Open Research Online

The Open University's repository of research publications and other research outputs

Large Deviation Analysis of Rapid Onset of Rain Showers

Journal Item

How to cite:

Wilkinson, Michael (2016). Large Deviation Analysis of Rapid Onset of Rain Showers. Physical Review Letters, 116(1), article no. 018501.

For guidance on citations see FAQs.

© 2016 American Physical Society

Version: Version of Record

Link(s) to article on publisher's website:
http://dx.doi.org/doi:10.1103/PhysRevLett.116.018501

Copyright and Moral Rights for the articles on this site are retained by the individual authors and/or other copyright owners. For more information on Open Research Online's data policy on reuse of materials please consult the policies page.

oro.open.ac.uk

Large Deviation Analysis of Rapid Onset of Rain Showers

Michael Wilkinson*
Department of Mathematics and Statistics, The Open University, Walton Hall, Milton Keynes MK7 6AA, United Kingdom
(Received 9 June 2015; revised manuscript received 29 September 2015; published 8 January 2016)

Rainfall from ice-free cumulus clouds requires collisions of large numbers of microscopic droplets to create every raindrop. The onset of rain showers can be surprisingly rapid, much faster than the mean time required for a single collision. Large-deviation theory is used to explain this observation.

DOI: 10.1103/PhysRevLett.116.018501

The dynamics of the onset of rainfall from ice-free ("warm") cumulus clouds is poorly understood [1–3]. A rain drop grows by collisions of microscopic water droplets. A large number of microscopic droplets must combine to make one rain drop: the volume increase is a factor of approximately 1×10^6. The collision rates in the early stages of the growth process are low (typically of order 1 collision/h). Given the large number of collisions required, it is very hard to understand why rain showers can be initiated over short periods, of perhaps 20 min.

One possible resolution is a consequence of the large number of microscopic droplets which must combine to make a raindrop. This implies that only very few drops are required to undergo rapid growth, and perhaps there are sufficient rare combinations of rapid multiple collisions to explain rainfall: this has previously been emphasized by various authors [1,4,5]. Kostinski and Shaw [6] introduced an elegant model for this runaway growth process. They presented numerical evidence that the model can lead to a rapid development of showers, but a transparent theoretical approach is required. Because this problem involves the analysis of rare events, methods based upon large deviation theory [7,8] are appropriate. In this Letter these methods are used to investigate the hypothesis that rare combinations of rapid collisions trigger showers. It is shown that a rain shower can develop over a time scale which is a small fraction of the mean time scale for one collision.

This Letter will start by discussing some observations and estimates [1–3] that illustrate the difficulties in making a quantitative description of rainfall. These will be followed by describing a model for runaway droplet growth, Eqs. (4)–(6), introduced in Ref. [6]. This model will then be analyzed using a large deviation theoretic approach.

A convecting cumulus cloud may have droplets of mean radius radius $a_0 = 10\ \mu$m, which result from condensation onto aerosol nuclei. Raindrops have a much larger size, typically 1 mm. The volume of a droplet that becomes a raindrop therefore increases by a very large factor, denoted by $\mathcal{N}$, which is typically $\mathcal{N}\approx10^6$. The number density of microscopic droplets is typically of order $N_0 = 2.5\times10^8\ \mathrm{m}^{-3}$, which gives a liquid water content, expressed as a volume fraction, $\Phi_l \approx 4\pi N_0 a_0^3/3 \approx 10^{-6}$. The cloud depth may be $h = 2\times10^3$ m and the typical vertical velocity of air inside the cloud has magnitude $U \approx 2\ \mathrm{ms}^{-1}$, so that the turnover time for convection is approximately $\tau_h = 10^3$ s. Rainfall from this type of cloud can develop over a time scale of approximately $20\,\mathrm{min}\approx10^3$ s.

Collisions between droplets arise principally from different terminal velocities. The Stokes law for the drag on a sphere at low Reynolds number indicates that the terminal velocity is

$$v = \tau_p g = \alpha a^2, \qquad \alpha = \frac{2}{9}\frac{\rho_l}{\rho_g}\frac{g}{\nu}, \tag{1}$$

where τ_p is the response time characterizing the Stokes drag on a droplet, ρ_l is the density of liquid water, and ρ_g and ν are, respectively, the density and kinematic viscosity of air. Inserting values for air and water at 5 °C gives $\alpha\approx1.4\times10^8\ \mathrm{m}^{-1}\,\mathrm{s}^{-1}$, so that when $a_0 = 10\ \mu$m the terminal velocity is $v\approx1.4\times10^{-2}\ \mathrm{ms}^{-1}$ and the response time is 1.4×10^{-3} s. The collision rate of a drop of radius a_1 with a gas of droplets of radius a_0 is

$$R_1 = \pi\varepsilon N_0(a_0+a_1)^2\alpha(a_1^2-a_0^2), \tag{2}$$

where ε is the coalescence efficiency [1,2]. The coalescence efficiencies ε of small droplets are somewhat uncertain, but it is widely accepted that they are low for typical cloud droplets [1,2]. If the larger droplet has radius below 20 μm, it is believed that $\varepsilon\le0.1$, and that for radius 10 μm, $\varepsilon\le0.03$ [2]. For droplets of size $a = 50\ \mu$m colliding with droplets of size $a = 10\ \mu$m, however, the efficiencies are expected to be close to unity [1,2]. Setting $a_1 - a_0 = 2.5\ \mu$m and $\varepsilon = 0.03$ in addition to the parameters defined above gives $R_1\approx2\times10^{-5}\ \mathrm{s}^{-1}$. The rate of coalescence of typical sized water droplets due to collisions is therefore very small.

Cumulus clouds are turbulent because of convective instability. Saffman and Turner [9] investigated the role of turbulence in facilitating collisions between water droplets. In the case of very small droplets, the collision rate due to turbulence is a consequence of shearing motion. The shear

 © 2016 American Physical Society

rate of small-scale motions in turbulence is the inverse of the Kolmogorov time scale, $\tau_K = \sqrt{\nu/\epsilon}$, where ϵ is the rate of dissipation per unit mass. According to the Saffman-Turner model, shear induces a collision speed of order a_0/τ_K. They argue that the corresponding collision rate is

$$R_{\rm turb} = \sqrt{\frac{8\pi}{15}}\frac{N_0\varepsilon(2a)^3}{\tau_K}. \tag{3}$$

For the parameters of the cloud model, the rate of dissipation is $\epsilon \sim U^2/\tau_h \approx 2\times 10^{-3}\ {\rm m^2\,s^{-3}}$, giving $\tau_K \approx$ 70 ms, so that $R_{\rm turb} \approx 10^{-6}\ {\rm s^{-1}}$, which is negligible. The effects of turbulence are dramatically increased when the effects of droplet inertia are significant: this was noticed in numerical experiments by Sundaram and Collins [10], who ascribed the effect to a clustering effect termed "preferential concentration" [11]. More recent work has proposed an alternative mechanism, which has been termed the "sling effect" [12], and which has been explained in terms of the existence of caustics in the velocity field of the droplets [13]. Inertial effects are measured by the Stokes number, ${\rm St} \equiv \tau_p/\tau_K$. Recent numerical studies [14] (see also Ref. [15]) show that while the collision rate is greatly enhanced by effects due to caustics for ${\rm St} > 0.3$, Eq. (3) is a good estimate when ${\rm St} \ll 1$. Although it is in principle possible for turbulence to increase the collision rate of water droplets due to inertial effects, the parameters of the cloud model discussed above yield ${\rm St} \approx 2\times 10^{-2}$, where there is no significant effect. While there is a consensus that turbulence is important for the formation of rain showers [16], turbulent enhancement of collision rates does not appear to be sufficient.

Now consider how to model the onset of a shower, developing the approach discussed in Ref. [6]. It has already been remarked that showers occur on a time scale which may be smaller than the typical time scale for one collision. It is, therefore, reasonable to assume that the runaway droplets are falling through a background of droplets which have not yet coalesced, and which are all of similar size. As a runaway droplet falls it collides with a large number $\mathcal{N}$ of small droplets of size a_0. The time between successive collisions may be assumed to be independent Poisson processes. If the time between the collision with index n and the previous collision is t_n, the time for a droplet to experience runaway growth is

$$T = \sum_{n=1}^{\mathcal{N}} t_n, \tag{4}$$

where the t_n are independent random variables with an exponential distribution

$$P_n(t_n) = R_n \exp(-R_n t_n). \tag{5}$$

The problem is to determine the statistics of T in the limit as $\mathcal{N} \to \infty$. Note that, because the droplets grow by collisions, the droplet volume increases by a factor of approximately $\mathcal{N}$ as a result of these collisions, so that its radius increases by a factor of $\mathcal{N}^{1/3}$. Equation (2) shows that the rates for successive collisions increase as the size of the falling drop grows. Because all of the collision rates R_n scale in the same way as a function of the droplet size a_0 and the number density N_0, write $R_n = R_1 f(n)$. Here R_1 depends upon the properties of the cloud but the function $f(n)$ is the same for all clouds. In order to identify the form of $f(n)$, consider the rate of collision of a large droplet resulting from n previous collisions with a gas of small droplets of radius a_0. The radius of the large droplet is $a_n = n^{1/3}a_0$. When n is large it may be assumed that the collision efficiency is $\varepsilon \approx 1$ and $a_n \gg a_0$, so that $R_n \sim \pi N_0 \alpha a_n^4 \propto n^{4/3}$, which suggests setting $f(n) = n^{4/3}$. However, during the early stages of droplet growth, the collision efficiency for the first few collisions is small, but increases rapidly with n. In what follows $f(n)$ is assumed to be a power law, so that

$$R_n = R_1 n^\gamma. \tag{6}$$

If the collision efficiency of droplets were unity, it would be appropriate to set $\gamma = 4/3$. Because the collision efficiency of droplets at the crucial initial stage of their growth is small, the collision rate increases more rapidly as the size of the falling droplet increases. When the droplets are between 10 and 50 μm it is reasonable to model the product of the collision rate and the collision efficiency as being proportional to a^6, that is to n^2, where n is the number of collisions [6]. In other cases, such as solid precipitation (snow), other values of γ may be appropriate. In the following, γ is left as an adjustable parameter, but special consideration is given to $\gamma = 2$, because it gives a good approximation to terrestrial rainfall, and to $\gamma = 4/3$, because this may be a good approximation for atmospheres on other planets where the collision efficiency might not limit the rate of coalescence. The remainder of this Letter is concerned with using large deviation theory to analyze the consequences of the model contained in Eqs. (4)–(6).

It is necessary to determine the probability density for the time T being a very small fraction of its mean value, $\langle T\rangle$. Inspired by large deviation theory [7,8], the probability density of T is written in an exponential form:

$$P(T) = \frac{1}{\langle T\rangle}\exp[-J(\tau)], \qquad \tau = \frac{T}{\langle T\rangle}. \tag{7}$$

When R_n is given by Eq. (6), the mean time for explosive growth converges as $\mathcal{N} \to \infty$ when $\gamma > 1$:

$$\lim_{\mathcal{N}\to\infty} \langle T\rangle = \lim_{\mathcal{N}\to\infty} \frac{1}{R_1}\sum_{n=1}^{\mathcal{N}} n^{-\gamma} = \frac{1}{R_1}\zeta(\gamma). \tag{8}$$

where ζ is the Riemann zeta function. The function $J(\tau)$ in Eq. (7) is often termed the entropy in texts on large deviation theory [7,8]. It will be necessary to determine the entropy function $J(\tau)$ from the rates R_n.

After a drop has grown to a size where it is much larger than the typical droplets, and where the collision efficiency is approximately unity, it falls rapidly and collects other droplets in its path. Consider a drop of size a_1 falling through a "gas" of much smaller droplets, with liquid volume fraction Φ_l. The larger drop falls with velocity v and sweeps out a volume $\pi a_1^2 v$ per unit time, so that it grows in volume at a rate $\dot{V} = \pi\epsilon a_1^2 \Phi_l v = 4\pi a_1^2 \dot{a}_1$, where $\dot{a}_1$ is the rate of increase of the drop radius. The rate of increase of the radius of the "collector" drop as a function of the distance x through which it has fallen is, therefore,

$$\frac{da}{dx} = \frac{\epsilon \Phi_l}{4}. \tag{9}$$

Note that this expression is valid whether or not the terminal velocity is given by the small Reynolds number approximation, Eq. (1). In the case of droplets which reach a radius of approximately 1 mm, the collision efficiency ϵ is close to unity throughout most of the fall. The droplet radius after falling through a cloud of depth h is therefore $a(h) \sim \Phi_l h/4$. It will be assumed that the most relevant collector drops are those that started at the top of the cloud. Because droplets grow by coalescence of microscopic droplets as they fall, the ratio of the final volume to the initial volume is equal to the number of collisions: this is

$$\mathcal{N} = \left(\frac{h}{4a_0}\right)^3 \Phi_l^3. \tag{10}$$

Using the representative values given above gives $\mathcal{N} \approx 10^5$. Kostinski [17] has used related arguments to explain observations that the rate of production of drizzle from maritime stratus clouds is proportional to the cube of its depth.

The fraction of droplets undergoing runaway growth between time t and $t + \delta t$ is $P(t)\delta t$. Because the volume of the runaway growth droplets has increased by a factor of $\mathcal{N}$, when these raindrops fall out of the cloud they reduce the liquid water content Φ_l by $\mathcal{N}\Phi_l P(t)\delta t$. The rate of change of the liquid water content of a cloud due to the runaway growth of droplets is, therefore,

$$\frac{d\Phi_l}{dt} = -\Phi_l \mathcal{N} P(t). \tag{11}$$

Note that the growth factor $\mathcal{N}$ and the probability density for runaway growth after time t are both functions of Φ_l, but if the objective is to understand the onset of a rain shower it suffices to evaluate these quantities with the initial value $\Phi_l(0)$. The onset of the shower is determined by the criterion that a significant fraction μ (typically a few percent) of the liquid water content of a cloud is removed by creating raindrops, resulting from $\mathcal{N}$ collision events: from Eq. (11) this occurs when $\mathcal{N}\int_0^t dt' P(t') = \mu$. Using Eq. (7) for $P(t)$, the condition for the time scale t^*, where there is a significant reduction in $\Phi_l(t)$ is approximated by $\mathcal{N}^* \exp[-J(t^*/\langle T\rangle)] = 1$, where $\mathcal{N}^* = \mathcal{N}/\mu$ is a large number, typically $\mathcal{N}^* \approx 10^6$. The condition for the onset of a shower is, therefore,

$$t^* = \tau^* \langle T\rangle, \qquad J(\tau^*) = \ln \mathcal{N}^*. \tag{12}$$

To determine the solution of Eq. (12) for t^*, it is necessary to determine the entropy function $J(\tau)$ for the random sum defined by Eqs. (4) and (5).

Now consider how to compute $J(T)$. A cumulant generating function $\lambda(k)$ is defined by writing

$$\exp[-\lambda(k)] = \langle \exp(-kT)\rangle = \int_0^\infty dT P(T) \exp(-kT). \tag{13}$$

Because the t_n are independent, with a distribution given by Eq. (5),

$$\lambda(k) = -\sum_{n=1}^{\mathcal{N}} \ln\langle \exp(-kt_n)\rangle = \sum_{n=1}^{\mathcal{N}} \ln\left(1 + \frac{k}{R_n}\right). \tag{14}$$

Consider how to obtain $P(T)$ from $\lambda(k)$. Noting that $\exp[-\lambda(k)]$ is the Laplace transform of $P(T)$, application of the Bromwich integral formula for inversion gives

$$P(T) = \frac{1}{2\pi \mathrm{i}} \int_{\mathcal{R}-\mathrm{i}\infty}^{\mathcal{R}+\mathrm{i}\infty} dz \exp[zT - \lambda(z)], \tag{15}$$

where $\mathcal{R} > -R_1$. The integral is dominated by contributions from the neighbourhood of a saddle at $z = k^*$ (on the real axis), where

$$T = \sum_{n=1}^{\mathcal{N}} \frac{1}{R_n + k^*}, \tag{16}$$

which is to be solved for the saddle point k^* given a value of the time T. The probability density $P(T)$ is then approximated by

$$P(T) = \frac{1}{R_1} \frac{1}{\sqrt{2\pi J''(k^*)}} \exp[-J(\tau)], \tag{17}$$

where $\tau = T/\langle T\rangle$ and $J''(k)$ is the magnitude of the second derivative of the exponent in Eq. (15). Equations (16) and (17) cannot be solved exactly and explicitly for $J(\tau)$. Consider how to write down a parametric representation of $J(\tau)$ using a scaled variable, κ, defined by $\kappa = k^*/R_1$. Imposing the requirement that the integrand of Eq. (15) is stationary with respect to z, in the limit as $\mathcal{N} \to \infty$, the dimensionless time for raindrop formation is

$$\tau(\kappa) = \frac{1}{\zeta(\gamma)} \sum_{n=1}^{\infty} \frac{1}{\kappa + n^{\gamma}} \tag{18}$$

[this is a dimensionless version of (16), expressed in terms of $\tau = T/\langle T \rangle = R_1 T/\zeta(\gamma)$]. The entropy function is the value of the exponent of (15) evaluated at the stationary point κ which solves (18): in the limit as $\mathcal{N} \to \infty$ this is

$$J(\kappa) = S(\kappa) - \zeta(\gamma)\kappa\tau(\kappa), \tag{19}$$

with

$$S(\kappa) = \sum_{n=1}^{\infty} \ln(1 + \kappa n^{-\gamma}). \tag{20}$$

In general, Eq. (18) must be solved numerically to determine κ for a given value of τ, but it is possible to extract asymptotic expressions which are valid for small and large τ. The small τ asymptotics are determined by the large κ asymptotics of the sums in Eqs. (18) and (20). These require some delicate analysis (detailed in the Supplemental Material [18]), but the methods are standard. The leading order behavior of $S(\kappa)$ is

$$S \sim \gamma A(\gamma)\kappa^{(1/\gamma)} - \frac{1}{2}\ln(\kappa) - \gamma C + O(\kappa^{-1}), \tag{21}$$

where C is a constant defined in the Supplemental Material [18], and

$$A(\gamma) = \frac{1}{\gamma}\int_0^{\infty} dx \frac{x^{-(\gamma-1/\gamma)}}{1+x} = \frac{1}{\gamma} B\left(1 - \frac{1}{\gamma}, \frac{1}{\gamma}\right) \tag{22}$$

[here $B(v, w)$ is the Euler beta function as defined in Ref. [19]]. Differentiation of $S(\kappa)$ yields an expression equivalent to Eq. (18) relating for τ to κ, which can be inverted to give an expression for the saddle point

$$\kappa^* = b\tau^{-(\gamma/\gamma-1)}[1 - c\tau^{(1/\gamma-1)}], \tag{23}$$

where b and c are functions of γ. In terms of τ, the leading order terms of the entropy are

$$J(\tau) = (\gamma - 1)Ab^{(1/\gamma)}\tau^{-(\gamma/\gamma-1)} + \frac{\gamma}{2(\gamma-1)}\ln\tau + D, \tag{24}$$

where D is another constant. Furthermore, in terms of τ the second derivative of the entropy J'' is proportional to $\tau^{(2\gamma-1/\gamma-1)}$. The probability density is therefore

$$P(\tau) = K\tau^{-[3\gamma-1/2(\gamma-1)]}\exp(-\mathcal{C}/\tau^{(1/\gamma-1)}), \tag{25}$$

where $\mathcal{C} = (\gamma - 1)Ab^{(1/\gamma)}$ and K is another constant which can be explicitly constructed. For completeness, the approximation of $P(\tau)$ for $\mathcal{N} \to \infty$ and $\tau \gg 1$ is

$$P(\tau) \sim \zeta(\gamma)\exp[\mathcal{S}(\gamma) - \zeta(\gamma)\tau],$$
$$\mathcal{S}(\gamma) = \sum_{n=2}^{\infty} \ln(1 - n^{-\gamma}). \tag{26}$$

Figure 1 shows the distribution of $\tau = T/\langle T \rangle$ for the case $\gamma = 2$ (the case which is most relevant to rain showers), with $\mathcal{N} = 10^4$ and $R_1 = 1$, comparing the results of simulation of Eq. (4), the Bromwich integral (15), the saddle-point approximation, Eqs. (17), (19), (20), and the explicit asymptotic formulae, Eqs. (25) and (26), which are all in excellent agreement. A small discontinuity in the asymptotic expression at $\tau = 1$ marks the switch between using Eqs. (25) and (26). The entropy function increases very rapidly as $\tau \to 0$, indicating that the value of $\tau^* = t^*/\langle T \rangle$ is quite insensitive to the value of $\ln\mathcal{N}$. It is clear from Fig. 1 that the solution of Eq. (25) gives small values of τ^* when $\mathcal{N}^*$ is large. Numerical evaluation of the solution of Eq. (12) using the Bromwich integral of $P(\tau)$ with $\gamma = 2$ gives $\tau^* \approx 0.077$ when $\mathcal{N}^* = 10^5$ and $\tau^* \approx 0.068$ when $\mathcal{N}^* = 10^6$. Alternatively, in terms of $\langle t_1 \rangle = \langle T \rangle/\zeta(\gamma)$, when $\gamma = 2$, the predicted time for onset of a shower is a small fraction of the mean time for the first collision: $t^* \approx 0.128\langle t_1 \rangle$ when $\mathcal{N}^* = 10^5$ and $t^* \approx 0.112\langle t_1 \rangle$, when $\mathcal{N}^* = 10^6$. These estimates for t^* are compatible with results reported by Kostinski and Shaw [6].

In the case where $\gamma = 4/3$, there is also excellent agreement between the exact evaluation of $P(\tau)$ using Eq. (15) and the saddle point approximation. While Eq. (25) is the precise asymptotic expression for $P(\tau)$ in the limit as $\tau \to 0$, for $\gamma = 4/3$ the convergence of this estimate is not as good as for $\gamma = 2$. Reasons for this, and

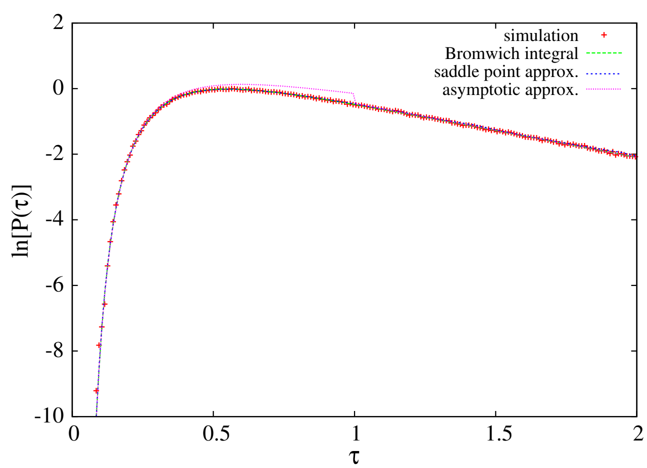

FIG. 1. Plot of $\ln[P(\tau)]$, for $\mathcal{N} = 10^4$ and $\gamma = 2$. The results of simulation, evaluation of the Bromwich integral, the saddle point approximation, and the explicit asymptotic formulas, Eq. (25) (used where $\tau \leq 1$) and Eq. (26) (used for $\tau > 1$) for all are in excellent agreement.

an improved asymptotic approximation that also gives excellent agreement with simulations at $\gamma = 4/3$, are discussed in the Supplemental Material [18].

Equations (12) and (25) imply that the time scale t^* for the initiation of a shower is smaller than the mean time for a single collision. Surprisingly, the time scale t^* *decreases* as the number of collisions required to make a raindrop increases: as $\mathcal{N}^* \to \infty$ the dominant term of Eq. (24) as $\tau \to 0$ gives

$$\tau^* \sim \frac{1}{R_1}[\ln \mathcal{N}^*]^{-(\gamma-1/\gamma)}. \tag{27}$$

Thus, large deviation theory has resolved an apparent paradox of meteorology, that rain showers can start very quickly, on time scales that are short compared to typical mean collision times.

This calculation does not resolve all of the uncertainties about initiation of rain showers. Clouds can exist for a long period without producing a rain shower, before depositing a significant fraction of their water content over a short time. Shower activity is associated with convective motion in clouds. For typical levels of turbulence, however, turbulent enhancement of collisions does not appear to be sufficient to trigger showers. It seems as if noncollisional mechanisms involving convection must play a role in initiating the cascade [20].

This work was initiated during a visit to the Kavli Institute for Theoretical Physics, Santa Barbara, and supported in part by the National Science Foundation under Grant No. NSF PHY11-25915.

*m.wilkinson@open.ac.uk

[1] B. J. Mason, *The Physics of Clouds*, 2nd. ed. (Oxford University Press, New York, 1971).
[2] H. R. Pruppacher and J. D. Klett, *Microphysics of Clouds and Precipitation*, 2nd ed. (Kluwer, Dordrecht, 1997).
[3] R. A. Shaw, Annu. Rev. Fluid Mech. **35**, 183 (2003).
[4] J. W. Telford, J. Meteorol. **12**, 436 (1955).
[5] S. Twomey, J. Atmos. Sci. **23**, 405 (1966).
[6] A. B. Kostinski and R. A. Shaw, Bull. Am. Meteorol. Soc. **86**, 235 (2005).
[7] M. I. Freidlin and A. D. Wentzell, *Random Perturbations of Dynamical Systems*, Grundlehren der Mathematischen Wissenschaften (Springer, New York, 1984), Vol. 260.
[8] H. Touchette, Phys. Rep. **478**, 1 (2009).
[9] P. G. Saffamn and J. S. Turner, J. Fluid Mech. **1**, 16 (1956).
[10] S. Sundaram and L. R. Collins, J. Fluid Mech. **335**, 75 (1997).
[11] M. R. Maxey, J. Fluid Mech. **174**, 441 (1987).
[12] G. Falkovich, A. Fouxon, and M. G. Stepanov, Nature (London) **419**, 151 (2002).
[13] M. Wilkinson, B. Mehlig, and V. Bezuglyy, Phys. Rev. Lett. **97**, 048501 (2006).
[14] M. Voßkuhle, A. Pumir, E. Lévêque, and M. Wilkinson, J. Fluid Mech. **749**, 841 (2014).
[15] W. W. Grabowski and L. P. Wang, Annu. Rev. Fluid Mech. **45**, 293 (2013).
[16] E. Bodenschatz, S. P. Malinowski, R. A. Shaw, and F. Stratmann, Science **327**, 970 (2010).
[17] A. B. Kostinski, Environ. Res. Lett. **3**, 045019 (2008).
[18] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevLett.116.018501 for details of the asymptotic analysis.
[19] M. Abramowitz and I. A Stegun, *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables* (Dover, New York, 1964).
[20] M. Wilkinson, Europhys. Lett. **108**, 49001 (2014).