Averaged coordination numbers of planar aperiodic tilings

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We consider averaged shelling and coordination numbers of aperiodic tilings. Shelling numbers count the vertices on radial shells around a vertex. Coordination numbers, in turn, count the vertices on coordination shells of a vertex, defined via the graph distance given by the tiling. For the Ammann-Beenker tiling, we find that coordination shells consist of complete shelling orbits, which enables us to calculate averaged coordination numbers for rather large distances explicitly. The relation to topological invariants of tilings is briefly discussed.

Key words: Aperiodic order; Shelling numbers, Coordination numbers; Averages

1. Introduction

Many combinatorial questions from lattice theory are best extended to aperiodic system by using an additional averaging process. In particular, this is the case for the shelling problem, where one asks for the number of vertices on spherical (circular) shells. In this contribution, we consider an extension of the shelling problem to the setting of more general distances, and give examples for the coordination number case [3, 6], which corresponds to the graph distance in a tiling.

To keep things simple, we explain our approach for cyclotomic model sets in the Euclidean plane $\mathbb{R}^2 \simeq \mathbb{C}$, with co-dimension 2 (referring to the so-called internal space). Here, following the algebraic setting of Pleasants [15], one starts from a set of cyclotomic integers, $L = \mathbb{Z}[\xi_n]$ with $\xi_n = e^{2\pi i/n}$ and suitable n, which is the set of all integer linear combinations of the regular n-star of unit length. Note that one can choose a $\mathbb{Z}$-basis of $\phi(n)$ elements, where ϕ is Euler's totient function [2].

This setting is equipped with a natural $\star$-map, defined by a suitable algebraic conjugation (such as $\xi_5 \mapsto \xi_5^2$, $\xi_8 \mapsto \xi_8^3$, and $\xi_{12} \mapsto \xi_{12}^5$ in the examples discussed below, together with the canonical extension to all elements of L). The set $\tilde{L} := \{(x, x^\star) \mid x \in L\}$ is then a *lattice* in $\mathbb{R}^{\phi(n)}$, the so-called Minkowski embedding [9] of L. A *model set* $\varLambda$ is now a set of the form

$$\varLambda = \{x \in L \mid x^\star \in \varOmega\}, \tag{1}$$

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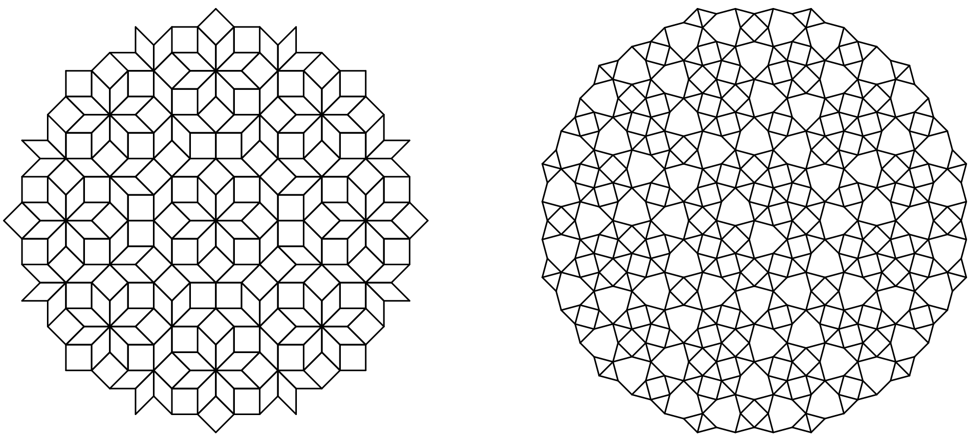

Figure 1: Patches of the Ammann-Beenker tiling (left) and the shield tiling (right).

or any translate of it, where the window Ω is a relatively compact subset of internal space with non-empty interior. A natural choice for Ω that preserves n-fold symmetry is a regular n-gon, which leads to *regular* model sets (i.e., the boundary $\partial\Omega$ has Lebesgue measure 0). More precisely, we focus on the *generic* case ($\partial\Omega \cap L^\star = \varnothing$), where Λ is repetitive and its LI-class (consisting of all locally indistinguishable patterns) defines a uniquely ergodic dynamical system [17]. This includes the examples of figure 1.

Let $d(x, y)$ denote any translation invariant distance between x and y. Due to unique ergodicity, combined with finite local complexity, we know [17, 5] that the averaged number $s_d(r)$ of vertices on a d-shell of radius r is determined by a sum over a *finite* number of patches, weighted by their frequencies, which exist uniformly. Since we work with model sets, this can be further reduced to a sum that involves only admissible pairs of vertices with the correct distance. The frequency of a pair (x, y) with $x, y \in L$ is given by the scale-independent autocorrelation coefficient $\nu(x - y)$, where [5]

$$\nu(z) \;=\; \frac{1}{\mathrm{vol}(\Omega)} \int_{\mathbb{R}^m} \mathbf{1}_\Omega(w)\, \mathbf{1}_\Omega(w + z^\star)\, \mathrm{d}w, \tag{2}$$

with $\mathbf{1}_\Omega$ denoting the characteristic function of the window. The normalisation is such that $\nu(0) = 1$, i.e., we count per point of Λ rather than per unit volume. Clearly, $\nu(z) = \nu(-z)$, and further identities may occur as a result of the symmetries of the window. In general, the averaged number reads

$$s_d(r) \;=\; \sum_{\substack{z \in \Lambda - \Lambda \\ d(0,z) = r}} \nu(z), \tag{3}$$

which can then be further simplified by means of a standard orbit analysis [5].

In view of this derivation, it is reasonable to consider a d-shell to be the collection of all pairs (x, y) of a given distance $d(x, y)$ *together* with their frequencies $\nu(x - y)$.

The coordination problem is now the extension of the shelling problem to a different distance concept, based on the graph distance in the tiling under consideration. The graph distance of two vertices is defined as the minimum number of edges in a path linking the two vertices. For simplicity, we restrict our discussion to examples with a single edge type (i.e., all edges have the same length), though various extensions are possible.

Table 1: Averaged coordination numbers of the Ammann-Beenker tiling.

k	$s_c(k)$	num. value	k	$s_c(k)$	num. value
1	4	4.000	21	$-3284 + 2392\sqrt{2}$	98.799
2	$32 - 16\sqrt{2}$	9.373	22	$2172 - 1464\sqrt{2}$	101.591
3	$-8 + 16\sqrt{2}$	14.627	23	$4164 - 2868\sqrt{2}$	108.036
4	$24 - 4\sqrt{2}$	18.343	24	$-8648 + 6196\sqrt{2}$	114.467
5	$40 - 12\sqrt{2}$	23.029	25	$6836 - 4752\sqrt{2}$	115.657
6	$40 - 8\sqrt{2}$	28.686	26	$3164 - 2152\sqrt{2}$	120.612
7	$-176 + 148\sqrt{2}$	33.304	27	$-7972 + 5728\sqrt{2}$	128.615
8	$444 - 288\sqrt{2}$	36.706	28	$1500 - 968\sqrt{2}$	131.041
9	$240 - 140\sqrt{2}$	42.010	29	$4716 - 3240\sqrt{2}$	133.948
10	$-648 + 492\sqrt{2}$	47.793	30	$792 - 460\sqrt{2}$	141.462
11	$232 - 128\sqrt{2}$	50.981	31	$-10216 + 7328\sqrt{2}$	147.357
12	$508 - 320\sqrt{2}$	55.452	32	$10500 - 7320\sqrt{2}$	147.957
13	$-272 + 236\sqrt{2}$	61.754	33	$7236 - 5008\sqrt{2}$	153.618
14	$-556 + 440\sqrt{2}$	66.254	34	$-18132 + 12936\sqrt{2}$	162.267
15	$1540 - 1040\sqrt{2}$	69.218	35	$5356 - 3672\sqrt{2}$	163.008
16	$980 - 640\sqrt{2}$	74.903	36	$7328 - 5064\sqrt{2}$	166.423
17	$-3064 + 2224\sqrt{2}$	81.211	37	$2800 - 1856\sqrt{2}$	175.220
18	$1424 - 948\sqrt{2}$	83.326	38	$-19444 + 13876\sqrt{2}$	179.627
19	$812 - 512\sqrt{2}$	87.923	39	$12416 - 8652\sqrt{2}$	180.224
20	$740 - 456\sqrt{2}$	95.119	40	$21932 - 15376\sqrt{2}$	187.052

There is an important connection between averaged coordination and shelling numbers, which stems from the relation between graph and Euclidean distances. On a given coordination shell, the vertices appear in symmetry orbits of the underlying tiling; these vertices are orbitwise distributed over finitely many Euclidean shells. Within one orbit, every vertex contributes the same amount to the averaged numbers. Conversely, the vertices on a Euclidean shell orbitwise belong to (possibly different) coordination shells. Here, the term orbit simply refers to the orbit of a point under the point symmetry group of the pattern under consideration.

2. Examples

In what follows, we concentrate on three particular examples, with 8-, 10-, and 12-fold symmetry. For $n = 8$, we consider the Ammann-Beenker tiling, obtained by the above construction with a regular octagon of edge length 1 as the window. The special role of $\sqrt{2}$ reflects the fact that $\mathbb{Z}[\sqrt{2}] = \mathbb{R} \cap \mathbb{Z}[\xi_8]$, see [5] for details. This reference also contains the known results on the circular shelling.

For this tiling, the connection between shelling and coordination numbers is rather advantageous, because coordination shells comprise only *complete* circular shells. This is a consequence of the fact that the four directions of the edges in the tiling form a $\mathbb{Z}$-basis of the underlying module $\mathbb{Z}[\xi_8]$, which is possible because $\phi(n) = n/2$ for $n = 8$. This means that for a given distance vector, the number of steps along each direction is uniquely determined. Moreover, there always exists at least one path along edges of the actual tiling that is admissible (in the sense that never has to 'backtrack' along the path). This is a higher-dimensional analogue of the corresponding (trivial) situation for

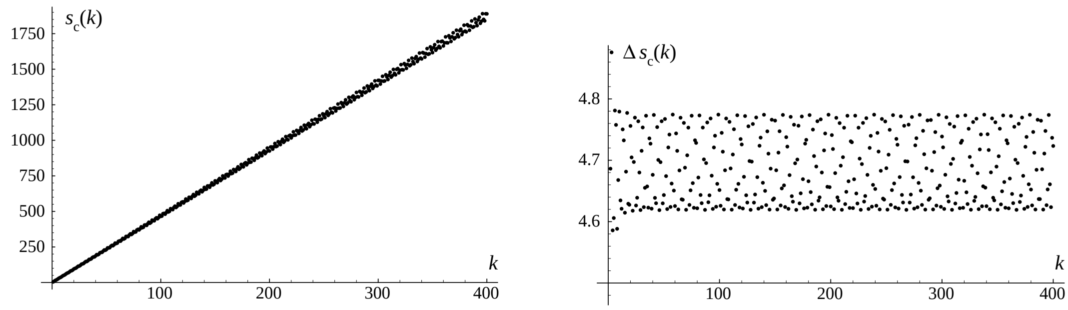

Figure 2: The first 400 coordination numbers for the Ammann-Beenker tiling (left) and the fluctuations in the differences of successive coordination numbers (right).

the silver mean chain (when viewed as a cut and project set obtained from a rectangular lattice), which fails for other tilings.

The averaged coordination numbers can therefore be calculated by identifying the contributing circular shells, and summing the corresponding averaged shelling numbers, which can be obtained as described in [5]. Following this approach, we calculated the first few hundred averaged coordination numbers for the Ammann-Beenker tiling, see table 1 and figure 2.

As for periodic planar lattices, the averaged coordination numbers $s_c(k)$ grow, on the average, linearly with the number of steps k. However, a closer inspection of the left part of figure 2 shows that the growth rate fluctuates, and that the data points do not lie on a single line, but inside a sector bounded by two lines of slightly different slopes. An even closer inspection reveals that the fluctuations of $s_c(k)$ follow a sophisticated pattern, as displayed in the right part of figure 2 which shows the differences $\Delta s_c(k) = s_c(k+1) - s_c(k)$ of consecutive averaged shelling numbers. The resulting pattern appears to comprise a number of sinusoidally varying curves. This might be caused by the variations of overlap areas of the window with shifted copies of itself, which enters the computation of patch frequencies; a closer investigation of this phenomenon might lead to interesting results.

A construction of the rhombic Penrose tiling as a cyclotomic model set with four components is described in [7]. Due to the necessity of four windows (which are also present in the non-minimal embedding via the lattice $\mathbb{Z}^5$), the determination of averaged quantities is technically more involved. Table 2 recalls some results obtained earlier in [6]. In this case, $\mathbb{Z}[\xi_5] \cap \mathbb{R} = \mathbb{Z}[\tau]$, where $\tau = (1+\sqrt{5})/2$ is the golden ratio, which is the relevant algebraic integer here.

Table 2: Averaged coordination numbers of the rhombic Penrose tiling.

k	$s_c(k)$	num. value	k	$s_c(k)$	num. value
1	4	4.000	6	$980 - 588\tau$	28.596
2	$58 - 30\tau$	9.459	7	$-1614 + 1018\tau$	33.159
3	$-128 + 88\tau$	14.387	8	$2688 - 1638\tau$	37.660
4	$288 - 166\tau$	19.406	9	$-3840 + 2400\tau$	43.282
5	$-374 + 246\tau$	24.036	10	$4246 - 2594\tau$	48.820

Table 3: Averaged coordination numbers of the twelvefold symmetric shield tiling, split into the contributions from circular shells of radius r.

k	$s_c(k)$	num. value	r^2	contribution
1	$8 - 2\sqrt{3}$	4.536	$2 - \sqrt{3}$	$8 - 2\sqrt{3}$
2	$20 - 6\sqrt{3}$	9.608	$4 - 2\sqrt{3}$	2
			$6 - 3\sqrt{3}$	$4 - 2\sqrt{3}$
			1	$14 - 4\sqrt{3}$
3	$64 - 28\sqrt{3}$	15.503	1	$-6 + 4\sqrt{3}$
			$5 - 2\sqrt{3}$	$10 - 4\sqrt{3}$
			2	$48 - 24\sqrt{3}$
			$4 - \sqrt{3}$	$12 - 4\sqrt{3}$
4	$-46 + 38\sqrt{3}$	19.818	$4 - \sqrt{3}$	$-6 + 4\sqrt{3}$
			$8 - 3\sqrt{3}$	$-76 + 44\sqrt{3}$
			3	$-4 + \frac{16}{3}\sqrt{3}$
			$7 - 2\sqrt{3}$	$20 - \frac{32}{3}\sqrt{3}$
			$2 + \sqrt{3}$	$12 - 2\sqrt{3}$
			4	$8 - \frac{8}{3}\sqrt{3}$

As an example with 12-fold symmetry, we consider the so-called shield tiling [11] of figure 1. It is obtained from $\mathbb{Z}[\xi_{12}]$ choosing a regular dodecagon of edge length 1 as window, see [4] for details. In this case, $\mathbb{Z}[\xi_{12}] \cap \mathbb{R} = \mathbb{Z}[\sqrt{3}]$; examples are listed in table 3. Note that a single circular shell can contribute to several coordination shells.

3. Frequency modules

It is remarkable that the averaged coordination numbers are special algebraic integers in all three examples. From the cut and project method, in conjunction with equations (2) and (3), it is clear that these numbers must be rational, i.e., elements of the corresponding cyclotomic field $\mathbb{Q}(\xi_n)$. This follows from the computability of the integrals in equation (2) within these number fields.

The further restriction to algebraic *integers* is due to a special structure of the frequency module of the tiling, i.e., the $\mathbb{Z}$-span of the frequencies of all finite patches in the tiling. Since our averaged quantities are simple integer linear combinations of patch frequencies, a 'quantisation' of the latter to integers implies the result for the former. This phenomenon has been observed before several times, and for various related problems [6, 16, 5, 12]. The proof relies on the topological structure of the compact LI-class, viewed as a dynamical system under the translation and/or inflation action [1, 8, 10].

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