Full proof of the existence of a degree 8 circulant graph of order \( L(8,k) \) of arbitrary diameter \( k \)

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Full proof of the existence of a degree 8 circulant graph of order $L(8, k)$ of arbitrary diameter $k$

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This is the full proof of Theorem 3 in the paper “The degree-diameter problem for circulant graphs of degree 8 and 9” by the author [2]. To avoid the paper being unduly long it includes only the exceptions for the orthant of $v_1$ for diameter $k \equiv 0 \pmod{2}$ and for $k \equiv 1 \pmod{2}$. In the version below the exceptions for all eight orthants for diameter $k \equiv 0$ and $k \equiv 1 \pmod{2}$ are included in full. This proof closely follows the approach taken by Dougherty and Faber in their proof of the existence of the degree 6 graph of order $DF(6, k)$ for all diameters $k \geq 2$ [1].

**Theorem 3.** For all $k \geq 2$, there is an undirected Cayley graph on four generators of a cyclic group which has diameter $k$ and order $L(8, k)$, where

$$L(8, k) = \begin{cases} 
(k^4 + 2k^3 + 6k^2 + 4k)/2 & \text{if } k \equiv 0 \pmod{2} \\
(k^4 + 2k^3 + 6k^2 + 6k + 1)/2 & \text{if } k \equiv 1 \pmod{2}
\end{cases}$$

Moreover for $k \equiv 0 \pmod{2}$ a generator set is \{1, $(k^3 + 2k^2 + 6k + 2)/2, (k^4 + 4k^2 - 8k)/4, (k^4 + 4k^2 - 4k - 4)/4\}$, and for $k \equiv 1 \pmod{2}$, \{1, $(k^3 + k^2 + 5k + 3)/2, (k^4 + 2k^2 - 8k - 11)/4, (k^4 + 2k^2 - 4k - 7)/4\}$.

**Proof.** We will show the existence of four-dimensional lattices $L_k \subseteq \mathbb{Z}^4$ such that $\mathbb{Z}^4/L_k$ is cyclic, $S_k + L_k = \mathbb{Z}^4$, where $S_k$ is the set of points in $\mathbb{Z}^4$ at a distance of at most $k$ from the origin under the $l^1$ (Manhattan) metric, and $|\mathbb{Z}^4 : L_k| = L(8, k)$ as specified in the theorem. Then, by Theorem 1 of [2], the resultant Cayley graph has diameter at most $k$.

Let $a = \begin{cases} 
 k/2 & \text{for } k \equiv 0 \pmod{2} \\
 (k + 1)/2 & \text{for } k \equiv 1 \pmod{2}.
\end{cases}$

For $k \equiv 0 \pmod{2}$ let $L_k$ be defined by four generating vectors as follows:

$$
\begin{align*}
\mathbf{v}_1 &= (-a - 1, a + 1, a, -a + 1) \\
\mathbf{v}_2 &= (a - 1, a + 1, a + 1, -a) \\
\mathbf{v}_3 &= (-a - 1, -a + 1, a + 1, -a) \\
\mathbf{v}_4 &= (-a, -a, a, a + 1)
\end{align*}
$$
Then the following vectors are in $L_k$:

$$-(2a^2 + 2a + 1)v_1 + (2a^2 + a + 2)v_2 - (a + 2)v_3 + v_4 = (4a^3 + 4a^2 + 6a + 1, -1, 0, 0),$$
$$-(2a^3 - 1)v_1 + (2a^3 - a^2 + 2a - 2)v_2 - (a^2 + a - 1)v_3 + (a - 1)v_4 = (4a^4 + 4a^2 - 4a, 0, -1, 0),$$
$$-2a^3v_1 + (2a^3 - a^2 + 2a - 1)v_2 - (a^2 + a - 1)v_3 + (a - 1)v_4 = (4a^4 + 4a^2 - 2a, 0, 0, -1)$$

Hence we have $e_2 = (4a^3 + 4a^2 + 6a + 1)e_1, e_3 = (4a^4 + 4a^2 - 4a)e_1$ and $e_4 = (4a^4 + 4a^2 - 2a)e_1$ in $\mathbb{Z}^4/L_k$, and so $e_1$ generates $\mathbb{Z}^4/L_k$.

Also $\det \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix} = \det \begin{pmatrix} 8a^4 + 8a^3 + 12a^2 + 4a & 0 & 0 & 0 \\ 4a^3 + 4a^2 + 6a + 1 & -1 & 0 & 0 \\ 4a^4 + 4a^2 - 4a & 0 & -1 & 0 \\ 4a^3 + 4a^2 - 2a & 0 & 0 & -1 \end{pmatrix} = -(8a^4 + 8a^3 + 12a^2 + 4a) = -(k^4 + 2k^3 + 6k^2 + 4k)/2 = -L(8, k)$, as in the statement of the theorem.

Thus $\mathbb{Z}^4/L_k$ is isomorphic to $\mathbb{Z}_{L(8,k)}$ via an isomorphism taking $e_1, e_2, e_3, e_4$ to $1, 4a^3 + 4a^2 + 6a + 1, 4a^4 + 4a^2 - 4a, 4a^4 + 4a^2 - 2a$. As $a = k/2$ this gives the first generator set specified in the theorem: $\{1, (k^3 + 2k^2 + 6k + 1)/2, (k^4 + 4k^3 - 8k)/4, (k^4 + 4k^2 - 4k)/4\}$.

Similarly for $k \equiv 1 \pmod{2}$ let $L_k$ be defined by four generating vectors as follows:

$$v_1 = (-a + 1, a + 1, -a + 1, a)$$
$$v_2 = (a + 1, a + 1, -a + 2, a - 1)$$
$$v_3 = (-a - 1, a - 1, a - 1, -a)$$
$$v_4 = (-a, a, a, a - 1)$$

In this case the following vectors are in $L_k$:

$$-(2a^2 + a + 2)v_1 + (2a^2 + 2a + 1)v_2 - a^2v_3 - v_4 = (4a^3 - 4a^2 + 6a - 1, -1, 0, 0),$$
$$-(2a^3 - a^2 - 2a - 2)v_1 + (2a^3 - 4a - 1)v_2 - (a^2 - a - 1)v_3 - (a - 1)v_4 = (4a^4 - 8a^3 + 8a^2 - 8a, 0, -1, 0),$$
$$-(2a^3 - a^2 - 2a - 1)v_1 + (2a^3 - 4a)v_2 - (a^2 - a - 1)v_3 - (a - 1)v_4 = (4a^4 - 8a^3 + 8a^2 - 6a, 0, 0, -1).$$

Hence we have $e_2 = (4a^3 + 4a^2 + 6a - 1)e_1, e_3 = (4a^4 - 8a^3 + 8a^2 - 8a)e_1$ and $e_4 = (4a^4 - 8a^3 + 8a^2 - 6a)e_1$ in $\mathbb{Z}^4/L_k$, and so $e_1$ generates $\mathbb{Z}^4/L_k$.

Also $\det \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix} = \det \begin{pmatrix} 8a^4 - 8a^3 + 12a^2 - 4a & 0 & 0 & 0 \\ 4a^3 - 4a^2 + 6a - 1 & -1 & 0 & 0 \\ 4a^4 - 8a^3 + 8a^2 - 8a & 0 & -1 & 0 \\ 4a^4 - 8a^3 + 8a^2 - 6a & 0 & 0 & -1 \end{pmatrix} = -(8a^4 - 8a^3 + 12a^2 - 4a) = -(k^4 + 2k^3 + 6k^2 + 4k + 1)/2 = -L(8, k)$, as in the statement of the theorem.

Thus $\mathbb{Z}^4/L_k$ is isomorphic to $\mathbb{Z}_{L(8,k)}$ with generators $1, 4a^3 - 4a^2 + 6a - 1, 4a^4 - 8a^3 + 8a^2 - 8a, 4a^4 - 8a^3 + 8a^2 - 6a$. As $a = (k + 1)/2$ in this case, this gives the second generator set specified in the theorem: $\{1, (k^3 + 5k^2 + 3k + 1)/2, (k^4 + 2k^3 - 4k^2 - 8k + 11)/4, (k^4 + 2k^2 - 4k - 7)/4\}$.

It remains to show that $S_k + L_k = \mathbb{Z}^4$. First we consider the case $k \equiv 0 \pmod{2}$.

For $k = 2$, it is straightforward to show directly that $\mathbb{Z}_{32}$ with generators $1, 4, 6, 15$ has
with \( v_1, v_2, v_3, v_4 \) as defined for \( k \equiv 0 \pmod{2} \). Then the 16 vectors \( \pm v_i \) for \( i = 1, \ldots, 8 \) provide one element of \( L_k \) lying strictly within each of the 16 orthants of \( \mathbb{Z}^4 \). Most of the coordinates of these vectors have absolute value at most \( a+1 \). Only \( \pm v_5 \) and \( \pm v_7 \) each have one coordinate with absolute value equal to \( a+2 \).

Now we consider the case \( k \equiv 1 \pmod{2} \). For \( k = 3 \) it may be shown directly that \( \mathbb{Z}_{104} \) with generators 1, 16, 20, 27 has diameter 3. So we assume \( k \geq 5 \), so that \( a \geq 3 \), and let

\[
\begin{align*}
v_5 &= v_1 - v_2 - v_4 = (-a, a, -a-1, -a+2) \\
v_6 &= v_2 + v_3 - v_4 = (a, a, -a+1, -a) \\
v_7 &= v_1 + v_3 - v_4 = (-a, a, a-1, -a+1) \\
v_8 &= v_1 - v_2 + v_3 = (-a+1, -a+1, -a, a+1)
\end{align*}
\]

with \( v_1, v_2, v_3, v_4 \) as defined for \( k \equiv 1 \pmod{2} \), so that the 16 vectors \( \pm v_i \) provide one element of \( L_k \) lying strictly within each of the orthants of \( \mathbb{Z}^4 \). In this case all the coordinates of these vectors have absolute value at most \( a+1 \).

We must show that each \( x \in \mathbb{Z}^4 \) is in \( S_k + L_k \), which means that for any \( x \in \mathbb{Z}^4 \) we need to find a \( w \in L_k \) such that \( x - w \in S_k \). However \( x - w \in S_k \) if and only if \( \delta(x, w) \leq k \), where \( \delta \) is the \( l^1 \) metric on \( \mathbb{Z}^4 \). If \( x, y, z \in \mathbb{Z}^4 \) and each coordinate of \( y \) lies between the corresponding coordinate of \( x \) and \( z \) or is equal to one of them, then \( \delta(x, y) + \delta(y, z) = \delta(x, z) \). In such a case we say that “\( y \) lies between \( x \) and \( z \)”.

For any \( x \in \mathbb{Z}^4 \), we reduce \( x \) by adding appropriate elements of \( L_k \) until the resulting vector lies within \( l^1 \)-distance \( k \) of \( 0 \) or some other element of \( L_k \). The first stage is to reduce \( x \) to a vector whose coordinates all have absolute value at most \( a+1 \). If \( x \) has a coordinate with absolute value above \( a+1 \), then let \( v \) be one of the vectors \( \pm v_i (1 \leq i \leq 8) \) such that the coordinates of \( v \) have the same sign as the corresponding coordinates of \( x \). If a coordinate of \( x \) is 0 then either sign is allowed for \( v \) as long as the corresponding coordinate of \( v \) has absolute value \( \leq a+1 \). So in the case \( k \equiv 0 \pmod{2} \) if the \( e_3 \) coordinate of \( x \) is 0 then we avoid \( v_7 \) and take \( v_5 \) instead. Also if the \( e_4 \) coordinate of \( x \) is 0 (or both \( e_3 \) and \( e_4 \) coordinates are 0) then instead of \( v_5 \) we take \( v_1 \).

Now consider \( x' = x - v \). If a coordinate of \( x \) has absolute value \( s, 1 \leq s \leq a+1 \), then the corresponding coordinate of \( x' \) will have absolute value \( s' \leq a+1 \) because of the sign matching and the fact that the coordinates of \( v \) have absolute value \( \leq a+2 \). If a coordinate of \( x \) has absolute value \( s = 0 \), then as indicated above, the corresponding value of \( x' \) will have absolute value \( s' \leq a+1 \) because \( v \) is chosen such that the corresponding coordinate has absolute value \( \leq a+1 \). If a coordinate of \( x \) has absolute value \( s > a+1 \), then the corresponding coordinate of \( x' \) will be strictly smaller in absolute value. Therefore repeating this procedure will result in a vector whose coordinates all have absolute value at most \( a+1 \).
If the resulting vector \( x' \) lies between 0 and \( v \), where \( v = \pm v_i \) for some \( i \), then we have \( \delta(0, x') + \delta(x', v) = \delta(0, v) \). For \( k \equiv 0 \pmod{2} \) all of the vectors \( v \) satisfy \( \delta(0, v) = 4a + 1 \), and for \( k \equiv 1 \pmod{2} \) they all satisfy \( \delta(0, v) = 4a - 1 \). So in either case we have \( \delta(0, v) = 2k + 1 \). Since \( \delta(0, x') \) and \( \delta(x', v) \) are both non-negative integers, one of them must be at most \( k \), so that \( x' \in S_k + L_k \). Hence we also have \( x \in S_k'' + L_k \). For both cases \( k \equiv 0 \pmod{2} \) the exceptions need to be considered for each orthant in turn. We first consider all eight orthants for the case \( k \equiv 0 \pmod{2} \) and then the same for \( k \equiv 1 \pmod{2} \).

**Orthant of \( v_1 \), \( k \equiv 0 \pmod{2} \)**

Suppose that \( k \equiv 0 \pmod{2} \) and \( x \) lies within the orthant of \( v_1 \), but not between 0 and \( v_1 \). Then as \( v_1 = (-a - 1, a + 1, a, -a + 1) \), the third coordinate of \( x \) is equal to \( a + 1 \) or the fourth coordinate equals \( -a \) or \( -a - 1 \). We distinguish three cases.

Case 1: \( x = (-r, s, a + 1, -u) \) where \( 0 \leq r, s \leq a + 1 \) and \( 0 \leq u \leq a + 1 \). Let \( x' = x - v_1 = (a + r - s, a - 1, a - 1 - u) \), which lies between 0 and \(-v_7\) unless \( r \leq 1 \) or \( s \leq 1 \). Let \( x'' = x' + v_7 = (2 - r, s - 2, -a - 1, 2a - u) \). If \( r \leq 1 \) and \( s \leq 1 \) then \( x'' \) lies between 0 and \(-v_1\) unless \( u = a \), in which case let \( x''' = x'' + v_1 = (1 - a - r - a - 1 + s, -1, a + 1 - u) \) which lies between 0 and \(-v_3\). If \( r \leq 1 \) and \( s \geq 2 \) then \( x'' \) lies between 0 and \(-v_3\). If \( r \geq 2 \) and \( s \leq 1 \) then \( x'' \) lies between 0 and \(-v_2\).

Case 2: \( x = (-r, s, a + 1, -u) \) where \( 0 \leq r, s \leq a + 1 \) and \( 0 \leq u \leq a - 1 \). Let \( x' = x - v_1 = (a + r - s, a - 1, a - 1 - u) \), which lies between 0 and \(-v_6\) unless \( r = 0 \) or \( s = 0 \). Let \( x'' = x' + v_6 = (1 - r, s - 1, -a, -u - 1) \). If \( r = 0 \) and \( s = 0 \) then \( x'' \) lies between 0 and \(-v_5\). If \( r = 0 \) and \( s \geq 1 \) then \( x'' \) lies between 0 and \(-v_4\). If \( r \geq 1 \) and \( s = 0 \) then \( x'' \) lies between 0 and \(-v_8\).

Case 3: \( x = (-r, s, t, -u) \) where \( 0 \leq r, s \leq a + 1 \) and \( 0 \leq t \leq a \) and \( 0 \leq u \leq a + 1 \). Let \( x' = x - v_1 = (a + r - s, a - 1, t - a, a - 1 - u) \), which lies between 0 and \(-v_5\) unless \( r = 0 \) or \( s = 0 \) or \( t = 0 \). If \( r = 0 \) and \( s = 0 \) and \( t = 0 \) then \( x' \) lies between 0 and \(-v_7\). Let \( x'' = x' + v_5 = (1 - r, s - 1, t - 1, 2a + 1 - u) \). If \( r = 0 \) and \( s \geq 1 \) and \( t \geq 1 \) then \( x'' \) lies between 0 and \(-v_8\). If \( r = 0 \) and \( s \geq 1 \) and \( t = 0 \) then \( x'' \) lies between 0 and \(-v_5\). If \( r \geq 1 \) and \( s = 0 \) and \( t = 0 \) then \( x'' \) lies between 0 and \(-v_4\). If \( r \geq 1 \) and \( s \geq 1 \) and \( t = 0 \) then \( x'' \) lies between 0 and \(-v_8\). If \( r \geq 1 \) and \( s = a + 1 \) and \( t = 0 \) then \( x'' \) lies between 0 and \(-v_4\).

This completes the cases for the orthant of \( v_1 \) for \( k \equiv 0 \pmod{2} \).
Orntant of $v_2$, $k \equiv 0 \pmod{2}$

Now suppose that $x$ lies in the orthant of $v_2$ but not between $0$ and $v_2$. Then the first coordinate of $x$ is equal to $a$ or $a+1$, or the fourth coordinate equals $-a-1$. We distinguish three cases.

Case 1: $x = (r, s, t, -a-1)$ where $0 \leq r \leq a + 1$ and $0 \leq s, t \leq a + 1$. Let $x' = x - v_2 = (r - a + 1, s - a - 1, t - a - 1, -1)$, which lies between $0$ and $-v_3$ unless $s = 0$ or $t \leq 1$, in which case let $x'' = x' + v_5 = (r - 2a + 1, s - 1, t - 2, a + 1)$. If $s = 0$ and $t \leq 1$ then let $x''' = x + v_5 = (r - a, a, t + a - 1, 1)$ which lies between $0$ and $v_8$. If $s = 0$ and $t \geq 2$ then let $x''' = x'' - v_8 = (r - 2a, 0, t - 1, -a)$ which lies between $0$ and $v_3$. If $s \geq 1$ and $t \leq 1$ then $x''$ lies between $0$ and $v_7$ unless $s = a + 1$, in which case let $x'' = x'' - v_7 = (r - a, 1, a + t, 0)$ which lies between $0$ and $v_2$.

Case 2: $x = (r, s, t, -u)$ where $a \leq r \leq a + 1$, $0 \leq s, t \leq a + 1$ and $0 \leq u \leq a$. Let $x' = x - v_2 = (r - a + 1, s - a - 1, t - a - 1, a - u)$, which lies between $0$ and $-v_1$ unless $t = 0$ or $u = 0$. If $t = 0$ and $u = 0$ then $x$ lies between $0$ and $-v_3$ unless $a \leq s \leq a + 1$. If $r = a + 1$, $a \leq s \leq a + 1$, $t = 0$ and $u = 0$ then let $x'' = x + v_3 = (r - a - 1, s - a + 1, t + a + 1, -u - a)$ which lies between $0$ and $v_2$. If $a \leq r \leq a + 1$, $s = a + 1$, $t = 0$ and $u = 0$ then let $x'' = x - v_2 = (r - a + 1, s - a - 1, t - a - 1, -u + a)$ which lies between $0$ and $-v_3$. If $r = a, s = a, t = 0$ and $u = 0$ then $x$ lies between $0$ and $v_8$. Now let $x''' = x' + v_1 = (r - 2a, s, t - 1, 1 - u)$. If $t = 0$ and $1 \leq u \leq a$ then $x'''$ lies between $0$ and $v_6$ unless $a = 0$. If $r = a + 1, s = a + 1, t = 0$ and $u = 0$ then let $x''' = x'' - v_6 = (r - a, s - a, t + a, a + 1 - u)$ which lies between $0$ and $v_8$. If $1 \leq t \leq a + 1$ and $u = 0$ then $x'''$ lies between $0$ and $v_5$ unless $s = a + 1$ or $t = a + 1$, in which case let $x''' = x'' - v_5 = (r - a, s - a, t - a, -a - 1 - u)$. If $s = a + 1$ and $t = a + 1$ then let $x''$ lies between $0$ and $v_8$. If $s = a + 1$ and $1 \leq t \leq a$ then let $x''$ lies between $0$ and $v_4$. If $0 \leq s \leq a$ and $t = a + 1$ then $x'''$ lies between $0$ and $v_7$ unless $s = 0$, in which case $x''$ lies between $0$ and $v_4$.

Case 3: $x = (r, s, t, -a - 1)$ where $0 \leq r \leq a - 1$ and $0 \leq s, t \leq a + 1$. Let $x' = x - v_2 = (r - a + 1, s - a - 1, t - a - 1, -1)$, which lies between $0$ and $-v_5$ unless $s = 0$ or $t = 0$. If $s = 0$ then $x$ lies between $0$ and $-v_7$. If $t = 0$ then $x$ lies between $0$ and $-v_4$ unless $s = a + 1$, in which case let $x'' = x + v_4 = (r - a, 1, a, 0)$ which lies between $0$ and $v_1$. This completes the cases for the orthant of $v_2$.

Orntant of $v_3$, $k \equiv 0 \pmod{2}$

Now suppose that $x$ lies in the orthant of $v_3$ but not between $0$ and $v_3$. Then the second coordinate of $x$ is equal to $-a$ or $-a - 1$, or the fourth coordinate equals $-a - 1$. We distinguish three cases.

Case 1: $x = (-r, -s, t, -a - 1)$ where $0 \leq r, t \leq a + 1$ and $a \leq s \leq a + 1$. Let $x' = x - v_3 = (a + 1 - r, a - 1 - s, t - a - 1, -1)$, which lies between $0$ and $-v_3$ unless $r = 0$ or $t \leq 1$, in which case let $x'' = x' + v_5 = (1 - r, 2a - 1 - s, t - 2, a + 1)$. If $r = 0$ and $t \geq 2$ then $x''$ which lies between $0$ and $v_8$. If $r = 0$ and $t \leq 1$ then let $x''' = x + v_5 = (-a, a - s, a - 1 + t, 1)$ which lies between $0$ and $v_4$. If $r \geq 1$ and $t \leq 1$ then $x''$ lies between $0$ and $v_7$ unless $r = a + 1$, in which case let $x''' = x'' - v_7 = (-1, -a - s, a + t, 0)$ which lies between $0$ and $v_3$. 


Case 2: \( x = (-r, -s, t, -a - 1) \) where \( 0 \leq r, t \leq a + 1 \) and \( 0 \leq s \leq a - 1 \). Let \( x' = x - v_3 = (a + 1 - r, a - 1 - s, t - a - 1, -1) \), which lies between \( 0 \) and \( -v_4 \) unless \( r = 0 \) or \( t = 0 \). Let \( x'' = x' + v_4 = (1 - r, -1 - s, t - 1, a) \). If \( r = 0 \) and \( t \geq 1 \) then \( x'' \) lies between \( 0 \) and \( -v_6 \). If \( r \geq 1 \) and \( t = 0 \) then \( x'' \) lies between \( 0 \) and \( -v_2 \) unless \( r = a + 1 \), in which case let \( x''' = x'' + v_2 = (-1, a - s, a, 0) \) which lies between \( 0 \) and \( v_1 \). If \( r = 0 \) and \( t = 0 \), then let \( x''' = x'' + v_1 = (-a, a - s, a - 1, 1) \) which lies between \( 0 \) and \( v_5 \).

Case 3: \( x = (-r, -s, t, -u) \) where \( 0 \leq r, t \leq a + 1 \), \( a \leq s \leq a + 1 \) and \( 0 \leq u \leq a \). Let \( x' = x - v_3 = (a + 1 - r, a - 1 - s, t - a - 1, a - u) \), which lies between \( 0 \) and \( -v_1 \) unless \( t = 0 \) or \( u = 0 \). Let \( x'' = x + v_8 = (a - r, a - s, a + t, a + 1 - u) \). If \( t = 0 \) and \( u = 0 \) then \( x'' \) lies between \( 0 \) and \( v_4 \) unless \( r \leq a - 1 \), in which case let \( x''' = x + v_2 = (a - 1 - r, a + 1 - s, a + 1 + t, -a - u) \) which lies between \( 0 \) and \( v_2 \). If \( t = 0 \) and \( u \geq 1 \) then \( x'' \) lies between \( 0 \) and \( -v_6 \) unless \( r = a + 1 \), in which case \( x'' \) lies between \( 0 \) and \( v_4 \). Let \( x''' = x - v_4 = (a - r, a - s, t - a, -a - 1) \). If \( u \geq 1 \), \( u = 0 \) and \( r \leq a \) then \( x''' \) lies between \( 0 \) and \( -v_5 \) unless \( t = a + 1 \). If \( x = a + 1, u = 0 \) and \( r \leq a \) then \( x''' \) lies between \( 0 \) and \( -v_5 \) unless \( r = 0 \) in which case \( x'' \) lies between \( 0 \) and \( v_6 \). If \( t = 1, u = 0 \) and \( r = a + 1 \) then \( x' \) lies between \( 0 \) and \( -v_2 \).

This completes the cases for the orthant of \( v_3 \).

**Orthant of \( v_4 \), \( k \equiv 0 \) (mod 2)**

Now suppose \( x \) lies in the orthant of \( v_4 \) but not between \( 0 \) and \( v_4 \). Then the first coordinate of \( x \) is equal to \(-a - 1\) or the second coordinate is equal to \(-a - 1\), or the third equals \(a + 1\). We distinguish seven cases.

Case 1: \( x = (-a - 1, -a - 1, a + 1, u) \) where \( 0 \leq u \leq a + 1 \). Let \( x' = x - v_4 = (-1, -1, 1, u - a - 1) \), which lies between \( 0 \) and \( v_4 \) if \( u = a + 1 \) and between \( 0 \) and \( v_3 \) if \( u \leq a \).

Case 2: \( x = (-a - 1, -a - 1, t, u) \) where \( 0 \leq t \leq a \) and \( 0 \leq u \leq a + 1 \). Let \( x' = x - v_4 = (-1, -1, t, u - a - 1) \), which lies between \( 0 \) and \( v_8 \).

Case 3: \( x = (-a - 1, -s, a + 1, u) \) where \( 0 \leq s \leq a \) and \( 0 \leq u \leq a + 1 \). Let \( x' = x - v_4 = (-1, -a - s, 1, u - a - 1) \), which lies between \( 0 \) and \( v_1 \) unless \( u \geq a \), in which case \( x'' = x' - v_1 = (a, -s - 1, -a + 1, u - 2) \) which lies between \( 0 \) and \( -v_1 \).

Case 4: \( x = (-r, -a - 1, a + 1, u) \) where \( 0 \leq r \leq a \) and \( 0 \leq u \leq a + 1 \). Let \( x' = x - v_4 = (a - r, -1, 1, u - a - 1) \), which lies between \( 0 \) and \( -v_7 \) unless \( r = 0 \), in which case \( x'' = x' + v_7 = (1, a - 2, -a - 1, u) \) which lies between \( 0 \) and \( -v_3 \) unless \( u = a + 1 \), in which case \( x''' = x'' + v_3 = (a, -1, 0, 1) \) which lies between \( 0 \) and \( v_4 \).

Case 5: \( x = (-r, -s, a + 1, u) \) where \( 0 \leq r, s \leq a \) and \( 0 \leq u \leq a + 1 \). Let \( x' = x - v_4 = (a - r, -s, 1, u - a - 1) \), which lies between \( 0 \) and \( v_2 \) unless \( r = 0 \) or \( u = 0 \) in which case \( x'' = x' = (1 - r, -s - 1, -a, u - 1) \). If \( r \geq 1 \) and \( u = 0 \) then \( x'' \) lies between \( 0 \) and \( -v_8 \) unless \( s = a \), in which case \( x''' = x'' + v_8 = (a + 1 - r, -1, 0, a) \) which lies between \( 0 \) and \( -v_6 \). If \( r = 0 \) then \( x'' \) lies between \( 0 \) and \( v_1 \) unless \( u = 0 \) or \( u = a + 1 \). If \( r = 0 \) and \( u = 0 \), then let \( x''' = x'' + v_5 = (1, a - 1, -1, a + 1) \) which lies between \( 0 \) and \( v_7 \) unless \( s = a \), in which case \( x''' = x'' + v_2 = (0, a, a, 1) \) which lies between \( 0 \) and \( v_8 \). If \( r = 0 \) and \( u = a + 1 \), then let \( x''' = x'' + v_1 = (-a, a - s, 0, 1) \) which lies between \( 0 \) and \( v_5 \).
Case 4: $x = (-r, -a - 1, t, u)$ where $0 \leq r, t \leq a$ and $0 \leq u \leq a + 1$. Let $x' = x - v_4 = (a - r, -1, t - a, u - a - 1)$, which lies between 0 and $-v_5$ unless $t = 0$, in which case let $x'' = x' + v_5 = (-r, a - 1, -1, u + 1)$ which lies between 0 and $v_7$ unless $r = a$ or $u = a + 1$. If $t = 0$ and $r = a$ then let $x''' = x'' - v_7 = (-1, 0, a + 1, u - a)$ which lies between 0 and $v_3$ unless $u = a + 1$. If $t = 0$ and $u = a + 1$ then let $x''' = x'' - v_7 = (a - 1 - r, 0, a + 1, 1)$ which lies between 0 and $-v_6$ unless $r = a$. If $t = 0, r = a$ and $u = a + 1$, then $x''' = (-1, 0, a + 1, 1)$. Let $x''' = x'' - v_4 = (a - 1, a, 1, -a)$ which lies between 0 and $v_2$.

Case 7: $x = (-a - 1, -s, t, u)$ where $0 \leq s, t \leq a$ and $0 \leq u \leq a + 1$. Let $x' = x - v_4 = (-1, a - s, t - a, u - a - 1)$, which lies between 0 and $v_4$ unless $u = 0$, in which case let $x'' = x' - v_6 = (a - 1, -s, t + 1, -1)$ which lies between 0 and $-v_7$ unless $s = a$, in which case let $x''' = x'' + v_7 = (0, -1, t - a - 1, a)$ which lies between 0 and $-v_2$.

This completes the cases for the orthant of $v_4$.

Orthant of $v_3$, $k \equiv 0 \pmod{2}$

Now suppose $x$ lies in the orthant of $v_3$ but not between 0 and $v_3$. Then the first coordinate of $x$ is equal to $-a - 1$ or the second coordinate is equal to $a + 1$, or the third equals $a$ or $a + 1$. We distinguish seven cases.

Case 1: $x = (-a - 1, a + 1, t, u)$ where $a \leq t \leq a + 1$ and $0 \leq u \leq a + 1$. Let $x' = x - v_5 = (-1, 1, t - a + 1, u - a - 2)$, which lies between 0 and $v_1$ unless $u \leq 2$, in which case let $x'' = x' - v_1 = (a, -a, t - 2a + 1, u - 3)$ which lies between 0 and $-v_5$.

Case 2: $x = (-a - 1, a + 1, t, u)$ where $0 \leq t \leq a - 1$ and $0 \leq u \leq a + 1$. Let $x' = x - v_5 = (-1, 1, t - a + 1, u - a - 2)$, which lies between 0 and $v_4$ unless $u \leq 1$, in which case let $x'' = x' - v_6 = (a - 1, -a, t + 2, u - 2)$ which lies between 0 and $-v_7$.

Case 3: $x = (-a - 1, s, t, u)$ where $0 \leq s \leq a$, $a \leq t \leq a + 1$ and $0 \leq u \leq a + 1$. Let $x' = x - v_5 = (-1, s - a, t - a + 1, u - a - 2)$, which lies between 0 and $v_3$ unless $u = 0$ or $u \leq 1$, in which case let $x'' = x' - v_3 = (a, s - 1, t - 2a, u - 2)$. If $s = 0$ and $u \leq 1$ then $x'$ lies between 0 and $-v_5$ unless $t = a$, in which case let $x''' = x'' + v_5 = (0, a - 1, -1, a + u)$ which lies between 0 and $v_7$. If $s = 0$ and $u \geq 2$ then $x''' = x'' + v_1 = (-1, a, t - a, u - a - 1)$ which lies between 0 and $v_1$. If $s \geq 1$ and $u \leq 1$ then $x'$ lies between 0 and $-v_4$.

Case 4: $x = (-r, a + 1, t, u)$ where $0 \leq r \leq a$, $a \leq t \leq a + 1$ and $0 \leq u \leq a + 1$. Let $x' = x - v_5 = (a - r, 1, t - a + 1, u - a - 2)$, which lies between 0 and $v_2$ unless $r = 0$ or $u \geq 2$, in which case let $x'' = x' - v_2 = (1 - r, -a, t - 2a, u - 2)$. If $r = 0$ and $u \geq 2$ then $x''$ lies between 0 and $-v_1$. If $r = 0$ and $u \leq 1$ then $x'$ lies between 0 and $-v_5$ unless $t = a$, in which case let $x''' = x'' + v_5 = (1 - a, 0, -1, a + u)$ which lies between 0 and $v_7$. If $r \geq 1$ and $u \leq 2$ then $x''$ lies between 0 and $-v_2$.

Case 5: $x = (-r, s, t, u)$ where $0 \leq r, s \leq a$, $a \leq t \leq a + 1$ and $0 \leq u \leq a + 1$. Let $x' = x - v_5 = (a - r, s - a, t - a + 1, u - a - 2)$, which lies between 0 and $-v_7$ unless $r = 0$ or $s = 0$ or $u = 0$. If $r = 0$ then $x$ lies between 0 and $v_8$ unless $t = a + 1$, in which case let $x' = x - v_8 = (-a, s - a, 1, u - a - 1)$ which lies between 0 and $v_3$ unless $u = 0$. If $r = 0, t = a + 1$ and $u = 0$ then $x$ lies between 0 and $v_2$. If $r \geq 1$ and $s = 0$ then $x$ lies between 0 and $v_4$ unless $t = a + 1$, in which case let $x' = x - v_4 = (a - r, a, 1, u - a - 1)$ which lies between 0 and $v_2$ unless $u = 0$. If $r \geq 1, s = 0$ and $u = 0$, then let $x'' = x' - v_2 = (1 - r, -1, -a, -1)$ which lies between 0 and $-v_2$.

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and $-v_8$. If $r \geq 1, s \geq 1$ and $u = 0$, then $x$ lies between $0$ and $v_1$ unless $t = a + 1$, in which case let $x' = x - v_1 = (a + 1 - r, s - a - 1, 1, a - 1)$ which lies between $0$ and $-v_7$.

Case 6: $x = (-r, a + 1, t, u)$ where $0 \leq r \leq a$, $0 \leq t \leq a - 1$ and $0 \leq u \leq a + 1$. Let $x' = x - v_5 = (a - r, 1, t - a + 1, u - a - 2)$, which lies between $0$ and $-v_4$ unless $u = 0$ in which case $x$ lies between $0$ and $v_1$.

Case 7: $x = (-a - 1, s, t, u)$ where $0 \leq s \leq a$, $0 \leq t \leq a - 1$ and $0 \leq u \leq a + 1$. Let $x' = x - v_5 = (-1, s - a, t - a + 1, u - a - 2)$, which lies between $0$ and $-v_8$ unless $u = 0$ in which case $x$ lies between $0$ and $v_1$.

This completes the cases for the orthant of $v_5$.

**Orthant of $v_6$, $k \equiv 0 \pmod{2}$**

Now suppose $x$ lies in the orthant of $v_6$ but not between $0$ and $v_6$. Then the first coordinate of $x$ is equal to $-a - 1$ or the second coordinate is equal to $a + 1$, or the fourth equals $-a - 1$. We distinguish seven cases.

Case 1: $x = (-a - 1, a + 1, -t, -a - 1)$ where $0 \leq t \leq a + 1$. Let $x' = x - v_6 = (-1, 1, a + 1 - t, -a - 1)$, which lies between $0$ and $v_1$ unless $t = 0$, in which case let $x'' = x' - v_1 = (a, -a, 1, a - 2)$ which lies between $0$ and $-v_6$.

Case 2: $x = (-a - 1, a + 1, -t, -u)$ where $0 \leq t \leq a + 1$ and $0 \leq u \leq a$. Let $x' = x - v_6 = (-1, 1, a + 1 - t, a - u)$, which lies between $0$ and $v_5$ unless $t \leq 1$, in which case let $x'' = x' - v_5 = (a - 1, 1 - a, 2 - t, -u - 2)$ which lies between $0$ and $-v_7$ unless $u = a$. If $t = 1$ and $u = a$ then $x'$ lies between $0$ and $v_1$. If $t = 0$ and $u = a$ then let $x'' = x' - v_1 = (a, -a, 1, a + 1)$ and $x''' = x'' + v_6 = (0, 0, -a, 1)$ which lies between $0$ and $v_7$.

Case 3: $x = (-a - 1, s, -t, -a - 1)$ where $0 \leq s \leq a$ and $0 \leq t \leq a + 1$. Let $x' = x - v_6 = (-1, s - a, a + 1 - t, -a - 1)$, which lies between $0$ and $v_3$ unless $s = a$ in which case let $x'' = x' - v_3 = (a, -a, 1, a - 1)$ which lies between $0$ and $-v_3$.

Case 4: $x = (-r, a + 1, -t, -a - 1)$ where $0 \leq r \leq a$ and $0 \leq t \leq a + 1$. Let $x' = x - v_6 = (a - r, 1, a + 1 - t, -a - 1)$, which lies between $0$ and $v_2$ unless $r = 0$, in which case let $x'' = x' - v_2 = (1 - a, -t, -a - 1)$ which lies between $0$ and $-v_1$ unless $t = a + 1$ in which case $x' = (a, 1, 0, -1)$ which lies between $0$ and $-v_4$.

Case 5: $x = (-r, s, -t, -a - 1)$ where $0 \leq r, s \leq a$ and $0 \leq t \leq a + 1$. Let $x' = x - v_6 = (a - r, s - a, a + 1 - t, -a - 1)$, which lies between $0$ and $-v_7$ unless $r = 0$ or $s = 0$, in which case let $x'' = x' + v_7 = (1 - r, s - 1, -t - 1, a)$. If $r = 0$ and $s \geq 1$ then $x''$ lies between $0$ and $-v_3$ unless $t = a + 1$, in which case let $x''' = x'' + v_3 = (-a, s - a, -1, 0)$ which lies between $0$ and $v_6$. If $r \geq 1$ and $s = 0$ then $x''$ lies between $0$ and $-v_2$ unless $t = a + 1$, in which case let $x''' = x'' + v_2 = (a - r, a, 1, 0)$ which lies between $0$ and $-v_4$. If $r = 0$ and $s = 0$ then $x$ lies between $0$ and $v_8$ unless $t = a + 1$, in which case let $x''' = x + v_8 = (a, a, -1, 0)$ which lies between $0$ and $-v_4$.

Case 6: $x = (-r, a + 1, -t, -u)$ where $0 \leq r, u \leq a$ and $0 \leq t \leq a + 1$. Let $x' = x - v_6 = (a - r, 1, a + 1 - t, a - u)$, which lies between $0$ and $v_8$ unless $t = 0$, in which case let $x = (-r, a + 1, 0, -u)$ which lies between $0$ and $v_1$ unless $u = a$. If $t = 0$ and $u = a$ then let $x'' = x - v_1 = (a + 1 - r, 0, -a, -1)$ which lies between $0$ and $-v_4$ unless $r = 0$ in
which case \( x = (0, a + 1, 0, -a) \) which lies between 0 and \( v_1 \).

Case 7: \( x = (-a - 1, s, -t, -u) \) where \( 0 \leq s, u \leq a \) and \( 0 \leq t \leq a + 1 \). Let \( x' = x - v_6 = (-1, s - a, a + 1 - t, a - u) \), which lies between 0 and \( v_4 \) unless \( t = 0 \) in which case \( x = (-a - 1, s, 0, -u) \) which lies between 0 and \( v_1 \) unless \( u = a \). If \( t = 0 \) and \( u = a \) then let \( x'' = x - v_1 = (0, s - a - 1, -a, -1) \) which lies between 0 and \( -v_8 \) unless \( s = 0 \) in which case let \( x''' = x'' + v_8 = (a, -1, 0, a) \) which lies between 0 and \( -v_6 \).

This completes the cases for the orthant of \( v_6 \).

Orthant of \( v_7 \), \( k \equiv 0 \pmod{2} \)

Now suppose \( x \) lies in the orthant of \( v_7 \) but not between 0 and \( v_7 \). Then the first coordinate of \( x \) is equal to \(-a\) or \(-a - 1\) or the second equals \( a \) or \( a + 1 \). We distinguish seven cases.

Case 1: \( x = (-r, s, -t, u) \) where \( a \leq r, s \leq a + 1, 2 \leq t \leq a + 1 \) and \( 0 \leq u \leq a + 1 \). Let \( x' = x - v_7 = (a - 1 - r, s - a + 1, a + 2 - t, u - a - 1) \), which lies between 0 and \( v_1 \) unless \( u \leq 1 \), in which case let \( x'' = x - v_6 = (a - r, s - a, a + 1 - t, a + u) \) which lies between 0 and \( v_5 \).

Case 2: \( x = (-a, a, -t, u) \) where \( 0 \leq t \leq 1 \) and \( 0 \leq u \leq a + 1 \). If \( u = 0 \) then \( x \) lies between 0 and \( v_6 \). If \( u \geq 1 \) then let \( x' = x - v_7 - v_3 = (a, a, 1 - t, u - 1) \), which lies between 0 and \( v_8 \).

Case 3: \( x = (-a - 1, a, -t, u) \) where \( 0 \leq t \leq 1 \) and \( 0 \leq u \leq a + 1 \). Let \( x' = x - v_1 = (0, -1, -a - t, a + 1 + u) \). If \( u \leq 1 \) then \( x' \) lies between 0 and \( -v_2 \). If \( u \geq 2 \) then let \( x'' = x' + v_2 = (a - 1, a - 1 - t, u - 1) \), which lies between 0 and \( v_8 \).

Case 4: \( x = (-a, a + 1, -t, u) \) where \( 0 \leq t \leq 1 \) and \( 0 \leq u \leq a + 1 \). Let \( x' = x - v_1 = (1, 0, -a - t, a - 1 + u) \). If \( u \leq 1 \) then \( x' \) lies between 0 and \( -v_3 \). If \( u \geq 2 \) then let \( x'' = x' + v_3 = (-a, -a + 1, 1 - t, u - 1) \), which lies between 0 and \( v_4 \).

Case 5: \( x = (-a - 1, a + 1, -t, u) \) where \( 0 \leq t \leq 1 \) and \( 0 \leq u \leq a + 1 \). Let \( x' = x - v_1 = (0, 0, -a - t, a + 1 + u) \). If \( u \leq 1 \) then \( x' \) lies between 0 and \( v_7 \). If \( u \geq 2 \) then let \( x'' = x' - v_7 = (a - 1, -a + 1, 2 - t, u - 2) \), which lies between 0 and \( -v_6 \).

Case 6: \( x = (-r, s, -t, u) \) where \( 0 \leq r \leq a - 1, a \leq s \leq a + 1 \) and \( 0 \leq t, u \leq a + 1 \). Let \( x' = x - v_7 = (a - 1 - r, s - a + 1, a + 2 - t, u - a - 1) \), which lies between 0 and \( v_2 \) unless \( t = 0 \) or \( u = 0 \), in which case let \( x'' = x' - v_2 = (-r, s - 2a, 1 - t, u - 1) \). If \( t = 0 \) and \( u \geq 1 \) then \( x'' \) lies between 0 and \( v_4 \). If \( t \geq 1 \) and \( u = 0 \) then \( x'' = (r, s - 2a, 1 - t, -1) \) which lies between 0 and \( -v_8 \).

Case 7: \( x = (-r, s, -t, u) \) where \( a \leq r \leq a + 1, 0 \leq s \leq a - 1 \) and \( 0 \leq t, u \leq a + 1 \). Let \( x' = x - v_7 = (a - 1 - r, s - a + 1, a + 2 - t, u - a - 1) \), which lies between 0 and \( v_3 \) unless \( t = 0 \) and \( u = 0 \) in which case let \( x'' = x' - v_3 = (2a - r, s, 1 - t, u - 1) \). If \( t = 0 \) and \( u = 0 \) then \( x \) lies between 0 and \( v_1 \). If \( t = 0 \) and \( u \geq 1 \) then \( x'' \) lies between 0 and \( v_8 \). If \( t \geq 1 \) and \( u = 0 \) then \( x'' \) lies between 0 and \( -v_4 \).

This completes the cases for the orthant of \( v_7 \).
Orthant of \( v_8, k \equiv 0 \pmod{2} \)

Finally suppose \( x \) lies in the orthant of \( v_8 \) but not between 0 and \( v_8 \). Then at least one of the first three coordinate of \( x \) is equal to \( a + 1 \). We distinguish seven cases.

Case 1: \( x = (a + 1, a + 1, a + 1, u) \) where \( 0 \leq u \leq a + 1 \). Let \( x' = x - v_8 = (1, 1, 1, u - a - 1) \), which lies between 0 and \( v_2 \) unless \( u = 0 \), in which case let \( x'' = x' - v_2 = (-a + 2, -a, -a, -1) \) which lies between 0 and \( -v_8 \).

Case 2: \( x = (a + 1, a + 1, t, u) \) where \( 0 \leq t \leq a \) and \( 0 \leq u \leq a + 1 \). Let \( x' = x - v_8 = (1, t - a, u - a - 1) \), which lies between 0 and \( v_4 \).

Case 3: \( x = (a + 1, s, a + 1, u) \) where \( 0 \leq s \leq a \) and \( 0 \leq u \leq a + 1 \). Let \( x' = x - v_8 = (1, s - a, 1, u - a - 1) \), which lies between 0 and \( -v_7 \) unless \( s = 0 \), in which case let \( x'' = x' + v_7 = (-a + 2, -a - 1, u) \) which lies between 0 and \( -v_2 \) unless \( u = a + 1 \). If \( s = 0 \) and \( u = a + 1 \) then \( x' = (1, -a, 1, 0) \) which lies between 0 and \( -v_6 \).

Case 4: \( x = (r, a + 1, a + 1, u) \) where \( 0 \leq r \leq a \) and \( 0 \leq u \leq a + 1 \). Let \( x' = x - v_8 = (r - a, 1, u - a - 1) \), which lies between 0 and \( v_1 \) unless \( u \leq 1 \), in which case let \( x'' = x' - v_1 = (r + 1, -a - 1, u - 2) \) which lies between 0 and \( -v_5 \) unless \( r = a \). If \( r = a \) and \( u \leq 1 \) then let \( x'' = x' + v_5 = (1, 0, u + a) \) which lies between 0 and \( v_8 \).

Case 5: \( x = (a + 1, s, t, u) \) where \( 0 \leq s, t \leq a \) and \( 0 \leq u \leq a + 1 \). Let \( x' = x - v_8 = (1, s - a, 1, t - a - 1) \), which lies between 0 and \( v_5 \) unless \( t = 0 \), in which case let \( x'' = x' + v_5 = (-a + 1, s - 1, u + 1) \) which lies between 0 and \( v_7 \) unless \( s = a \) or \( u = a + 1 \). If \( s = a \) and \( t = 0 \) then \( x' = (1, 0, u - a - 1) \) which lies between 0 and \( -v_4 \). If \( t = 0 \) and \( u = a + 1 \) then \( x' = (1, s - a, u - a, 0) \) which lies between 0 and \( -v_1 \).

Case 6: \( x = (r, a + 1, t, u) \) where \( 0 \leq r \leq a \) and \( 0 \leq u \leq a + 1 \). Let \( x' = x - v_8 = (r - a, 1, t - a - 1) \), which lies between 0 and \( v_6 \) unless \( u = 0 \), in which case \( x \) lies between 0 and \( v_2 \) unless \( r = a \). If \( r = a \) and \( u = 0 \) then let \( x'' = x' - v_2 = (1, 0, t - a - 1, a) \) which lies between 0 and \( -v_3 \).

Case 7: \( x = (r, s, a + 1, u) \) where \( 0 \leq r, s \leq a \) and \( 0 \leq u \leq a + 1 \). Let \( x' = x - v_8 = (r - a, s - a, 1, u - a - 1) \), which lies between 0 and \( v_3 \) unless \( s = 0 \) or \( u = 0 \), in which case let \( x'' = x' - v_3 = (r + 1, s - 1, -a - 1) \). If \( s = 0 \) and \( u = 0 \) then \( x \) lies between 0 and \( -v_6 \). If \( s = 0 \) and \( u \geq 1 \) then \( x'' \) lies between 0 and \( v_6 \), in which case \( x' \) lies between 0 and \( v_4 \). If \( s \geq 1 \) and \( u = 0 \) then \( x \) lies between 0 and \( v_2 \) unless \( r = a \), in which case \( x' \) lies between 0 and \( -v_7 \).

This completes the cases for the orthant of \( v_8 \).

This also completes the proof of the theorem for any \( k \equiv 0 \pmod{2} \).

Now we consider the eight orthants \( v_1, \ldots, v_8 \) in turn for the case \( k \equiv 1 \pmod{2} \).

Orthant of \( v_1, k \equiv 1 \pmod{2} \)

First suppose that \( x \) lies within the orthant of \( v_1 \), but not between 0 and \( v_1 \). Then the first coordinate of \( x \) is equal to \(-a\) or \(-a - 1\), or the third coordinate equals \(-a\) or \(-a - 1\), or the fourth equals \(a + 1\). We distinguish seven cases.

Case 1: \( x = (-r, s, t, a + 1) \) where \( a \leq r, t \leq a + 1 \) and \( 0 \leq s \leq a + 1 \). Let \( x' = x - v_1 = \ldots \)
(a - 1 - r, s - a - 1, a - 1 - t, 1), which lies between 0 and v₈ unless s ≤ 1 in which case let $x'' = x' - v₈ = (2a - 2 - r, s - 2, 2a - 1 - t, -a)$ which lies between 0 and $-v₁$.

Case 2: $x = (r, -s, -t, u)$ where $a ≤ r, t ≤ a + 1$ and $0 ≤ s ≤ a + 1$ and $0 ≤ u ≤ a$. Let $x' = x - v₁ = (a - 1 - r, s - a - 1, a - 1 - t, u - a)$, which lies between 0 and $v₅$ unless $u = 0$ or $u ≤ 1$, in which case let $x'' = x' - v₅ = (2a - 1 - r, s - 1, 2a - t, u - 2)$. If $s = 0$ and $u ≤ 1$ then $x''$ lies between 0 and $-v₁$, unless $t = a$, in which case let $x''' = x'' + v₁ = (a - r, a - 1, u + a - 2)$ which lies between 0 and $v₄$. If $s = 0$ and $u ≥ 2$ then $x''$ lies between 0 and $-v₇$. If $s ≥ 1$ and $u ≤ 1$ then $x''$ lies between 0 and $-v₈$ unless $s = a + 1$, in which case let $x''' = x'' + v₈ = (a - r, 1, a - t, a + u - 1)$ which lies between 0 and $v₁$.

Case 3: $x = (-r, s, -t, a + 1)$ where $a ≤ r ≤ a + 1, 0 ≤ s ≤ a + 1$ and $0 ≤ t ≤ a - 1$. Let $x' = x - v₁ = (a - 1 - r, s - a - 1, a - 1 - t, 1)$, which lies between 0 and $-v₃$ unless $s = 0$, in which case let $x'' = x' + v₂ = (2a - 1 - r, -1, -t, -a + 1)$ which lies between 0 and $-v₂$.

Case 4: $x = (-r, s, -t, a + 1)$ where $0 ≤ r ≤ a - 1, 0 ≤ s ≤ a + 1$ and $a ≤ t ≤ a + 1$. Let $x' = x - v₁ = (a - 1 - r, s - a - 1, a - 1 - t, 1)$, which lies between 0 and $-v₃$ unless $s ≤ 1$, in which case let $x'' = x' + v₃ = (-2 - r, s - 2, 2a - 2 - t, -a + 1)$ which lies between 0 and $-v₂$.

Case 5: $x = (-r, s, -t, a + 1)$ where $0 ≤ r, t ≤ a - 1$ and $0 ≤ s ≤ a + 1$. Let $x' = x - v₁ = (a - 1 - r, s - a - 1, a - 1 - t, 1)$, which lies between 0 and $-v₇$ unless $s = 0$, in which case let $x'' = x' + v₇ = (-r - 1, -1, -t - 1, -a + 2)$ which lies between 0 and $v₃$.

Case 6: $x = (-r, s, -t, u)$ where $0 ≤ r ≤ a - 1, 0 ≤ s ≤ a + 1, a ≤ t ≤ a + 1$ and $0 ≤ u ≤ a$. Let $x' = x - v₁ = (a - 1 - r, s - a - 1, a - 1 - t, u - a)$, which lies between 0 and $-v₄$ unless $s = 0$ or $u = 0$, in which case let $x'' = x' + v₄ = (-r - 1, s - 1, 2a - 1 - t, u - 1)$. If $s = 0$ and $u = 0$ then let $x''' = x'' + v₂ = (a - r, a, a - 1 - t, a - 2)$ which lies between 0 and $-v₁$. If $s = 0$ and $u ≥ 1$ then $x''$ lies between 0 and $v₆$. If $s ≥ 1$ and $u = 0$ then $x''$ lies between 0 and $v₃$ unless $s = a + 1$, in which case $x'$ lies between 0 and $v₆$.

Case 7: $x = (-r, s, -t, u)$ where $a ≤ r ≤ a + 1, 0 ≤ s ≤ a + 1, 0 ≤ t ≤ a - 1$ and $0 ≤ u ≤ a$. Let $x' = x - v₁ = (a - 1 - r, s - a - 1, a - 1 - t, u - a)$, which lies between 0 and $-v₂$ unless $t = 0$ or $u = 0$, in which case let $x'' = x' + v₂ = (2a - r, s, -t + 1, u - 1)$. If $t = 0$ and $u = 0$ then let $x''' = x'' + v₃ = (a + 1 - r, s - a + 1, a + 1 - t, u + 1)$ which lies between 0 and $-v₃$ unless $a ≤ s ≤ a + 1$, in which case let $x''' = x - v₇ = (a - r, s - a - 1, a - 1 - t, a - 2)$ which lies between 0 and $v₄$. If $t = 0$ and $u ≥ 1$ then $x''$ lies between 0 and $-v₅$ unless $s = a + 1$ or $u = a$ in which case let $x'' = x'' + v₅ = (a - r, s - a, -a, -a + u + 1)$. If $s = a + 1$ then $x'$ lies between 0 and $v₃$. If $1 ≤ s ≤ a$ and $u = a$ then $x''$ lies between 0 and $v₈$. If $s = 0$ and $u = a$ then $x''$ lies between 0 and $-v₇$. If $t ≥ 1$ and $u = 0$ then $x''$ lies between 0 and $v₆$ unless $s = a + 1$, in which case $x'$ lies between 0 and $v₃$.

This completes the cases for the orthant of $v₁$.

**Orthant of $v₂$, $k ≡ 1 \pmod{2}$**

Now suppose that $x$ lies in the orthant of $v₂$ but not between 0 and $v₂$. Then the third coordinate of $x$ is equal to $-a + 1, -a$ or $-a - 1$, or the fourth coordinate equals $a$ or $a + 1$. 

R. R. Lewis

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We distinguish three cases.

Case 1: \( x = (r, s, -t, u) \) where \( 0 \leq r, s \leq a + 1, a - 1 \leq t \leq a + 1 \) and \( a \leq u \leq a + 1 \). Let \( x' = x - v_2 = (r - a - 1, s - a - 1, a - 2 - t, u - a + 1) \), which lies between 0 and \( v_8 \) unless \( r \leq 1 \) or \( s \leq 1 \), in which case let \( x'' = x' - v_8 = (r - 2s, s - 2a - 2 - t, u - 2a) \). If \( r \leq 1 \) and \( s \geq 2 \) then \( x'' \) lies between 0 and \( v_3 \). If \( r \geq 2 \) and \( s \leq 1 \) then \( x'' \) lies between 0 and \( -v_1 \). If \( r \leq 1 \) and \( s \leq 1 \) then \( x'' \) lies between 0 and \( -v_2 \) unless \( t = a - 1 \) or \( u = a \). Let \( x''' = x'' + v_2 = (r + a - 1, s + a - 1, a - t, u - a - 1) \). If \( r \leq 1 \) and \( s \leq 1 \) and \( u = a \) then \( x''' \) lies between 0 and \( v_6 \) unless \( t = a - 1 \), in which case let \( x''' = x'' - v_6 = (r - 1, s - 1, 2a - 1 - t, u - 1) \) which lies between 0 and \( v_4 \) if \( s = 1 \), and between 0 and \( -v_7 \) if \( r = 1 \). If \( r = 0 \) and \( s = 0 \) then \( x \) lies between 0 and \( v_1 \). If \( r \leq 1 \) and \( s \leq 1 \) and \( t = a - 1 \) and \( u = a + 1 \) then \( x'' \) lies between 0 and \( -v_3 \).

Case 2: \( x = (r, s, -t, u) \) where \( 0 \leq r, s \leq a + 1, 0 \leq t \leq a - 2 \) and \( a \leq u \leq a + 1 \). Let \( x' = x - v_2 = (r - a - 1, s - a - 1, a - 2 - t, u - a + 1) \), which lies between 0 and \( -v_6 \) unless \( r = 0 \) or \( s = 0 \). Let \( x'' = x' + v_6 = (r - 1, s - 1, -t - 1, u - 2a + 1) \). If \( r = 0 \) and \( s = 0 \) then \( x'' \) lies between 0 and \( v_3 \) unless \( u = a \), in which case let \( x''' = x'' + v_5 = (r + a - 1, s + a - 1, a - t, u - a - 1) \) which lies between 0 and \( -v_8 \). If \( r = 0 \) and \( s \geq 1 \) then \( x''' \) lies between 0 and \( v_7 \). If \( r \geq 1 \) and \( s = 0 \) then \( x''' \) lies between 0 and \( -v_5 \).

Case 3: \( x = (r, s, -t, u) \) where \( 0 \leq r, s \leq a + 1, a - 1 \leq t \leq a + 1 \) and \( 0 \leq u \leq a - 1 \). Let \( x' = x - v_2 = (r - a - 1, s - a - 1, a - 2 - t, u - a + 1) \), which lies between 0 and \( v_5 \) unless \( r = 0 \) or \( s = 0 \) or \( u = 0 \), in which case let \( x'' = x' - v_5 = (r - 1, s - 1, 2a - 1 - t, u - 1) \). If \( r = 0 \) and \( s = 0 \) and \( u = 0 \) then \( x'' \) lies between 0 and \( -v_2 \) unless \(-a - 1 \leq t \leq a \), in which case let \( x''' = x'' + v_2 = (a + r, a + s, a + 1 - t, a - 2 + u) \) which lies between 0 and \( -v_5 \). If \( r = 0 \) and \( s \geq 1 \) and \( u = 0 \) then \( x'' \) lies between 0 and \( -v_6 \) unless \( t = a - 1 \), in which case let \( x''' = x'' + v_6 = (r + a - 1, s + a - 1, a - t, u - a - 1) \) which lies between 0 and \( -v_8 \). If \( r = 0 \) and \( s \geq 1 \) and \( u = 0 \) then \( x'' \) lies between 0 and \( v_3 \) unless \( s = a + 1 \) or \( t = a - 1 \), in which case let \( x''' = x' - v_3 = (r + a, s - a, a - t, u + a - 1) \). If \( s = a + 1 \) then \( x'' \) lies between 0 and \( v_2 \) unless \( t = a - 1 \). If \( t = a - 1 \) then \( x'' \) lies between 0 and \( -v_7 \) unless \( s = a + 1 \), in which case let \( x'' = x'' + v_5 = (r, s - 2a, -t - 1, u + 1) \) which lies between 0 and \( v_8 \). If \( r = 0 \) and \( s \geq 1 \) and \( u = 0 \) then \( x'' \) lies between 0 and \( v_4 \). If \( r \geq 1 \) and \( s = 0 \) and \( u = 0 \) then \( x'' \) lies between 0 and \( -v_1 \) unless \( r = a + 1 \) or \( t = a - 1 \), in which case let \( x''' = x'' + v_1 = (r - a, s - a, a - t, a + u - 1) \). If \( r = a + 1 \) and \( t \geq a \) then \( x'' \) lies between 0 and \( v_2 \). If \( r = a + 1 \) and \( t = a - 1 \) then \( x''' = x'' + v_5 = (r - 2a, s, -t - 1, u + 1) \) which lies between 0 and \( v_8 \). If \( 1 \leq r \leq a \) and \( t = a - 1 \) then \( x'' \) lies between 0 and \( v_3 \). If \( 1 \leq r \leq a 

Orphant of \( v_3 \), \( k \equiv 1 \pmod{2} \)

Now suppose that \( x \) lies in the orphant of \( v_3 \) but not between 0 and \( v_3 \). Then the second coordinate of \( x \) is equal to \( a \) or \( a + 1 \), or the
fourth equals \(-a - 1\). We distinguish seven cases.

Case 1: \(x = (-r, s, t, -a - 1)\) where \(0 \leq r \leq a + 1\) and \(a \leq s, t \leq a + 1\). Let \(x' = x - v_3 = (a + 1 - r, s - a + 1, t - a + 1, -1)\), which lies between \(0\) and \(-v_8\) unless \(r \leq 1\), in which case let \(x'' = x' + v_8 = (2 - r, s - 2a + 2, t - 2a + 1, a)\) which lies between \(0\) and \(-v_3\).

Case 2: \(x = (-r, s, t, -u)\) where \(0 \leq r \leq a + 1\) and \(a \leq s, t \leq a + 1\) and \(0 \leq u \leq a\). Let \(x' = x - v_3 = (a + 1 - r, s - a + 1, t - a + 1, a - u)\), which lies between \(0\) and \(-v_5\) unless \(r = 0\) or \(u = 1\), in which case let \(x'' = x' + v_5 = (1 - r, s - 2a + 1, t - 2a, 2 - u)\). If \(r = 0\) and \(u \leq 1\) then \(x''\) lies between \(0\) and \(-v_3\) unless \(t = a\), in which case let \(x''' = x'' + v_3 = (-a - r, s - a, t - a - 1, 2 - a - u)\) which lies between \(0\) and \(v_7\). If \(r = 0\) and \(u \geq 2\) then \(x''\) lies between \(0\) and \(-v_4\). If \(r \geq 1\) and \(u \leq 1\) then \(x''\) lies between \(0\) and \(v_8\) unless \(r = a + 1\), in which case let \(x''' = x'' - v_8 = (a - r, s - a, t - a, a - 1 - u)\) which lies between \(0\) and \(v_3\).

Case 3: \(x = (-r, s, t, -a - 1)\) where \(0 \leq r \leq a + 1\) and \(a \leq s \leq a + 1\) and \(0 \leq t \leq a - 1\). Let \(x' = x - v_3 = (a + 1 - r, s - a + 1, t - a + 1, -1)\), which lies between \(0\) and \(v_6\) unless \(r = 0\), in which case let \(x'' = x' - v_6 = (1 - r, s - 2a + 1, t, a - 1)\) which lies between \(0\) and \(-v_7\).

Case 4: \(x = (-r, s, t, -a - 1)\) where \(0 \leq r \leq a + 1\) and \(0 \leq s \leq a - 1\) and \(a \leq t \leq a + 1\). Let \(x' = x - v_3 = (a + 1 - r, s - a + 1, t - a + 1, -1)\), which lies between \(0\) and \(-v_1\) unless \(r \leq 1\), in which case let \(x'' = x' + v_4 = (2 - r, s + 2, t - 2a + 2, a - 1)\) which lies between \(0\) and \(v_2\).

Case 5: \(x = (-r, s, t, -a - 1)\) where \(0 \leq r \leq a + 1\) and \(0 \leq s, t \leq a - 1\). Let \(x' = x - v_3 = (a + 1 - r, s - a + 1, t - a + 1, -1)\), which lies between \(0\) and \(-v_4\) unless \(r = 0\), in which case let \(x'' = x' + v_4 = (1 - r, s + 1, t + 1, a - 2)\) which lies between \(0\) and \(-v_5\).

Case 6: \(x = (-r, s, t, -u)\) where \(0 \leq r \leq a + 1\), \(0 \leq s \leq a - 1\), \(a \leq t \leq a + 1\) and \(0 \leq u \leq a\). Let \(x' = x - v_3 = (a + 1 - r, s - a + 1, t - a + 1, a - u)\), which lies between \(0\) and \(-v_7\) unless \(r = 0\) or \(u = 0\), in which case let \(x'' = x' + v_7 = (1 - r, s + 1, t - 2a + 1, 1 - u)\). If \(r = 0\) and \(u = 0\) then \(x''\) lies between \(0\) and \(v_2\) unless \(t = a\), in which case let \(x''' = x'' - v_2 = (-a - r, s - a, t - a - 1, 2 - a - u)\) which lies between \(0\) and \(v_5\). If \(r = 0\) and \(u \geq 1\) then \(x''\) lies between \(0\) and \(v_6\). If \(r \geq 1\) and \(u = 0\) then \(x''\) lies between \(0\) and \(v_1\) unless \(r = a + 1\), in which case let \(x''' = x'' - v_1 = (a - r, s - a, t - a, 1 - a - u)\) which lies between \(0\) and \(-v_2\).

Case 7: \(x = (-r, s, t, -u)\) where \(0 \leq r \leq a + 1\), \(a \leq s \leq a + 1\), \(0 \leq t \leq a - 1\) and \(0 \leq u \leq a\). Let \(x' = x - v_3 = (a + 1 - r, s - a + 1, t - a + 1, a - u)\), which lies between \(0\) and \(v_2\) unless \(t = 0\) or \(u = 0\), in which case let \(x'' = x' - v_2 = (-r, s - 2a, t - 1, 1 - u)\). If \(t = 0\) and \(u = 0\) then \(x''\) lies between \(0\) and \(v_8\) unless \(r \leq 1\), in which case let \(x''' = x'' - v_8 = (a - 1 - r, s - a - 1, t + a - 1, -a - u)\) which lies between \(0\) and \(-v_1\). If \(t = 0\) and \(u \geq 1\) then \(x''\) lies between \(0\) and \(v_5\) unless \(r = 0\) or \(u = a\), in which case let \(x''' = x'' - v_5 = (a - r, s - a, t + a + 1, a - u - 1)\). If \(r = 0\), \(t = 0\) and \(u = a\) then let \(x'' = x''' + v_8 = (1 - r, s - 2a + 1, t, 2a - u)\) which lies between \(0\) and \(-v_3\). If \(r = 0\), \(t = 0\) and \(1 \leq u \leq a - 1\) then \(x'''\) lies between \(0\) and \(-v_5\). If \(1 \leq r \leq a\), \(t = 0\) and \(u = a\) then \(x''\) lies between \(0\) and \(-v_8\). If \(r = a + 1\), \(t = 0\) and \(u = a\) then \(x''\) lies between \(0\) and \(-v_6\). If \(t \geq 1\) and \(u = 0\) then \(x''\) lies between \(0\) and \(-v_6\).

This completes the cases for the orthant of \(v_3\).
Orthant of $v_4$, $k \equiv 1 \text{ (mod 2)}$

Now suppose $x$ lies in the orthant of $v_4$ but not between 0 and $v_4$. Then the first coordinate of $x$ is equal to $-a-1$ or the second coordinate is equal to $a+1$, or the third equals $a+1$ or the fourth equals $a$ or $a+1$. We distinguish fifteen cases.

Case 1: $x = (-a-1, a+1, a+1, u)$ where $a \leq u \leq a+1$. Let $x' = x - v_4 = (-1, 1, 1, u-a+1)$, which lies between 0 and $v_4$.

Case 2: $x = (-a-1, a+1, a+1, u)$ where $0 \leq u \leq a-1$. Let $x' = x - v_4 = (-1, 1, 1, u-a+1)$, which lies between 0 and $v_4$.

Case 3: $x = (-a-1, a+1, t, u)$ where $0 \leq t \leq a$ and $a \leq u \leq a+1$. Let $x' = x - v_4 = (-1, 1, t-a, u-a+1)$, which lies between 0 and $v_4$.

Case 4: $x = (-a-1, s, a+1, u)$ where $0 \leq s \leq a$ and $a \leq u \leq a+1$. Let $x' = x - v_4 = (-1, s-a, 1, u-a+1)$, which lies between 0 and $-v_2$.

Case 5: $x = (r, a+1, a+1, u)$ where $0 \leq r \leq a$ and $a \leq u \leq a+1$. Let $x' = x - v_4 = (a-r, 1, 1, u-a+1)$, which lies between 0 and $-v_5$.

Case 6: $x = (-r, s, a+1, u)$ where $0 \leq r, s \leq a$ and $a \leq u \leq a+1$. Let $x' = x - v_4 = (a-r, 1, s-a, u-a+1)$, which lies between 0 and $-v_7$.

Case 7: $x = (-r, a+1, t, u)$ where $0 \leq r, t \leq a$ and $a \leq u \leq a+1$. Let $x' = x - v_4 = (-a-1, t-a, u-a+1)$, which lies between 0 and $v_2$ unless $t \leq 1$, in which case let $x'' = x' - v_2 = (-r-1, -a, t-2, u)$ and $x''' = x'' - v_8 = (a-r-2, -1, a+t-2, u-a+1)$. $x''$ lies between 0 and $-v_1$ unless $a-1 \leq r \leq a$, in which case let $x''' = x'' + v_2 = 2(a-r-1, a, t, u-2a)$, which lies between 0 and $-v_5$ unless $u = a+1$. If $a-1 \leq r \leq a$, $t \leq 1$ and $u = a+1$, then let $x^v = x''' + v_5 = (a-r-1, 0, t-a-1, u-a)$ and $x^v = x^v - v_5 = (2a-r-2, a-1, t-1, u-2a-1)$ which lies between 0 and $v_6$.

Case 8: $x = (-a-1, s, t, u)$ where $0 \leq s, t \leq a$ and $a \leq u \leq a+1$. Let $x' = x - v_4 = (-1, s-a, t-a, u-a+1)$, which lies between 0 and $v_8$ unless $s = 0$, in which case let $x'' = x' - v_8 = (-a-2, s-1, t, u-2a)$ which lies between 0 and $-v_1$ unless $t = a$. If $s = 0$ and $t = a$ then let $x''' = x'' + v_1 = (-1, s+a, t-1, u-a)$, which lies between 0 and $v_4$.

Case 9: $x = (-r, a+1, a+1, u)$ where $0 \leq r \leq a$ and $0 \leq u \leq a-1$. Let $x' = x - v_4 = (a-r, 1, 1, u-a+1)$, which lies between 0 and $-v_8$ unless $r = 0$, in which case let $x'' = x' + v_8 = (1-r, 2-a, 1-a, u+2)$ which lies between 0 and $v_8$.

Case 10: $x = (-a-1, s, a+1, u)$ where $0 \leq s \leq a$ and $0 \leq u \leq a-1$. Let $x' = x - v_4 = (-1, s-a, 1, u-a+1)$, which lies between 0 and $-v_2$.

Case 11: $x = (-a-1, a+1, t, u)$ where $0 \leq t \leq a$ and $0 \leq u \leq a-1$. Let $x' = x - v_4 = (-1, 1, t-a, u-a+1)$, which lies between 0 and $v_7$.

Case 12: $x = (-r, s, t, u)$ where $0 \leq r, s, t \leq a$ and $a \leq u \leq a+1$. Let $x' = x - v_4 = (a-r, s-a, t-a, u-a+1)$, which lies between 0 and $-v_5$ unless $s = 0$ or $t = 0$. If $s = 0$ and $t = 0$ then let $x'' = x' + v_3 = (-1-r, s-1, t-1, u-2a+1)$ which lies between 0 and $v_5$ unless $r = a$ or $u = a$ in which case let $x''' = x'' - v_5 = (a-1-r, s+a-1, t+a, u-a-1)$.

If $r = a$ and $u = a$, then $x$ lies between 0 and $-v_6$. If $r = a$ and $u = a+1$, then $x''$
lies between 0 and $v_3$. If $r \leq a - 1$ and $u = a$, then $x'''$ lies between 0 and $-v_8$. If $s = 0$ and $1 \leq t \leq a$ then let $x''$ lies between 0 and $-v_2$ unless $t = a$, in which case let $x'' = x'' + v_2 = (a - r, a + s, t - a + 1, u - a)$ which lies between 0 and $-v_5$. If $1 \leq s \leq a$ and $t = 0$ then $x''$ lies between 0 and $v_7$ unless $r = a$, in which case let $x'' = x'' - v_7 = (a - r, s - a + 1, t + a - 1, u - a)$ which lies between 0 and $-v_6$.

Case 13: $x = (-r, s, a + 1, u)$ where $0 \leq r, s \leq a$ and $0 \leq u \leq a - 1$. Let $x' = x - v_4 = (a - r, s - a, 1, u - a + 1)$, which lies between 0 and $-v_6$ unless $r = 0$, in which case let $x'' = x' + v_1 = (1 - r, s + 1, a + 2, u + 1)$ which lies between 0 and $v_2$ unless $u = a - 1$. If $r = 0$ and $u = a - 1$ then let $x''' = x'' - v_2 = (-a - r, s - a, 0, u - a + 2)$ which lies between 0 and $v_5$.

Case 14: $x = (-r, a + 1, t, u)$ where $0 \leq r, t \leq a$ and $0 \leq u \leq a - 1$. Let $x' = x - v_4 = (a - r, 1, t - a, u - a + 1)$, which lies between 0 and $v_6$ unless $t = 0$, in which case let $x'' = x' - v_6 = (-r, 1 - a, t - 1, u + 1)$ which lies between 0 and $v_8$ unless $r = a$. If $r = a$ and $t = 0$ then let $x''' = x'' - v_8 = (a - 1 - r, 0, t + a - 1, u - a)$ which lies between 0 and $v_3$.

Case 15: $x = (-a - 1, s, t, u)$ where $0 \leq s, t \leq a$ and $0 \leq u \leq a - 1$. Let $x' = x - v_4 = (-1, -a, t - a, u - a + 1)$, which lies between 0 and $v_5$ unless $t = 0$ or $u = 0$ in which case let $x'' = x' - v_5 = (a - 1, s, t + 1, u - 1)$. If $t = 0$ and $u = 0$, then $x''$ lies between 0 and $-v_8$ unless $s = a$, in which case let $x''' = x'' + v_8 = (0, s - a + 1, t - a + 1, u + 1)$ which lies between 0 and $v_1$. If $t = 0$ and $1 \leq u \leq a - 1$, then $x''$ lies between 0 and $-v_5$. If $1 \leq t \leq a - 1$ and $u = 0$, then $x''$ lies between 0 and $-v_8$ unless $s = a$, in which case $x'''$ lies between 0 and $v_1$. If $t = a$ and $u = 0$ then $x'''$ lies between 0 and $-v_6$ unless $s = a$, in which case let $x''' = x'' - v_4 = (a, s - 2a + 1, t - 2a + 1, u + 1)$ which lies between 0 and $-v_3$.

This completes the cases for the orthant of $v_4$.

Orthant of $v_5$, $k \equiv 1 \pmod{2}$

Now suppose $x$ lies in the orthant of $v_5$ but not between 0 and $v_5$. Then the first coordinate of $x$ is equal to $-a - 1$ or the second coordinate is equal to $-a - 1$, or the fourth equals $-a + 1$, $-a + 1$. We distinguish seven cases.

Case 1: $x = (-a - 1, -a - 1, -t, -u)$ where $0 \leq t \leq a + 1$ and $a - 1 \leq u \leq a + 1$. Let $x' = x - v_5 = (-1, -1, a + 1 - t, a - 2 - u)$, which lies between 0 and $-v_2$ unless $t \leq 2$, in which case let $x'' = x' + v_2 = (a, a, 3 - t, 2a - 3 - u)$ which lies between 0 and $-v_5$.

Case 2: $x = (-a - 1, -a - 1, -t, -u)$ where $0 \leq t \leq a + 1$ and $0 \leq u \leq a - 2$. Let $x' = x - v_5 = (-1, -1, a + 1 - t, a - 2 - u)$, which lies between 0 and $-v_6$ unless $t \leq 1$, in which case let $x'' = x' + v_6 = (a, a, 1 - 2 - t, 2a - 2 - u)$ which lies between 0 and $-v_8$.

Case 3: $x = (-a - 1, -s, -t, -u)$ where $0 \leq s \leq a$, $0 \leq t \leq a + 1$ and $a - 1 \leq u \leq a + 1$. Let $x' = x - v_5 = (-1, a - s, a + 1 - t, a - 2 - u)$, which lies between 0 and $v_3$ unless $s = 0$ or $t \leq 1$, in which case let $x'' = x' - v_3 = (a, 1 - s, 2 - t, 2a - 2 - u)$. If $s = 0$ and $t \leq 1$ then $x''$ lies between 0 and $-v_5$ unless $u = a - 1$, in which case let $x''' = x'' + v_5 = (0, 1 - a - s, 1 - a - t, a - u)$ which lies between 0 and $v_8$. If $s = 0$ and $t \geq 2$ then $x''$ lies between 0 and $v_2$ unless $t = a + 1$, in which case let
\[x'' = x'' - v_2 = (-1, -a - s, a + t, a - 1 - u)\] which lies between \(0\) and \(v_8\). If \(1 \leq s \leq a\) and \(t \leq 1\) then \(x''\) lies between \(0\) and \(-v_7\).

**Case 4:** \(x = (-r, -a - 1, -t, -u)\) where \(0 \leq r \leq a\), \(0 \leq t \leq a + 1\) and \(a - 1 \leq u \leq a + 1\). Let \(x' = x - v_5 = (a - r, -1, a + 1 - t, a - 2 - u)\), which lies between \(0\) and \(-v_1\) unless \(r = 0\) or \(t \leq 1\), in which case let \(x'' = x' + v_1 = (1 - r, a, 2 - t, 2a - a - 2 - u)\). If \(r = 0\) and \(t \leq 1\) then \(x''\) lies between \(0\) and \(-v_5\) unless \(u = a - 1\), in which case let \(x''' = x'' + v_5 = (1 - a - r, 0, 1 - a - t, a - u)\) which lies between \(0\) and \(v_8\). If \(r = 0\) and \(2 \leq t \leq a + 1\) then \(x''\) lies between \(0\) and \(v_2\) unless \(t = a + 1\), in which case let \(x''' = x'' - v_2 = (-a - r, -1, a - t, a - 1 - u)\) which lies between \(0\) and \(v_5\). If \(r \geq 1\) and \(t \leq 1\) then \(x''\) lies between \(0\) and \(v_4\).

**Case 5:** \(x = (-r, -s, -t, -u)\) where \(0 \leq r, s \leq a\), \(0 \leq t \leq a + 1\) and \(a - 1 \leq u \leq a + 1\). Let \(x' = x - v_5 = (a - r, -s, a + 1 - t, a - 2 - u)\), which lies between \(0\) and \(-v_5\) unless \(r = 0\) or \(s = 0\) or \(t = 0\), in which case let \(x'' = x' + v_8 = (1 - r, 1 - s, 1 - t, 2a - a - 1 - u)\). If \(r = 0\), \(s = 0\) and \(t = 0\) then \(x\) lies between \(0\) and \(-v_8\). If \(r = 0\), \(s = 0\) and \(1 \leq t \leq a + 1\) then \(x''\) lies between \(0\) and \(v_2\) unless \(a \leq t \leq a + 1\) or \(u = a - 1\), in which case let \(x''' = x'' - v_2 = (-a - r, -s, a + 1 - t, a - u)\). If \(a \leq t \leq a + 1\) and \(a - 1 \leq u \leq a + 1\) then \(x''\) lies between \(0\) and \(v_3\). If \(1 \leq a - 1\) and \(u = a - 1\), then \(x''\) lies between \(0\) and \(-v_6\).

**Case 6:** \(x = (-r, -a - 1, -t, -u)\) where \(0 \leq r \leq a\), \(0 \leq t \leq a + 1\) and \(0 \leq u \leq a - 2\). Let \(x' = x - v_5 = (a - r, -1, a + 1 - t, a - 2 - u)\), which lies between \(0\) and \(-v_7\) unless \(t = 0\), in which case let \(x'' = x' + v_7 = (-r, a - 1 - t, -1 - u)\) which lies between \(0\) and \(v_3\).

**Case 7:** \(x = (-a - 1, -s, -t, -u)\) where \(0 \leq s \leq a\), \(0 \leq t \leq a + 1\) and \(0 \leq u \leq a - 2\). Let \(x' = x - v_5 = (-1, a - s, a + 1 - t, a - 2 - u)\), which lies between \(0\) and \(v_4\) unless \(t = 0\), in which case let \(x'' = x' - v_4 = (a, -s, 1 - t, -1 - u)\) which lies between \(0\) and \(-v_1\).

This completes the cases for the orthant of \(v_5\).

**Orthant of \(v_6\), \(k \equiv 1 \mod 2\)**

Now suppose \(x\) lies in the orthant of \(v_6\) but not between \(0\) and \(v_6\). Then the first coordinate of \(x\) is equal to \(a + 1\) or the second coordinate is equal to \(a + 1\), or the third equals \(-a\) or \(-a - 1\) or the fourth equals \(-a - 1\). We distinguish fifteen cases.

**Case 1:** \(x = (a + 1, a + 1, -t, -a - 1)\) where \(a \leq t \leq a + 1\). Let \(x' = x - v_6 = (1, 1, a - 1 - t, -1)\), which lies between \(0\) and \(v_6\).

**Case 2:** \(x = (a + 1, a + 1, -t, -u)\) where \(a \leq t \leq a + 1\) and \(0 \leq u \leq a\). Let \(x' = x - v_6 = (1, 1, a - 1 - t, a - u)\), which lies between \(0\) and \(v_2\) unless \(u = 0\), in which case let \(x'' = x' - v_2 = (-a, -a, 2a - 3 - t, 1 - u)\) which lies between \(0\) and \(-v_6\).

**Case 3:** \(x = (a + 1, a + 1, -t, -a - 1)\) where \(0 \leq t \leq a + 1\). Let \(x' = x - v_6 = (1, 1, a - 1 - t, -1)\), which lies between \(0\) and \(-v_8\).

**Case 4:** \(x = (a + 1, s, -t, -a - 1)\) where \(0 \leq s \leq a\) and \(a \leq t \leq a + 1\). Let \(x' = x - v_6 = (1, s - a, a - 1 - t, -1)\), which lies between \(0\) and \(-v_4\).

**Case 5:** \(x = (r, a + 1, -t, -a - 1)\) where \(0 \leq r \leq a\) and \(a \leq t \leq a + 1\). Let \(x' = x - v_6 = (1, r, -a - 1 - t, -1)\), which lies between \(0\) and \(-v_6\).
(r - a, 1, a - 1 - t, -1), which lies between 0 and \(v_7\).

Case 6: \(x = (r, s, -t, -a - 1)\) where \(0 \leq r, s \leq a\) and \(a \leq t \leq a + 1\). Let \(x' = x - v_0 = (r - a, s - a, a - 1 - t, -1)\), which lies between 0 and \(v_5\).

Case 7: \(x = (r, a + 1, -t, -a - 1)\) where \(0 \leq r \leq a\) and \(0 \leq t \leq a - 1\). Let \(x' = x - v_0 = (r - a, 1, a - 1 - t, -1)\), which lies between 0 and \(v_3\).

Case 8: \(x = (a + 1, s, -t, -a - 1)\) where \(0 \leq s \leq a\) and \(0 \leq t \leq a - 1\). Let \(x' = x - v_0 = (1, s - a, a - 1 - t, -1)\), which lies between 0 and \(-v_1\).

Case 9: \(x = (r, a + 1, -t, u)\) where \(0 \leq r, u \leq a\) and \(a \leq t \leq a + 1\). Let \(x' = x - v_0 = (r - a, 1, a - 1 - t, a - u)\), which lies between 0 and \(-v_3\) unless \(r = 0\), in which case let \(x'' = x' - v_1 = (r - 1, -a, 2a - 2 - t, -u)\) which lies between 0 and \(-v_2\) unless \(u = a\) in which case \(x'\) lies between 0 and \(v_7\).

Case 10: \(x = (a + 1, s, -t, -u)\) where \(0 \leq s, u \leq a\) and \(a \leq t \leq a + 1\). Let \(x' = x - v_0 = (1, s - a, a - 1 - t, a - u)\), which lies between 0 and \(-v_5\) unless \(s = 0\), in which case let \(x'' = x' + v_3 = (-a, s - 1, 2a - 2 - t, -u)\) which lies between 0 and \(-v_2\) unless \(u = a\). If \(s = 0\) and \(u = a\) then let \(x'' = x'' + v_2 = (1, s + a, a - t, a - 1 - u)\) which lies between 0 and \(v_2\).

Case 11: \(x = (a + 1, a + 1, -t, -u)\) where \(0 \leq t \leq a - 1\) and \(0 \leq u \leq a\). Let \(x' = x - v_0 = (1, 1, a - 1 - t, a - u)\), which lies between 0 and \(-v_5\) unless \(a - 1 \leq u \leq a\), in which case let \(x'' = x' + v_5 = (1, 1, a - 1 - t, 2 - u)\) which lies between 0 and \(v_5\).

Case 12: \(x = (r, s, -t, -a - 1)\) where \(0 \leq r, s \leq a\) and \(0 \leq t \leq a - 1\). Let \(x' = x - v_0 = (r - a, s - a, a - 1 - t, -1)\), which lies between 0 and \(-v_2\) unless \(t = 0\), in which case let \(x'' = x' + v_2 = (r + 1, s + 1, 1 - t, -a - 2)\) and \(x''' = x'' + v_5 = (r - a + 1, s - a, a - 1 - t, 0)\). Then \(x'''\) lies between 0 and \(v_8\).

Case 13: \(x = (r, s, -t, -u)\) where \(0 \leq r, s \leq a\) and \(a \leq t \leq a + 1\). Let \(x' = x - v_0 = (r - a, s - a, a - 1 - t, a - u)\), which lies between 0 and \(v_8\) unless \(r = 0\) or \(s = 0\), in which case let \(x'' = x' - v_8 = (r - 1, s - 1, 2a - 1 - t, -1 - u)\). If \(r = 0\) and \(s = 0\) then \(x''\) lies between 0 and \(-v_2\) unless \(t = a - 1\) in which case \(x'' = x' - v_1 = (a - a, a - t, a - u)\) which lies between 0 and \(-v_7\). If \(r = 0\) and \(1 \leq s \leq a\) then let \(x'' = x - v_7 = (r + a, s - a, a - t, -1 - u)\) which lies between 0 and \(-v_3\) unless \(u = a\) in which case \(x''\) lies between 0 and \(-v_4\). If \(1 \leq r \leq a\) and \(s = 0\) then \(x'' = x + v_4 = (r - a, s + a, a - t, -1 - u)\) which lies between 0 and \(v_1\) unless \(u = a\), in which case \(x''\) lies between 0 and \(v_7\).

Case 14: \(x = (r, a + 1, -t, -u)\) where \(0 \leq r, u \leq a\) and \(0 \leq t \leq a - 1\). Let \(x' = x - v_0 = (r - a, 1, a - 1 - t, a - u)\), which lies between 0 and \(v_4\) unless \(u = 0\), in which case let \(x'' = x' - v_4 = (r - 1, a, -1 - t, 1 - u)\) which lies between 0 and \(-v_3\) unless \(t = a - 1\). If \(t = a - 1\) and \(u = 0\) then \(x'' = x'' + v_3 = (r - a - 1, 0, a - 2 - t, 1 - a - u)\) which lies between 0 and \(v_7\) unless \(r = 0\), in which case \(x\) lies between 0 and \(v_7\).

Case 15: \(x = (a + 1, s, -t, -u)\) where \(0 \leq s, u \leq a\) and \(0 \leq t \leq a - 1\). Let \(x' = x - v_0 = (1, s - a, a - 1 - t, a - u)\), which lies between 0 and \(-v_7\) unless \(u = 0\), in which case let \(x'' = x' + v_7 = (1, a, s - 1, 1 - t, 1 - u)\) which lies between 0 and \(v_1\) unless \(t = a - 1\). If \(t = a - 1\) and \(u = 0\) then \(x'' = x'' + v_1 = (0, s - a - 1, a - 2 - t, 1 - a - u)\) which lies
between 0 and $-v_4$ unless $s = 0$, in which case $x$ lies between 0 and $-v_3$.

This completes the cases for the orthant of $v_6$.

**Orthant of $v_7$, $k \equiv 1 \pmod{2}$**

Now suppose $x$ lies in the orthant of $v_7$ but not between 0 and $v_7$. Then the first coordinate of $x$ is equal to $-a - 1$ or the second is equal to $a + 1$ or the third equals $-a - 1$, or the fourth equals $-a$ or $-a - 1$. We distinguish fifteen cases.

**Case 1:** $x = (-a - 1, a + 1, -a - 1, -u)$ where $a \leq u \leq a + 1$. Let $x' = x - v_7 = (-1, 1, 1, a - 1 - u)$, which lies between 0 and $v_7$.

**Case 2:** $x = (-a - 1, a + 1, -a - 1, -u)$ where $0 \leq u \leq a - 1$. Let $x' = x - v_7 = (-1, 1, 1, a - 1 - u)$, which lies between 0 and $v_1$.

**Case 3:** $x = (-a - 1, a + 1, -t, -u)$ where $0 \leq t \leq a$ and $a \leq u \leq a + 1$. Let $x' = x - v_7 = (-1, 1, a - t, a - 1 - u)$, which lies between 0 and $v_3$ unless $t = 0$, in which case let $x'' = x' - v_3 = (a, 2a - 1, 2a - 1 - u)$, which lies between 0 and $-v_7$.

**Case 4:** $x = (-a - 1, s, -a - 1, -u)$ where $0 \leq s \leq a$ and $a \leq u \leq a + 1$. Let $x' = x - v_7 = (-1, s - a, -1, a - 1 - u)$, which lies between 0 and $v_5$.

**Case 5:** $x = (-r, a + 1, -a - 1, -u)$ where $0 \leq r \leq a$ and $a \leq u \leq a + 1$. Let $x' = x - v_7 = (-a - r, 1, a - 1, -u)$, which lies between 0 and $-v_4$.

**Case 6:** $x = (-r, s, -a - 1, -u)$ where $0 \leq r, s \leq a$ and $a \leq u \leq a + 1$. Let $x' = x - v_7 = (-a - r, s - a, -1, a - 1 - u)$, which lies between 0 and $-v_4$.

**Case 7:** $x = (-r, a + 1, -t, -u)$ where $0 \leq r, t \leq a$ and $a \leq u \leq a + 1$. Let $x' = x - v_7 = (-a - r, 1, a - t, a - 1 - u)$ which lies between 0 and $-v_7$.

**Case 8:** $x = (-a - 1, s, -a - 1, -u)$ where $0 \leq s, t \leq a$ and $a \leq u \leq a + 1$. Let $x' = x - v_7 = (-1, s - a, -t, a - 1 - u)$ lies between 0 and $-v_2$ unless $t \leq 1$, in which case let $x'' = x' + v_2 = (a, s + 1, 2a - t, 2a - 2 - u)$ lies between 0 and $v_5$ unless $s = 0$. If $s = a$ and $t \leq 1$ then let $x''' = x'' + v_5 = (0, s - a + 1, -a - 1, -t, a - u)$ which lies between 0 and $v_7$.

**Case 9:** $x = (-r, a + 1, -a - 1, -u)$ where $0 \leq r \leq a$ and $0 \leq u \leq a - 1$. Let $x' = x - v_7 = (-r, a - 1, a - 1, a - 1 - u)$, which lies between 0 and $v_2$.

**Case 10:** $x = (-a - 1, s, -a - 1, -u)$ where $0 \leq s \leq a$ and $0 \leq u \leq a - 1$. Let $x' = x - v_7 = (-1, s - a, -t, a - 1 - u)$ lies between 0 and $v_3$ unless $s = 0$, in which case let $x'' = x' - v_3 = (a, 2a - 1, 2a - 2 - u)$ lies between 0 and $v_1$ unless $u = a - 1$. If $s = a$ and $u = a - 1$ then let $x''' = x'' + v_1 = (-1, s + a, a - 2 - u)$ which lies between 0 and $v_7$.

**Case 11:** $x = (-a - 1, a + 1, -t, -u)$ where $0 \leq t \leq a$ and $0 \leq u \leq a - 1$. Let $x' = x - v_7 = (-1, a - t, a - 1 - u)$, which lies between 0 and $v_4$.

**Case 12:** $x = (-r, s, -a - 1, -u)$ where $0 \leq r, s \leq a$ and $a \leq u \leq a + 1$. Let $x' = x - v_7 = (-a - r, s - a, -t, a - 1 - u)$ lies between 0 and $-v_1$ unless $r = 0$ or $t = 0$, in which case let
\[ x'' = x' + v_1 = (1-r, s+1, 1-t, 2a-1-u). \]

If \( r = 0 \) and \( t = 0 \) then \( x'' \) lies between 0 and \(-v_5\) unless \( s = a \) or \( u = a \), in which case let \( x''' = x'' + v_5 = (1-a-r, s-a+1, -a-t, a+1-u) \).

If \( u = a \) then \( x \) lies between 0 and \( v_6 \). If \( s = a \) and \( u = a+1 \) then \( x'' \) lies between 0 and \( v_7 \). If \( r = 0 \) and \( 1 \leq t \leq a \) then \( x'' \) lies between 0 and \( v_2 \) unless \( t = a \), in which case \( x' \) lies between 0 and \(-v_4 \). If \( 1 \leq r \leq a \) and \( t = 0 \) then \( x'' \) lies between 0 and \( v_4 \) unless \( s = a \), in which case \( x' \) lies between 0 and \(-v_5 \).

**Case 13:** \( x = (-r, s, -a - 1, -u) \) where \( 0 \leq r, s \leq a \) and \( 0 \leq u \leq a - 1 \). Let \( x' = x - v_7 = (a-r, s-a, -1, a-1-u) \) lies between 0 and \(-v_3 \) unless \( s = 0 \), in which case let \( x'' = x' + v_3 = (1-r, s-1, a-2, -1-u) \) which lies between 0 and \(-v_2 \) unless \( u = a-1 \). If \( s = 0 \) and \( u = a-1 \) then let \( x''' = x'' + v_2 = (a-r, a+s, 0, a-2-u) \) which lies between 0 and \( v_6 \).

**Case 14:** \( x = (-r, a+1, -t, -u) \) where \( 0 \leq r, t \leq a \) and \( 0 \leq u \leq a - 1 \). Let \( x' = x - v_7 = (a-r, 1-a-t, a-1-u) \) which lies between 0 and \(-v_5 \) unless \( u = 0 \), in which case let \( x'' = x' + v_5 = (1-r, a-1, -t-1, 1-u) \) which lies between 0 and \( v_8 \) unless \( r = a \) or \( t = a \). If \( r = a \) and \( u = 0 \) then \( x' \) lies between 0 and \( v_4 \). If \( 0 \leq r \leq a-1 \), \( t = a \) and \( u = 0 \) then let \( x'' = x'' - v_8 = (a-1-r, 0, a-1-t, -a-u) \) which lies between 0 and \( v_6 \).

**Case 15:** \( x = (-a-1, s, -t, -u) \) where \( 0 \leq s, t \leq a \) and \( 0 \leq u \leq a - 1 \). Let \( x' = x - v_7 = (-1, s-a, a-t, a-1-u) \) which lies between 0 and \(-v_6 \) unless \( t = 0 \), in which case let \( x'' = x' + v_6 = (a-1, s, 1-t, -1-u) \) which lies between 0 and \(-v_8 \) unless \( s = a \). If \( s = a \) and \( t = 0 \) then let \( x''' = x'' + v_8 = (0, s-a+1, 1-a-t, a-u) \) which lies between 0 and \( v_1 \).

This completes the cases for the orphant of \( v_7 \).

**Orphant of \( v_8 \), \( k \equiv 1 \pmod{2} \)**

Finally suppose \( x \) lies in the orphant of \( v_8 \) but not between 0 and \( v_8 \). Then the first coordinate of \( x \) is equal to \(-a \) or \(-a-1 \), or the second is equal to \(-a \) or \(-a-1 \), or the third is equal to \(-a-1 \). We distinguish seven cases.

**Case 1:** \( x = (-r, -s, -a-1, u) \) where \( a \leq r, s \leq a+1 \) and \( 0 \leq u \leq a+1 \). Let \( x' = x - v_8 = (a-1-r, a-1-s, 0, a-u-1) \), which lies between 0 and \(-v_5 \) unless \( u \leq 2 \), in which case let \( x'' = x' + v_5 = (2a-1-r, 2a-1-s, a, u-3) \) which lies between 0 and \(-v_8 \).

**Case 2:** \( x = (-r, -s, -t, u) \) where \( a \leq r, s \leq a+1, 0 \leq t \leq a \) and \( 0 \leq u \leq a+1 \).

Let \( x' = x - v_8 = (a-1-r, a-1-s, a-t, u-a-1) \), which lies between 0 and \(-v_2 \) unless \( t \leq 1 \) or \( u \leq 1 \), in which case let \( x'' = x' + v_2 = (2a-r, 2a-s, 2-t, u-t) \). If \( t \leq 1 \) and \( u \leq 1 \) then \( x'' \) lies between 0 and \(-v_8 \) unless \( r = a \) or \( s = a \), in which case let \( x''' = x'' + v_8 = (2a-1-r, 2a-1-s, 2-a-t, u+a-1) \). If \( t \leq 1 \), \( u \leq 1 \), \( r = a \) and \( s = a \) then \( x''' \) lies between 0 and \( v_2 \) unless \( t = 1 \) or \( u = 1 \). If \( r = a \), \( s = a \), \( t = 1 \) and \( u \leq 1 \) then let \( x''' = x - v_5 = (a-r, a-s, a+1-t, a+2) \) which lies between 0 and \( v_4 \).

If \( r = a \), \( s = a \), \( t = 0 \) and \( u = 1 \) then \( x \) lies between 0 and \(-v_6 \). If \( t \leq 1 \), \( u \leq 1 \), \( r = a \) and \( s = a+1 \) then \( x''' \) lies between 0 and \(-v_3 \). If \( t \leq 1 \), \( u \leq 1 \), \( r = a+1 \) and \( s = a \) then \( x''' \) lies between 0 and \( v_1 \). If \( t \leq 1 \) and \( 2 \leq u \leq a+1 \) then \( x''' \) lies between 0 and \( v_5 \) unless \( u = a+1 \), in which case let \( x''' = x'' + v_5 = (a-r, a-s, 1-a-t, u-a) \) which lies between 0 and \( v_8 \). If \( 2 \leq t \leq a+1 \), and \( u \leq 1 \) then \( x'' \) lies between 0 and \( v_6 \).
Case 3: \( x = (-r, -s, -a - 1, u) \) where \( a \leq r \leq a + 1, 0 \leq s \leq a - 1 \) and \( 0 \leq u \leq a + 1 \).

Let \( x' = x - v_8 = (a - 1 - r, a - 1 - s, -1, u - a - 1) \), which lies between \( 0 \) and \( v_7 \) unless \( u \leq 1 \), in which case let \( x'' = x' - v_7 = (2a - 1 - r, -1 - s, a - 1, u - 2) \) which lies between \( 0 \) and \( -v_1 \).

Case 4: \( x = (-r, -s, -a - 1, u) \) where \( 0 \leq r \leq a - 1, a \leq s \leq a + 1 \) and \( 0 \leq u \leq a + 1 \).

Let \( x' = x - v_8 = (a - 1 - r, a - 1 - s, -1, u - a - 1) \), which lies between \( 0 \) and \( -v_4 \) unless \( u \leq 1 \), in which case let \( x'' = x' + v_4 = (-1 - r, 2a - 1 - s, a - 1, u - 2) \) which lies between \( 0 \) and \( v_3 \).

Case 5: \( x = (-r, -s, -a - 1, u) \) where \( 0 \leq r, s \leq a - 1 \) and \( 0 \leq u \leq a + 1 \). Let \( x' = x - v_8 = (a - 1 - r, a - 1 - s, -1, u - a - 1) \), which lies between \( 0 \) and \( v_6 \) unless \( u = 0 \), in which case \( x \) lies between \( 0 \) and \( v_5 \).

Case 6: \( x = (-r, -s, -s, -a - 1, u) \) where \( 0 \leq r \leq a - 1, a \leq s \leq a + 1, 0 \leq t \leq a \) and \( 0 \leq u \leq a + 1 \).

Let \( x' = x - v_8 = (a - 1 - r, a - 1 - s, -a - t, u - a - 1) \), which lies between \( 0 \) and \( -v_1 \) unless \( t = 0 \) or \( u = 0 \), in which case let \( x'' = x' + v_1 = (-r, 2a - 2 - s, 1 - t, u - 1) \). If \( t = 0 \) and \( u = 0 \) then \( x \) lies between \( 0 \) and \( -v_2 \). If \( t = 0 \) and \( 1 \leq u \leq a + 1 \) then \( x'' \) lies between \( 0 \) and \( v_4 \) unless \( u = a + 1 \), in which case let \( x'' = x'' - v_4 = (a - r, a - 2 - s, 1 - a - t, u - a) \) which lies between \( 0 \) and \( -v_3 \). If \( 1 \leq t \leq a \) and \( u = 0 \) then \( x'' \) lies between \( 0 \) and \( v_7 \).

Case 7: \( x = (-r, -s, -s, -t, u) \) where \( a \leq r \leq a + 1, 0 \leq s \leq a - 1, 0 \leq t \leq a \) and \( 0 \leq u \leq a + 1 \).

Let \( x' = x - v_8 = (a - 1 - r, a - 1 - s, -a - t, u - a - 1) \), which lies between \( 0 \) and \( v_3 \) unless \( t = 0 \) or \( u = 0 \), in which case let \( x'' = x' - v_3 = (2a - r, -s, 1 - t, u - 1) \). If \( t = 0 \) and \( u = 0 \) then \( x \) lies between \( 0 \) and \( -v_2 \). If \( t = 0 \) and \( 1 \leq u \leq a + 1 \) then \( x'' \) lies between \( 0 \) and \( -v_7 \) unless \( u = a + 1 \), in which case let \( x'' = x'' + v_7 = (a - r, a - s, 1 - a - t, u - a) \) which lies between \( 0 \) and \( v_1 \). If \( 1 \leq t \leq a \) and \( u = 0 \) then \( x'' \) lies between \( 0 \) and \( -v_4 \).

This completes the cases for the orthant of \( v_8 \).

This also completes the proof of the theorem for any \( k \equiv 1 \pmod{2} \), and therefore for all \( k \geq 2 \).

\[ \blacksquare \]

References
