Staircases, dominoes, and the growth rate of 1324-avoiders

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David Bevan [a] Robert Brignall [b] Andrew Elvey Price [c]
Jay Pantone [d]

[a] *Department of Computer and Information Sciences, University of Strathclyde, Glasgow, UK*

[b] *School of Mathematics and Statistics, The Open University, Milton Keynes, UK*

[c] *School of Mathematics and Statistics, University of Melbourne, Melbourne, Australia*

[d] *Department of Mathematics, Dartmouth College, Hanover, NH, USA*

Abstract

We establish a lower bound of 10.271 for the growth rate of the permutations avoiding 1324, and an upper bound of 13.5. This is done by first finding the precise growth rate of a subclass whose enumeration is related to West-2-stack-sortable permutations, and then combining copies of this subclass in particular ways.

Keywords: Permutation, patterns, enumeration, growth rate.

1 Introduction

The class of 1324-avoiding permutations is famously hard to count. Whereas every other permutation class that avoids a single length 4 permutation was enumerated in the 1990s (see Bóna [5] and Gessel [14]), not even the first-order asymptotics (the "growth rate") of Av(1324) is yet known.

Let π and σ be permutations of lengths n and m respectively, written in one-line notation. We say that σ is *contained* in π if there exists a subsequence $i_1 < i_2 < \cdots < i_m$ of $1, \ldots, n$ such that $\sigma(j) < \sigma(k)$ if and only if $\pi(i_j) < \pi(i_k)$, for all $1 \leq j, k \leq m$. If σ is not contained in π, then it *avoids* π. We write $\mathrm{Av}(\pi)$ to mean the set consisting of all permutations that avoid π, and note that it forms a hereditary class, or *permutation class*, in the sense that whenever $\sigma \in \mathrm{Av}(\pi)$ and τ is contained in σ, then $\tau \in \mathrm{Av}(\pi)$.

Given any permutation π, let $S_n(\pi)$ denote the number of permutations of length n that avoid π. The *growth rate* of the class $\mathrm{Av}(\pi)$ is

$$\mathrm{gr}(\mathrm{Av}(\pi)) = \lim_{n\to\infty} \sqrt[n]{S_n(\pi)},$$

and is known to exist by a result of Arratia [3], combined with the celebrated resolution of the Stanley-Wilf conjecture by Marcus and Tardos [15]. [1] More generally, for an infinite sequence $s_1, s_2, \ldots$ of positive integers, the *growth rate* of (s_n) is $\lim_{n\to\infty} \sqrt[n]{s_n}$, if this exists.

In the same paper, Arratia [3] conjectured that $\mathrm{gr}(\mathrm{Av}(\pi)) \leq (|\pi| - 1)^2$, where $|\pi|$ denotes the length of π. However this conjecture was refuted in 2006 by Albert, Elder, Rechnitzer, Westcott and Zabrocki [1], by proving that $\mathrm{gr}(\mathrm{Av}(1324)) \geq 9.47$, thereby cementing $\mathrm{Av}(1324)$ as the *bête noire* of permutation classes. Indeed, during a conference in 2004 when the result of [1] was announced, Doron Zeilberger famously declared that "not even God knows $S_{1000}(1324)$". Humans, with the help of computers, now know $S_{36}(1324)$, and Conway and Guttman's analysis [13] of their computation provides an estimate for $\mathrm{gr}(\mathrm{Av}(1324))$ of 11.60 ± 0.01, and they conjecture that $S_n(1324) \sim B \cdot \mu^n \cdot \mu_1^{n^\sigma} \cdot n^g$ where $\sigma = \frac{1}{2}$, which would imply that this sequence does not have an algebraic singularity.

The history of rigorous lower and upper bounds for $\mathrm{gr}(\mathrm{Av}(1324))$ now spans several papers, and is summarised in Table 1. In addition to these, Claesson, Jelínek and Steingrímsson [12] make a conjecture regarding the number of permutations with a fixed number of inversions of each length, which if resolved would give an improved upper bound of $e^{\pi\sqrt{2/3}} \approx 13.001954$.

Our contribution to the growth rate study of $\mathrm{Av}(1324)$ is to provide new lower and upper bounds on $\mathrm{gr}(\mathrm{Av}(1324))$. This relies on a structural characterisation of $\mathrm{Av}(1324)$, together with some precise asymptotic analysis of a specific subclass, both of which are presented in the next section. In the third

[1] The existence of growth rates for general permutation classes (i.e. those avoiding one or more permutations) remains open: Marcus and Tardos [15] only guarantees that lim sup exists.

Reference	**Lower**	**Upper**
2004: Bóna [6]		288
2005: Bóna [7]	9	
2006: Albert et al [1]	9.47	
2012: Claesson, Jelínek and Steingrímsson [12]		16
2014: Bóna [8]		13.93
2015: Bóna [9]		13.74
2015: Bevan [4]	9.81	
This article	10.27	13.5

Table 1
A chronology of record upper and lower bounds for gr(Av(1324)).

and final section, we take multiple copies of this domino class and 'glue' them together in ways that are consistent with the structural characterisation, and thus establish the claimed bounds on the growth rate of Av(1324).

2 Staircase Structure and Dominoes

The following structural characterisation has implicitly been used in the literature (notably the red and blue subtrees of Bevan [4], and the colouring approaches taken by [8,9,12] to express Av(1324) as the merge of a permutation in Av(132) with a permutation in Av(213)), but it does not seem to have been stated explicitly before now. The characterisation is illustrated in Figure 1(a).

Proposition 2.1 *The 1324-avoiders are contained in a decreasing staircase whose cells alternate between* Av(132) *and* Av(213)*. Furthermore, any copy of 1324 in this staircase is contained in a pair of adjacent cells, with two points in each cell.*

Note that this characterisation is consistent with the decomposition due to Claesson, Jelínek and Steingrímsson [12], since the cells that avoid 132 (respectively, 213) collectively also avoid 132 (resp., 213). Furthermore, momentarily dropping the 1324-condition, the class comprising all permutations that lie in the staircase illustrated in Figure 1(a) has growth rate equal to 16, which can be deduced immediately from [2, Theorem 3]. This is, of course, essentially the method employed in [12] to achieve the same upper bound.

Now consider the set of gridded permutations lying in the two-cell staircase

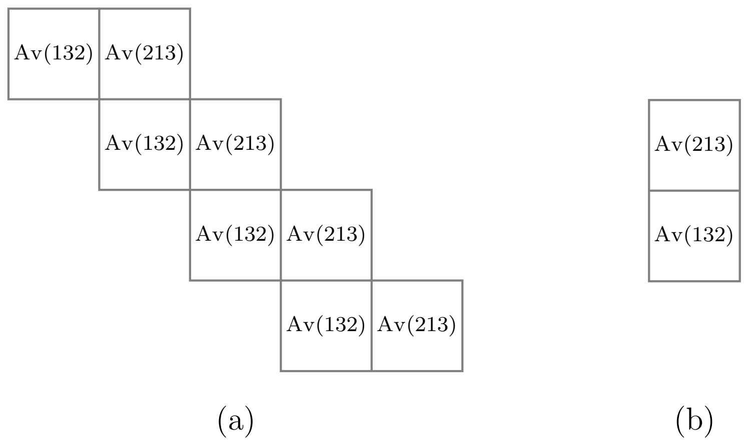

Fig. 1. (a) A portion of the infinite staircase containing the 1324-avoiders; (b) A two-cell domino.

shown in Figure 1(b) that is composed of an upper cell whose permutations avoid 213 and a lower cell whose permutations avoid 132. The intersection of this set with Av(1324) forms a permutation class, which we call the *domino class*. A *domino permutation* is a permutation that lies in the domino class, together with a horizontal dividing line that witnesses its membership in the domino class. Thus, an underlying permutation from the domino class may correspond to more than one domino permutation, but note that both the domino class and the set of domino permutations still share the same growth rate (assuming it exists, which we will establish in a moment).

By exploiting connections with interleavings of 'arch systems', and solving the resulting functional equation via iterated discriminants (see Bousquet-Mélou and Jehanne [10]) to obtain a minimal polynomial for the generating function of domino permutations, we obtain the following result.

Theorem 2.2 *The number of domino permutations on n points is*

$$\frac{2(3n+3)!}{(n+2)!(2n+3)!}.$$

The growth rate of this sequence is $27/4$.

The sequence matches the OEIS sequence A000139, and is also the counting sequence for West-two-stack-sortable permutations [16] and rooted non-separable planar maps [11].

This theorem on its own gives us enough information to establish record upper and lower bounds for Av(1324) (see Proposition 3.1, but we can improve

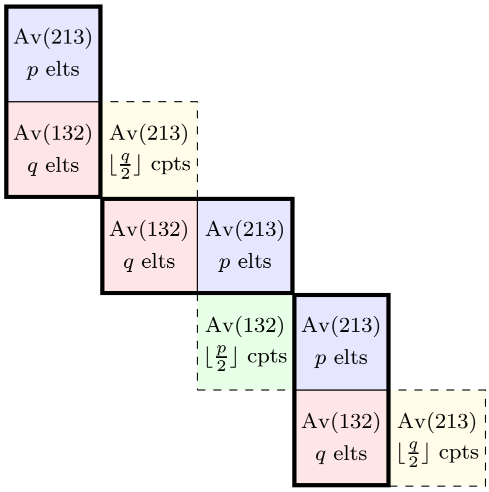

Fig. 2. Decomposing a staircase into dominoes (red and blue cells, surrounded by thick black lines) and connecting cells (yellow and green cells, surrounded by dashed lines).

the lower bound by refining the analysis used in proving Theorem 2.2 to obtain the following result. A *leaf* in a domino permutation is a point which is either a right-to-left maximum in the upper cell, or a left-to-right minimum in the lower cell.

Theorem 2.3 *The expected number of leaves in a domino permutation on* n *points is asymptotically* $5n/9$*, with standard deviation* $O(\sqrt{n})$*.*

3 Bounds on the growth rate of Av(1324)

Since we know the growth rate of domino permutations is 27/4 from Theorem 2.2, we can now consider ways to 'glue' dominoes together inside a staircase to form upper and lower bounds.

Proposition 3.1 $10.125 \leq \mathrm{gr}(\mathrm{Av}(1324) \leq 13.5$.

The upper bound is found by taking a sequence of domino permutations (which can readily be shown also to have growth rate 27/4), together with a sequence over two letters of an appropriate length (of growth rate 2) to govern how the points in adjacent dominoes may interleave. The combined growth rate is $2 \times 27/4 = 13.5$.

The scheme used to construct the lower bound is more complicated, and is illustrated in Figure 2. In this figure, pairs of adjacent red and blue cells contain domino permutations (or a symmetry of a domino permutation), while

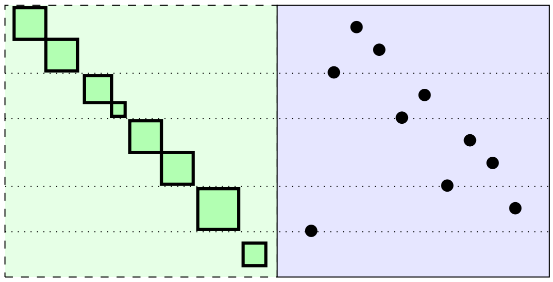

Fig. 3. Interleaving skew indecomposable components in a green cell with the leaves in a blue cell to its right.

the yellow and green cells connect these domino permutations together, and have a specified number of *skew indecomposable* permutations. Critically, we can guarantee that no copy of 1324 is created, providing the skew indecomposable components do not interact with any points in the red or blue (domino) cells.

To improve our lower bound, we observe that the leaves of a domino permutation in such a staircase *can* interleave the points of a skew indecomposable component in a yellow or green cell without creating a copy of 1324 – see Figure 3. Theorem 2.3 allows us to consider domino permutations for which 5/9 of all points are leaves whilst maintaining the growth rate of 27/4, thus offering some scope for interleaving.

We implement this observation in a construction that is based on the one given in Figure 2, and after some further analysis and numerical optimization, we establish the following result.

Theorem 3.2 *The growth rate of* Av(1324) *is at least* 10.2710129... *(the root of a polynomial of degree 104).*

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