \( z \in \mathbb{Z}_p^{k-1} \) and \( i \in \mathbb{Z}_p \). This is shown schematically in Figure 1 with the row and column labels omitted.

\[
A_p^k = \begin{array}{cccc}
(A_p^k,0,0) & (A_p^k,0,1) & \ldots & (A_p^k,0,p-1) \\
(A_p^k,1,1) & (A_p^k,1,2) & \ldots & (A_p^k,1,p-1) \\
& \vdots & \ddots & \vdots \\
(A_p^k,p-1,0) & (A_p^k,p-1,1) & \ldots & (A_p^k,p-1,p-2)
\end{array}
\]

\[
\begin{array}{cccc}
A_{0,0} & A_{0,1} & \ldots & A_{0,p-1} \\
A_{1,0} & A_{1,1} & \ldots & A_{1,p-1} \\
& \vdots & \ddots & \vdots \\
A_{p-1,0} & A_{p-1,1} & \ldots & A_{p-1,p-1}
\end{array}
\]

Figure 1: Partitioning \( A_p^k \).

Taken without the row and column labels inherited from \( A_p^k \), the subarrays \( A_{ij} \) and \( A_{i',j'} \) are identical when \( i + j = i' + j' \) in \( \mathbb{Z}_p \). However, we will associate each of these subarrays with their original row and column labels.

Our transversal trades will be based on copies of \( A_p^k \), each having precisely one entry from each \( A_{ij} \). Specifically, one (row, column, entry) triple is selected from the leading diagonal of \( A_{0,0} \), say \(((a,0),(a,0),(2a,0))\), and one triple is selected from \( A_{0,1} \) having the same row entry, say \(((a,0),(b,1),(a+b,1))\). These two choices are sufficient to determine a copy of \( A_p^k \), denoted by \( A(a,b) \), as shown in Figure 2, which also shows the inherited row and column labels.

\[
\begin{array}{cccc}
(a,0) & (b,1) & (2b-a,2) & \ldots & (2a-b,p-1) \\
(2a,1) & (a+b,1) & (2b,2) & \ldots & (3a-b,p-1) \\
& \vdots & \ddots & \vdots & \vdots \\
(3a-b,p-1) & (2a,0) & (a+b,1) & \ldots & (4a-2b,p-2)
\end{array}
\]

Figure 2: The array \( A(a,b) \).

Note that the row and column labels of \( A(a,b) \), inherited from \( A_p^k \), have the form \( (rb-(r-1)a,r) \) and the entries have the form \( (rb-(r-2)a,r) \), both for \( r = 0,1,\ldots,p-1 \).

The leading diagonal of \( A(a,b) \) lies in the leading diagonal of \( A_p^k \) and therefore this diagonal of \( A(a,b) \) forms a part of the transversal \( T \). There are \( T^* \) transversals in \( A(a,b) \) that do not contain the triple \(((a,0),(a,0),(2a,0))\). If the diagonal transversal of \( A(a,b) \) in \( T \) is traded for any one of these \( T^* \) transversals, then a new transversal in \( A_p^k \) is obtained that does not contain the triple \(((a,0),(a,0),(2a,0))\). Hence, for each given \( a \in Z_p^{k-1} \), \( T^* \) distinct transversals of \( A_p^k \) may be obtained for each \( b \in Z_p^{k-1} \). Furthermore, for two different
values $b, b' \in \mathbb{Z}_p^{k-1}$, the arrays $A(a, b)$ and $A(a, b')$ only intersect in the cell $((a, 0), (a, 0))$, and so by varying $b$, a total of $p^{k-1}T^*$ distinct transversals of $A_p^k$ may be obtained that do not contain the triple $((a, 0), (a, 0), (2a, 0))$.

In principle, we wish to carry out these trades sequentially for as many values of $a$ as is possible. The obstacle is that having carried out a trade using $A(a, b)$, and having chosen $a' \neq a$, the choice of $b'$ is constrained by the need to ensure that $A(a', b')$ avoids the rows, columns and entries of $A(a, b)$. So suppose that trades have already been made using $c-1$ choices of $(a, b)$ and that a $c^{th}$ choice is to be made. If $(a, b)$ defines one of the previous choices and $(a', b')$ is the proposed $c^{th}$ choice, with $a' \neq a$, then to ensure that rows and columns do not clash it is necessary and sufficient that $((r'b' - (r' - 1)a', r')$ and $(rb - (r - 1)a, r)$ are unequal for all $r, r' = 0, 1, \ldots, p-1$. But these two quantities can only be equal if $r' = r$, and then only if $r b' - (r - 1)a' = rb - (r - 1)a$. Hence the rows and columns of $A(a, b)$ and $A(a', b')$ are distinct provided that $b' \neq b + \frac{1}{p-1}(a' - a)$ for $r = 1, 2, \ldots, p-1$. As $r$ varies from 1 to $p - 1$, \( \frac{1}{p-1} \) takes all values in $\mathbb{Z}_p$, apart from the value 1. Hence in selecting $b'$ it is necessary to avoid the $p - 1$ values $b + \rho(a' - a)$ for $\rho = 0, 2, 3, \ldots, p - 1$ for each previous choice of $(a, b)$.

By arguing in a similar fashion regarding entries, we obtain exactly the same condition to avoid entry clashes between $A(a, b)$ and $A(a', b')$. It follows that at the $c^{th}$ choice, there are at least $p^{k-1} - (c - 1)(p - 1)$ choices for $b'$ (rather more if there is multiple counting of excluded rows, columns and entries).

Now put $C = \left( \frac{p^{k-1}}{p-1} \right)$ and let $c \leq C$ be a positive integer. Then it is possible to choose $c$ subarrays of the form $A(a, b)$ that are pairwise disjoint as regards rows, columns and entries. Suppose that the subarrays chosen are $A(a_i, b_i)$ for $i = 1, 2, \ldots, c$. Then the number of transversals in $A_p^k$ that do not contain any of the triples $((a_i, 0), (a_i, 0), (2a_i, 0))$ for $i = 1, 2, \ldots, c$, and which can be constructed by trades on these arrays is at least

$$(T^*)^c(p^{k-1} - (p - 1)) \left( p^{k-1} - 2(p - 1) \right) \left( p^{k-1} - (c - 1)(p - 1) \right) > (T^* - (p - 1))^c \left( \frac{C!}{(C - c)!} \right)^c$$

To see that these transversals are all distinct, consider any one of them, say $T^*$. Each $a_i$ for $i = 1, 2, \ldots, c$ can be identified from those diagonal entries of $A_0, 0$ that do not form part of $T^*$. Having identified an $a_i$, there will be a triple of $T^*$ of the form $((a_i, 0), (rb - (r - 1)a_i, r), (rb - (r - 2)a_i, r))$ where $r \neq 0$. From this triple, $r$ can be identified and hence also $b_i$. Thus the subarrays $A(a_i, b_i)$ can be recovered from $T^*$, and the distinctness of the transversals follows. In fact any distinct choices of up to $C$ values for $a_i$ will yield distinct transversals. Hence we obtain the following theorem.

**Theorem 2.1** If $p$ is an odd prime and $k$ is a positive integer, then the number of transversals in the Latin square $A_p^k$, denoted by $T(A_p^k)$, satisfies the inequality

$$T(A_p^k) > \sum_{c=0}^{C} \left( \frac{p^{k-1}}{c} \right) (T^* - (p - 1))^c \left( \frac{C!}{(C - c)!} \right)$$

(1)
where \( C = \frac{p^{k-1} - 1}{p-1} \) and \( T^* = T(A_p)(1 - \frac{1}{p}) \).

The final term in the summation (1) gives

\[
T(A_p^k) > \left( \frac{p^{k-1}}{C} \right) (T^*(p-1))^C! \frac{(p^{k-1})!}{((p^{k-1} - C)!)} (T^*(p-1))^C
\]

Applying Stirling’s Theorem in the form \( r! = r^{r+\frac{1}{2}}e^{-r}\sqrt{2\pi}e^{o(1)} \) (as \( r \to \infty \)) to this expression for large \( k \) gives

\[
T(A_p^k) > \left[ p^{k-1}T^*(p-1)e^{-1} \right]^C \cdot \left( 1 - \frac{C}{p^{k-1}} \right)^{-(p^{k-1}-C+\frac{1}{2})}.e^{o(1)}. \quad (2)
\]

For \( p \geq 3 \) and \( k \geq 2 \) we have \( 1 - \frac{C}{p^{k-1}} \leq (1 - \frac{1}{p}) \) and \( p^{k-1} - C + \frac{1}{2} > (p-2)C \). Hence

\[
T(A_p^k) > \left[ p^{k} \left( \frac{p}{p-1} \right)^{p-4} T(A_p)e^{-1} \right]^C. e^{o(1)}.
\]

The square \( A_p^k \) has order \( n = p^k \) and \( C = \frac{n}{p(p-1)} = \frac{1}{p-1} \), so taking \( Q \) to be slightly less than \( \left( \frac{p}{p-1} \right)^{p-4} T(A_p)e^{-1} \) gives the following corollary.

**Corollary 2.1** If \( p \) is an odd prime, there exists \( Q > 0 \) such that for all sufficiently large \( k \),

\[
T(A_p^k) > (nQ)^{\frac{n}{p(p-1)}},
\]

where \( n = p^k \).

In fact if \( p \) is also sufficiently large, then using the result of [1], we may take \( Q = (3.246)^p \). However, the bound is clearly best when \( p \) is small. In the case \( p = 3 \), inequality (2) simplifies as follows. Firstly \( T(A_3) = 3 \), so \( T^* = 2 \). Also \( C = \frac{3^{k-1} - 1}{2} \) and \( 3^{k-1} - C + \frac{1}{4} = 3^{k-1}/2 + 1 \). Hence

\[
T(A_3^k) > (4 \cdot 3^{k-1} \cdot e^{-1})^C \cdot \left( \frac{1}{2} + \frac{1}{3^{k-1}} \right)^{-(\frac{3^{k-1} - 1}{2})}.e^{o(1)}
\]

\[
= \left( \frac{4n}{3e} \right)^C \cdot 2^{C+\frac{1}{2}} \left( 1 + \frac{1}{3^{k-1}} \right)^{-(\frac{3^{k-1} - 1}{2})}.e^{o(1)}
\]

\[
= \left( \frac{8n}{3e} \right)^C \cdot 2^{\sqrt{2}} \cdot \frac{1}{\sqrt{e}}.e^{o(1)},
\]

since \((1 + \frac{1}{r})^{-r} \to e^{-1} \) as \( r \to \infty \). Noting that \( 8/3e > 0.981 \) and that \( C = \frac{n}{6} - \frac{1}{2} \), we obtain
Corollary 2.2 For all sufficiently large \( k \), \( T(A_k^3) > (0.981n)^{3/6} \), where \( n = 3^k \).

Finally we remark that, by transitivity, the number of orthogonal mates of the Latin square \( A_k^p \) is \( T(A_k^p)/n \) (where \( n = p^k \)) and so Theorem 2.1 and its corollaries also provide lower bounds for this quantity.

References


