Chiral maps of given hyperbolic type

Journal Item

Chiral maps of given hyperbolic type

Marston D. E. Conder
University of Auckland, Auckland, New Zealand

Veronika Hucíková
Slovak University of Technology, Bratislava, Slovakia

Roman Nedela
Matej Bel University, Banská Bystrica, Slovakia

Jozef Širáň
Open University, Milton Keynes, U.K., and
Slovak University of Technology, Bratislava, Slovakia

Abstract

We prove the existence of infinitely many orientably-regular but chiral maps of every given hyperbolic type $\{m, k\}$, by constructing base examples from suitable permutation representations of the ordinary $(2, k, m)$ triangle group, and then taking abelian p-covers. The base examples also help to prove that for every pair (k, m) of integers with $1/k + 1/m \leq 1/2$, there exist infinitely many regular and infinitely many orientably-regular but chiral maps of type $\{m, k\}$, each with the property that both the map and its dual have simple underlying graph.

1 Introduction

A *map* is a 2-cell embedding of a connected graph or multigraph on a closed surface, where '2-cell' means that the embedding breaks up the surface into simply-connected regions, called the *faces* of the map. An *automorphism* of a map is a permutation of its edges, preserving incidence with vertices and faces. Except in degenerate cases (which will not be relevant in this paper), every automorphism is uniquely determined by its effect on a given incident vertex-edge-face triple, sometimes called a *flag* of the map. If the group of all automorphisms acts transitively (and hence regularly) on such triples, then the map is said to be *regular*. Intuitively, regular maps exhibit the highest possible level of symmetry among all maps.

The study of regular maps began with the work of Brahana, Burnside and Dyck and others in the late 1800s and early 1900s. But in fact these authors were interested mainly in what we now call *orientably-regular* maps, which are maps on orientable surfaces with the property that the group of all orientation-preserving automorphisms of the map is transitive (and hence regular) on the set of all *arcs* (ordered edges) in the map. Examples include the Platonic maps (embeddings of the 1-skeletons of the Platonic solids on the sphere), and triangular, quadrangular and honeycomb tessellations of the torus.

In a regular or orientably-regular map, all vertices have the same valency, say k, and all faces are bounded by closed walks of the same length, say m, and then we say the map has *type* $\{m, k\}$. (Technically this should be an ordered pair (m, k), but the notation $\{m, k\}$ is historic, due to Schläfli, and now widely accepted.) For example, the tetrahedral, octahedral, cube, icosahedral and dodecahedral maps on the sphere have have types $\{3, 3\}$, $\{3, 4\}$, $\{4, 3\}$, $\{3, 5\}$ and $\{5, 3\}$, while triangular, quadrangular and honeycomb tessellations of the torus have types $\{3, 6\}$, $\{4, 4\}$ and $\{6, 3\}$, respectively.

The geometric dual of an orientably-regular map with genus g and type $\{m, k\}$ is also orientably-regular, with genus g and type $\{k, m\}$. There is a close connection between orientably-regular maps of given type $\{m, k\}$ and smooth quotients of the ordinary $(2, k, m)$ triangle group $\Delta(2, k, m) = \langle\, x_1, x_2 \mid x_1^2 = x_2^k = (x_1x_2)^m = 1\,\rangle$, which we will describe in the next section. This connection was known to Dyck *et al* (over 100 years ago).

The *genus* of a regular or orientably-regular map is the genus of the carrier surface. The genus g and type $\{m, k\}$ of an orientably-regular map are related via the Euler-Poincaré formula by $2 - 2g = \chi = |G|(1/k - 1/2 + 1/m)$, where χ is the Euler characteristic of the carrier surface and G is the group of all orientation-preserving automorphisms. Hence in particular, the type satisfies $1/k + 1/m > 1/2$ when the map is spherical (genus 0), or $1/k + 1/m = 1/2$ when the map is toroidal (genus 1). The remaining types (with $1/k + 1/m < 1/2$) are called *hyperbolic*, since they lie on surfaces of genus $g > 1$. We will also use the same adjective (namely spherical, toroidal or hyperbolic) for a pair (k, m), depending on whether $1/k + 1/m$ is greater than equal to, or less than $1/2$, respectively.

Some orientably-regular maps are (fully) regular, since they admit orientation-reversing automorphisms (which are often called 'reflections'), while others are not. The latter kind of maps are irreflexible, or *chiral*. All orientably-regular maps on the sphere are reflexible (and hence fully regular), while those on the torus include infinitely many reflexible and infinitely many chiral maps of each of the three types $\{3, 6\}$, $\{4, 4\}$ and $\{6, 3\}$; see [9].

The objective of this article is to prove the existence of infinitely many orientably-regular but chiral maps of every given hyperbolic type.

By the Hurwitz bound of $84(g - 1)$ for the (largest) number of automorphisms of a compact Riemann surface of genus $g > 1$, it is known that every compact orientable surface of genus $g > 1$ supports just a finite number of orientably-regular maps. Those with genus 2 to 6 were completely classified by Brahana (1927), Coxeter (1957), Sherk (1959) and Garbe (1969); see [9]. These classifications were extended up to genus 15 by Conder and Dobcsányi (2001), with the help of computational methods, and then more recently by Conder much further, up to genus 301; see [4, 5].

Determination of the genera of surfaces supporting an orientably-regular but chiral map is an open problem. All orientably-regular maps on surfaces of genus between 2 and 6 are reflexible, but for genus 7 there are two chiral examples, with types $\{6, 9\}$ and $\{7, 7\}$. Based on the above classifications of orientably-regular maps of small genus, it was noticed and then proved by Conder, Širáň and Tucker [7] that on any orientable surface of genus

$p+1$ where p is prime, there is no orientably-regular but chiral map unless $p-1$ is divisible by 3, 5 or 8. This is currently the only known infinite family that gives an infinite set of gaps in the genus spectrum of orientably-regular but chiral maps.

In contrast, for every $g \geq 1$ there is a rather trivial fully regular map of type $\{4g, 4g\}$ on an orientable surface of genus g. The underlying graphs of these maps, however, have multiple edges, each being a bouquet of $2g$ circles. Another open problem is to determine the genera of surfaces supporting an orientably-regular map with simple underlying graph; some progress was made recently in [6], where it was shown that these genera account for over 83% of all positive integers.

It is known that for every hyperbolic pair (k, m) there exist infinitely many reflexible orientably-regular maps of type $\{m, k\}$. This follows from work by Macbeath [16], as will be explained in Section 4, or by residual finiteness of hyperbolic triangle groups (see [21]), or by other methods (see [18]).

The question of extending this statement to orientably-regular but *chiral* maps was posed by David Singerman in [20], and this question was wide open until recently. The case where one of k and m is 3 was dealt with by Bujalance, Conder and Costa in [2], proving that for every integer $\ell \geq 7$, all but finitely many of the alternating groups A_n are the automorphism group of an orientably-regular but chiral map of type $\{3, \ell\}$.

In this paper we prove there are infinitely many orientably-regular but chiral maps of every given hyperbolic type $\{m, k\}$. We do this by constructing base examples from suitable permutation representations of the corresponding triangle groups, and then taking abelian p-covers. At the time of preparing this article, it was communicated to us that Gareth Jones has proven the same thing (for both maps and hypermaps), by using group representations and the theory of differentials on Riemann surfaces [12]. Our base examples also help to prove that for every non-spherical pair (k, m), there exist infinitely many regular and infinitely many orientably-regular but chiral maps of type $\{m, k\}$, each with the property that both the map and its dual have simple underlying graph.

We give some further background on orientably-regular maps and triangle groups, together with a brief description of our approach, in Section 2. Then we construct some infinite families of chiral maps in Section 3, prove our main theorems in Section 4, and make some final remarks in Section 5.

2 Further background

We begin with some further background on orientably-regular maps. Much of this can be found in [13] and other references on regular maps.

Let M be an orientably-regular map of type $\{m, k\}$, and let G be the group of all orientation-preserving automorphisms of M. Also let (v, e, f) be any incident vertex-edge-face triple (or *flag*) of M. Then it is well known that there exist orientation-preserving automorphisms R and S which generate the stabilisers in G of the face f and vertex v respectively, such that R induces a single-step rotation of the edges (and vertices) incident with f, while S induces a single-step rotation of the edges incident with v, and their product RS reverses the edge e (and so generates the stabiliser in G of e).

By conjugation (and connectedness of M), the automorphisms R and S generate the group G. Also R and S have orders m and k, while RS has order 2. In particular, R and S satisfy the relations $R^m = S^k = (RS)^2 = 1$, and therefore G is a smooth quotient of the ordinary $(2, k, m)$ triangle group $\Delta(2, k, m) = \langle x_1, x_2 \mid x_1^2 = x_2^k = (x_1x_2)^m = 1 \rangle$, under the epimorphism ψ that takes x_1, x_2 and x_1x_2 to $x = RS$, $y = S^{-1}$ and $xy = R$, respectively. Here 'smooth' means that the orders of the elements x_1, x_2 and x_1x_2 are preserved in the quotient; equivalently (for triangle groups), this means that the kernel K of the epimorphism ψ is torsion-free.

In fact, $\Delta(2, k, m)$ is a Fuchsian group, and $K = \ker\psi$ is the surface kernel, or in other words, the fundamental group of the orientable surface X carrying M. As such, K has a presentation of the form $\langle\, a_1, b_1, \ldots, a_g, b_g \mid [a_1, b_1][a_2, b_2] \ldots [a_g, b_g] = 1 \,\rangle$, where g is the genus of X (and hence of M).

Conversely, if $\psi\colon \Delta(2, k, m) \to G$ is any smooth homomorphism from $\Delta(2, k, m)$ onto a finite group G, then we may construct an orientably-regular map M of type $\{m, k\}$ on which G acts as a group of orientation-preserving automorphisms. If R and S are the images of x_1x_2 and x_2^{-1}, then the vertices can be taken as the right cosets of $V = \langle S\rangle$ in G, edges as the right cosets of $E = \langle RS\rangle$, and faces as the right cosets of $F = \langle R\rangle$, with incidence given by non-empty intersection. The group G then acts on M by right multiplication of cosets, and hence in particular, G acts transitively on the arcs of M.

Note that the underlying graph of the resulting map M might or might not be simple. An edge from a given vertex v to one of its neighbours can be repeated, with multiplicity m_v equal to the order of the core of $\langle S\rangle$ in G (the largest normal subgroup of G contained in S). By arc-transitivity, this multiplicity is the same for every vertex v. Similarly, the multiplicity m_f of repeated edges incident with every face f is equal to order of the core of $\langle R\rangle$ in G. Hence in particular, the underlying graph of M is simple if and only if $\langle S\rangle$ is core-free in G, while the graph of its dual M^* is simple if and only if $\langle R\rangle$ is core-free in G.

Next, the map M is reflexible if and only if it has the property that for a given flag (v, e, f), there exists an orientation-reversing automorphism T that fixes the vertex v and preserves the face f, but moves the edge e (to another edge of f incident with v). In this case, R, S and T generate the full automorphism group of M, and satisfy the relations $R^m = S^k = (RS)^2 = T^2 = (RT)^2 = (TS)^2 = 1$, with R and S generating a subgroup of index 2. In particular, conjugation by T induces an automorphism of $G = \langle R, S\rangle$ which takes R and S to R^{-1} and S^{-1}. Equivalently, conjugation by TS is an automorphism θ of order 2 that preserves the image $x = RS$ and inverts the image $y = S^{-1}$ of the generators x_1 and x_2 of the ordinary $(2, k, m)$ triangle group. In particular, if G admits such an automorphism θ, then M is reflexible, and if not, then M is chiral.

There are many ways in which one can attempt to prove or disprove the existence of such an automorphism. Many of these involve knowing the automorphism group of G, for example $\mathrm{Aut}(\mathrm{PSL}(2, p)) \cong \mathrm{PGL}(2, p)$ for every prime $p > 4$, and $\mathrm{Aut}(A_n) \cong S_n$ and $\mathrm{Aut}(S_n) \cong S_n$ for $n = 5$ and all $n > 6$. We could use the last two of these in what follows, but there are other ways that do not require any knowledge of $\mathrm{Aut}(G)$.

For example, the permutations $x = (1,2)(3,8)(4,9)(6,7)$ and $y = (2,3,4,5)(7,8,9,10)$ generate an imprimitive permutation group G of degree 10 and order 1920, which acts regularly on the arcs of a map M of type $\{4,5\}$, with $xy = (1,3,9,5,2)(4,10,7,6,8)$. Now if θ were an automorphism of G taking (x,y) to (x,y^{-1}), then the elements $(xy)^2(xy^{-1})^2xyxy^{-1}$ and $(xy^{-1})^2(xy)^2xy^{-1}xy$ would be interchanged by θ and therefore have the same order. But these elements are $(1,8,5,9,4,6)(3,7)$ and $(1,7,6)(3,5,9)$, with orders 6 and 3, and therefore no such θ exists. Hence the map M is chiral.

Another way that works easily occurs when we are given a coset diagram that depicts a faithful representation of group generated by the automorphisms $x = RS$ and $y = S^{-1}$, as used in [2, 8, 9] for example. If the corresponding map $M = M(x,y)$ is reflexible, then also this coset diagram must admit a reflection (or 'mirror symmetry'), as explained in [8], and if that does not happen, then the map M is chiral.

We will construct examples in which the automorphism group is A_n or S_n for some n. In doing this, we will make extensive use of a recent strengthening of Jordan's theorem:

Proposition 1 (Jones [11]) *Let G be a primitive permutation group of degree n that contains a cycle of length not exceeding $n-3$. Then G is isomorphic to the symmetric or the alternating group of degree n.*

Jordan's theorem (see [22]) required the cycle length to be prime. (On the other hand, the proof of the above generalisation uses the classification of finite simple groups.)

3 Infinite families

In this section we construct some infinite families of orientably-regular but chiral maps, with types satisfying certain conditions. By an earlier remark about duality, we may restrict our attention to hyperbolic types $\{m,k\}$ for which $k \le m$.

In each case we define two carefully chosen permutations x and y on a set of n points, where n depends on k and m (in a way that is unique for that family), and x, y and xy have orders 2, k and m, respectively.

We illustrate the effect of each of x and y by a coset diagram. The cycles of y are represented by polygons, the vertices of which are cyclically permuted (anti-clockwise) by y, and the fixed points of y are represented by heavy dots. The 2-cycles of the involution x are represented by additional edges, the vertices of which are interchanged by x. We omit the loops corresponding to fixed points of x. Each such diagram is connected, which means that the permutation group generated by x and y is transitive on the given n points. Moreover, our choice of x and y in each case ensures that either y or xy is a single cycle, fixing at least three points.

We then use various properties of the two chosen permutations to show that the subgroup G of S_n generated by x and y is primitive. It then follows from Proposition 1 that $G = A_n$ or S_n. Finally, we use the coset diagram to show that there is no automorphism of G taking (x,y) to (x,y^{-1}), and hence that the corresponding map of type $\{m,k\}$ is chiral.

3.1 Type $\{m,k\}$ with $k+2 \le m \le 2k-4$

In this case we construct a family of examples in which $n = m+1 = k+r$, where the parameter r satisfies $3 \le r \le k-3$. For the permutation y, we take a single k-cycle $(1,2,\ldots,k)$, fixing the r points $k+1, k+2, \ldots, k+r$, and for the involutory permutation x we take $(1,k+1)(2,k+2)\ldots(r,k+r)(k-1,k)$, which is the product of $r+1$ transpositions. These two permutations are illustrated by the coset diagram in Figure 1.

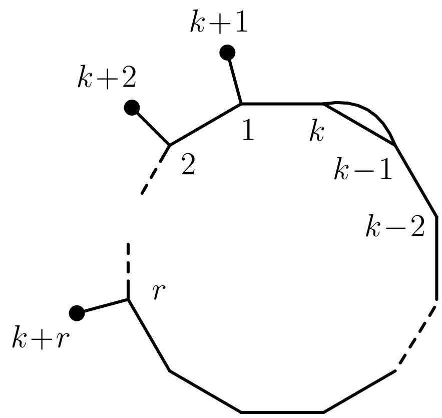

Figure 1: Generators for type $\{m,k\}$ with $k+2 \le m \le 2k-4$

Note that xy is a single cycle $(1, k+1, 2, k+2, \ldots, r, k+r, r+1, \ldots, k-2, k-1)$ of length $m = n-1$, fixing only the point k. In particular, $G = \langle x, y\rangle$ is 2-transitive, and therefore primitive. Since also y is a single cycle, fixing at least three points, it follows from Proposition 1 that $G = A_n$ or S_n.

Now suppose that the corresponding map $M = M(x,y)$ of type $\{m,k\}$ is reflexible. Then there exists an automorphism θ of G taking (x,y) to (x,y^{-1}), and this induces a mirror symmetry of the coset diagram, say $\bar{\theta}$. In that case, the unique fixed point k of xy must be taken by $\bar{\theta}$ to the unique fixed point of $(xy)^{\theta} = xy^{-1}$, namely $k-1$, and vice versa; and then since θ inverts y, it follows that $\bar{\theta}$ interchanges j with $k-j-1$ for $1 \le j < k/2$. In particular, $\bar{\theta}$ interchanges 1 with $k-2$. But then 1 is moved by x while $1^{\bar{\theta}} = k-2$ is fixed by x, so this cannot happen. Hence the map M is chiral, as required.

3.2 Type $\{m,k\}$ with $k+1 \le m \le 2k-7$

In this case we take $n = m+2 = k+r$, where the parameter r satisfies $3 \le r \le k-5$, and we take the same permutation for y as in §3.1, but we add another transposition $(k-3,k-2)$ to x, to give $x = (1,k+1)(2,k+2)\ldots(r,k+r)(k-3,k-2)(k-1,k)$. The associated coset diagram is in Figure 2.

This time, xy is a single cycle $(1, k+1, 2, k+2, \ldots, r, k+r, r+1, \ldots, k-3, k-1)$ of length $m = n-2$, with two fixed points $k-2$ and k. Its conjugate $y^{-2}(xy)y^2$ is another m-cycle, fixing only k and 2, so the stabiliser in $G = \langle x, y\rangle$ of the point k is transitive on

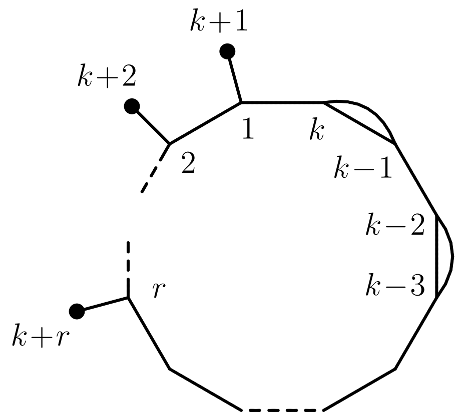

Figure 2: Generators for type $\{m, k\}$ with $k + 1 \leq m \leq 2k - 7$

the remaining $n - 1$ points, and again G is 2-transitive, and hence primitive. Also y is a single cycle fixing at least three points, so again we find that $G = A_n$ or S_n.

Next, suppose the corresponding map $M = M(x, y)$ of type $\{m, k\}$ is reflexible. Then the mirror symmetry $\bar{\theta}$ of the coset diagram induced by the reflecting automorphism θ must interchange the fixed points $k - 2$ and k of xy with the fixed points of xy^{-1}, namely $k - 3$ and $k - 1$, in some order. Since θ inverts y, we find that $\bar{\theta}$ interchanges $k - 2$ with $k - 1$, and $k - 3$ with k, and it follows that $\bar{\theta}$ interchanges 1 with $k - 4$. But again 1 is moved by x, while $1^{\bar{\theta}} = k - 4$ is fixed by x, so this cannot happen, and M is chiral.

3.3 Type $\{m, k\}$ with $k \leq m \leq 2k - 10$

In this case we take $n = m + 3 = k + r$, where $3 \leq r \leq k - 7$, and we take the same permutation for y as in §3.2, but we add yet another transposition $(k - 5, k - 4)$ to x, to give $x = (1, k+1)(2, k+2) \ldots (r, k+r)(k-5, k-4)(k-3, k-2)(k-1, k)$. The associated coset diagram is in Figure 3.

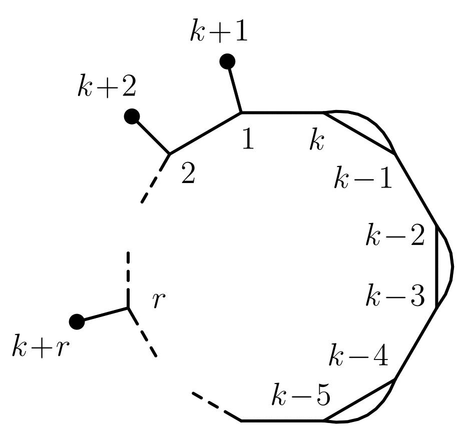

Figure 3: Generators for type $\{m, k\}$ with $k \leq m \leq 2k - 10$

This time, xy is a single cycle of length $m = n - 3$, with three fixed points $k-4$, $k-2$ and k. Its conjugate $y^{-4}(xy)y^4$ is another m-cycle, fixing only k, 2 and 4, so again G is 2-transitive, and hence primitive. Again y is a single cycle fixing at least three points, so we find that $G = A_n$ or S_n. Also if the corresponding map M is reflexible, then the mirror symmetry $\bar{\theta}$ must interchange the fixed points k, $k-2$ and $k-4$ of xy with the fixed points $k-5$, $k-3$ and $k-1$ of xy^{-1}, but then $\bar{\theta}$ interchanges 1 with $k-6$, which is impossible. Hence M is chiral.

3.4 Type $\{m, k\}$ with $2k - 3 \leq m \leq 4k - 11$ for $k \geq 5$

In this case we take $n = m + 3 = 2k + r + s$, where $0 \leq r \leq k - 3$ and $0 \leq s \leq k - 5$, and we define

$$y = (1, 2, \dots, k)(k+1, k+2, \dots, 2k), \text{ fixing all the other } r+s \text{ points, and}$$

$$x = (1, 2k+1)(2, 2k+2)\dots(r, 2k+r)(k-2, k-1)(k, k+1)(2k-s-3, 2k+r+s)$$
$$\dots(2k-5, 2k+r+2)(2k-4, 2k+r+1)(2k-3, 2k-2)(2k-1, 2k).$$

These two permutations are illustrated by the coset diagram in Figure 4, where we have deleted some labels to avoid clutter.

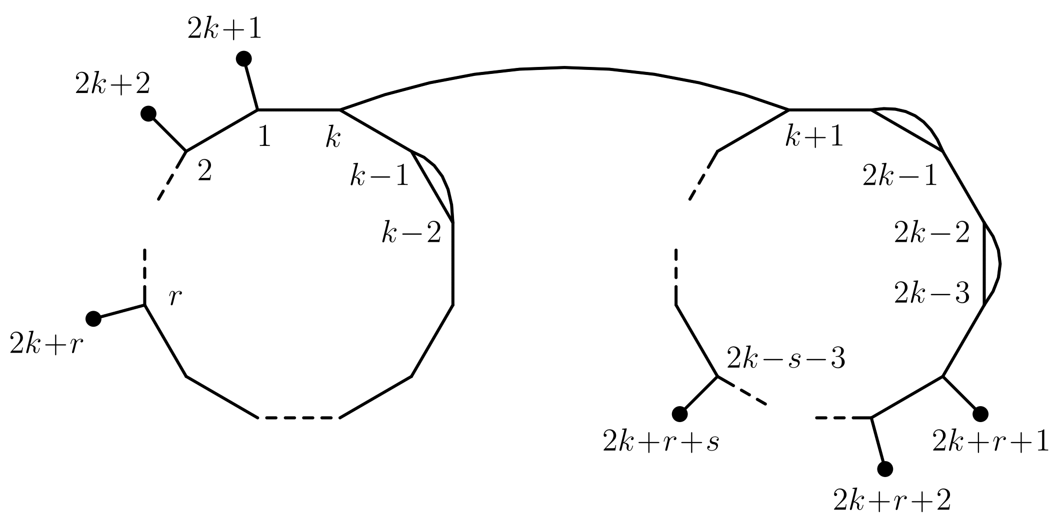

Figure 4: Generators for type $\{m, k\}$ with $2k - 3 \leq m \leq 4k - 11$ for $k \geq 5$

Here y is a product of two k-cycles, while xy is a single cycle of length $n - 3 = m$, with fixed points $k-1$, $2k-2$ and $2k$. The conjugate $y^{-2}(xy)y^2$ of xy is another m-cycle, fixing only 1, $k+2$ and $2k$, so the stabiliser of the point $2k$ is transitive on the remaining points, and therefore G is 2-transitive, and hence primitive. Then since xy is a single cycle fixing at least three points, we have $G = A_n$ or S_n. Also if the corresponding map M is reflexible, then the mirror symmetry $\bar{\theta}$ must interchange the fixed points $k-1$, $2k-2$ and $2k$ of xy with the fixed points $k-2$, $2k-3$ and $2k-1$ of xy^{-1}, in some order, and so must induce a reflection of each of the polygons representing the k-cycles of y; indeed $\bar{\theta}$ must interchange $2k$ with $2k-3$, and $2k-2$ with $2k-1$, and $k-1$ with $k-2$. But then $\bar{\theta}$ takes the transposition $(k, k+1)$ of x to $(k-3, 2k-4)$, which is impossible. Hence M is chiral.

3.5 Type $\{m, k\}$ with $m \geq 2k-3$ for $k \geq 8$

In this and the final two cases, we allow for an arbitrarily large number of k-cycles for the permutation y. Here we take $n = m+3 = ck+r$, where $c \geq 2$ and $0 \leq r < k$, and define

$$y = (1, 2, \ldots, k)(k+1, k+2, \ldots, 2k) \ldots ((c-1)k+1, (c-1)k+2, \ldots, ck), \text{ and}$$
$$x = (1, 2)(3, 4)(6, 7)(k, k+1)(2k, 2k+1) \ldots ((c-1)k, (c-1)k+1)(ck-r+1, ck+r)$$
$$(ck-r+2, ck+r-1) \ldots (ck-1, ck+2)(ck, ck+1).$$

These two permutations are illustrated by the coset diagram in Figure 5, where we have again deleted some labels to avoid clutter. Note that for $1 \leq j \leq c-2$, the permutation x fixes all but the first and last points of the k-cycle $(jk+1, jk+2, \ldots, (j+1)k)$ of y.

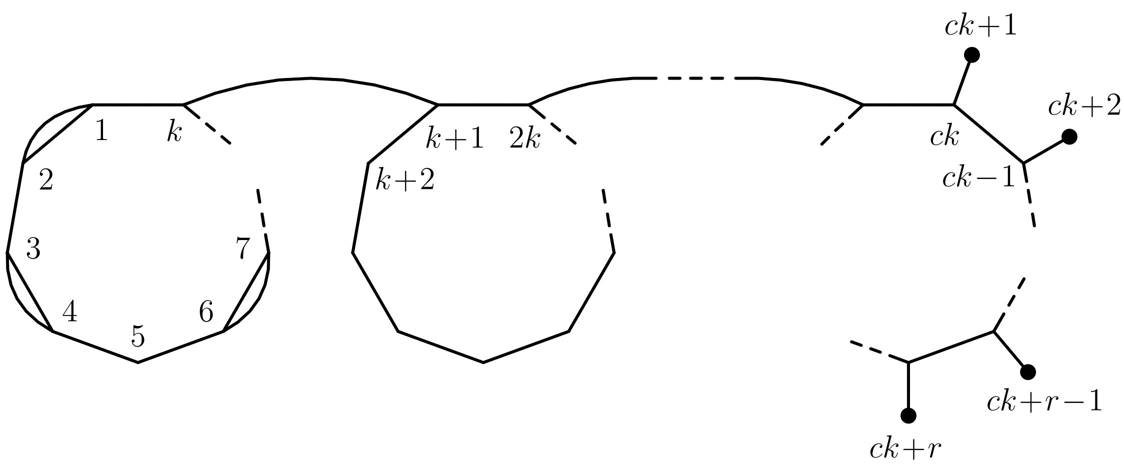

Figure 5: Generators for type $\{m, k\}$ with $m \geq 2k-3$ for $k \geq 8$

The permutation y is a product of c disjoint k-cycles, fixing the other $n-ck = r$ points, while xy is a single cycle of length $n-3 = m$, with fixed points 2, 4 and 7. The conjugate $y^{-2}(xy)y^2$ of xy is another m-cycle, fixing only 4, 6 and 9 (or 4, 6 and 1 when $k = 8$), so the stabiliser of the point 4 is transitive on the remaining points, and therefore G is 2-transitive, and hence primitive. Then since xy is a single cycle fixing at least three points, we have $G = A_n$ or S_n. Also if the corresponding map M is reflexible, then the mirror symmetry $\bar{\theta}$ must interchange the fixed points 2, 4 and 7 of xy with the fixed points 1, 3 and 6 of xy^{-1}, in some order, and so must induce a reflection of the polygon representing the k-cycle $(1, 2, \ldots, k)$ of y. The only such reflection interchanging $\{2, 4, 7\}$ with $\{1, 3, 6\}$ would be $(1, 4)(2, 3)(5, 8)(6, 7)$, occurring only when $k = 8$, but this interchanges the fixed point 5 of x with the point 8 which is not fixed by x, so there is no possibility for $\bar{\theta}$. Thus M is chiral.

3.6 Type $\{m, k\}$ with $m \geq 3k - 3$ for $k \geq 5$

In this case, again we take $n = m + 3 = ck + r$, where $c \geq 2$ and $0 \leq r < k$, and we define y exactly the same way as in §3.5, but we take

$$x = (1,2)(3,4)(k,k+1)(k+2,k+3)(2k,2k+1)\ldots((c-1)k,(c-1)k+1)(ck-r+1,ck+r)$$
$$(ck-r+2,ck+r-1)\ldots(ck-1,ck+2)(ck,ck+1).$$

These two permutations are illustrated by the coset diagram in Figure 6. Note that if $2 \leq j < c$, then all but the first and last points of the j-th k-cycle $((j-1)k+1, \ldots jk-1, jk)$ are fixed by x.

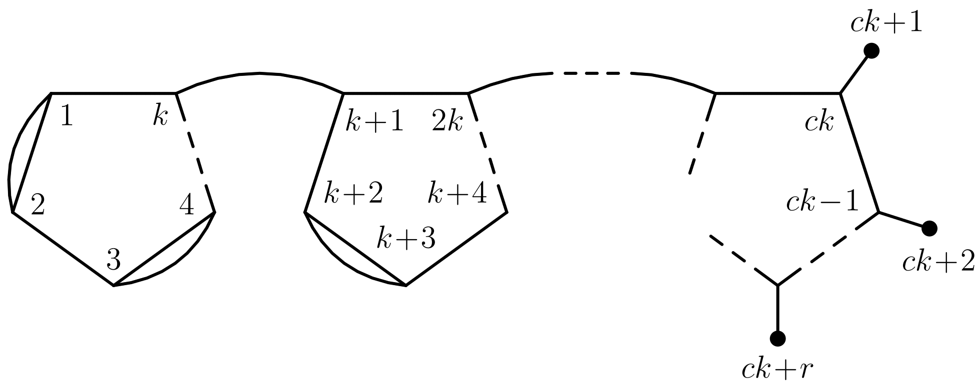

Figure 6: Generators for type $\{m, k\}$ with $m \geq 3k - 3$ for $k \geq 5$

In this case xy is still a single cycle of length $n - 3 = m$, but with fixed points 2, 4 and $k + 3$. The conjugate $y^{-2}(xy)y^2$ of xy is another m-cycle, fixing only 4, 6 (or 1 when $k = 5$) and $k + 5$, so the stabiliser of the point 4 is transitive on the remaining points, and therefore G is 2-transitive, and hence primitive. Again xy is a single cycle fixing at least three points, so $G = A_n$ or S_n. Also if the corresponding map M is reflexible, then the mirror symmetry $\bar{\theta}$ must interchange the fixed points 2, 4 and $k + 3$ of xy with the fixed points 1, 3 and $k + 2$ of xy^{-1}, in some order. It follows that $\bar{\theta}$ induces a reflection of each of the two polygons representing the k-cycles $(1, 2, \ldots, k)$ and $(k + 1, k + 2, \ldots, 2k)$ of y, and in particular, $\bar{\theta}$ must interchange $k + 2$ and $k + 3$, but then $\bar{\theta}$ interchanges the fixed point $k + 4$ of x with the point $k + 1$ not fixed by x. Hence there is no possibility for $\bar{\theta}$, and M is chiral.

3.7 Type $\{m, k\}$ with $m \geq 4k - 3$ for $k \geq 4$

In this final case, again we take $n = m + 3 = ck + r$, where $c \geq 2$ and $0 \leq r < k$, and we define y the same way as in §3.5 and §3.6, but take

$$x = (2,3)(k,k+1)(k+2,k+3)(2k,2k+1)(2k+2,2k+3)(3k,3k+1)$$
$$\ldots((c-1)k,(c-1)k+1)(ck-r+1,ck+r)\ldots(ck-1,ck+2)(ck,ck+1).$$

These two permutations are illustrated by the coset diagram in Figure 7. Once again xy is a single cycle of length $n-3=m$, but this time with fixed points 3, $k+3$ and $2k+3$.

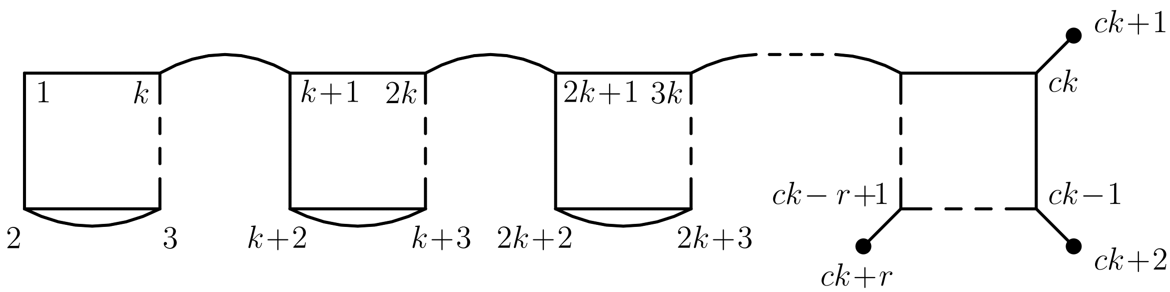

Figure 7: Generators for type $\{m, k\}$ with $m \geq 4k-3$ for $k \geq 4$

Here we can use a different argument to establish primitivity of $G = \langle x, y\rangle$. If G is imprimitive, then the blocks moved by xy must be permuted in a cycle by xy, and the remaining three points 3, $k+3$ and $2k+3$ must then form a single block B. Hence in particular, the block size is 3. But then y^{-2} takes B to $C = \{1, k+1, 2k+1\}$, and the latter cannot be a block since $1^x = 1 \in C$ while $(k+1)^x = k \notin C$. Hence G is primitive, and again since xy is a single cycle fixing three points, we have $G = A_n$ or S_n.

Also if the corresponding map M is reflexible, then the mirror symmetry $\bar{\theta}$ must interchange the fixed points 3, $k+3$ and $2k+3$ of xy with the fixed points 2, $k+2$ and $2k+2$ of xy^{-1}, in some order. It then follows that also the points 1, $k+1$ and $2k+1$ are taken by $\bar{\theta}$ to the points 4, $k+4$ and $2k+4$, in some order. But if $k > 4$, then each of those points is fixed by x, while $k+1$ and $2k+1$ are not, so this cannot happen. Similarly, if $k = 4$ then those three points are k, $2k$ and $3k$, which are all moved by x, but the point 1 is not. Hence no such $\bar{\theta}$ exists, and M is chiral.

4 Main theorems

We are now ready to prove our first main theorem.

Theorem 1 *There exists an orientably-regular but chiral map of type* $\{m, k\}$ (*with automorphism group* A_n *or* S_n *for some* n), *for every hyperbolic pair* (k, m).

Proof. By an earlier remark about duality, we may consider only pairs (k, m) with $k \leq m$. Also the case $k = 3$ was dealt with in [2], so we may suppose that $4 \leq k \leq m$.

If $k \geq 10$, then the family given in §3.3 shows that there exists such a map whenever $k \leq m \leq 2k-10$, and then those in §3.2 and §3.1 do the same for $k+1 \leq m \leq 2k-7$ and $k+2 \leq m \leq 2k-4$, respectively, and the family in §3.5 gives one for all $m \geq 2k-3$. Similarly, the families in §3.2, §3.1 and §3.5 do the same for all types $\{m, 9\}$ with $m \geq 10$, and all types $\{m, 8\}$ with $m \geq 9$. The families in §3.1, §3.4 and §3.6 give such maps of all types $\{m, 7\}$ with $9 \leq m \leq 10$, and $11 \leq m \leq 17$, and $m \geq 18$, respectively, and types

$\{m,6\}$ with $m = 8$, and $9 \le m \le 13$, and $m \ge 15$, respectively. Similarly, the families in §3.4 and §3.6 do the same for types $\{m,5\}$ with $7 \le m \le 9$ and $m \ge 12$. The family in §3.7 on its own does the same for all types $\{m,4\}$ with $m \ge 13$.

This leaves only 19 cases, namely those with $k = 4$ and $5 \le m \le 12$, or $k = 5$ and $m \in \{5,6,10,11\}$, or $k = 6$ and $m \in \{6,7,14\}$, or $k = 7$ and $m \in \{7,8\}$, or $(k,m) = (8,8)$ or $(9,9)$. These cases are easily handled. Below is a list that gives for each remaining pair (k,m) a symmetric or alternating group, and a pair of generators x and y of orders 2 and k such that the product xy has order m, but there is no (involutory) automorphism θ taking x and y to x and y^{-1}. (These properties can be checked in the same kind of way as for the families in the previous section, or with the help of the MAGMA system [1].)

$$
\begin{array}{lll}
(k,m) = (4,5): & S_{11} & = \langle\,(1,2)(3,5)(6,8)(7,9)(10,11), (1,2,3,4)(5,6,7,8)(9,10)\,\rangle \\
(k,m) = (4,6): & S_7 & = \langle\,(1,2)(3,5)(6,7), (1,2,3,4)(5,6)\,\rangle \\
(k,m) = (4,7): & S_8 & = \langle\,(1,2)(3,5)(6,7), (1,2,3,4)(5,6)(7,8)\,\rangle \\
(k,m) = (4,8): & S_8 & = \langle\,(3,7)(4,5)(6,8), (1,2,3,4)(5,6)\,\rangle \\
(k,m) = (4,9): & S_9 & = \langle\,(3,5)(4,7)(8,9), (1,2,3,4)(5,6)(7,8)\,\rangle \\
(k,m) = (4,10): & S_7 & = \langle\,(2,5)(3,6)(4,7), (1,2,3,4)(5,6)\,\rangle \\
(k,m) = (4,11): & A_{11} & = \langle\,(3,5)(4,7)(8,9)(10,11), (1,2,3,4)(5,6)(7,8)(9,10)\,\rangle \\
(k,m) = (4,12): & A_9 & = \langle\,(2,5)(3,8)(4,6)(7,9), (1,2,3,4)(5,6,7,8)\,\rangle \\
(k,m) = (5,5): & A_{11} & = \langle\,(2,3)(4,6)(5,8)(9,11), (1,2,3,4,5)(6,7,8,9,10)\,\rangle \\
(k,m) = (5,6): & S_7 & = \langle\,(2,3)(4,6)(5,7), (1,2,3,4,5)\,\rangle \\
(k,m) = (5,10): & A_{10} & = \langle\,(1,2)(3,5)(4,6)(7,9), (1,2,3,4,5)(6,7,8,9,10)\,\rangle \\
(k,m) = (5,11): & A_{11} & = \langle\,(5,6)(8,11), (1,2,3,4,5)(6,7,8,9,10)\,\rangle \\
(k,m) = (6,6): & S_6 & = \langle\,(2,4)(3,6), (1,2,3)(4,5)\,\rangle \\
(k,m) = (6,7): & S_7 & = \langle\,(1,4)(3,6)(5,7), (1,2,3)(4,5)\,\rangle \\
(k,m) = (6,14): & S_9 & = \langle\,(3,5)(6,7)(8,9), (1,2,3,4,5,6)(7,8)\,\rangle \\
(k,m) = (7,7): & A_8 & = \langle\,(5,6)(7,8), (1,2,3,4,5,6,7)\,\rangle \\
(k,m) = (7,8): & S_8 & = \langle\,(2,5)(3,6)(7,8), (1,2,3,4,5,6,7)\,\rangle \\
(k,m) = (8,8): & S_8 & = \langle\,(4,6)(5,8), (1,2,3,4,5,6,7,8)\,\rangle \\
(k,m) = (9,9): & A_9 & = \langle\,(1,3)(2,4)(5,8)(7,9), (1,2,3,4,5,6,7,8,9)\,\rangle.
\end{array}
$$

Hence in each case we have a generating pair for the automorphism group of an orientably-regular but chiral map of type $\{m,k\}$, completing the proof. □

Next, we have the following consequence.

Corollary 1 *For every non-spherical pair (k,m), there exist at least two orientably-regular maps of type $\{m,k\}$, one reflexible and one chiral, such that both the map and its dual have simple underlying graph.*

Proof. This is already known for the Euclidean (or toroidal) types $\{3,6\}$, $\{4,4\}$ and $\{6,3\}$, via triangular, quadrangular and honeycomb tessellations of the torus; for details, see [9].

In the hyperbolic case, the existence of an orientably-regular map of type $\{m, k\}$ follows from work by Macbeath in [16] on two-element generation of the projective special linear groups $\mathrm{PSL}(2, q)$. From the content of [16] it can be seen that for any hyperbolic pair (k, m) other than $(5, 5)$, the group $\mathrm{PSL}(2, p)$ can be generated by elements A, B and C of orders 2, k and m such that $ABC = 1$, whenever p is a prime such that $\mathrm{PSL}(2, p)$ contains elements of orders k and m. Also it is easy to show that the same thing holds for $\mathrm{PSL}(2, 11)$ when $(k, m) = (5, 5)$. (This case is treated as 'exceptional' in [16] because the exceptional subgroup A_5 is also $(2, 5, 5)$-generated.) Hence for every hyperbolic pair (k, m), there exists at least one orientably-regular map of type $\{m, k\}$ with automorphism group $\mathrm{PSL}(2, p)$ for some p. Moreover, by a theorem of Singerman [19, Thm 3], which follows from [16, Thm 3], every orientably-regular map with orientation-preserving automorphism group $\mathrm{PSL}(2, q)$ for some prime-power q is reflexible, so all these maps are reflexible.

Similarly, for each hyperbolic pair (k, m), we know from Theorem 1 that there exists an orientably-regular but chiral map of type $\{m, k\}$ with automorphism group isomorphic to an alternating or symmetric group of degree $n \geq 5$.

In both cases, we have an orientably-regular map M of type $\{m, k\}$, such that the group G of all orientation-preserving automorphisms of M is almost simple. In particular, G has no non-trivial cyclic normal subgroups, and it follows that in each case the core in $G = \mathrm{Aut}(M)$ of the stabiliser of every vertex of the map M is trivial, and so the underlying graph is simple. The same thing happens also for face-stabilisers, and hence the underlying graph of the dual of M is simple too. $\square$

Next, we can use what is now known as the 'Macbeath trick' to construct infinitely many such maps (of every hyperbolic type), in both the reflexible and chiral cases. This is a method used by Macbeath in [15] to construct infinite families of covers of a given Hurwitz surface, with abelian covering group. It has also been used for maps (as in [14], for example). Here we use the Macbeath trick to prove the following:

Proposition 2 *Let (k, m) be any hyperbolic pair, and let M be any orientably-regular map of genus g and type $\{m, k\}$, with orientation-preserving automorphism group H. Then for every prime p that does not divide the order of H, there exists a regular map $M^{(p)}$ of type $\{m, k\}$ which covers M, such that the covering group is an elementary abelian p-group of order p^{2g} (that is, the group G of orientation-preserving automorphisms of $M^{(p)}$ has an elementary abelian normal subgroup P of order p^{2g} such that $G/N \cong H$). Moreover, the covering map $M^{(p)}$ is reflexible if and only if the base map M is reflexible. Similarly, if the underlying graph of M is simple, then so is the underlying graph of $M^{(p)}$, and the same holds for the duals of M and $M^{(p)}$.*

Proof. Let Δ be the ordinary $(2, k, m)$ triangle group. Then there exists a smooth epimorphism $\psi \colon \Delta \to H$, such that the kernel K of ψ is isomorphic to the fundamental group of the orientable surface supporting the map M. In particular, K has a presentation of the form $\langle\, a_1, b_1, \ldots, a_g, b_g \mid [a_1, b_1][a_2, b_2] \ldots [a_g, b_g] = 1 \,\rangle$. Now let $J = K'K^{(p)}$, the subgroup of K generated by all commutators and p-th powers of elements of K. Then the quotient

K/J is the largest elementary abelian p-quotient of K, and so has order p^{2g}. Moreover, J characteristic in K and hence normal in Δ, and it follows that $G = \Delta/J$ is the automorphism group of an orientably-regular map $M^{(p)}$ of type $\{m, k\}$. This map is a cover of the given map M, with abelian covering group $K/J \cong {\mathbb{Z}_p}^{2g}$.

The latter subgroup K/J is a normal Sylow p-subgroup of Δ/J, and hence characteristic in Δ/J. It follows that if θ is an involutory automorphism of Δ/J inverting the images of the canonical generators of Δ, then θ preserves K/J, and so induces an analogous automorphism $\tilde{\theta}$ of $(\Delta/J)(K/J) \cong \Delta/K \cong H$. Hence if $M^{(p)}$ is reflexible, then so is M. Conversely, if M is reflexible then the involutory automorphism of Δ that inverts its two canonical generators must preserve the normal subgroup K, and hence also preserves J (again since J characteristic in K). Thus $M^{(p)}$ is reflexible if and only if M is reflexible.

Finally, we note that if the core in H of the stabiliser of a vertex or face of M is trivial, then the same is true of the core in $G = \Delta/J$ of a the stabiliser of a vertex or face of $M^{(p)}$, because the latter stabilisers have trivial intersection with the normal Sylow p-subgroup of G, and under the natural epimorphism from $G = \Delta/J$ to $(\Delta/J)(K/J) \cong \Delta/K \cong H$, every normal subgroup of G that intersects K/J trivially is taken to to a normal subgroup of H of the same order. Hence if the map M or its dual has simple underlying graph, then so does the map $M^{(p)}$ or its dual, respectively. $\square$

Note: The converse of the last part of the above proposition is not true — the underlying graph of the covering map $M^{(p)}$ can often be simple, when that of the base map M is not. (For example, this happens with 5-coverings of the genus 2 regular map of type $\{4, 6\}$.)

Our second main theorem is a stronger form of Theorem 1, obtained with the help of Proposition 2:

Theorem 2 *For every pair (k, m) of integers with $1/k + 1/m \le 1/2$, there exist infinitely many regular and infinitely many orientably-regular but chiral maps of type $\{m, k\}$, each with the property that both the map and its dual have simple underlying graph.*

Proof. First, this is well known for pairs with $1/k + 1/m = 1/2$; the maps are triangular, quadrangular and honeycomb tessellations of the torus. Hence we may suppose that the pair (k, m) is hyperbolic. Now let M_1 be a regular map of type $\{m, k\}$ with automorphism group $\mathrm{PSL}(2, p)$ for some p, as provided by Macbeath's work, and let M_2 be an orientably-regular but chiral map of type $\{m, k\}$ with automorphism group A_n or S_n for some n, as given by Theorem 1. Then by Proposition 2, there exist infinitely many regular coverings of M_1 of type $\{m, k\}$, one for each prime p that does not divide $|\mathrm{Aut}(M_1)|$, and with each having the property that both the map and its dual have simple underlying graph. The same holds for p-coverings of M_2 for each prime p that does not divide $|\mathrm{Aut}(M_2)|$, and this gives infinitely many orientably-regular but chiral coverings of M_2, each of type $\{m, k\}$, and again with simple graphs underlying both the map and its dual. $\square$

In addition, since the underlying graphs of these maps are simple and 2-connected (that is, they have no bridges), every edge has two vertices and every edge lies in two faces, and therefore the maps are abstract polyhedra. Thus we have the following as well:

Corollary 2 *For every pair (k,m) of integers with $1/k+1/m \le 1/2$, there exist infinitely many regular and infinitely many orientably-regular but chiral polyhedra of type $\{m,k\}$.*

5 Final remarks

The simplicity of our approach to proving Theorem 1 may be thought to be overshadowed by the fact that one of the key results we used, namely the generalisation (by Jones [11]) of Jordan's theorem, was proved with the help of the classification of finite simple groups. The use of Jones's theorem can be avoided, however, with some additional work, to find in each case considered in Section 3 a cycle of *prime* length fixing at least 3 points, and then using the original version of Jordan's theorem. Further details of an alternative approach using Jordan's theorem will be available in the second author's PhD thesis [10].

We can also prove an analogue of Theorem 1 for regular maps of given hyperbolic type. This is known for cases where $k = 3$ or $m = 3$, by a theorem of the first author in [3], and for almost all cases where both k and m are odd primes, by a theorem of Mushtaq and Servatius in [17]. The following theorem extends these known facts to all hyperbolic pairs, but we give only a sketch proof, because the details are similar to those used earlier in this paper for the chiral case.

Theorem 3 *For every hyperbolic pair (k,m), there exists a fully regular map of type $\{m,k\}$ with orientation-preserving automorphism group isomorphic to A_n or S_n for some n.*

Sketch proof. As we did earlier for chiral maps, we can construct regular maps of type $\{m,k\}$ for all pairs in one of a number of classes of hyperbolic pairs (k,m) with $4 \le k \le m$. In fact we can do this using just two broad classes:

Class (a) Almost all (k,m) with $4 \le k \le m \le 2k-1$

In this case we take $n = k+r = m+s$ where $0 \le r \le k-1$, $0 \le s < k/2$, and $r+2s \le k$. For the generator y, we take a single k-cycle fixing r points, and for the generator x, we take a product of r disjoint transpositions, each swapping a point of the k-cycle of y with a fixed point of y, together with a product of s transpositions, each swapping two neighbouring points of the k-cycle of y (different from those used already), in a way similar to the choice of x and y in the families in §§3.1–3.3. The product xy is then a single cycle of length $n-s = m$, fixing s points. If we take $r \ge 3$ or $s \ge 3$, then the resulting permutations generate A_n or S_n. Moreover, provided it is not the case that k is even while both r and s are odd, the fixed points of y and xy can be arranged in such a way that the coset diagram is symmetric about some axis through the centre of the k-cycle of y, and it follows that the corresponding orientably-regular map of type $\{m,k\}$ is reflexible.

This works for type $\{k,k\}$ with $r = s = 3$ when k is odd and $k \ge 9$, and with $r = s = 4$ when k is even and $k \ge 12$, and for type $\{k,k+1\}$ with $(r,s) = (3,2)$ when $k \ge 7$, and so on, up to type $\{k,2k-1\}$ with $(r,s) = (k-1,0)$ when $k \ge 4$.

Class (b) Almost all (k,m) with $m \ge 2k$ for $k \ge 4$

In this case we take $n = ck + r = m + s$ where $c \geq 2$, $0 \leq r \leq k-1$, $3 \leq s < 2k$, and $r + 2s \leq c(k-1)$. For the generator y, we take c disjoint k-cycles, fixing r points, and use transpositions of x to string them together, and join each fixed point of y to a point of one of the k-cycles of y, and join s pairs of distinct points of the form $\{j, j^y\}$, as in §§3.4–3.7. Again this makes the product xy a single cycle of length $n - s = m$, fixing s points, and since $s \geq 3$, it follows (after verifying primitivity) that the resulting permutations generate A_n or S_n. Moreover, if the fixed points of y and xy are arranged in such a way that the coset diagram is symmetric about some axis through the centre of the middle k-cycle of y (when c is odd) or between the two middle k-cycles of y (when c is even), then the corresponding orientably-regular map of type $\{m, k\}$ is reflexible.

To achieve this, there are some cases to avoid, namely where $c = 2$ and $k = s + 1$ with s even, or $(c, k, r) = (3, 4, 0)$, or c is odd, $k \equiv 0 \bmod 4$, $r = 0$ and $2s = 3k$.

On the other hand, it works for type $\{k, 2k\}$ with $(c, r, s) = (2, 4, 4)$ whenever $k \geq 7$, and for type $\{k, 2k+1\}$ with $(c, r, s) = (3, 0, k-1)$ whenever $k \geq 5$, and for type $\{k, 2k+2\}$ with $(c, r, s) = (2, 3, 4)$ whenever $k \geq 8$, and for type $\{k, 2k+3\}$ with $(c, r, s) = (3, 0, k-3)$ whenever $k \geq 6$, and so on.

The constructions in the above two classes leave only 21 hyperbolic pairs (k, m) with $k \leq m$ to consider, namely $(4, 5)$, $(4, 6)$, $(4, 8)$, $(4, 9)$, $(4, 10)$, $(4, 13)$, $(4, 16)$, $(5, 5)$, $(5, 6)$, $(5, 10)$, $(5, 15)$, $(5, 17)$, $(6, 6)$, $(6, 7)$, $(6, 12)$, $(6, 14)$, $(6, 16)$, $(7, 7)$, $(8, 8)$, $(8, 20)$ and $(10, 10)$, and it is easy to use Magma to show that in these cases, there is a regular map of type $\{m, k\}$ with automorphism group S_5, S_5, S_8, S_9, S_7, S_{13}, S_{16}, A_5, S_5, S_7, A_8, A_{17}, S_5, S_7, S_7, S_9, S_{16}, A_7, S_8, S_9 and S_7, respectively. □

In fact, we conjecture that for every hyperbolic pair (k, m), all but finitely many of the alternating groups A_n occur as the orientation-preserving automorphism group of some (fully) regular map of type $\{m, k\}$, and that the same holds in the orientably-regular but chiral case as well. This is known when one of k or m is 3 (see [3] and [2], respectively), and in the regular case whenever both k and m are sufficiently large primes (see [17]), but proving it for all hyperbolic pairs is likely to be difficult.

Acknowledgements

The first author's work was supported by a James Cook Fellowship and a Marsden Fund grant, awarded by the Royal Society of New Zealand. The second and fourth authors acknowledge support from the VEGA Research Grants 1/0065/13 and 1/0007/14 and the APVV Research Grants 0223-10 and 0136-12, as well as the APVV support as part of the EUROCORES Programme EUROGIGA, project GREGAS, ESF-EC-0009-10, financed by the European Science Foundation. In addition, we acknowledge significant use of the Magma system [1] in helping us to make and test many of the observations described in this paper.

References

[1] W. Bosma, J. Cannon and C. Playoust, The MAGMA Algebra System I: The User Language, *J. Symbolic Computation* 24 (1997), 235–265.

[2] E. Bujalance, M.D.E. Conder and A.F. Costa, Pseudo-real Riemann surfaces and chiral regular maps, *Trans. Amer. Math. Soc.* 362 (2010), 3365–3376.

[3] M.D.E. Conder, More on generators for alternating and symmetric groups, *Quart. J. Math. Oxford Ser. (2)* 32 (1981) 126, 137–163.

[4] M.D.E. Conder, Regular maps and hypermaps of Euler characteristic −1 to −200, *J. Combin. Theory Ser. B* 99 (2009), 455–459, with lists of all such maps and hypermaps available at `http://www.math.auckland.ac.nz/~conder/hypermaps.html`.

[5] M.D.E. Conder, Regular maps of Euler characteristic −1 to −600, see webpages linked from `http://www.math.auckland.ac.nz/~conder`.

[6] M.D.E. Conder and J. Ma, Regular maps with simple underling graph, preprint.

[7] M.D.E. Conder, J. Širáň and T.W. Tucker, The genera, reflexibility and simplicity of regular maps, *J. Europ. Math. Soc.* 12 (2010), 343–364.

[8] M.D.E Conder and S.E. Wilson, Inner reflectors and non-orientable regular maps, *Discrete Math.* 307 (2007), 367–372.

[9] H.S.M. Coxeter and W.O.J. Moser, *Generators and Relations for Discrete Groups*, 4th Edition, Springer, Berlin, 1984.

[10] V. Hucíková, PhD Dissertation, in preparation.

[11] G.A. Jones, Primitive permutation groups containing a cycle, arXiv:1209.5169 (2013).

[12] G.A. Jones, Chiral covers of hypermaps, arXiv:1311.4355 (2013).

[13] G.A. Jones and D. Singerman, Theory of maps on orientable surfaces, *Proc. London Math. Soc.* 37 (1978), 273–307.

[14] G.A. Jones and D.B. Surowski, Regular cyclic coverings of the Platonic maps, *Europ. J. Combin.* 21 (2000), 333–345.

[15] A.M. Macbeath, On a theorem of Hurwitz, *Proc. Glasgow Math. Assoc.* 5 (1961), 90–96.

[16] A.M. Macbeath, Generators of the fractional linear groups, *Number Theory*, Proc. Sympos. Pure Math., vol. XII, Amer. Math. Soc., 1969, pp. 14–32.

[17] Q. Mushtaq and H. Servatius, Permutation representations of the symmetry groups of regular hyperbolic tessellations, *J. London Math. Soc. (2)* 48 (1993), 77–86.

[18] R. Nedela and M. Škoviera, Regular maps on surfaces with large planar width, *European J. Combin.* 22 (2001), 243–261.

[19] D. Singerman, Symmetries of Riemann surfaces with large automorphism group, *Math. Ann.* 210 (1974), 17–32.

[20] D. Singerman, Reflexibility and symmetry, preprint, Univ. of Southampton, 1992.

[21] A. Vince, Regular combinatorial maps, *J. Combin. Theory Ser. B* 35 (1983), 256–277.

[22] H. Wielandt, *Finite Permutation Groups*, Academic Press (New York and London), 1964.