Sound Propagation in Inhomogeneous Media

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Sound Propagation in Inhomogeneous Media

Thesis Submitted by

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for the degree of
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Contents

Summary..........................................................................................................1
Chapter 1 - Wave Propagation in Media With Variable Sound Speed............3
  1.1 Review...........................................................................................3
  1.2 General form of the wave potential above an impedance boundary......6
  1.3 Bilinear sound speed profile...........................................................8
  1.4 Wentzel-Kramers-Brillouin approximation.......................................13
  1.5 WKB-type approximation with Airy functions...............................16
    1.5.1 Practical Considerations......................................................18
  1.6 A new solution for a constant gradient profile...............................20
    1.6.1 Asymptotic Approximations to Hankel Functions of Large,
          Complex Order.......................................................................22
  1.7 Conclusion....................................................................................24

Chapter 2 - Fast Field Calculation for Propagation Above Porous
          Elastic Media................................................................................25
  2.1 Introduction..................................................................................25
  2.2 Biot theory of poroelasticity..........................................................26
  2.3 Fast Field Program.....................................................................31
    2.3.1 Inherent Assumption- Open pore Condition.........................36
  2.4 Boundary conditions......................................................................37
  2.5 Source terms...............................................................................41
  2.6 Numerical techniques...................................................................43
  2.7 de la Cruz and Spanos theory of seismic wave propagation in a porous
          medium....................................................................................45
  2.8 Conclusion...................................................................................48

Appendix : list of symbols.........................................................................49

Chapter 3 - Predicted Effects of Porosity on Propagation in a Fluid-
          Saturated Porous Elastic Medium...................................................51
  3.1 Introduction..................................................................................51
  3.2 The Biot-Stoll model.....................................................................54
  3.3 Pore size distribution....................................................................65
  3.4 Propagation in the fluid medium...................................................71
  3.5 Conclusion...................................................................................73

Appendix : List of symbols......................................................................74

Chapter 4 - On the Influence of Ground Porosity on Acoustic/Seismic
          Coupling Spectra...........................................................................77
  4.1 Introduction..................................................................................77
  4.2 Theory..........................................................................................78
  4.3 Measurement of acoustic-to-seismic coupling ratio.........................88
  4.4 Conclusion...................................................................................92

Chapter 5 - Sound Field From an Arbitrarily Oriented Dipole Source
          Above Rigid Porous and Poroelastic Ground.................................93
  5.1 Introduction..................................................................................93
  5.2 Theory..........................................................................................95
List of Figures and Captions

Figure 1.1 Sound field above a rigid (dashed line) and a rough rigid (solid line) surface in a shadow zone. Source and receiver are placed on the surface, and $\frac{dc}{dz}=-0.1$. The frequency is 100 Hz.

Figure 1.2 Sound field above a rigid (dashed line) and a rough rigid (solid line) surface in a shadow zone. Source height ($hs$)=10.m, receiver height ($hr$)=10.m, and $\frac{dc}{dz}=-0.1$.

Figure 1.3 Path of integration of the phase function when the turning point is complex.

Figure 3.1 Predicted fluid bulk modulus in air filled pores as a function of frequency for porosity values of 0.1..0.7 from right to left.

Figure 3.2 As in figure 3.1 but for water-saturated pores.

Figure 3.3 Predicted viscosity correction function vs. Frequency for porosity values of 0.7, 0.5, 0.3, 0.2, and 0.1 from left to right.

Figure 3.4 Viscosity correction function plotted against $\lambda$.

Figure 3.5 Fast wave speed in air-filled soils. Very little change with either frequency or porosity is predicted.

Figure 3.6 Fast wave specific attenuation in air-filled soils. The large peak corresponds to porosity value of 0.5.

Figure 3.7 Fast wave speed for water-saturated sediments corresponding to porosity values of (from top to bottom) 0.1, 0.2, 0.3, 0.5 and 0.7.

Figure 3.8 Fast wave specific attenuation for water-saturated sediments. Values of porosity increase from right to left.

Figure 3.9 Shear wave speed in air-filled soils.

Figure 3.10 Shear wave specific attenuation in air-filled soils.

Figure 3.11 Shear wave speed in water-saturated sediments. Porosity increases from top to bottom.

Figure 3.12 Shear wave specific attenuation in water-saturated sediments. Porosity increases from right to left.

Figure 3.13 Slow wave speed in air-filled soils. Porosity increases from right to left.
Figure 3.14 Slow wave specific attenuation in air-filled soils. Porosity increases from right to left.

Figure 3.15 Slow wave speed in water-saturated sediments. Porosity increases from top to bottom.

Figure 3.16 Slow wave specific attenuation in water-saturated sediments. Porosity increases from right to left.

Figure 3.17 The viscosity correction function for three values of (σ) in air-filled soils.

Figure 3.18 Pore fluid bulk modulus for three values of (σ).

Figure 3.19 Slow wave speed and specific attenuation for three values of (σ) in air-filled soils.

Figure 3.20 Slow wave speed and specific attenuation for three values of (σ) in water-saturated sediments.

Figure 3.21 Predicted Excess Attenuation of sound above surface of air-filled solid. The soil pore size distribution is assumed to have standard deviation, in ø units, of (from left to right) 0.5, 1 and 2. Source and receiver heights are 5.0m and 1.0 m respectively and separation is 50.0m.

Figure 3.22 Predicted Excess Attenuation of sound produced by a point source above the surface of an air-filled porous elastic ground. The source and receiver heights are assumed to be 5. m and 1. m and the separation is 400m. Porosity increases from right to left.

Figure 3.23 Predicted Excess Attenuation of sound under water above surface of water-saturated sediment. Source and receiver heights are 5.0m and 1.0 m respectively and separation is 50.0m. Porosity increases from left to right.

Figure 4.1 Reflection coefficient above poro-elastic layers of various thickness with Vp=100.m/s and Vs=70.7 m/s.

Figure 4.2 An example of Excess Attenuation of sound above a poro-elastic soil layer. The small minima are due to layer resonances but are usually too small to detect above the experimental error and uncertainty.

Figure 4.3 Reflection coefficient above poro-elastic layers of various thickness with Vp=270.m/s and Vs=190 m/s.

Figure 4.4 Acoustic/seismic coupling ratio above a 2.0m thick porous elastic layer. Other parameters are given in Table 4.1.
Figure 4.5 The geometry assumed in calculating the ground layer resonances.

Figure 4.6 measured (crosses and circles) and predicted (solid line) Acoustic to seismic coupling ratio measured at bondville. The range is 15.0m and the source height is 31.0m.

Figure 4.7 measured (crosses and circles) and predicted (solid line) Acoustic to seismic coupling ratio measured at bondville. The range is 18.0m and the source height is 31.0m

Figure 5.1 Pressure (dB) contour plot of a horizontal dipole above a porous elastic ground with azimuthal angle of 45° at \( f = 200 \) Hz. The source and receiver heights are 5.0 and 1.0m respectively.

Figure 5.2 Sound field of a vertical dipole source compared with superposition of two out of phase monopoles separated by a small distance along z-axis. The lines are coincident. Also plotted is the field of a monopole source (broken line). Frequency is 1 kHz and source and receiver heights are 1. And 5 m respectively.

Figure 5.3 Predicted Transmission Loss of a vertical dipole at 1 kHz above an impedance plane. Results of the DipoleFFP and the asymptotic expression are coincident (solid line), Generalov’ solution (dash dot line) and monopole field (dotted line). \( H_s = 2.0m, H_r = 1.0m \).

Figure 5.4 Predicted Transmission Loss of an horizontal dipole at 1 kHz above an impedance plane. Results of the DipoleFFP and the asymptotic expression are indistinguishable above 1. m (solid line), monopole field (dotted line). \( H_s = 2.0m, H_r = 1.0m \).

Figure 5.5 As in figure 5.3 for frequency of 100 Hz.

Figure 5.6 As in figure 5.4 but at \( f = 100 \) Hz.

Figure 5.7 Predicted excess attenuation due to a monopole (solid), horizontal dipole(dotted) and a vertical dipole(dashed line). \( H_s = 1.0m, h_r = 0.5m, R = 5.0m \).

Figure 5.8 Predicted Excess attenuation of sound due to a monopole(solid line), and vertical dipole (dotted line) with \( hs = 0.25m \) and \( hr = 0.1m \). The dashed line represents a vertical dipole with source and receiver positions reversed. The shift in ground effect dip is due to the extra phase difference.

Figure 5.9 As in figure 5.7 but range=1.0m.
Figure 5.10 Predicted Excess attenuation of sound due to a monopole (solid) and a vertical dipole (dashed). The dotted line is the ground wave component of the dipole field. \( H_s=0.1 \text{m}, H_r=0.025 \text{m}, R=2.0 \text{m} \).

Figure 5.11 Sensitivity of monopole (graph a) and vertical dipole (graph b) fields to effective flow resistivity.

Figure 5.12 Measured (dotted) and predicted (solid) field of an unbaffled loudspeaker. The reference is the field above a solid plane.

Figure 5.13 Drawing of the transducer disc and the supporting rod.

Figure 5.14 Directivity pattern of transducer disc (circles) compared with directivity of a dipole (solid line). The disc was positioned perpendicular to the plane of the paper and parallel to 90° axis, as indicated by a thick line.

Figure 5.15 Measured (solid line) and fitted (broken line) excess attenuation spectra due to a monopole source above a rigid-backed 1.6 cm thick felt: a) \( H_s=H_r=0.05 \text{m}, R=2.0 \text{m} \). b) \( H_s=0.05 \text{m}, H_r=0.15 \text{m}, R=2.0 \text{m} \). c) \( H_s=H_r=0.05 \text{m}, R=1.0 \text{m} \). d) \( H_s=H_r=0.05 \text{m}, H_r=0.15 \text{m}, R=1.0 \text{m} \). In all four cases the parameters were 38 kPa sm⁻¹ and 15 m⁻¹.

Figure 5.16 Measured (crosses) and predicted sound fields from a horizontal dipole at 4096 Hz. Asymptotic form (solid line) and monopole field (dashed line) normalised to coincide at first measured point (at 0.1 m in this case). \( H_s=0.03 \text{m}, H_r=0.02 \text{m} \).

Figure 5.17 Measured (crosses) and predicted sound fields from a vertical dipole at 4096 Hz. Asymptotic form (solid line), Generalov's solution (dash-dot line) and monopole field (dashed line) normalised to coincide at first measured point (at 0.0 m in this case). \( H_s=0.03 \text{m}, H_r=0.02 \text{m} \).

Figure 5.18 Measured (crosses) and predicted sound fields from a vertical dipole at 2915 Hz. Asymptotic form (solid line), Generalov's solution (dash-dot line) and monopole field (dashed line) normalised to coincide at first measured point (at 0.0 m in this case). \( H_s=0.035 \text{m}, H_r=0.02 \text{m} \).

Figure 6.1 Diagramatic view of a quadrupole ant its dipole constituents. The vectors d and m are the dipole and quadrupole amplitude vectors respectively (see eqns. 6.5 and 6.6).

Figure 6.2 Transmission Loss of a vertical quadrupole (solid line), vertical dipole (dotted line) and a monopole (dashed line). Frequency is 500 Hz, and source and receiver heights are 1.5 and 1.0 m respectively.
Figure 6.3  Excess Attenuation of sound due to a vertical quadrupole (solid line), a vertical dipole (dotted line) and a monopole (dashed line) above an impedance ground. The source and receiver heights are 0.1m and 0.025m respectively and their separation is 2.0m.

Figure 6.4  a) Sound field, as a function of range, of a longitudinal quadrupole (solid line), a vertical dipole (dotted line), and a monopole (dashed line). Source and receiver heights are 1.0m and 2.0m respectively and frequency= 1 kHz. b) same as (a) above but for a lateral quadrupole (solid line) the dotted line and dashed line represent the vertical dipole and monopole respectively.

Figure 6.5  Same as figure 6.3 but for $f=100$ Hz.

Figure 6.6  a) The excess attenuation spectra for a longitudinal quadrupole (solid line), a vertical dipole (dotted line) and a monopole (dashed line). Source and receiver heights are 1.0m and 0.5m respectively and the separation is 5.0m. b) Same as (a) above but for a lateral quadrupole.

Figure 6.7  Same as figure 6.5 but for a source height of 0.1m, receiver height 0.025m and separation of 2.0m.

Figure 6.8  The polar plot in the $x-z$ plane of an ensemble average of far field intensity of an arbitrarily-oriented quadrupole above a rigid boundary (dashed line) and above an impedance surface (solid line). The frequency is 1 kHz and the source height is 0.11m. a) a longitudinal quadrupole; and b) a lateral quadrupole.

Figure 6.9  The polar plot in the $x-z$ plane of an ensemble average of far field intensity of an arbitrarily-oriented quadrupole above a rigid boundary (dashed line) and above an impedance surface (solid line). The frequency is 1 kHz and the source height is 0.11m. a) Ensemble average of longitudinal quadrupole; and b) ensemble average of lateral quadrupoles.
**List of Tables**

Table 3.1  List of parameters used in chapter 3

Table 4.1  List of parameters used in chapter 4

Table 4.2  Measured and predicted peak frequencies of acoustic/seismic coupling ratio. Asterisks refer to major peaks, Sn and Pn to the nth harmonics of s- and p-waves respectively.

Table 4.3  list of measured environment parameters at Bondville.
Summary

Outdoor sound propagation is of interest in community noise problems, for example noise prediction. The interaction of sound with the ground is also of interest in remote surveillance by acoustic and seismic sensors. In this thesis, previous work concerned with sound propagating in a layered refracting atmosphere, and with acoustic-to-seismic coupling is reviewed and extended. The thesis makes contributions in several areas concerned with these topics. Specifically

- methods of predicting propagation in a variable sound velocity gradients are reviewed and a new formulation is advanced;
- a program FFLAGS (Fast Field program for Layered Air Ground Systems) is reviewed and revised to improve numerical efficiency and enable wider application;
- the Biot theory of wave propagation in poro-elastic media is reviewed and contrasted with a recent theory advanced by de la Cruz and Spanos, particularly in respect of boundary conditions;
- the influence of porosity on wave propagation in fluid-saturated porous elastic media is investigated using predictions based on Biot’s theory. It is found that there are large differences between predictions for air-filled soils and water-saturated sediments. It is also predicted that porosity plays an important role in determining the wave speed, especially in water-saturated sediments;
Summary

- the influence of porosity on acoustic-to-seismic coupling is investigated. It is found that a) it is possible to obtain improved agreement with data on the frequencies of maximum acoustic-to-seismic coupling, and, b) it is possible to obtain tolerable agreement with new data on acoustic-to-seismic coupling spectra;

- a new method of measuring excess attenuation spectrum of sound using MLSSA (Maximum length Sequence System Analyser)\textsuperscript{[82]} is described and used to investigate the field due to a dipole source over an impedance plane;

- directivity effects over an impedance or poroelastic plane are explored using the FFP formulation, asymptotic approximations and laboratory experiments.

The over all theme of this thesis is the study of the influence of inhomogeneities in the medium on sound wave propagation. Because of the multiplicity of topics, relevant literature reviews are included in each of the chapters.
Chapter 1

Wave Propagation in Media With Variable Sound Speed

1.1 Review

Wind and temperature variations with altitude in the atmosphere produce conditions under which the speed of sound in the medium varies with altitude only. One then assumes that the medium parameters are functions of one co-ordinate only, namely $z$. A more general case involves range-dependent variation of sound speed as well as vertical variation in which case the parameters are functions of all three co-ordinates. This is a very complicated problem and requires solving the most general form of the wave equation. This case will not be covered here.

Let a point monopole source be radiating spherical waves in an inhomogeneous fluid medium in which the physical parameters are functions of altitude only. In addition let the source be situated a distance $h$ above a plane interface. If the fluid density varies sufficiently little in the range of interest, we have the following wave equation for the pressure in cylindrical co-ordinates:

$$\nabla^2 P(r,z) + k^2(z) P(r,z) = -2\pi \delta(r) \delta(z - h)$$

where $k = \omega c(z)$

By applying a Hankel transform

$$\varphi(z, k_r) = \int_0^\infty P(r, z) J_0(k_r r) r dr$$

(1.2)
to equation (1.1) we arrive at the Helmholtz Equation:

$$\frac{\partial^2}{\partial z^2} \varphi + [k^2(z) - k_e^2] \varphi = \delta(z - h)$$

(1.3)

the pressure can then by found by applying the inverse Hankel transform to $\varphi$

$$P(r,z) = \int_0^\infty \varphi(z,k) J_0(kr)k,dk,$$

(1.4)

In general, the problem is solved numerically, for example by assuming a system comprising of horizontally stratified but otherwise homogeneous, layers each with constant, but different, sound velocity. The solution within each layer is then trivial. The method is described in chapter 2 where the fast field method is discussed. On the other hand, for certain sound speed profiles, one can find analytical solutions to the above equation with varying degrees of simplicity and computability.

Pekeris\(^{(2)}\) solved the problem for several sound speed profiles including constant gradient and a 'parabolic $k$' profile:

$$c(z) = az \text{ and } c(z) = \frac{a}{\sqrt{1+az+bz^2}}$$

(1.5)

For the first case, he expressed the wave potential in terms of Whittaker's function $W_{0,a}(\xi)$ and Bessel functions of complex order. He derived simplified expressions for sound propagation under a constant sound speed gradient in an unbounded medium. For the second case, the solution was derived in terms of Hankel functions of one third order. He also derived the general form of the solution for the wave equation in a semi-infinite fluid medium bounded by a pressure release boundary with an arbitrary sound
speed profile. In the next section his procedure will be modified and extended to derive the general form of the sound field potential in a fluid above an impedance boundary. The most widely used solution at present was derived by Pierce\textsuperscript{[3]} who considered a sound speed profile (similar to the second case of Pekeris) approximated by,

$$c(z) = \frac{c_0}{\sqrt{1 + \frac{2z}{R}}} \quad (1.6)$$

where $R$ is the approximate radius of curvature of sound rays. He transformed the Helmholtz equation into the Airy differential equation and expressed the potential in terms of Airy functions.

Tolstoy\textsuperscript{[4]} also considered a sound pressure field in a fluid with a similar sound velocity profile above a rigid rough boundary. Like Pekeris, he based his solution to the resulting equation in terms of Hankel functions of one-third order.

Refractive index profiles of the form $n = n_0(1 - be^{-az})^v$, where $b$, $a$ and $v$ are constants, have been considered and the corresponding sound fields solved by Osborne\textsuperscript{[6]} in terms of confluent hypergeometric and Papperitz functions. These are less amenable for efficient computation.

Other profiles studied by various workers in the past include\textsuperscript{[6]}:

$$c(z) = \frac{c_0}{\sqrt{a^2 - b^2/z^2}} \quad (1.7)$$

the solution of which is $\sqrt{2} H_p^{(1,2)}(Bz)$

where $H$ is the Hankel function, $p = \sqrt{k_0^2 b^2 + \frac{1}{4}}$ and $B = k_0^2 a^2 - k_0^2$
and

$$k^2(z) = k_0^2 \left[ 1 - N \frac{e^{mz}}{1 + e^{mz}} - 4M \frac{e^{mz}}{(1 + e^{mz})^2} \right]$$

where $N, M,$ and $m$ are constant parameters. This profile was investigated by Epstein\textsuperscript{[7,8]} and is used in the study of propagation in waveguides.

Recently, Y Li\textsuperscript{[9]} has also given approximate analytical expression for sound field under an arbitrary sound speed profile using a modified form of the WKB approximation.

In the next subsection the general form of the Green’s Function for pressure potential above an impedance boundary for an arbitrary sound speed profile is derived. This approach is an extension of one taken by Pekeris. In the subsequent subsections the solutions by Pierce, Tolstoy, Li and the WKB approximation are discussed in some detail. Finally, a new solution to propagation in linear sound speed profile \((1.5)\), arrived at independently of Pekeris, and in terms of Hankel functions of complex order, is offered and discussed.

**1.2 General form of the wave potential above impedance boundary**

Pekeris has derived the sound pressure potential for an arbitrary sound speed profile above a pressure release boundary given that the wave differential equation can be solved. Here, his method is extended for the more general case of an impedance boundary. The results of this section will be used later when various solutions to the wave equation are discussed.
Let \( c = c(z) \) be the sound speed profile and let the differential equation resulting from the wave equation have two independent solutions \( N(\zeta) \) and \( M(\zeta) \) where \( \zeta = \zeta(z) \). Let us assume further that of the two solutions \( N \) satisfies the Sommerfeld radiation condition, namely, that \( N \to 0 \) as \( z \to \infty \). The general solution for (1.3) is then:

\[
\varphi(z) = \begin{cases} 
A_i N(\zeta) & \text{if } z \geq h \\
A_i N(\zeta) + A_j M(\zeta) & \text{if } z \leq h
\end{cases}
\]

The boundary conditions to determine \( A_i \) are:

1. the continuity of pressure at source height,
2. unit step discontinuity of particle velocity at source height, and
3. impedance boundary at \( z=0 \):

\[
A_i N(\zeta) - [A_i N(\zeta) + A_j M(\zeta)] = 0 \quad \text{at } z = h
\]

\[
\frac{d\zeta}{dz} [A_i N(\zeta) - (A_i N'(\zeta) + A_j M'(\zeta))] = 1 \quad \text{at } z = h
\]

\[
\frac{d\zeta}{dz} [A_i N'(\zeta) + A_j M'(\zeta)] + \beta [A_i N(\zeta) + A_j M(\zeta)] = 0 \quad \text{at } z = 0
\]

where \( \beta \) is the surface admittance and the prime indicates differentiation with respect to the argument. The resulting set of linear equations for \( A_i \) are easily solved to give

\[
\varphi(z) = \begin{cases} 
\frac{1}{l(h)W} [M(\zeta) - N(\zeta)R]N(\zeta) & \text{if } z \geq h \\
\frac{1}{l(h)W} [M(\zeta) - N(\zeta)R]N(\zeta_h) & \text{if } z \leq h
\end{cases}
\]

where \( W \) is the Wronskian of \( N \) and \( M \) (\( = MN' - NM' \)), \( l(z) = d\zeta/dz \), and

\[
R = \frac{M'(\zeta_0) - qM(\zeta_0)}{N'(\zeta_0) - qN(\zeta_0)}
\]

(1.10)
and \( q = \frac{ik_0 B}{l(0)} \). (1.12)

In the preceding expressions the subscripts \( 0 \) and \( h \) denote evaluation of the term at \( z=0 \) and \( z=h \) respectively. The choice of \( M \) and \( N \) depends not only on the profile of sound speed but usually also on its sign. \( N \) must satisfy Sommerfeld radiation condition and can be considered an up-going wave. \( M \) can then be thought of as the corresponding down-going wave, possibly satisfying the radiation condition at \( z\to-\infty \). Equation (1.10) will prove useful in the following sections where various solutions to the inhomogeneous wave equation are discussed.

### 1.3 Bilinear sound speed profile

Pierce considered a bilinear sound velocity (1.6) where a positive \( R \) indicates a negative gradient for the sound speed profile.

We then have from (1.3):

\[
\frac{\partial^2}{\partial z^2} \varphi + \left[ k_0^2 - k_0^2 + \frac{2k_0^2 z}{R} \right] \varphi = \delta(z - h)
\]

(1.13)

This is an Airy differential equation the general solution of which is in terms of \( Ai(\zeta) \) and \( Bi(\zeta) \). where

\[
\zeta(z) = -i^{1/2}(k_0^2 - k_0^2 + i^2 z)
\]

(1.14)

and \( l = \left( \frac{2k_0^2}{R} \right)^{1/2} \)

(1.15)

Pekeris originally considered this profile and gave the solution for an unbounded medium in terms of Hankel functions of one-third order. Pierce,
on the other hand, chose \( Ai(\zeta) \) and Fock's function, \( w_1(\zeta) \) (=\( 2\pi^{1/2} e^{i\pi d} Ai(\zeta e^{2i\pi j}) \)) as the two independent solutions. Here we choose \( Ai \) and \( Ai-iBi \) functions. The choice is insignificant as long as the two functions are linearly independent and satisfy the boundary conditions. In fact the Fock's function is equivalent to \( 2i^d\pi [Ai(\zeta) - iBi(\zeta)] \) which for the sake of brevity we will call \( Ci(\zeta) \). Therefore

\[
Ci(\zeta) = Ai(\zeta) - iBi(\zeta) 
\]  

(1.16)

The boundary conditions for a semi-infinite fluid above an impedance plane boundary here are the Sommerfeld's radiation condition at \( z \to \infty \) and the impedance condition at \( z=0 \). For a sound profile with a positive gradient (i.e. negative \( R \)) \( Ai(\zeta) \) satisfies the radiation condition since as \( z \to \infty, \zeta \to \infty \) then \( Ai(\zeta) \to 0 \).

Therefore for a positive sound speed gradient (-ve \( R \)) we have:

\[
\varphi = \begin{cases} 
\frac{Ai(\zeta) - Ci(\zeta) - qCi(\zeta)}{I} & z \leq h \\
\frac{Ci(\zeta) - Ai(\zeta) - qAi(\zeta)}{I} & z \geq h 
\end{cases}
\]  

(1.17)

For the case of negative sound speed gradient (positive \( R \)) the choice for \( N(\zeta) \) is \( Ci(\zeta) \) since both \( Ai \) and \( Bi \) show oscillatory behaviour for \( \zeta \to -\infty \) and their asymptotic series representations do not converge but \( Ai-iBi \) is convergent. In this case we have:

\[
\varphi = \begin{cases} 
\frac{Ci(\zeta) - Ai(\zeta) - qAi(\zeta)}{I} & z \leq h \\
\frac{Ai(\zeta) - Ci(\zeta) - qCi(\zeta)}{I} & z \geq h 
\end{cases}
\]  

(1.18)
where \( q = ik_0 \beta l \) and \( W \) is the Wronskian of \( Ai \) and \( Bi \). It is independent of the argument and is equal to \( \pi^1 \). Pierce then proceeds to evaluate the Hankel integral by residue series method. Raspet et al.\(^{[10]}\) investigated the residue series solution and identified the terms relating to the surface waves that transport some acoustic energy into the shadow zone created by upward refracting sound speed profile. The Airy function solution has become the established method for tackling the problem of propagation through a medium with a sound velocity profile.

Tolstoy\(^{[4,5]}\) also considered a sound pressure field in a fluid with a similar sound velocity profile above a rigid rough boundary. The boundary condition is then\(^{[5]}\):

\[
\frac{d}{dz} \varphi = -\varepsilon \left( k^2 \varphi + \sigma \frac{d^2}{dz^2} \varphi \right) \quad \text{at } z = 0
\]

where \( \varepsilon \) & \( \sigma \) are parameters relating to the roughness of the boundary.

This can be rearranged to resemble the impedance boundary condition with an effective impedance.

\[
\frac{d}{dz} \varphi + \varepsilon k_0^2 \left( 1 - \sigma \frac{k_z^2}{k_0^2} \right) \varphi = 0 \quad \text{at } z = 0
\]

This suggests an effective admittance of

\[
\beta = -i \varepsilon k_0 \left( 1 - \sigma \frac{k_z^2}{k_0^2} \right)
\]

He bases his solution to the resulting Helmholtz equation in terms of Hankel functions of one third order:
\[
\varphi = \begin{cases} 
\gamma H^{(1)}_{1/3}(s) - \gamma H^{(1)}_{1/3}(s_h) + \gamma H^{(1)}_{1/3}(s) \\
\gamma H^{(2)}_{1/3}(s) + \gamma H^{(2)}_{1/3}(s) \\
H^{(1)}_{1/3}(s) - \gamma H^{(1)}_{1/3}(s_h) + \gamma H^{(1)}_{1/3}(s) \end{cases} \begin{cases} 
H^{(1)}_{-2/3}(s_h) - \gamma H^{(1)}_{-2/3}(s) \\
H^{(2)}_{-2/3}(s) - \gamma H^{(2)}_{-2/3}(s) \end{cases} \begin{cases} 
\gamma H^{(1)}_{1/3}(s) - \gamma H^{(1)}_{1/3}(s_h) + \gamma H^{(1)}_{1/3}(s) \end{cases} \begin{cases} 
\gamma H^{(2)}_{1/3}(s) + \gamma H^{(2)}_{1/3}(s) \end{cases} 
\]

where \( \gamma^2 = k_0^2 - k_r^2 + l^2 z = \zeta l \); \( s = \frac{2}{3l^2} \gamma^2 = \frac{2}{3} \zeta \gamma^2 \) and subscripts \( h \) and \( 0 \) refer to the value of the parameter at the source and ground altitudes respectively. By using the identity relations between the Airy functions and the Hankel functions of one third order \(^{[11]}\), it is easily seen that the two solutions are - apart from a numerical factor - identical, if we can assume that the rough boundary acts as a surface with an effective impedance \(^{[12,13]}\).

Given that \( z = (\frac{1}{3} \xi)^{2/3} \) we have:

\[
H^{(1)}_{1/3}(\xi) = e^{\frac{\mu \xi}{6\sqrt{2}}}[Ai(-z) - iBi(-z)]
\]

\[
H^{(2)}_{1/3}(\xi) = e^{\frac{\mu \xi}{6\sqrt{2}}}[Ai(-\xi) + iBi(-\xi)]
\]

\[
H^{(1)}_{2/3}(\xi) = e^{\frac{\mu \xi}{6\sqrt{2}}}[Ai'(-z) - iBi'(-z)]
\]

and

\[
H^{(2)}_{2/3}(\xi) = e^{-\mu \xi \sqrt{2}}[Ai'(-z) + iBi'(-z)]
\]

We note that \( \xi \) above is identical to \( s \) in Tolstoy's notation, and \( z \) above can be identified as \( \zeta \) in Pierce's expression.

Tolstoy predicted that the effect of roughness on a rigid surface is to produce surface waves which transfer energy into the shadow zone. This is shown in Figure 1.1 & Figure 1.2 where the predicted transmission loss of sound from a point source in an upward refracting atmosphere above a rough rigid plane is compared to the predicted loss above a rigid surface. The sound speed profile is assumed to be a bilinear profile. The roughness elements are
assumed to be closely-packed semi-cylindrical bosses of 5cm radius and a density of 10 per meter. The frequency is 100 Hz. The parameters $\varepsilon$ and $\delta$ are defined in reference [5] and the effective impedance (in fact admittance) was calculated from equation (1.21). In Figure 1.1 the source and receiver are assumed to lie on the surface and Figure 1.2 shows the same curves with source and receiver both at a height of 10m. clearly there is much less enhancement of the sound field in the case of elevated receiver which indicates that the predominant mechanism for transmission of energy into the shadow zone is the predicted surface wave.

![Figure 1.1 Sound field above a rigid (dashed line) and a rough rigid (solid line) surface in a shadow zone. Source and receiver are placed on the surface, and $dc/dz=-0.1$. The frequency is 100Hz.](image-url)
Chapter 1 - Wave Propagation...

Figure 1.2  Sound field above a rigid (dashed line) and a rough rigid (solid line) surface in a shadow zone. Source height (hs)=10.0 m, receiver height (hr)=10.0 m, and dc/dz=-0.1.

1.4 Wentzel-Kramers-Brillouin approximation

In a medium in which the speed of sound varies slowly over a distance compared to the wavelength of the sound one can use the geometrical acoustics or WKB approximation to solve the wave equation. The analysis given here, is due to Brekovskikh\textsuperscript{[7]} but a more rigorous and general derivation can be found in Olver\textsuperscript{[14]}, who refers to the method as the 'Liouville-Green approximation'. We write (1.3) in the form:

\[ \frac{\partial^2}{\partial z^2} \psi + F(z) \psi = \delta(z - h) \]  

(1.23)

If \( F(z) \) changes sufficiently little over distances greater than a wavelength, (1.23) can be satisfied by a solution of the form

\[ \psi = \Psi(z)e^{i\phi(z)} \]  

(1.24)
The amplitude ($\Psi$) and the phase($\Phi$) functions are determined by substituting into and equating terms of different orders to zero separately:

$$[\Psi'''] + [2i\Phi'\Psi' + i\Phi''\Psi] + [(F^2(z) - \Phi'')\Psi] = 0$$

The terms in each square bracket are put equal to zero separately. We will then have

$$\Phi = \pm \int_{z_0}^{z} F(z)\,dz$$

(1.26)

and

$$\Psi = \frac{A}{\sqrt{F(z)}}$$

(1.27)

$z$ is the observer height and $z_0$ is some reference height which can be taken to be the source height or zero. The exact value is unimportant save for numerical stability considerations. Olver states that the condition for the validity of the approximation is that $|\frac{\Phi'''}{\Phi''}|$ is small and can be neglected.

The solution has a pole at $F(z)=0$. This can be tackled by making $k_r$ complex. This is equivalent to changing path of integration from the real axis to a line below the real axis. The integral in the phase function may be evaluated numerically by appropriate means. However, for certain sound speed profile functions the integral has a closed form solution. One such profile function that resembles the linear sound speed gradient is:

$$c(z) = \frac{c_0}{1 + \alpha z}$$

(1.28)
This profile is equivalent to having the refractive index increasing linearly with altitude. In this case we have

\[ k(z) = k_0(1 + \alpha z) \]

\[ F(z) = \sqrt{k_0^2(1 + \alpha z)^2 - k_r^2} \]

(1.29)

and

\[ \Phi = \int_{z_0}^{z} \sqrt{k_0^2(1 + \alpha z)^2 - k_r^2} \, dz \]

\[ = \frac{1}{k_0 \alpha} \left\{ \frac{1}{2} k_0 (1 + \alpha z) \sqrt{k_0^2(1 + \alpha z)^2 - k_r^2} - k_r^2 \frac{1}{2} \ln \left| \sqrt{k_0^2(1 + \alpha z)^2 - k_r^2} + k_0 (1 + \alpha z) \right| \right\} - \Phi_0 \]

(1.30)

We now have two independent solutions to the wave equation which can be invoked to find the general solution in this case. For upward refraction case (negative a) we have the following choices for \( N(z) \) and \( M(z) \):

\[ N(z) = \frac{1}{\left( k_0^2(1 + \alpha z)^2 - k_r^2 \right)^{1/2}} e^{i\phi(z)} \]

\[ M(z) = \frac{1}{\left( k_0^2(1 + \alpha z)^2 - k_r^2 \right)^{1/2}} e^{+i\phi(z)} \]

and \( \Phi \) is determined by (1.30).

This method has limited usefulness and has been superseded by a modified WKB method which is valid even for \( F(z) = 0 \). This will be described in the next section.

1.5 WKB-type approximation with Airy functions

Recently, Y Li\(^9\) has given an approximate solution to the transformed wave equation with an arbitrary sound speed profile in terms of Airy functions. The solution is valid even at the turning points \( (k(z) = 0) \) and Y Li has shown that it agrees very well with output of an FFP program for benchmark linear
and bilinear profiles. This approximation was first arrived at by Langer\textsuperscript{[16]}. A more relevant form of it can be found in Olver\textsuperscript{[14]}, Goyal\textsuperscript{[17]} and Abram\textsuperscript{[11]}.

Here we give the mathematical background to the problem based on references [17], [11] and [14]. The transformed wave equation is written as;

\[
\frac{\partial^2}{\partial z^2} \varphi + F^2(z) \varphi = \delta(z - h)
\]  

(1.31)

with \( F^2(z) = k^2(z) - k^2 \)

A solution to the differential equation of this type can be given in WKB approximation which was described earlier. The alternative is as follows. If \( F(z) \) can be written in the form

\[
F^2(z) = \lambda^2 p(z) + q(z, \lambda)
\]  

(1.32)

where \( \lambda \) is some parameter, possibly complex, then the zeros of \( p(z) \) are said to be the turning points of the differential equation. In all the cases discussed here we will put \( q(z, \lambda) \) equal to zero. The special case of \( p \equiv z \) reduces to the Airy differential equation, described in section 1.3, which has a solution in terms of \( \text{Ai}(\lambda^{2/3} z) \). The precise series solutions for the special case are given in references [11] and [14]. The general case can easily be transformed into this special case if we make the substitution

\[
w(\zeta) = \left[ \frac{p(z)}{\zeta} \right]^{1/4} \varphi(z) \quad \text{with} \quad \zeta(z) \quad \text{being the solution of}
\]

\[
\zeta(z)\zeta' = p(z).
\]

The final expression is then

\[
\varphi(z) = \left[ \frac{\zeta}{p(z)} \right]^{1/4} \text{Ai}(\lambda^{2/3} \zeta)
\]  

(1.33)
with $\zeta(z)=\left\{\frac{3}{2} \int_{z_0}^{z} \sqrt{F^2(z)} dz \right\}^{\frac{1}{2}} \quad z \geq z_0$

and $\zeta(z)=-\left\{\frac{3}{2} \int_{z}^{z_0} \sqrt{-F^2(z)} dz \right\}^{\frac{1}{2}} \quad \text{for } z_0 \geq z$

(1.34a & b)

The expression is valid even at turning points (i.e. $F^2(z)=0$) since the term in square brackets tends to a limiting value when $p$ goes to zero. Conditions under which (1.33) is valid are that

- there is only one turning point ([8] & [14]),
- $F$ is a slowly varying function of $z$ ([17]),
- $q(z)$ can be neglected, and that
- $|\lambda|$ is large.

This last condition is violated here (it is equal to 1) but since $q(z,\lambda)$ is set equal to zero it seems to be unimportant. The two independent solutions for the potential are again chosen to be $Ai$ and $Ci$ (see (1.16) above). These two functions are substituted in (1.10) to obtain the field potential above an impedance boundary. $Z_0$, the turning point, is the solution of $F^2(z)=0$. In terms of ray acoustics, every positive turning point represents a direct ray from source to the receiver. In the following section the applicability of this approach to the logarithmic sound speed profile is considered and some practical points in calculating the phase function are discussed.

1.5.1 Practical considerations

An interesting consequence of the turning point theory is that under certain circumstances there may be more than one direct ray present. Consider a
logarithmic sound speed profile. This type of profile is very common in wind speed profiles.

\[ c(z) = c_0 \left( 1 + a \ln(1 + bz) \right) \]  

(1.35)

where \( a \) is +ve for down-wind and -ve for up-wind conditions. The turning point is the solution of

\[ k(z) = k_0 \frac{1}{1 + a \ln(1 + bz)} = \pm k, \]

(1.36)

which gives

\[ z_0 = \frac{1}{\beta} \left\{ e^{\frac{i}{1 - \sin \mu}} - 1 \right\} \]

(1.37)

where \( \sin \mu = \frac{k_z}{k_0} \) represents the incident angle of the ray which is real for downward refracting direct rays and complex for upward refracting ones (since they do not reach the ground). For real \( \mu, \sin \mu < 1 \) and it is clear that for down-wind case only the +ve sign in the exponent gives a positive \( z \), therefore having only one turning point, while for up-wind case (-ve \( a \) and \( \sin \mu > 1 \)) both + and - signs give valid (i.e. positive) values for \( z \) which effectively means two direct rays in an upwind condition. Since this has already been predicted in ray tracing simulations, it shows that the numerical simulation agrees with analysis rather than being a numerical error. The same applies to an ‘inverse square’ wind profile. Clearly, the fact that there are two turning points, renders the modified WKB method unsuitable for this important class of wind speed profiles. The positive log gradient case is not valid either. This is because near the ground the rate of change of sound speed with height is large (~ 2 m\(^1\)) and the condition that the parameters
should change little within one wavelength is not true for most audio frequencies. This is seen by comparing the results of the above method with that of an FFP program.

![Figure 1.3 Path of integration of the phase function when the turning point is complex.](image)

The integrals in the calculation of the arguments for the Airy functions, eqns. (1.34), must be performed numerically in general, although for certain profiles such as the linear profile or the bilinear refractive index profile described in section 1.3, the integral can be evaluated analytically. In this case the evaluation of the amplitude function (square bracket in (1.33)) is straightforward even at the turning points. For other profiles, the integral must be evaluated numerically. Y Li has suggested a novel method proposed by Perez et al.[18,19] based on Gauss-Chebychev quadrature which is both efficient and accurate. In this case the amplitude function must be rewritten as $(\zeta')^{-1/2}$ where the prime denotes derivative with respect to $z$. At turning points this must be evaluated numerically as a finite difference approximation to the differentiation. In any case, since the horizontal wavenumber is complex, the turning point, $z$, will be a complex quantity. Integration then is performed along the path shown in Figure 1.3.
1.6 A solution for a constant gradient profile

In this section an analytical solution to the wave equation with a linear sound speed profile is discussed. As mentioned earlier, Pekeris\(^{[2]}\) has offered a solution for this problem in an unbounded medium in terms of Bessel functions of complex order. The derivation in this section, was formulated before the author was aware of Pekeris' work and therefore obtained independently. However this section extends Pekeris' work by attempting to find approximate asymptotic form for the Hankel function suitable for numerical computations.

Let speed of sound vary with altitude as:

\[
c(z) = c_0 \left(1 + \frac{z}{R}\right)
\]

(1.38)

with \(c_0\) and \(R\) being speed of sound at the ground level and the radius of curvature of rays respectively. Changing the variable to \(x = z + R\), the Helmholtz equation becomes:

\[
\frac{\partial^2}{\partial x^2} \varphi + \left[\left(\frac{k_0 R^2}{x}\right)^2 - k_r^2\right] \varphi = \delta(z - h)
\]

(1.39)

This is a transform of the Bessel Differential equation the general solution of which is:

\[
\psi = \sqrt{x} \mathcal{H}_\nu \left(ik_r x\right)
\]

where \(\nu = (1/4 - k_r^2 R^2)^{1/2} = ik R\) and \(\mathcal{H}_\nu \left(\right)\) is a linear combination of Hankel functions. We have then:

---

\(^{[2]}\) A version of this section was presented at the Fifth International Symposium on Long Range Sound Propagation, Milton Keynes, 1992\(^{[3]}\).
Chapter 1 - Wave Propagation

\[
\varphi = \begin{cases} 
A_1 \sqrt{x} H^1_\nu (ikx) & z \geq h \\
A_2 \sqrt{x} H^1_\nu (ikx) + B_2 \sqrt{x} H^2_\nu (ikx) & z \leq h 
\end{cases}
\]  \hspace{1cm} (1.40)

where \( H^1_\nu() \) has been chosen for the field above the source height so as to satisfy the radiation condition. The unknown amplitudes are again determined from the impedance boundary condition at the ground surface and continuity equation at the source position. Writing for simplicity:

\[
H_1(x) = \sqrt{x} H^1_\nu (ikx) \\
H_2(x) = \sqrt{x} H^2_\nu (ikx)
\]  \hspace{1cm} (1.41)

we have

\[
\varphi = \begin{cases} 
Q H_1(x_0) \left[ H_2(x) - H_1(x) \frac{k_r H'_2(x_0) + k_0 \beta H_2(x_0)}{k_r H'_1(x_0) + k_0 \beta H_1(x_0)} \right] & z \leq h \\
Q H_1(x) \left[ H_2(x) - H_1(x) \frac{k_r H'_2(x_0) + k_0 \beta H_2(x_0)}{k_r H'_1(x_0) + k_0 \beta H_1(x_0)} \right] & z \geq h 
\end{cases}
\]  \hspace{1cm} (1.42)

where \( Q = (\pi S)/(4k_0) \), \( S \) is the source strength, and subscripts \( h \) and \( 0 \) refer to the value of the parameter at the source and ground altitudes respectively. This solution introduces Hankel functions of complex order. Consequently, it is necessary to consider the computation of such functions. This is discussed in the next subsection.

1.6.1 Asymptotic Approximations to Hankel functions of large, complex order

Hankel or Bessel functions of complex order cannot be evaluated from the standard approximations to the Bessel functions of integer order. Specific series approximations are needed to evaluated these functions correctly. The following derivations are mainly due to Debye. For reference see Watson’s
comprehensive Treatise on Theory of Bessel Functions\textsuperscript{[20]}. There are two different approximations to $H_v(u)$ according to whether difference between argument and the order of the Hankel function is small or large.

1.6.1.1 $|u-v|>>1$

Let $|u-v|$ be large. Then we have:

$$H^{(1)}_v(u) = \frac{1}{i\pi} \int_{-\infty}^{\infty} e^{-u(w \cosh y - \sinh w)} dw$$

(1.43)

and

$$H^{(2)}_v(u) = \frac{1}{i\pi} \int_{-\infty}^{\infty} e^{-u(w \cosh y - \sinh w)} dw$$

(1.44)

where $\gamma = \cosh^{-1}\left(\frac{\sqrt{\gamma}}{|u|}\right)$ and $u$, $v$ and $\gamma$ are complex.

To evaluate the above integrals we can utilise their asymptotic series form provided that: $\text{Re}(u) > 0$ and $0 < \text{Im}(\gamma) < \pi$. Both these conditions are satisfied if $k$, is complex with a small negative imaginary part. Then, we will have;

$$H^{(1)}_v(u) = F(\tanh \gamma) e^{v(\tanh \gamma - \gamma)} \sum_{m=0}^{\infty} \frac{\Gamma(m + \frac{1}{2})}{\Gamma(\frac{1}{2})} \frac{A_m}{(0.5v)^m}$$

(1.45)

and

$$H^{(2)}_v(u) = F(\tanh \gamma) e^{v(\tanh \gamma - \gamma)} \sum_{m=0}^{\infty} \frac{\Gamma(m + \frac{1}{2})}{\Gamma(\frac{1}{2})} \frac{A_m}{(0.5v)^m}$$

(1.46)

where $F(\tanh \gamma) = (-0.5i\pi v \tanh \gamma)^{-\gamma}$, and $A_0=1$,

$$A_1 = \frac{(3 \coth \gamma - 5 \coth^3 \gamma)}{24}, A_2 = \frac{3}{128} \coth^2 \gamma - \frac{77}{576} \coth^4 \gamma + \frac{385}{3456} \coth^6 \gamma, \ldots$$

and $\Gamma(\cdot)$ is the Gamma function.
1.6.1.2 \(|v| = |u|\)

If the order and the argument of the function are of the same magnitude, we need to evaluate the following integral form:

\[
H_{v}^{(1,2)}(u) = \frac{\pm 1}{i\pi} \int_{-\infty}^{\infty} e^{-i(w(u-i\epsilon)-u\omega)}dw
\]

where \(\epsilon = \left(\frac{u-v}{u}\right)\) is a small quantity. The resulting asymptotic series is

\[
H_{v}^{(1,2)}(u) = \frac{2}{\pi} \sum_{m=0}^{\infty} e^{\pm \frac{1}{2}(m+1)\pi} \sin \frac{1}{2}(m+1)\pi \cdot E_{u}(eu) \frac{\Gamma \left( \frac{1}{2} \frac{m+1}{2} \right)}{\left( \frac{u}{6} \right)^{\frac{m+1}{2}}}
\]

where

\[
B_0 = 1,
\]

\[
B_1 = eu,
\]

\[
B_2 = \frac{(eu)^2}{2} - \frac{1}{20},
\]

\[
B_3 = \frac{(eu)^3}{6} - \frac{1}{15},
\]

\[
B_4 = \frac{(eu)^4}{24} - \frac{(eu)^2}{24} + \frac{1}{280}, \ldots
\]

This is valid if \(eu \sim O(u^{13})\).

No detailed computations have been shown for this new solution since the recent developments (see section 1.5) have given us a more general form of the wave potential in the presence of sound speed gradients.

1.7 Conclusion

This chapter reviews the existing analytical solutions to the wave equation for propagation in a variable sound speed profile. An expression for sound field in an inhomogeneous medium above an impedance boundary is offered
(extension of work by Pekeris). Application of a recent development in WKB approximation\textsuperscript{[9]} (WKB-Airy approximation) to the important case of logarithmic sound speed profile is considered. It is shown that this type of sound speed profile (both positive and negative gradient) violates the underlying restrictions of the approximation and therefore cannot be solved by this method.

Finally, an analytical solution to the wave equation with a linear sound speed profile is offered and asymptotic forms for the resulting functions are derived and discussed.
Chapter 2

Fast Field Calculations for Propagation Above Porous Elastic Media

2.1 Introduction

The main purpose of this chapter is to describe the theory of wave propagation in fluid-saturated porous elastic media and the details of a program to simulate such propagation namely, the Fast Field Program for Layered Air-Ground Systems (FFLAGS). Although the code was developed previously,\[20\] important modifications to the implementation of the theory are described in this thesis.

The basis for FFLAGS and its formulation have been described elsewhere\[20,21\], however, certain modifications have been made here to the original formulation. Consequently the relevant equations and boundary conditions are repeated explicitly here. The main changes involve:

- corrections to the implementation of the boundary conditions
- extension to allow for sources within the poroelastic layers
- improved efficiency of the numerical methods used in the original version of FFLAGS and described elsewhere\[20,21\].

These alterations are described in the subsequent sections of this chapter.
2.2 Biot theory of poroelasticity

In a series of papers, Biot\cite{22,25} developed a theory of propagation of waves in a porous elastic medium by considering stresses and strains on fluid and solid components. He assumed existence of a potential energy $W$ so that stress-strain relations can be written in terms of derivatives of $W$. By writing down the kinetic energy, $T$, and the Lagrange’s equations for the aggregate, he derived the equations of coupled motion for the propagation of waves. He allowed for viscous effects by adding a viscosity correction function to his equations to compensate for the breakdown of Poiseulle flow in the pores. In the following section his notation of ref. \cite{24} is used. The Biot equations of motion in two-phase media, as modified by Stoll\cite{26}, are:

\begin{align}
\nabla^2 (\bar{H}\Phi_s - \bar{C}\Phi_f) &= \frac{\partial^2}{\partial t^2} (\rho_s \Phi_s - \rho_f \Phi_f), \\
\nabla^2 (\bar{C}\Phi_s - \bar{M}\Phi_f) &= \frac{\partial^2}{\partial t^2} (\rho_f \Phi_s - m \Phi_f) - \frac{\eta}{\kappa} F(\lambda) \frac{\partial \Phi_f}{\partial t} 
\end{align}

(2.1)

(2.2)

and assuming

\[ \Phi_{s,f}(r,t) = e^{-i\omega t} \phi_{s,f}(r,k), \]

the time independent potentials may be written

\begin{align}
\nabla^2 (\bar{H}\phi_s - \bar{C}\phi_f) &= -\omega^2 (\rho_s \phi_s - \rho_f \phi_f), \\
\nabla^2 (\bar{C}\phi_s - \bar{M}\phi_f) &= -\omega^2 (\rho_f \phi_s - \rho' \phi_f) 
\end{align}

(2.3)

(2.4)
where \( m \) is a factor that accounts for extra inertia due to the fact that not all fluid flows along the axis of pores and \( \rho' \) can be considered as a complex fluid density:

\[
\rho' = m - \frac{i \eta}{\omega \kappa} F(\lambda)
\]  

(2.5)

and

\[
m = \frac{q^2 \rho_f}{\Omega}
\]

(2.6)

\( H, C & M \) are effective bulk moduli of elasticity. \( H \) can be thought of as the corresponding effective modulus of the solid, while \( C & M \) are the coupling terms. The three moduli are complex quantities to allow for inelastic properties of the medium and are determined from the solid and fluid bulk moduli of the constituent parts\(^{[26,27]}\):

\[
\bar{H} = \left[ \frac{(k_s - k_b)^2}{D - k_b} + k_b + \frac{\rho}{\eta} \mu \right]
\]

(2.7)

\[
\bar{C} = k_s \frac{(k_s - k_b)}{D - k_b}
\]

(2.8)

and

\[
\bar{M} = \frac{k_s^2}{D - k_b}
\]

(2.9)

with

\[
D = k_s \left[ 1 + \Omega \left( \frac{\kappa}{\mu} - 1 \right) \right]
\]

(2.10)

\( k_s \) = Bulk modulus of the solid grains.
The corresponding equations of motion for the rotational motion are:

\[ \mu \nabla^2 \chi_1 = -\omega^2 (\rho \chi_1 - \rho_f \chi_2), \]
\[ 0 = (\rho_f \chi_1 - m \chi_2) - \frac{i \eta}{\kappa \omega} F(\lambda) \chi_2 \]

with \( \mu \) being the rigidity of the material.

One can write the vector potentials \( \chi_{1,2} \) in terms of a scalar potential \( \phi_1 \). In cylindrical coordinates (which is used here) then:

\[ \tilde{x}_1 = -\frac{\partial \phi_1}{\partial r}, \]

\[ \tilde{x}_2 = m_3 \tilde{x}_1 \]

The fact that fluid rotation is proportional to the solid rotational movement and not independent motion, stems from eqn. (2.12) and is the result of the assumption, made by Biot, that the fluid is an ideal fluid and does not support vorticity. The solid displacement \( (u) \) and relative fluid displacement \( (w) \) can be expressed in terms of the three potentials:

\[ \tilde{u} = \nabla \phi_2 + \nabla \times \chi_1, \]
\[ \tilde{w} = \Omega (\tilde{u} - \tilde{U}) = \nabla \phi_f + m_3 \nabla \times \chi_1 \]
the sign of \( \phi_r \) in eqns. (2.1) & (2.2) and subsequent equations should also be changed to preserve the consistency.

If one assumes a plane wave propagating normal to the yz-plane we have for the potentials:

\[
\begin{align*}
\phi_x &= A e^{(k_x x - \omega t)}, \\
\phi_f &= B e^{(k_x x - \omega t)}, \\
X_1 &= X_1 e^{(k_x x - \omega t)} \quad (2.17) \\
X_2 &= X_2 e^{(k_x x - \omega t)}
\end{align*}
\]

then substitution in (2.3) and (2.4) and some rearrangement results in the following equations for the amplitudes:

\[
\begin{align*}
A(\bar{H}k^2 - \rho \omega^2) + B(\rho_f \omega^2 - \bar{C}k^2) &= 0 \\
A(\bar{C}k^2 - \rho_f \omega^2) + B(\rho' \omega^2 - \bar{M}k^2) &= 0
\end{align*}
\]

(2.18)  
(2.19)

To satisfy non-triviality condition for \( A \) & \( B \), one puts the coefficient determinant equal to zero to obtain the following dispersion equation for the dilatational phase velocities:

\[
(\rho_f^2 - \rho \rho') v_i^4 + (\bar{H} \rho' + \rho \bar{M} - 2 \rho_f \bar{C}) v_i^2 + (\bar{C}^2 - \bar{H} \bar{M}) = 0
\]

(2.20)

where \( v_i \ (= \omega / k_i) \) are the phase velocities. The resulting quartic equation has two roots. The two dilatational waves are called "fast" and "slow" waves. Both have components in the fluid as well as in the solid. The "fast" wave travels chiefly in the solid with little attenuation and is usually faster than the other wave (hence the name). It is very similar to the familiar P-wave. The
"slow" wave, on the other hand, is highly attenuated wave with a low phase speed that travels mainly in the fluid. At audio frequencies in high flow resistivity soils, it is diffusive in nature. Biot explains this wave as the wave with solid and fluid moving out of phase. Attenborough \(^{[28]}\) has explored conditions under which the slow wave is similar to the 'pore wave' predicted in the rigid porous ground models. There are, however, circumstances under which the slow wave also becomes a true propagating wave. This occurs at high porosity and high frequencies. The characteristics of these waves and their dependence on ground porosity and on frequency will be explored in chapter 3. If either \(k_3=0\) (pure fluid) or \(k_e=0\) (elastic limit) then the above equation has only one solution and the corresponding dispersion equation for fluid or elastic media is retrieved.

Upon substituting \(\chi_1\) & \(\chi_2\) in equations (2.11) & (2.12) and solving for the non-trivial case we obtain for the shear wave speed:

\[
\nu_s^2 = \frac{\mu \left( m - \frac{i\eta}{\omega \kappa} F(\lambda) \right)}{\rho \left( m - \frac{i\eta}{\omega \kappa} F(\lambda) \right) - \rho_f^2}
\]

(2.21)

As mentioned, there is only one shear wave propagating in the porous solid which is very similar to the elastic S-wave.

The viscosity correction function, \(F(\lambda)\), arises from including viscous drag of the fluid in the pores. It accounts for the breakdown of the Poiseuille flow in the pores and depends on a dimension-less parameter relating to the thickness of the boundary layer at the pore walls. Thus \(\eta F(\lambda)\) acts as a dynamic viscosity factor.
Chapter 2 • Fast Field

Biot’s theory of wave propagation in porous elastic media has been used extensively to model the propagation both in atmosphere and under water. (See for example references [27-29, 74-77])

2.3 Fast Field Program

In the FFP formulation\(^{[78-80]}\), we assume that the two inhomogeneous media in contact (a fluid above a poro-elastic ground) consist of vertically stratified homogeneous layers, and that the whole system bounded above and below by homogeneous half-spaces. The wave equation in each layer- assuming a time dependence of \( \exp(-i\omega t) \) is

\[
\nabla^2 \Psi_i(r, z) + k_i^2 \Psi_i(r, z) = \delta_i(r, z)
\]

(2.22)

where \( \Psi_i \) are the scalar displacement potentials for various wave types propagating in the medium, and \( k_i(=\omega/\sigma_i) \) are the corresponding wave numbers in layer \( i \). There is one compressional wave propagating in the fluid and two compressional and one shear wave traveling in the porous elastic ground. The wave numbers for the ground waves are determined from the dispersion equations. These, in turn, are derived from the Biot-Stoll equations of motion for a two-phase medium (see Chapter 3 for a detailed discussion). Here we use subscripts \( i=0 \) for the fluid wave, \( i=1,2 \) for the two compressional waves and \( i=3 \) for the shear wave in the solid. We also use a cylindrical system of co-ordinates throughout.

We note that there is an radial symmetry in our problem as set out here. To separate the radial and vertical variables in the equation and, thus, to reduce
this partial differential equation to an ordinary one, we use a pair of Hankel Transform integrals:

\[ \Psi_i(r,z) = \int_0^\infty \Psi_i(z,k_r) J_0(k_r r) k_r dk_r \]

and

\[ \psi_i(z,k_r) = \int_0^\infty \Psi_i(r,z) J_0(k_r r) r dr \]

where \( J_0(z) \) is the zero order Bessel function and \( k_r \) is the variable of integration and can be thought of as the horizontal or radial component of the wave number. Applying the second of these to the wave equation we obtain the transformed Helmholtz Equation:

\[ \frac{\partial^2}{\partial z^2} \psi_i + \beta_i^2 \psi_i = S_0 \delta(z) \]

where

\[ \beta_i^2 = k_i^2 - k_r^2 \]

and the right hand side is the source term. In this way the problem of determining the wave amplitudes is reduced to one of solving a set of ordinary differential equations (ODE's). The boundary condition equations (BCE's) are put in the form of a Global Matrix equation:

\[ A \cdot X = B \]

where \( X \) is a vector containing the wave amplitudes( \( A_i \) and \( R_i \) ), \( A \) an \( N \times N \) matrix containing the coefficients from the BCE’s and \( B \) is the source term.
vector. The order of the matrix, \( N \), is related to the number of fluid layers, \( n_f \), and the number of solid layers, \( n_s \), both including the half-space by:

\[
N = 6(n_s - 1) + 2n_f + 2
\]

(2.27)

The matrix equation can then be solved by a variety of methods including Gaussian elimination with pivoting or Crouts decomposition (the preferred method here).

Subsequently the forward Hankel transform is applied to obtain the full wave solutions. The essence of the FFP technique is that once the Green's functions (the range-independent \( \psi \)) are known as a function of \( k_r \), the transform can be replaced by a Fast Fourier Transform. This may be calculated in the far field by substituting its large argument approximation for the Bessel function. The integral can then be evaluated very quickly and efficiently using Discrete Fourier transform techniques available in signal processing. The inherent limitation in this process is that \( k_r \gg 1 \). Typically this translates to ranges greater than a couple of wavelengths. The solution to the equation (2.24) is trivial within each layer. Writing the solutions for each layer explicitly, we have:

\[
\psi_0 = R^+ e^{j(k_r-d_1)\beta_1} + R^- e^{-j(k_r-d_2)\beta_2}
\]

(2.28)

\[
\psi_1 = A_1^+ e^{j(k_r-d_1)\beta_1} + A_1^- e^{-j(k_r-d_2)\beta_1} + A_2^+ e^{j(k_r-d_1)\beta_2} + A_2^- e^{-j(k_r-d_2)\beta_2}
\]

(2.29)

\[
\psi_2 = m_1[A_1^+ e^{j(k_r-d_1)\beta_1} + A_1^- e^{-j(k_r-d_2)\beta_1}] + m_2[A_2^+ e^{j(k_r-d_1)\beta_2} + A_2^- e^{-j(k_r-d_2)\beta_2}]
\]

(2.30)

\[
\psi_3 = A_3^+ e^{j(k_r-d_1)\beta_2} + A_3^- e^{-j(k_r-d_2)\beta_2}
\]

(2.31)
where in (2.28), \( h1 \) & \( h2 \) are lower and upper fluid layer boundaries \( (h2 > h1) \) and \( z \) is positive away from the fluid-solid interface; and in equations (2.29)-(2.31), \( d1 \) & \( d2 \) are upper and lower solid boundaries \( (d2 > d1) \) and \( z \) is positive downwards from the interface. \( R^\pm \) and \( A_n^\pm \) \( (n = 1,2,3) \) are the amplitudes to be determined from the boundary condition equations. Each potential consists of upgoing and downgoing terms. Also, because the two compressional wave types can exist simultaneously in solid and pore fluid phases, the potentials are a linear superposition of the two wave solutions with \( m_i \) being the appropriate ratios of solid wave to pore wave (see Chapter 3 for more details).

The required parameters for the boundary conditions are the solid & fluid displacements and stresses.

The fluid displacement is simply

\[
\nabla \Psi_0 = \left( \frac{\partial \Psi_0}{\partial r}, \frac{\partial \Psi_0}{\partial z} \right)
\]

(2.32)

and the pressure is \( \rho \omega^2 \Psi_0 \)

In the ground the solid phase displacement, \( u \), is:

\[
\bar{u} = \nabla \Psi_1 + \nabla \times \bar{\chi}_1
\]

\[
u_r = \frac{\partial \Psi_1}{\partial r} + \frac{\partial^2 \Psi_1}{\partial \bar{r} \partial z},
\]

\[
u_s = \frac{\partial \Psi_1}{\partial z} - \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \Psi_3}{\partial r} \right)
\]

(2.33)

The pore fluid motion is given, most conveniently, by the relative fluid motion defined as:
\[ w = \nabla \psi_2 + \nabla \times \chi_2 \]
\[ w_r = \frac{\partial \psi_2}{\partial r} + m_3 \frac{\partial^2 \psi_3}{\partial r \partial z}, \]
\[ w_z = \frac{\partial \psi_2}{\partial z} - m_3 \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \psi_3}{\partial r} \right) \]

(2.34)

In (2.33) & (2.34), \( \chi_i \) are vector potentials representing the transverse motion, and

\[ \chi_2 = m_3 \chi_1 \]
\[ \tilde{\chi}_1 = -\frac{\partial \psi_3}{\partial r} \]

(2.35)

\( m_3 \) is the ratio of fluid rotational motion to the solid one.

It should be noted that the boundary conditions apply to the full range-dependent potentials and therefore care should be taken in determining the correct forms for the Green's functions. For example:

\[ \frac{\partial}{\partial r} \psi = \frac{\partial}{\partial r} \int \psi(z) J_0(k_r r) k_r dk_r, \]
\[ = \int \psi(z) J_1(k_r r) k_r^2 dk_r = \int (-j k_r) \psi(z) J_0(k_r r) k_r dk_r, \]

(2.36)

as \( \frac{\partial}{\partial z} J_0(z) = -j J_0(z) \) for large \( z \).

and

\[ \frac{\partial^2}{\partial r^2} \psi = \frac{\partial^2}{\partial r^2} \int \psi(z) J_0(k_r r) k_r dk_r, \]
\[ = \int (-k_r^2) \psi(z) J_0(k_r r) k_r dk_r, \]

(2.37)
2.3.1 Inherent Assumptions- Open Pore Condition

In deriving the linear wave equation, it has been assumed that changes in the fluid density in atmosphere are sufficiently small that they may be ignored. Some researchers have investigated the validity of assuming that pores of contiguous poro-elastic layers are microscopically connected at the boundary.\textsuperscript{[29]} A standard procedure has been to include an additional parameter (T) to allow for variations in the extent to which pore fluid flow between adjacent layers is restricted. Clearly in an extreme case where say the lower boundary has pores that are much smaller than those in the upper layer an open pore condition will not be satisfactory. On the other hand such a situation might not be far removed from a closed pore condition and an adequate approximation would be to assume that the lower layer is elastic and not porous. Of more interest are cases where the pores in adjacent layers are partially connected or open to each other. However there is no convenient way of measuring the degree of openness. In the remainder of this thesis therefore the boundaries are treated as either open pore or closed pore (i.e. the lower layer is elastic and not porous). An additional parameter to allow for intermediate conditions is not included. It should be added that if a need for calculation of such partially open pores arises, it should be possible to simulate this case by inserting a thin porous layer of small porosity (depending on the degree of 'openness') and similar constitution to the upper or lower layer between the relevant layers. However, this would punish the computation time severely and is not recommended.
2.4 Boundary conditions

We here discuss the required boundary conditions in both fluid and poroelastic solid layers and give the resulting BC Equations explicitly. These are essentially as reported elsewhere\(^{[20]}\) but various errors (typographical or otherwise) are corrected and several necessary modifications are made. In the following, the subscript \(n\) signifies the layer, with \(n = 1\) at the upper solid-fluid interface and increasing in the positive \(z\) direction (i.e. away from the surface) in each medium. We denote \(\exp(jD_n\beta_{i,n})\) by \(Q_{i,n}\), where \(D\) is the layer thickness, and \(\beta_{i,n}\) is the vertical component of wavenumber of \(i\)th wave type. \(S\) is the relevant source term (if any) in that particular layer. The source term will be discussed in the next section.

2.4.1 Fluid-fluid interfaces

The boundary conditions for fluid-fluid interfaces in a layered fluid system are well known\(^{[7,30]}\), Two conditions are required as there are two unknown amplitudes in each layer. These, expressed for the range-independent potentials, are:

1. Continuity of pressure, \(P (=\rho\omega^2\psi_0)\)

\[
\rho_n\omega^2 [R_n^1 + R_n^i Q_{0,n}] + S_n = \rho_{n1}\omega^2 [R_{n1}^i Q_{0,n+1} + R_{n1}^1] + S_{n+1} \tag{2.38}
\]

2. Continuity of normal fluid displacement, \(u_z = \frac{\partial \psi_0}{\partial z}\).

\[-j\beta_{0,n} R_n^1 + j\beta_{0,n} R_n^i Q_{0,n} + S_n = -j\beta_{0,n+1} R_{n1}^i Q_{0,n+1} + j\beta_{0,n+1} R_{n1}^1 + S_{n+1} \tag{2.39}\]
2.4.2 The fluid-solid interface

Boundary conditions and continuity equations for interface between a fluid and an elastic solid are also fairly well known and used extensively\textsuperscript{10}. The BC's for fluid/solid and solid/solid boundaries in a layered poro-elastic solid medium are derived by Deresiewicz & Skalak\textsuperscript{11}. They considered the rate of work done across a boundary using Biot's equations and determined the conditions under which there would be a unique solution to his equations of motion. Although their BC's are specific to the Biot theory, they are what one would expect from intuition, namely, that they are essentially continuity of volume averaged stresses and displacements (or velocities) of solid and fluid components normal to the interface and parallel to it. There are three wave types in the porous layer and one wave in the fluid layer so that four boundary conditions are required for uniqueness of the solutions at the fluid/porous-elastic solid interface:

3. Continuity of total normal stress, $-P = \sigma_{zz}$

$$-\rho \omega^2 \psi \rho = \bar{H} \nabla \cdot \bar{u} - \bar{C} \nabla \cdot \bar{w} - 2\mu \frac{\partial u}{\partial r}$$

(2.40)

where $u$ & $w$ are the solid and relative fluid displacements defined above.

$$-\rho \omega^2 [R^1Q_0 + R^1] + S_f = -[k_1^2 (\bar{H} - m_1 \bar{C}) - 2\mu k^2] (A_1^i + A_1^b Q_i)$$

$$-k_2^2 (\bar{H} - m_2 \bar{C} - 2\mu k^2) (A_2^i + A_2^b Q_i)$$

$$+(j\beta, k^2)(A_3^i - A_3^b Q_i) + S_i$$

(2.41)

We note that $\nabla \cdot (\nabla \times \chi) = 0$
4. Continuity of fluid pressure, $P = P_f$

$$\rho \omega^2 \psi_0 = \nabla \cdot \mathbf{u} - \mathbf{M} \cdot \nabla \cdot \mathbf{\bar{w}}$$

(2.42)

$$\rho \omega^2 [R^t Q_0 + R^r] + S_f = -[k_t^2 (\mathbf{C} - m_1 \mathbf{M})](A_i^t + A_i^r Q_i)$$

$$+ [k_r^2 (\mathbf{C} - m_2 \mathbf{M})](A_2^t + A_2^r Q_2) + S,$$

(2.43)

5. Continuity of normal fluid displacement, $\frac{\partial \psi_0}{\partial r} = u_r - \omega_s$

Deresiewicz et al.\[31\] state this as continuity of $u_r + \omega_s$, which is clearly different. This stems from their choice of definition for $\omega_s$. (compare eqn. (2.16) with eqn 14 of ref. 31) Our statement of this boundary condition is consistent with (2.16) so that eqn. 20 of ref. [31] is retrieved.

$$j \beta_0 (R^t Q_0 - R^r) + S_f = j \beta_1 (1 - m_i) (A_i^t - A_i^r Q_i)$$

$$+ j \beta_2 (1 - m_2) (A_2^t - A_2^r Q_2)$$

$$+ k_r^2 (1 - m_2) (A_2^t + A_2^r Q_2) + S,$$

(2.44)

and finally,

6. Continuity of tangential stress, $\sigma = \sigma_{rz}$

The fluid cannot support shear stresses so at the surface the solid shear stresses must be zero.

$$0 = \frac{1}{2} \mu \left( \frac{\partial u_r}{\partial x} + \frac{\partial u_z}{\partial r} \right)$$

(2.45)

$$0 = -2 j \mu k_r \left[ j \beta_1 (A_i^t - A_i^r Q_i) + j \beta_2 (A_2^t - A_2^r Q_2) \right] + j \mu k_r (\beta_r^2 - k_r^2) (A_i^t + A_i^r Q_i) + S,$$

(2.46)
2.4.3 Porous elastic solid-solid interfaces

The boundary between two poroelastic layers require six continuity equations as there are three waves in each layer. These are

7. Continuity of total normal stress, $\sigma_{zz}$

\[
\begin{align*}
-\left[k_{1,1}^2 (\overline{H}_n - m_{1,n} \overline{C}_n) - 2\mu_n k_r^2\right] (A_{1,n}^I Q_{1,n} + A_{1,n}^* \right) \\
-\left[k_{2,2}^2 (\overline{H}_n - m_{2,n} \overline{C}_n) - 2\mu_n k_r^2\right] (A_{2,n}^I Q_{2,n} + A_{2,n}^* \right) + \left(j \beta_{3,n} k_r^2\right) (A_{3,n}^I Q_{3,n} - A_{1,n}^*) + S_n \\
= -\left[k_{1,n+1}^2 (\overline{H}_{n+1} - m_{1,n+1} \overline{C}_{n+1}) - 2\mu_{n+1} k_r^2\right] (A_{1,n+1}^I + A_{1,n+1}^* Q_{1,n+1}) \\
-\left[k_{2,n+1}^2 (\overline{H}_{n+1} - m_{2,n+1} \overline{C}_{n+1}) - 2\mu_{n+1} k_r^2\right] (A_{2,n+1}^I + A_{2,n+1}^* Q_{2,n+1}) + \\
\left(j \beta_{3,n+1} k_r^2\right) (A_{3,n+1}^I - A_{3,n+1}^* Q_{3,n+1}) + S_{n+1}
\end{align*}
\]

(2.47)

8. Continuity of fluid pressure, $P_f$

We assume implicitly the 'open pore' condition here. Otherwise, $P_{f,a} = P_{f,a+1} = Tw$, where $T$ is a parameter. Here we assume $T=0$:

\[
\begin{align*}
-\left[k_{1,n}^2 (\overline{C}_n - m_{1,n} \overline{M}_n)\right] (A_{1,n}^I Q_{1,n} + A_{1,n}^*) - \left[k_{2,n}^2 (\overline{C}_n - m_{2,n} \overline{M}_n)\right] (A_{2,n}^I Q_{2,n} + A_{2,n}^*) + S_n \\
= -\left[k_{1,n+1}^2 (\overline{C}_{n+1} - m_{1,n+1} \overline{M}_{n+1})\right] (A_{1,n+1}^I + A_{1,n+1}^* Q_{1,n+1}) \\
-\left[k_{2,n+1}^2 (\overline{C}_{n+1} - m_{2,n+1} \overline{M}_{n+1})\right] (A_{2,n+1}^I + A_{2,n+1}^* Q_{2,n+1}) + S_{n+1}
\end{align*}
\]

(2.48)

9. Continuity of normal solid frame displacement, $u_z$

\[
\begin{align*}
\left(j \beta_{1,n} (A_{1,n}^I Q_{1,n} - A_{1,n}^*) + j \beta_{2,n} (A_{2,n}^I Q_{2,n} - A_{2,n}^*) + k_r^2 (A_{3,n}^I Q_{3,n} + A_{3,n}^*) + S_n \\
= j \beta_{1,n+1} (A_{1,n+1}^I - A_{1,n+1}^* Q_{1,n+1}) + j \beta_{2,n+1} (A_{2,n+1}^I - A_{2,n+1}^* Q_{2,n+1}) \\
+ k_r^2 (A_{3,n+1}^I + A_{3,n+1}^* Q_{3,n+1}) + S_{n+1}
\end{align*}
\]

(2.49)
Chapter 2 - Fast Field ...

10. Continuity of tangential stress, \( \sigma_{r\phi} \)

\[
-j \mu k_r \left[ j \beta_1 \left( A_{1,a}^* Q_{1,a} - A_{1,a}^r \right) + j \beta_2 \left( A_{2,a}^* Q_{2,a} - A_{2,a}^r \right) \right] + S_a \\
= -j \mu k_r \left[ j \beta_{1,a+1} \left( A_{1,a+1}^* Q_{1,a+1} - A_{1,a+1}^r \right) + j \beta_{2,a+1} \left( A_{2,a+1}^* Q_{2,a+1} - A_{2,a+1}^r \right) \right] + S_{a+1}
\]

(2.50)

11. Continuity of normal relative fluid displacement, \( w_z \)

\[
j m_1 \beta_1 \left( A_{1,a}^* Q_{1,a} - A_{1,a}^r \right) + j m_2 \beta_2 \left( A_{2,a}^* Q_{2,a} - A_{2,a}^r \right) \\
+ m_3 k_r^2 \left( A_{3,a}^* Q_{3,a} + A_{3,a}^r \right) + S_a \\
= j m_{1,a+1} \beta_{1,a+1} \left( A_{1,a+1}^* Q_{1,a+1} - A_{1,a+1}^r \right) + j m_{2,a+1} \beta_{2,a+1} \left( A_{2,a+1}^* Q_{2,a+1} - A_{2,a+1}^r \right) \\
+ m_{3,a+1} k_r^2 \left( A_{3,a+1}^* + A_{3,a+1}^r \right) + S_{a+1}
\]

(2.51)

12. Continuity of tangential frame displacement, \( u_r \)

\[
-j k_r \left( A_{1,a}^* Q_{1,a} + A_{1,a}^r \right) - j k_r \left( A_{2,a}^* Q_{2,a} + A_{2,a}^r \right) + k_r \beta_{3,a} \left( A_{3,a}^* Q_{3,a} - A_{3,a}^r \right) + S_a \\
= -j k_r \left( A_{1,a+1}^* + A_{1,a+1}^r Q_{1,a+1} \right) - j k_r \left( A_{2,a+1}^* - A_{2,a+1}^r Q_{2,a+1} \right) \\
+k_r \beta_{3,a+1} \left( A_{3,a+1}^* Q_{3,a+1} - A_{3,a+1}^r \right) + S_{a+1}
\]

(2.52)

On the whole there are 12 unknown amplitudes in each porous elastic layer. In the last solid layer (lower half-space) there are no 'up-going' waves so that these three amplitudes should be set to zero.

2.5 Source terms

The source terms \( S_{fs} \) are added to the right hand side of the Global matrix equation at interfaces above and below the source. They are derived from
source potentials. For a spherical point source in the solid (such as an explosive source inside a solid or a pressure source on the interface) these are:

\[
\varphi_{s1} = S \left( \frac{e^{i \beta_1 H}}{\beta_1} + L \frac{e^{i \beta_2 H}}{\beta_2} \right) \tag{2.53}
\]

\[
\varphi_{s2} = S \left( m_1 \frac{e^{i \beta_1 H}}{\beta_1} + m_2 L \frac{e^{i \beta_2 H}}{\beta_2} \right) \tag{2.54}
\]

\[
\varphi_{s3} = 0 \tag{2.55}
\]

the last potential is zero since a pressure point source only produces longitudinal waves. Here \( H \) is the source height, \( S \) the source strength, and \( L \) is given by:

\[
L = -\frac{k_1^2 (H - m_1 C - \tilde{C} + m_2 \tilde{M}) - 2 \mu k_1^2}{k_2^2 (H - m_2 C - \tilde{C} + m_2 \tilde{M}) - 2 \mu k_2^2} \tag{2.56}
\]

the resulting terms are then:

\[
S_{\text{stress}} = S \left[ -H \left( k_1^2 \frac{e^{\eta_1}}{\beta_1} + L \cdot k_2^2 \frac{e^{\eta_2}}{\beta_2} \right) + 2 \mu k_1^3 \left( \frac{e^{\eta_1}}{\beta_1^2} + \frac{e^{\eta_2}}{\beta_2^2} \right) + \tilde{C} \left( m_1 k_1^2 \frac{e^{\eta_1}}{\beta_1^2} + m_2 L k_2^2 \frac{e^{\eta_2}}{\beta_2^2} \right) \right] \tag{2.57}
\]

\[
S_{\text{fluid pressure}} = S \left[ (\tilde{C} - m_1 \tilde{M}) k_1^2 \frac{e^{\eta_1}}{\beta_1} + (\tilde{C} - m_2 \tilde{M}) k_2^2 \frac{e^{\eta_2}}{\beta_2} \right] \tag{2.58}
\]

\[
S_{\text{tangential stress}} = S \left[ 2 \mu k_1 (e^{\eta_1} + L e^{\eta_2}) \right] \tag{2.59}
\]

\[
S_{\text{normal displacement}} = i S \left[ (1 - m_1) e^{\eta_1} + (1 - m_2) L e^{\eta_2} \right] \tag{2.60}
\]
\begin{align}
S_{\text{normal fluid displacement}} &= iS \left[ m_1 e^{\frac{q_1 t}{\beta_1}} + m_2 Le^{\frac{q_2 t}{\beta_2}} \right] \\
S_{\text{tangential displacement}} &= iSk \left[ \frac{e^{\frac{q_1 t}{\beta_1}}}{\beta_1} + L \frac{e^{\frac{q_2 t}{\beta_2}}}{\beta_2} \right]
\end{align}

(2.61) 

where $q_i = j\beta_i H$. If several sources are present, additional source terms can be superimposed in the appropriate positions. Another type of source which is used sometimes is a force couple acting vertically (i.e. a vertical dipole). The potentials for a vertical point force couple are given by\cite{332}:

\begin{align*}
\varphi_{s1} &= S \left( e^{\frac{q_1 t}{\beta_1}} + Le^{\frac{q_2 t}{\beta_2}} \right) \\
\varphi_{s2} &= S \left( m_1 e^{\frac{q_1 t}{\beta_1}} + m_2 Le^{\frac{q_2 t}{\beta_2}} \right) \\
\text{and} \\
\varphi_{s3} &= Sk \left( \frac{e^{\frac{q_1 t}{\beta_3}}}{\beta_3} \right)
\end{align*}

2.6 Numerical techniques

The numerical methods used in an FFP program are involved a) in the calculation of the global matrix, and b) in the efficient and accurate calculation of the Hankel transform integral.

The matrix in equation (2.26) is a block-diagonal matrix (same as any matrix formed from BCE's). To solve the matrix equation, the Crout's decomposition (or \textit{LU} decomposition) from a commercial mathematical package has been used previously. This would pose serious difficulties when simulating a refracting atmosphere at frequencies higher than a few hundred Hertz. In these cases, hundreds, even thousands, of air layers would be
needed to model the atmosphere accurately since, for the best results, each layer thickness should be less than one quarter of the wavelength. The matrix $A$ would be a sparse band-diagonal matrix. Since inverting a matrix involves $O(N^3)$ operations, solving such a matrix by conventional decomposition methods does take an extremely long time even using very powerful computers. Therefore an alternative method was chosen\footnote{69}. This method involves storing the coefficient matrix $A$ in a more compact form in an $N \times M$ matrix where $M$ is the sum of number of non-zero elements to the right of the diagonal and number of non-zero elements to the left of it plus 1. For fluid layers above poroelastic solid strata, the value for $n$ is 17 and for fluid above rigid porous soil layers it is 5. The matrix inversion is then an $O(NM)$ operation. This is a substantial saving in computing time and has in fact made it possible to simulate the sound field in a refracting atmosphere with hundreds of layers using a desktop workstation.

Once the coefficient matrix is known then of course any number of receiver positions can be selected and the desired quantity (sound pressure or particle velocity of solid stress) calculated. Of course, each receiver requires calculating its own Green's function and performing FFT on it, but the computation time taken by the FFT is negligible compared to the time taken for forming and solving the Global Matrix equation.

Since the latter point has been covered in sufficient detail in previous publications\footnote{70,20,21} and no substantial changes have been made to the original work, it is omitted here.
2.7 de la Cruz & Spanos theory of seismic wave propagation in porous medium

An alternative theory of wave propagation to that of Biot's has been proposed by de la Cruz & Spanos\cite{33-35}. The main difference from Biot theory is the prediction of two rotational waves as well as two dilatational waves in the material. This is because these authors allow for fluid vorticity. Another important difference is that porosity is treated as a dynamic variable rather than a constant property of the system. They start by considering equations of motion and equations of continuity at the pore scale and also the boundary conditions at the pore walls. Then they use averaging theorems to obtain the mass-averaged macroscopic equations of state for the porous medium. Porosity is a dynamic variable to be determined from the system of equations. The equations of state for their theory can be summarised as the following\cite{37}:

\begin{align}
\rho_s \frac{\partial^2}{\partial t^2} \mathbf{u} &= k_s \nabla (\nabla \cdot \mathbf{u}) - \frac{k_s}{1-\Omega_0} \nabla \Omega + \mu \left[ \nabla^2 \mathbf{u} + \frac{1}{3} \nabla (\nabla \cdot \mathbf{u}) \right] \\
&\quad + \frac{\eta}{\kappa} \times \frac{\Omega_0^2}{1-\Omega_0} (\nabla \Omega - \nabla \Omega) - \frac{\rho_{12}}{1-\Omega_0} \frac{\partial}{\partial t} (v - V) \\
&\quad \quad \text{(2.63)}
\end{align}

\begin{align}
\rho_f \frac{\partial}{\partial t} \mathbf{V} &= -\nabla P_f + \eta \left[ \nabla^2 \mathbf{V} + \frac{1}{3} \nabla (\nabla \cdot \mathbf{V}) \right] \\
&\quad - \frac{\eta \Omega_0}{\kappa} (\nabla \mathbf{u}) + \rho_{12} \frac{\partial}{\partial t} (V - \mathbf{u}) \\
&\quad \quad \text{(2.64)}
\end{align}

and

\begin{align}
\frac{1}{k_s} (P_s - P_0) &= -\nabla \cdot \mathbf{u} + \frac{\Omega - \Omega_0}{1-\Omega_0} \\
&\quad \quad \text{(2.65)}
\end{align}
\[
\frac{1}{k_f} \frac{\partial}{\partial t} P_f = -\nabla \cdot V - \frac{1}{\Omega_0} \frac{\partial}{\partial t} \Omega
\]

(2.66)

and finally, equation for the porosity

\[
\frac{\partial}{\partial t} \Omega = \delta_f \nabla \cdot \nu - \delta_s \nabla \cdot \nabla
\]

(2.67)

where \( \nu \) and \( \nu \) are the solid and fluid particle velocity vectors respectively, \( \Omega_0 \) is the ambient value of porosity and \( \rho_{ij} \) is the mass coupling coefficient introduced by Biot. The variables to be determined from the five equations above are \( V, \ u, \ P_f, \ P_s, \) and \( \Omega \). If a harmonic time dependence \( (e^{iat}) \) is assumed, equation (2.67) can be used to eliminate the porosity from the other equations. de la Cruz and Spanos state that their equations are indistinguishable from Biot's except the second term in the right hand side of (2.20) which is fluid vorticity term [and also ignoring the viscosity correction function of Biot] provided the following is true:

\[
\frac{\delta_f}{\delta_s} = \frac{k_f}{k_s}
\]

(2.68)

They then proceed to show a correspondence between coefficients in Biot's 1956 paper and above equations. These equations have been generalized to include fluid bulk viscosity and permit treatment of macroscopic shear modulus and heat conductivities of solid and fluid as phenomenological parameters. For a summary of equations see reference [37]. In their 1989 paper, de la Cruz and Spanos derived a set of boundary conditions for fluid-saturated porous media which are independent of one's choice of equations of motion. By defining a boundary surface (which is unclear in
Chapter 2 - Fast Field

Deresiewicz & Skalak treatment) together with equations of continuity for a porous medium they arrive at the boundary conditions. A condition of no-slip is also considered to complete the set of BC's. Hickey and Sabatier\textsuperscript{[38]} have compared the two sets of boundary conditions (de la Cruz et al\textsuperscript{[35]} and Deresiewicz et al\textsuperscript{[31]} for the special case of a boundary between a fluid-saturated porous medium and a fluid half-space and show that all the BC's are comparable except one; continuity of particle velocity which in de la Cruz treatment is mass averaged while in the other one it is a volume averaged quantity that is continuous. Thus, we have for the first case:

$$U = \frac{\Omega \rho_f V_t + (1 - \Omega) \rho_s v_s}{\Omega \rho_f + (1 - \Omega) \rho_s}$$

while in the second case we have:

$$\dot{U} = \Omega V_t + (1 - \Omega) v_s = \dot{u}_s - \dot{w}_s$$ (c.f. BC no. 5)

where $U$ is the fluid velocity in the half-space. They show that these lead to different reflection coefficients and specific impedances. In this section we will concentrate on the same boundary condition but between two porous elastic media.

Deresiewicz and Skalak separate out the solid and fluid velocity conditions across the boundary. They state continuity of solid velocity (displacement) as one BC and continuity of relative fluid velocity (displacement) as a second BC (c.f. BC no. 11 and eqn. (2.16)). This is based on the physical requirement that there should exist no gap at the boundary during the passage of the wave and, therefore, that the solid frame has the same velocity (displacement) component either side of the boundary. But the BC for the material velocity
set by de la Cruz & Spanos in eqn. (2.69) is also compatible with this requirement provided the fluid velocity is mass averaged, as is shown below. If we use subscripts 1 and 2 for the two porous media, we have the following BC for the material velocity component normal to the interface;

\[
\frac{\Omega_1 \rho_f V_{n1} + (1 - \Omega_1) \rho_s v_{n1}}{\Omega_1 \rho_f + (1 - \Omega_1) \rho_s} = \frac{\Omega_2 \rho_f V_{n2} + (1 - \Omega_2) \rho_s v_{n2}}{\Omega_2 \rho_f + (1 - \Omega_2) \rho_s}
\]  

(2.70)

where we have assumed the same solid and fluid densities in both media. Setting \( v_{n1} = v_{n2} \) we obtain for the fluid velocities:

\[
\frac{\dot{w}_{z1}}{\Omega_1 \rho_f + (1 - \Omega_1) \rho_s} = \frac{\dot{w}_{z2}}{\Omega_2 \rho_f + (1 - \Omega_2) \rho_s}
\]  

(2.71)

in other words a mass averaged relative fluid velocity is conserved.(c.f. BC no 11 above) The other boundary conditions in their theory is basically continuity of different components of stresses. The exact implications of this formulation of the boundary conditions for interface between porous elastic media are left for future work in this field.

2.8 Conclusion

In this chapter Biot's theory of wave propagation in a porous elastic medium was reviewed together with the formulation of the boundary conditions between two such media. A Fast Field Program for calculating wave propagation above and within a stratified porous elastic soil, developed originally by S Tooms is outlined, extended to allow for sources in the solid medium and improved with respect to the efficiency of the numerical methods used in the calculation of the Global Matrix. An alternative theory
of poroelasticity, proposed by de la Cruz and Spanos is reviewed also. The boundary conditions for continuity of fluid displacement across the boundary suggested in the latter theory are shown to differ somewhat from those suggested by Deresiewicz and Skalak\cite{131} for implementation in the Biot theory. In the former theory the conserved quantities are volume averaged while in the latter one they are mass averaged (see eqn. (2.71)). The implications of this formulation of the boundary conditions is not discussed here but could be the basis of further work.

**Appendix: list of symbols**

- $F(\omega)$ and $F(\lambda)$: viscosity correction function
- $H, C, M$: moduli of elasticity of a poro-elastic solid in Biot's theory
- $J_v(z)$: Bessel function of order $v$
- $\mu$: rigidity modulus
- $k_b$: bulk modulus of the drained solid frame
- $k_f$: pore fluid bulk modulus
- $k_s$: bulk modulus of the solid grains
- $k_r$: horizontal wavenumber (variable of Hankel Transform)
- $k_{1,2,3}$: wavenumber corresponding to the three wave types in the solid
- $m_{1,2,3}$: ratios of the fluid-to-solid displacements for the three wavetypes
- $P_0$: ambient fluid pressure
- $q^2$: tortuosity
- $\Omega$: porosity
- $V_f$ and $V_p$: fast wave speed
- $V_2$: slow wave speed
- $V_j$ and $V_s$: shear wave speed
- $u$: solid component displacement vector
- $U$: pore fluid absolute displacement vector
- $w$: pore fluid displacement vector relative to the solid motion
- $\rho$: bulk medium density
- $\rho_0$ and $\rho_f$: fluid density
- $\rho_s$: grain solid density
- $\rho'$: complex pore fluid density (see eqn. 2.5)
<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\kappa$</td>
<td>permeability</td>
</tr>
<tr>
<td>$\eta$</td>
<td>dynamic fluid viscosity</td>
</tr>
<tr>
<td>$\lambda$</td>
<td>dimension-less parameter</td>
</tr>
</tbody>
</table>
Chapter 3

Predicted Effects of Porosity on Propagation in a Fluid-Saturated Porous Elastic Medium†

3.1 Introduction

The problem of wave propagation in a two-phase medium in general, and in porous-elastic materials in particular, has been addressed by many authors. The classical work in this field is that by Biot who has derived the dynamical equations of state and predicted the existence of three wave types in the porous-elastic medium. Two compressional waves, and one shear wave are predicted. The first compressional wave, usually called the “fast wave”, is similar to the P-wave with somewhat higher attenuation and is mainly carried in the solid frame. The second wave, called the “slow wave” is normally slower than the other wave, highly attenuated and propagates mainly in the pore fluid. Finally, the shear wave is very similar to the rotational S-wave and is carried in the solid frame.

Recently, a comparison of the wave motions predicted in air-filled and water-saturated materials has been made by Don Albert.[39] In this chapter‡,

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† An earlier version of this chapter has been published in reference [84]
‡ The work reported here was in fact carried out independently of ref. [39]
in a similar way to reference [39], the characteristics of the three wave types predicted in Biot theory in air-filled soils and water-saturated sediments are contrasted, but particularly, the influence of the porosity of the bulk medium is examined. In addition, FFLAGS is used to predict the above-ground propagation. The influence of pore size distribution on the wave characteristics is also investigated in a typical air-filled soil. It is predicted that porosity has considerable effect on the Slow wave and also on the wave propagation above the surfaces of both air-filled and water-saturated media.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Solid density</td>
<td>2650.0 kg/m³</td>
</tr>
<tr>
<td>P-wave speed</td>
<td>270.0 m/s</td>
</tr>
<tr>
<td>S-wave speed</td>
<td>190.0 m/s</td>
</tr>
<tr>
<td>grain bulk modulus ($K_r$)</td>
<td>$4.6 \times 10^{11}$</td>
</tr>
<tr>
<td>solid grain size</td>
<td>$0.49 \times 10^{-3}$ m</td>
</tr>
<tr>
<td>grain shape factor ratio</td>
<td>0.5</td>
</tr>
<tr>
<td>Specific Attenu. $[2\times\text{Im}(V)/\text{Re}(V)]$</td>
<td>0.0125</td>
</tr>
<tr>
<td>porosity</td>
<td>0.1 ... 0.7</td>
</tr>
<tr>
<td>air density</td>
<td>1.20 kg/m³</td>
</tr>
<tr>
<td>speed of sound in air</td>
<td>3440 m/s</td>
</tr>
<tr>
<td>dynamic viscosity of air</td>
<td>$1.81 \times 10^{-5}$ Nsm⁻²</td>
</tr>
<tr>
<td>water density</td>
<td>1100.0 kg/m³</td>
</tr>
<tr>
<td>speed of sound in water</td>
<td>1500.0 m/s</td>
</tr>
<tr>
<td>dynamic viscosity of water</td>
<td>$1.0 \times 10^{-3}$ Nsm⁻²</td>
</tr>
</tbody>
</table>

Table 3.1 list of parameters used in this chapter.

Don Albert’s calculations and conclusions are broadly in line with the conclusions reached in this chapter, although he does not consider the influence of porosity and uses a fixed value of 0.3 for his calculations (see table 1. of ref. [39]). He calculates the transmitted and reflected energy
coefficients for a plane wave in a fluid half-space incident on a poroelastic interface. He concludes that for air-filled soil the energy is mainly transmitted across the interface to a slow wave (pore wave) while in the case of water-saturated sediment the main energy conversion at the interface is to the fast waves.

A list of symbols used in this chapter appears in the Appendix to the chapter.

### 3.1.1 Default parameters

The ground parameters used in these calculations are listed in Table 3.1. The ground is modelled as a porous elastic layer of soil 2.0m thick over a porous elastic substrate. The pores are assumed to have cylindrical cross-section. Many of these values have been measured at Wezep in Holland and have been used elsewhere as benchmark parameters. The measured porosity was 0.27. Here I study the effects of varying porosity between 0.1 and 0.7.

The corresponding flow resistivity, $R_s$, is calculated from

$$R_s = \frac{180\eta}{d^2} \times \frac{(1-\Omega)^2}{\Omega^3}$$

(3.1)

where $d$ is the solid grain size and $\eta$ is the kinematic fluid viscosity; and the solid grains are assumed to be spherical. The tortuosity is found from

$$\tau^2 = \Omega^{-n'}$$

(3.2)
with \( n' \) being the grain shape factor which is assumed to be constant. The fluid in the pores and above the surface is assumed to be one and the same, either air or water.

### 3.2 The Biot-Stoll model

Biot equations of motion, modified by Stoll produce the following dispersion equation for the compressional wave speeds for the fluid-saturated porous medium in terms of three elastic moduli \( H, C \) and \( M; \) fluid and solid densities and viscosity terms:

\[
\left( \rho_f^2 - \rho_m + i \rho \frac{F(\lambda)}{\omega} R_t \right) v_f^2 + \left( m H - i \frac{F(\lambda)}{\omega} R_t \right) v_s^2 + \left( C^2 H M \right) = 0
\]

(3.3)

There are two solutions to this quartic equation which are the aforementioned “fast” and “slow” compressional waves. The corresponding shear wave equation is:

\[
v_j^2 = \frac{G_p \left( m - i \frac{F(\lambda)}{\omega} R_s \right)}{\rho m - i \rho \frac{F(\lambda)}{\omega} R_t - \rho_f^2}
\]

(3.4)

The ratios of the fluid-to-solid displacement amplitudes for the three waves are found from

\[
m_{1,2} = \left| - \frac{1}{\Omega} \times \frac{H - \rho V_{1,2}^2}{C - \rho_j V_{1,2}^2} \right|
\]

(3.5)

\[
m_3 = \frac{\rho V_s^2 - G_p}{\rho_j V_s^2}
\]

(3.6)
where subscripts 1, 2 and 3 refer to the 'fast', 'slow' and shear waves respectively. Other symbols are defined in the previous chapter and in the Appendix to this chapter.

In water-saturated sediments the pore fluid bulk modulus ($K_f$) can be approximated by the free fluid bulk modulus ($\rho c^2$). This is because thermal effects do not play a significant role in fluid motion in the pores. This is not the case in air-filled porous medium. $K_f$ cannot be considered constant. It is also complex and frequency dependent. For cylindrical pores of arbitrary shape and uniform size, $K_f$ is given by:

$$K_f(\omega)=\left[\gamma^2\left(1+2(\gamma-1)\frac{T(\lambda\sqrt{iN_{pr}})}{\lambda\sqrt{iN_{pr}}}\right)\right]$$

(3.7)

where $T(x)=\frac{J_1(x)}{J_0(x)}$, and

$$\lambda=\left(\frac{8q^2\rho_s\omega}{\Omega R_s}\right)^{\frac{1}{2}}$$

(3.8)

is a dimensionless parameter.

The viscosity correction function, introduced by Biot, accounts for viscous drag of fluid in the pores when the Poiseuille flow breaks down. Thus $\lambda F(\omega)$ acts as a dynamic viscosity factor. For cylindrical pores of arbitrary shape and uniform size:

$$F(\lambda)=\left[\frac{-(\lambda\sqrt{i})T(\lambda\sqrt{i})}{4[1-2T(\lambda\sqrt{i}/(\lambda\sqrt{i}))]}\right]$$
Chapter 3 - Predicted Effects ...

Figure 3.1 Predicted fluid bulk modulus in air filled pores as a function of frequency for porosity values of 0.1..0.7 from right to left.

Figure 3.2 As in Figure 3.1 but for water-saturated pores.

Figures 3.1, 3.2 & 3.3 show the predicted dependence of these functions (see (3.7) & (3.9)) on porosity over a frequency range of 10 Hz to 10 MHz. It is
clear that the air-filled pores show a dependence on porosity in the intermediate, dispersive range of frequency. This can be explained by considering that at the limit of small porosity, most of the fluid is affected by the fluid thermal boundary layer on the pore walls and the pore fluid modulus becomes less than the free fluid value, while at the higher porosity values (i.e. larger pore sizes) and lower frequencies the bulk of the fluid would be unaffected and the process is isothermal. The low- and high-frequency asymptotes of $K_f$ correspond to the isothermal and adiabatic limits respectively. The bulk modulus in water-saturated sediments is practically constant reflecting the fact that the ratio of specific heats for water is close to 1 ($\gamma=1.01$). The $F(\lambda)$ function is also plotted against the dimension-less parameter ($\lambda$).

Figure 3.3 Predicted viscosity correction function vs. Frequency for porosity values of 0.7, 0.5, 0.3, 0.2, and 0.1 from left to right.
Equations (3.3) and (3.4) were used to calculate the complex wave speeds for all three wavetypes in the poroelastic medium. The predicted dependence of the three wave speeds and the corresponding specific attenuation are shown in figures 3.5-3.16. The fast and shear wave speeds in air filled soils are independent of the porosity of the soil and are effectively non-dispersive. The reason for this is that the large difference between the densities of the solid and the fluid effectively decouples the displacements in them. The same pattern is evident in the predicted specific attenuation of the two waves types. On the other hand, the fast waves are predicted to show a strong dependence on porosity in water-filled sediments although their predicted dependence on frequency is small. The wave speed decreases from a value of over 3000 m/s at porosity of 0.1 to a value close to the speed of sound in water at higher porosity. The specific attenuation shows a dependence on both porosity and frequency, showing a peak at frequency...
range of 10 - 1000 Hz depending on the porosity. It is generally lower than the corresponding attenuation in air-filled soils.

Figure 3.5 Fast wave speed in air-filled soils. Very little change with either frequency or porosity is predicted.

Figure 3.6 Fast wave specific attenuation in air-filled soils. The large peak corresponds to porosity value of 0.5.
Figure 3.7 Fast wave speed for water-saturated sediments corresponding to porosity values of (from top to bottom) 0.1, 0.2, 0.3, 0.5 and 0.7.

Figure 3.8 Fast wave specific attenuation for water-saturated sediments. Values of porosity increase from right to left.
The shear wave in air-filled soils is predicted not to depend on either porosity or frequency, while in water-saturated sediments its speed varies slightly with porosity and exhibits a variation with frequency over the range 10 to 100 Hz at higher porosity. The specific attenuation of shear waves in water-saturated sediments exhibit strong dependence on both porosity and frequency and is generally higher than is the case for air-filled soil.

The slow wave is predicted to be strongly dependent on porosity and on frequency in both cases under investigation here. In particular the frequency at which it starts to be a propagating wave (i.e. when the specific attenuation falls below value of 1) is dependent on the porosity.

![Figure 3.9 Shear wave speed in air-filled soils.](image)
Figure 3.10  Shear wave specific attenuation in air-filled soils.

Figure 3.11  Shear wave speed in water-saturated sediments. Porosity increases from top to bottom.
At high frequencies, the slow wave speed approaches a limiting value which agrees reasonably well with predictions of a pore wave model. In this model the sound waves are assumed to travel in the fluid-filled pores with a limiting value of \( \frac{c_0}{q} \). The assumption, in such models, that the solid frame is rigid and does not support wave motion is tolerably true for air-filled soils but not valid for water-saturated sediments. In the former case the motion of the two phases of the media are effectively decoupled but in water-saturated sediments this is not the case. It is seen that the predicted high frequency limit of the slow waves in water-saturated solids falls far short of the pore wave prediction.
Chapter 3 - Predicted Effects...

Figure 3.13 Slow wave speed in air-filled soils. Porosity increases from right to left.

Figure 3.14 Slow wave specific attenuation in air-filled soils. Porosity increases from right to left.
Figure 3.15 Slow wave speed in water-saturated sediments. Porosity increases from top to bottom.

Figure 3.16 Slow wave specific attenuation in water-saturated sediments. Porosity increases from right to left.

3.3 Pore size distribution

The viscosity correction function was originally defined by Biot, for slit-like pores as
where
\[ U_{av} \text{ is the fluid velocity averaged over the pore cross-section; and } \tau \text{ is the friction stress at the pore walls.} \]

Using the above expression Yamamoto & Turgut\(^{[14]} \) have derived an expression for the viscosity correction function in terms of a distribution of pore sizes of slit-like pores of semi-width \( b \):

\[ F(\lambda) = \frac{\sqrt{-i\eta\rho\rho_{\text{f}}}}{\Omega R_{s}} \times \frac{\int b^{-1}e(b) \tanh(\lambda\sqrt{-i}) \, db}{\int e(b) \left[ 1 - \frac{\tanh(\lambda\sqrt{-i})}{(\lambda\sqrt{-i})} \right] \, db} \]

\[ (3.12) \]

in the above form a misprint in the original reference has been corrected. It has also been shown by Attenborough\(^{[14]} \) that the shape of the pores does not play a significant role in the impedances predicted from the correction function. In eqn. (3.12) we have:

\[ \lambda = b \sqrt{\frac{\omega\rho}{\eta}} \]

\[ (3.13) \]

\( b=2^\varphi \) (b in mm) and assuming a log-normal distribution of pores:

\[ \int_{0}^{\infty} e(b) \, db = \int_{-\infty}^{\infty} f(\varphi) \, d\varphi \]
Chapter 3 - Predicted Effects ...

\[ f(\varphi) = \frac{1}{\sigma \sqrt{2\pi}} \exp \left[ -\frac{(\varphi - \varphi_a)^2}{2\sigma^2} \right] \]

(3.15)

All the parameters here are quantities related to the distribution of pores:

The parameter \( \sigma \) is the standard deviation and \( \varphi_a \) is the mean value of pore size which can be found from:

\[ b_a = 2^{-\varphi_a} = \frac{3\eta}{\sqrt{\Omega R_s}} e^{-\left(\sigma \ln 2\right)^2} \]

(3.16)

The above integrals, in general, cannot be evaluated analytically and a quadrature technique is required. A simple Gaussian quadrature is, in most cases, adequate for the purpose. On the other hand, approximate forms in the limits of large and small \( \lambda \) have been derived by Attenborough.\(^{41}\) The viscosity correction function tends to following values at the limits of small and large \( \lambda \):\(^{41}\)

\[ F(\omega) \xrightarrow{\lambda \to 0} 1 + \frac{i\omega \rho_0}{R_s \Omega} \]

(3.17)

and

\[ F(\omega) \xrightarrow{\lambda \to \infty} \frac{\lambda_a \exp\left[\frac{1}{2} (\sigma \ln 2)^2\right]}{\exp\left[\frac{1}{2} (\sigma \ln 2)^2\right]} \frac{1}{1 - \frac{\lambda_a}{\sqrt{-i}}} \]

(3.18)
with \( \lambda_a = h_a \sqrt{\frac{\omega_0}{\eta}} \). These asymptotic forms are valid for \( \lambda < 0.1 \) and \( \lambda > 10 \).

The intermediate values, which happen to correspond to the typical outdoor soil at audio frequencies, can be evaluated from a Padé approximation to \( F(\omega) \) .

Figure 3.17 show plots of real and imaginary parts of \( F(\omega) \) for three values of the standard deviation of the distribution (\( \sigma \)). It is seen that the large \( \lambda \) limit of the viscosity function depends on the value of \( \sigma \).

For air-filled soils the complex bulk modulus of the pore fluid can also be derived in terms of the pore size distribution function using the following two relationships, first one due to Biot and the second due to Stinson[42]:

\[
\rho_\nu(\lambda) = \left( \frac{\omega_0}{\omega} \right) \rho_\nu + \frac{iR\Omega}{\omega\zeta} F(\lambda)
\]

(3.19)
the factor \( \left( \frac{q^2}{Q} \right) \) accounting for averaging over multiple pores; and

\[
K_f(\omega) = C^{-1}(\omega) = \left[ \gamma \left( \gamma - 1 \right) \frac{\rho_o}{q^2 \rho_c (\lambda \sqrt{N_p})} \right]^{-1}
\]

(3.20)

which gives a relationship between complex density and complex compressibility of pore fluids. Figure 3.18 is a plot of \( K_f \) vs. frequency for three values of the standard deviation of pores. A wider spread of the pore sizes extends the dispersive region of \( K_f \) to higher frequencies but not affecting the two limiting values. The variation of \( K_f \) with porosity, on the other hand, does not so much extend as to shift the dispersive range to higher \( f \) as can be seen by comparing figures 3.18 and 3.1. Similar plots for slow wave are shown in figures 3.19 and 3.20. The fast and shear waves change very little with changes in standard deviation of pore sizes and are not shown here.

Figure 3.18 Pore fluid bulk modulus for three values of \( \sigma \).
Chapter 3 - Predicted Effects ...

Figure 3.19 Slow wave speed and specific attenuation for three values of (σ) in air-filled soils.

Figure 3.20 Slow wave speed and specific attenuation for three values of (σ) in water-saturated sediments.
Finally, the influence of varying the standard deviation of pore size distribution function on the Excess Attenuation spectrum of sound (defined as $20 \log(\text{total sound pressure with reference to the direct component of the sound field})$) above such a solid medium is shown for three standard deviation values of 0.5, 1 and 2 is shown in Figure 3.21. This figure illustrates the importance of considering pore size distribution as well as the 'standard' bulk parameters (porosity, flow resistivity and tortuosity) when predicting excess attenuation spectra.

![Figure 3.21 Predicted Excess Attenuation of sound above surface of air-filled solid. The soil pore size distribution is assumed to have standard deviation, in $\phi$ units, of (from left to right) 0.5, 1 and 2. Source and receiver heights are 5.0m and 1.0 m respectively and separation is 50.0m.](image)

### 3.4 Propagation in the fluid medium

To predict the influence of ground porosity on propagation of sound waves above ground the FFP program (FFLAGS) was used. The predictions for
sound waves in atmosphere above an air-filled poroelastic ground are shown in figure 3.22 where Excess Attenuation spectra for varying values of porosity are plotted. The source and receiver are assumed to be 5.0m and

1.0 m above interface respectively, and separation is taken to be 100m. Increasing porosity shifts the ground effect dip to lower frequencies. This prediction is in line with predictions of the rigid porous ground model and measurements. Similar predictions for sound propagation in water above a water-saturated sediment (ignoring reflections from water surface) are shown in figure 3.22. The ground effect minimum can also be seen in this case, although it does not change its position significantly. At porosities
A series of experiments measuring high frequency sound field of a spark source in a water tank above water-saturated sand has been performed at Oldenburg University\textsuperscript{811} and the predictions using FFLAGS are in good agreement with the measurements above a pure sediment except at lower frequency end of the spectrum where the wavelength was less than the separation of the source and receiver\textsuperscript{811}.

### 3.5 Conclusion

This chapter has investigated predictions of wave propagation in a porous elastic medium. The Biot theory of poroelasticity was used to contrast propagation characteristics of a water-filled porous solid with those of an
air-filled solid. In addition, the consequences of varying the porosity of the medium were investigated.

It was shown that, in an air-saturated medium at audio-frequencies, the pore fluid bulk modulus is frequency-dependent. The range of frequency dependence increases to lower frequencies with increasing porosity. The imaginary part of the modulus, - a measure of the effects of viscosity - is predicted to be significant in air-saturated materials, and is about 10% of its real part at its maximum. In contrast, in a water-saturated soil, the pore fluid bulk modulus changes very little with frequency and porosity. Moreover, its imaginary part is negligible, (of the order of 0.1 % of its real part) so that it is a good approximation to assume that the bulk modulus of the pore fluid is constant.

Biot theory predicts that porosity has a significant effect on the ‘slow’ wave speed but little influence on the shear wave speed in air-saturated materials. The ‘fast’ wave speed is predicted to depend on porosity in a water-saturated soil (figure 3.7) but to be independent of porosity when the pore fluid was air (figure 3.5).

Appendix: List of symbols

\( b \) \hspace{1cm} \text{pore semi-width}
\( C_J(\omega) \) \hspace{1cm} \text{complex compressibility of the fluid}
\( F(\omega) \) and \( F(\lambda) \) \hspace{1cm} \text{viscosity correction function}
\( J_v(z) \) \hspace{1cm} \text{Bessel function of order} \ v
\( K_h \) \hspace{1cm} \text{bulk modulus of the drained solid frame}
\( K_f \) \hspace{1cm} \text{pore fluid bulk modulus}
Chapter 3 - Predicted Effects ...

- $K_r$: bulk modulus of the solid grains
- $n'$: grain shape factor ratio
- $P_0$: ambient fluid pressure
- $q^2$: tortuosity
- $Q'$: specific attenuation
- $R_s$: flow resistivity (dynamic fluid viscosity/permeability)
- $V_i$ and $V_p$: fast wave speed
- $V_2$: slow wave speed
- $V_j$ and $V_s$: shear wave speed
- $s_p$: pore shape factor ratio
- $\rho$: bulk medium density
- $\rho_0$ and $\rho_f$: fluid density
- $\rho_c$: complex pore fluid density
- $\rho_s$: grain solid density
- $\kappa$: permeability
- $\sigma$: pore distribution standard deviation (in $\phi$ units)
- $\gamma$: adiabatic constant of the fluid
- $\eta$: dynamic fluid viscosity
- $\lambda$: dimension-less parameter
Chapter 4

On the Influence of Ground Porosity on Acoustic-to-Seismic Coupling Spectra

4.1 Introduction

An airborne sound wave striking the ground surface can produce seismic excitation in the ground. The acoustic-to-seismic coupling spectra show distinctive structure. The structure is mainly attributed to wave resonance within the ground layers. This phenomenon has been extensively studied in the past. The earliest work was, probably, by Press et al\cite{43} who made measurements of the acoustically excited seismic waves to study the errors in seismic exploration for oil. Mooney and Kaasa\cite{44} also studied the effect and found that considerable energy is transferred to the ground from an acoustic pulse above it. Other incentives to study this effect were sonic boom effects\cite{45} and the possibility of rocket launch detection\cite{46,47}. In all these cases the main excited wave-type were considered to be a constant-frequency Rayleigh wave with the phase velocity of sound. In a series of papers Sabatier et al\cite{48,49} and Attenborough et al\cite{50} also studied the problem and used a plane wave version of Biot-Stoll theory of wave propagation in fluid saturated porous media. Attenborough and Richards,\cite{51} using spherical wave theory and the light fluid limit, have predicted two different types of surface waves produced at the interface of air and porous elastic ground.

\* A paper based on an earlier version of this chapter was presented at the 127th meeting of the Acoustical Society of America, MIT, Cambridge, MA, USA, June 1994\cite{48}.
depending on the elasticity of the solid. In this thesis, the extent to which
certain parameters influence the position of maxima of seismic-to-acoustic
coupling and its amplitude are investigated. The ground has been modelled
as a vertically layered porous-elastic medium using the modified Biot-Stoll
theory. Predictions, based on this theory and some previously derived
expressions are compared with data. It is suggested that for some typical
ground types the full Biot-Stoll theory including porosity and elasticity,
rather than a low-velocity elastic layer approximation is required to predict
the acoustic-to-seismic spectra adequately.

4.2 Theory

To explore the influence of ground elasticity and layering on propagation of
sound above surface and also on acoustically induced seismic waves, we
need to solve the interface boundary conditions on the Hankel-transformed
wave equation. We assume spherical waves propagating from a monopole
source situated at position \((0,0,h_s)\) above ground with no azimuthal
dependence. We further assume a homogeneous atmosphere. The ground is
modelled as a single porous-elastic layer over a non-porous elastic substrate.
Biot\textsuperscript{[22,23]} hydrodynamic equations of poroelasticity are assumed to hold
which determine the complex wave velocities.

The three displacement potentials corresponding to fast, slow and shear
waves that satisfy the equations of state are:

\[
\Phi_i(r,z,k) = \int_0^\infty \phi_i(z,k_r) J_0(k_r, r) k_r dk_r, \quad i=0,1,2,3
\]  

\text{(4.1)}
where subscripts 1 & 2 refer to the two dilatational waves and 3 refers to the rotational wave potentials in the porous solid and 0 refers to the displacement potential in the air.

\( \varphi_i \) are depth-separated Green's functions which are solutions of the transformed Helmholtz wave equations:

\[
\frac{d^2 \varphi_i}{dz^2} - \beta_i^2 \varphi_i = S \delta(z - h_i)
\]  

(4.2)

where \( \beta_i^2 = k_i^2 - k_r^2 \), \( k_i = \omega/c_i \) \( i=0,1,2,3 \)

\( k_r \) is the horizontal component of wavenumber and \( c_i \) are the corresponding wave speeds. \( \beta_i \) represent the vertical components of the complex wave number and \( S \) is the source strength.

We assume the following forms for the displacement potentials:

\[
\varphi_0 = R_k e^{i \beta_0 z} + S \frac{e^{i \beta_0 (z - h_1)}}{\beta_0}
\]  

(4.3)

where the first term represents the reflected wave and the second, the direct wave. We also write:

\[
\varphi_1 = (A e^{i \beta_1} + B e^{i (d-z) \beta_1}) + (C e^{i \beta_2} + D e^{i (d-z) \beta_2})
\]  

(4.4)

\[
\varphi_2 = m_1(A e^{i \beta_1} + B e^{i (d-z) \beta_1}) + m_2(C e^{i \beta_2} + D e^{i (d-z) \beta_2})
\]  

(4.5)

\[
\varphi_3 = (E e^{i \beta_1} + F e^{i (d-z) \beta_1})
\]  

(4.6)

\[
\varphi_1^d = A_1 e^{i (d-z) \beta_1} + C_1 e^{i (d-z) \beta_2}
\]  

(4.7)
\[ \varphi_{2i} = m_i A_i e^{i(d-z)\beta_i} + m_z C_z e^{i(d-z)\beta_z} \]  
\[ \varphi_{3i} = E_z e^{i(d-z)\beta_i} \]  

where \( d \) is the layer thickness and \( z \) the positive distance from the ground surface. Subscript \( s \) denotes the amplitude values corresponding to the substrate, \( m_i \) are the ratios of relative fluid displacement to the solid one, for the three body waves and are derived from the Biot equations.

The wave amplitudes \( R_p, A, B, C, D, E, F \) are to be derived from solving the air-ground boundary equations. In all, there are 10 boundary equations for this system; four at the fluid-solid interface, and six at the ground layer-substrate interface \(^{[31]}\). These are listed in detail in chapter 2. The set of 10 simultaneous linear equations arising from the boundary equations can be cast in matrix form and solved numerically:

\[ A \cdot X = B \]  

\( A \) is the coefficient matrix and \( B \) is the source term vector.

Once the amplitudes are known, an inverse Hankel Transform is performed, either by direct numerical integration or by FFP method, to obtain the final displacement potentials. This solution matrix is used to explore the influence of ground elastic parameters on wave propagation. We explore the ratio of ground reflected term and direct wave which gives the reflected wave amplitude. The values of ground parameters used in the example are listed in Table 4.1. Many of these values have been measured at a site near
Wezep in Holland and have been used elsewhere as benchmark parameters \cite{15}. They are typical of a sandy soil.

We would expect that p- or s-wave resonances within the layer affect the reflected wave, with constructive interference at some frequencies producing minima. The condition for constructive interference within an elastic layer of thickness $H$, over a rigid substrate is well known \cite{30}:

$$f_n = \frac{2n+1}{4H} \frac{c}{\sqrt{1 - \left( \frac{c}{c_0} \sin \theta \right)^2}}$$

(4.11)

where $c$ is the wave velocity in the layer (dilatational or shear), $c_0$ is the velocity of sound in the air, and, $\theta$ is the angle of incidence. The expression assumes a pressure release upper boundary and total reflection at the lower interface.

![Figure 4.1 Reflection coefficient above poro-elastic layers of various thickness with $V_p=100.\text{m/s}$ and $V_s=70.7 \text{ m/s}$.

We calculate the magnitude of the reflection coefficient ($R_p$) for ground parameters given in Table 4.1 but for lower values of $p$ and $s$ wave speeds.
(100.0 and 70.7 m/s respectively) for various layer thickness. The angle of incidence is dictated by the source and receiver positions, being 83° for the data considered. Figure 4.1 shows the reflection coefficient $R$ plotted against frequency times the layer thickness $(fH)$ for five cases. The predicted minima correspond to layer resonances. The set of minima indicated by $f_\text{sw}$ are associated with the first s-wave resonance, while the deepest minima indicated by $f_\text{pw}$ occur for the first p-wave resonance. One does not expect to observe a resonance due to slow waves because of very high attenuation of these waves in the frequency range of interest. These structures represent maxima in acoustic-to-seismic coupling and therefore indicate frequencies at which energy will be taken out of waves propagating above the poroelastic surface. They should also be apparent, therefore, in plots of excess attenuation spectra as extra dips (or peaks depending on the definition of excess attenuation). Figure 4.2 shows an example. Except for the very thin layers, the largest dips occur at low frequencies (below 100 Hz). The low p and s wave speeds and the large difference between these values and substrate velocities suggest total reflection from the substrate. This means that the assumptions leading to equation (4.10) are valid. The situation alters if one substitutes higher values for p- and s-wave speeds in the layer (270.0 and 190.0 m/s respectively) while keeping the ratio of the two bulk moduli constant. These values are nearer to the actual measured speeds. There are no clearly definable minima, and because the layer and substrate speeds are comparable, it is no longer possible to treat the substrate as rigid. We can use higher values for the substrate velocities and recalculate the reflection
coefficient (Figure 4.3). In Figure 4.3, the minimum marked ‘1’ corresponds to a wave speed of 200 m/s, i.e. close but not equal to the layer shear speed of 190.0 m/s; and ‘2’ refers to a speed of 250.0 m/s, again close but not equal to the layer p-wave speed of 270.0 m/s. It indicates that in this case, the upper boundary (air-soil layer) is no longer a pressure release boundary.

As an example, we compare predictions of the two models with measurements of acoustic-to-seismic transfer function in Wezep, Holland reported by Van Hoof and Doorman[51]. The measured ground parameters are those listed in Table 4.1 with a layer thickness of 2.0 m. The predicted acoustic/seismic coupling function using the full Biot theory is given in Figure 4.4. In Table 4.2 the frequencies corresponding to the maxima are compared with the measured peaks together with the predictions using equation (4.10).

![Figure 4.2 An example of Excess Attenuation of sound above a poro-elastic soil layer. The small minima are due to layer resonances but are usually too small to detect above the experimental error and uncertainty.](image)
Figure 4.3 Reflection coefficient above poro-elastic layers of various thickness with $V_p=270\, \text{m/s}$ and $V_s=190\, \text{m/s}$.

Figure 4.4 Acoustic/seismic coupling ratio above a 2.0m thick porous elastic layer. Other parameters are given in Table 4.1.
Attenborough\cite{attenborough1995} has used an expression due to Hastrup\cite{hastrup1983} for resonance frequencies in a non-hard-backed ground layer to calculate the peak frequencies:

\[ f_n = \frac{c}{4H} \frac{(2n + 1)\pi - S\mu_0}{\pi \sin \mu_0} \quad n=0,1,2,... \]

(4.12)

where
Chapter 4 - On the Influence ...

\[
S = \frac{2(\rho_s/\rho_l)}{\sqrt{1-(c_l/c_0)^2}}
\]  

(4.13)

and \(\mu_0\) is \(\text{arccos}(c/c_0)\). Subscript 0 refers to air, \(l\) to the layer and \(s\) to the substrate.

Figure 4.5 The geometry assumed in calculating the ground layer resonances.

Hastrup considered a soft water-saturated sediment such as mud above a hard sub-bottom. The geometry is shown in Figure 4.5. For the three layers, \(C_0, C_l\) and \(C_s\) are compressional wave velocities and \(\rho\) are the densities in each layer. Shear waves are assumed to be negligible in this case. It is further assumed that \(C_s > C_0 > C_l\) in the case under consideration. Furthermore, only small gazing angle is considered so that there is total reflection in the interface 2-3. For a constructive interference in the layer, the total acoustic path difference between two down-going waves must be an even multiple of \(\pi\), in other words:

\[
(AB+ BC) \cdot k + \varphi_{23} + \varphi_{31} = 2n\pi
\]

(4.14)
where $k$ is the wavenumber, $\varphi_{21}$ is the phase shift at the upper interface, assumed to be zero, and $\varphi_{23}$ is the phase shift at the lower interface which is equal to $S\mu_0 \pi$. This is defined as above. Taking into account that $(AB+BC)=2H\sin\mu_0$ we have then:

$$S\mu_0 + 2Hk \sin \mu_0 = (2n+1)\pi$$

(4.15)

which can be rearranged to produce eqn. (4.14). The term for the lower interface phase shift was derived from the expression for the reflection coefficient:

$$R_{23} = e^{-S\mu} = e^{i(s\mu-k)}$$

(4.16)

where $S' = \frac{2\rho_3/\rho_2}{\sqrt{\left(C_2/C_3\right)^2 - 1}} = -iS$

(4.17)

The measured peak frequencies and those predicted by the three methods are also listed in Table 4.2. It is clear that predictions based on the poroelastic theory are superior to the other two. It is also clear that the non-hard-back resonance formula of Hastrup is superior to that of a simple rigid-backed one even when applied to shear wave resonances. Note that no attempt has been made to fit the amplitudes of the peaks to the measured data for this case since the intention has been to indicate the peak frequencies.

There are measured maxima which are not predicted by FFLAGS (notably at 235, 275, 303 and 335 Hz). These are probably due to finer ground structure within the 2 metre top soil layer indicated here. Therefore the calculation
have been based on only a single layer. A more detailed refraction survey at other sites (see section 4.3) indicates that the ground is normally more structured. Note that data are available only above 100 Hz.

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
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<tbody>
<tr>
<td>105*</td>
<td></td>
<td>108.3(s2)</td>
<td>102*</td>
</tr>
<tr>
<td>116*</td>
<td></td>
<td>122.9(p1)</td>
<td>116</td>
</tr>
<tr>
<td>155</td>
<td>142.5(s2)</td>
<td></td>
<td>155*</td>
</tr>
<tr>
<td></td>
<td>164.1(p1)</td>
<td>165.36(s3)</td>
<td></td>
</tr>
<tr>
<td>174*</td>
<td></td>
<td></td>
<td>170</td>
</tr>
<tr>
<td></td>
<td>199.5(s3)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>220*</td>
<td>222.4(s4)</td>
<td>220*</td>
<td></td>
</tr>
<tr>
<td>235</td>
<td></td>
<td>231.6(p2)</td>
<td></td>
</tr>
<tr>
<td>250</td>
<td>256.6(s4)</td>
<td></td>
<td>255</td>
</tr>
<tr>
<td>275</td>
<td>273.5(p2)</td>
<td>279.5(s5)</td>
<td></td>
</tr>
<tr>
<td>303</td>
<td>313.5(s5)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>330*</td>
<td></td>
<td></td>
<td>330*</td>
</tr>
<tr>
<td>335</td>
<td></td>
<td>340.95(p3)</td>
<td></td>
</tr>
</tbody>
</table>

Table 4.2 Measured and predicted peak frequencies of acoustic/seismic coupling ratio. Asterisks refer to major peaks, Sn and Pn to the nth harmonics of s- and p-waves respectively.

4.3 Measurement of acoustic-to-seismic coupling ratio

This section is a report of the measurement and subsequent analysis of the acoustic-seismic coupling ratio in outdoor conditions. The measurements were performed in June of 1993 in Bondville, Ill., USA. The participants were S Taherzadeh and Prof. K Attenborough of the Open University, and

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*The measurements reported in this section were carried out during a period in June 1993 which the author spent at NCPA. He is grateful to Dr. J.M. Sabatier and Dr. H.E. Bass for providing this opportunity and enlightening discussions.*
Chapter 4 - On the Influence ...

Dr. J.M. Sabatier and H. Eswaran of the National Center for Physical Acoustics (NCPA), MS, USA.

The environment was characterised by measuring the flow resistivity and porosity of a number of soil samples from the site and also by carrying out a shallow seismic refraction survey of the site. These were analysed at NCPA and the results are listed in Table 4.3.

<table>
<thead>
<tr>
<th>flow resistivity</th>
<th>300000 Pa sm⁻²</th>
</tr>
</thead>
<tbody>
<tr>
<td>porosity</td>
<td>0.31</td>
</tr>
<tr>
<td>number of layers</td>
<td>4</td>
</tr>
<tr>
<td>layer thickness</td>
<td>P wave speed (m/s)</td>
</tr>
<tr>
<td>layer 1 0.06m</td>
<td>140.0</td>
</tr>
<tr>
<td>layer 2 0.2m</td>
<td>186.0</td>
</tr>
<tr>
<td>layer 3 0.43</td>
<td>186.0</td>
</tr>
<tr>
<td>layer 4 2.0m</td>
<td>386.0</td>
</tr>
<tr>
<td>substrate</td>
<td>1800.0</td>
</tr>
</tbody>
</table>

Table 4.3 list of measured environment parameters at Bondville.

All layers except the substrate were assumed to be porous with cylindrical pores. The value for the Grain Shape Factor \(n'\) were assumed to be 0.5 for all layers. The Attenuation rate for the soil layers were not measured either and assumed to be 0.05. This gave the best fit for the data without varying the value for each layer individually.

The source, a loudspeaker, was place on top of a wooden tower 32.0 m high. The transfer function between a microphone placed 0.05 above the surface and a geophone buried 0.1m below it was recorded by a Hitachi Spectrum Analyser. A sweep frequency noise generated by the spectrum analyser was
fed onto the loudspeaker. Nine positions at ranges from 15 m to 39 m away from the tower were selected and two measurements were taken in each position at places 25 cm apart. The predictions were made by the candidate at the Open University using the FFP program FFLAGS. Results for the 15 m range are shown in Figure 4.6 where the two measurements are shown by '+' and 'o' and the prediction by a solid line. It is interesting to note that the two measured spectra are quite different even though they were measured at the same range and only about 25 cm apart. The cause of the difference (which seem to be absence of one minimum or the other) is not clear. One possibility is that the geophones were placed in position differently at the two locations and hence this implies a source of experimental error. On the other hand Dr. Sabatier suggests that these indicate real differences in the structure of the ground between the two points rather than being an error of measurement procedure.

The corresponding results for geophone and microphone at a distance of 18.0 m are given Figure 4.7. The agreement between predictions of FFLAGS and the measurements are tolerably good particularly at the frequency range of 100 - 300 Hz. However, the predictions show a large peak at about 50 Hz which does not appear in measurements. This is a p-wave reflection from the rocky substrate and therefore sensitive to exact thickness of the top soil.
Figure 4.6 measured (crosses and circles) and predicted (solid line) Acoustic to seismic coupling ratio measured at bondville. The range is 15.0m and the source height is 31.0m.

Figure 4.7 measured (crosses and circles) and predicted (solid line) Acoustic to seismic coupling ratio measured at bondville. The range is 18.0m and the source height is 31.0m.
4.4 Conclusion

Biot's theory of wave propagation in porous elastic media, discussed in previous chapters, was used to calculate the reflection coefficient of sound impinged upon a porous elastic layer of soil overlaying a elastic substrate as well as the coupling of the airborne sound excitation to seismic ground motions. These were contrasted with two simpler models. One is a simple elastic layer resonance with total reflection from interfaces. The other, developed by Hastrup[53] for underwater acoustics, assumes a non-hard-backed elastic layer. These two models have been used by Sabatier[48] and Attenborough[52] to predict the peak frequencies of the Acoustic-to-Seismic coupling spectra measured by Van Hoop et al in Holland (see Table 4.2). It was shown in this section that the poroelastic theory of Biot offers better agreement with data. However, the error estimates in the measurements are not known.

Acoustic-to-Seismic coupling spectra have been measured also by the author and others at Bondville, USA. These measurements show significant differences in the spectra taken in two positions only 0.3 m apart. Although the Biot theory can predict the overall features of the spectra in a satisfactory manner, the cause for the spread in the data is not clear and requires further investigation, possibly in a controlled environment.
Chapter 5

Sound Field From an Arbitrarily Orientated Dipole Source Above Rigid Porous and Poroelastic Ground

5.1 Introduction

In all the analysis so far, I have assumed the source to be an idealised monopole point source generating spherically symmetric waves. In real life, however, some sources of sound, such as a jet engine or an unbaffled loudspeaker are more complicated. These noise sources exhibit directional output patterns and can be modelled as dipoles or quadrupoles. Such sources do not produce spherically symmetric radiation. The sound field due to a radiating dipole in an unbounded medium is well known and is given for example by Pierce.\textsuperscript{[3]} Generalov\textsuperscript{[54]} gives an asymptotic expression for sound pressure level from a multipole above a locally reacting rigid porous surface. Hu and Bolton\textsuperscript{[55]} treat the same problem by expressing the dipole field in terms of a two-dimensional Hankel Transform and carrying out a numerical integration of the resulting Hankel integral. They report that the solution by Generalov is not in good agreement with their results. It is possible also to reproduce the sound pressure field due to a dipole by superimposing the sound fields of two closely-spaced, out-of-phase monopole sources. However accurate calculation of the resulting field depends on the value used for the monopole separation, and the correct choice of a source

\textsuperscript{[A paper based on section 2 of this chapter has been published in Proc. Symposium on Computational Acoustics, ed. C.A. Brebbia, southampton, 1995\textsuperscript{[83]}.]}

93
strength. Moreover, the reflection coefficients for the two constituent sources must be calculated separately however closely-spaced they may be. It is therefore preferable to determine the sound field from a dipole source directly.

In this chapter, first, an expression is derived for the sound field from an arbitrarily oriented dipole in a stratified fluid above a layered poroelastic solid using a Fast Field Program to determine the one-dimensional Hankel Transform based on FFLAGS. The model is validated by comparing the resulting predictions for the field due to a vertical dipole with that predicted from two out of phase, vertically-separated monopoles and it is shown that the predictions are identical. It will also be shown that the field due to a horizontal dipole is very similar to that of a monopole except when the separation is comparable to the source height. An efficient technique for calculating the 2D contour field lines of a dipole in a plane parallel to the poroelastic boundary (x-y plane) is demonstrated. In the second part of the chapter a simpler version is given where the solid is modelled as porous but rigid and the boundary between the fluid and the solid may be treated as an impedance boundary. In this case, the field is the sum of direct and plane reflected terms in the wavenumber domain. By direct differentiation of the monopole field in the wavenumber domain and utilising existing approximations to evaluate the resulting integrals, an asymptotic closed-form expression for sound pressure level from a vertical dipole above an impedance boundary is derived. Similar techniques are used in a later chapter to formulate the corresponding expressions for sound field due to an
arbitrarily oriented quadrupole. Finally, in this chapter, the predictions of the models are compared with measurements of the dipole field due to a piezoceramic transducer element above a boundary of known impedance.

5.2 Theory

5.2.1 Hankel Transform of a Dipole

We model the dipole as the sum of two out of phase monopoles spaced a small distance $2d$ apart:

$$\phi_d = S \left( \frac{e^{-iR_A}}{R_A} - \frac{e^{-iR_B}}{R_B} \right)$$

(5.1)

where $R_A$ and $R_B$ are the distances of the two monopoles from the receiver. Then we have

$$R_A = \sqrt{(x - dsin\mu cos\varepsilon)^2 + (y - dsin\mu sin\varepsilon)^2 + (h - z - d cos\mu)^2}$$

(5.2)

and

$$R_B = \sqrt{(x + dsin\mu cos\varepsilon)^2 + (y + dsin\mu sin\varepsilon)^2 + (h - z + d cos\mu)^2}$$

(5.3)

in the above equations $\mu$ and $\varepsilon$ are the polar and azimuth inclination of the line joining the two monopoles (dipole axis). We then move to the wavenumber domain using a Hankel transform:

$$f(r,z) = \int_0 f(r) J_0(k,r) dk,$$

(5.4)
If we let:
\[ \delta_z = d \cos \mu \]
\[ \delta_r = d \sin \mu \cos \epsilon \]

where \( \epsilon \) is now the azimuth angle between receiver and the dipole orientation, then
\[
\varphi = \frac{S}{k_z} e^{ik_z(z-h)} \left\{ e^{ik_z \delta_z - i \delta_r \epsilon} - e^{-ik_z \delta_z + i \delta_r \epsilon} \right\}
\]

(5.5)

As \( e^{ix} - e^{-ix} = 2i \sin x = 2ix \) for small \( x \), we have:
\[
\varphi = 2id \frac{S}{k_z} e^{ik_z(z-h)} \left\{ -k_z \partial \cos \mu - k_z \sin \mu \cos \epsilon \right\}
\]

(5.6)

where \( \partial \) is \( \text{sign}(h-z) \), or with some rearrangement
\[
\varphi_d = (-i)S_d \frac{e^{ik_z}[z-h]}{k_z} \left\{ \text{sign}(h-z)k_z \cos \mu + k_z \sin \mu \cos \epsilon \right\}
\]

(5.7)

This is the wavenumber domain potential for a dipole with an arbitrary orientation. Note that the Hu and Bolton’s expression is valid only for a source elevation greater than the receiver height while eqn. (5.7) is valid for all cases. The first term is the vertical component and the second term is the horizontal component of the dipole field.

5.2.2 Boundary conditions and source terms

The total field is calculated by the global matrix method with the above potential forming the source terms (right-hand side of the matrix equation).

Thus, for the layer containing the source:
\[
\varphi = A e^{ik_z(z-d_a)} + B e^{-ik_z(z-d_a)} + \varphi_d.
\]

(5.8)
From this point on I will handle the two components separately for reasons that are explained later.

The boundary conditions on the fluid-fluid interface are: 1) continuity of pressure, and 2) continuity of particle displacement. Therefore the source terms are:

vertical dipole \[ \rho \omega^2 (-i) e^{ik_z|z-h|} \text{sign}(h-z) \]

horizontal dipole \[ (-i) \frac{\zeta}{k_z} e^{ik_z|z-h|} \]

\[ k_z e^{ik_z|z-h|} \quad \text{sign}(h-z) k_z e^{ik_z|z-h|} \]

\[ (5.9a,b) \]

(5.10a,b)

For the poroelastic solid-fluid interface we have

1. continuity of total Normal Stress \((-\rho \alpha^2 \phi)\);
2. continuity of Normal Displacement \((d\phi/dz)\),
3. continuity of Tangential displacement \((= 0)\), and
4. continuity of Fluid Pressure \((\rho \alpha^2 \phi)\). These have the same form as above terms.

The boundary conditions for the solid-solid interfaces are as reported in chapter 3.

For an impedance boundary condition at the solid-fluid interface (instead of the full continuity conditions) we then have instead:

\[ e^{ik_z|z-h|} (k_z - k_0 \beta) \]

\[ k_z e^{ik_z|z-h|} \left(1 - \frac{k_0 \beta}{k_z}\right) \]

\[ (5.11a,b) \]

the global matrix equations are then formed:
\[ A \mathbf{X}_v = \mathbf{B}_r \quad \quad A \mathbf{X}_h = \mathbf{B}_s \]

(5.12a,b)

where subscripts \( v \) and \( h \) refer to the vertical and horizontal dipoles respectively. The two equations can be solved simultaneously with very little extra computational effort if an \( LU \) decomposition technique such as Crout’s Method is used, where the main computational time is spent in decomposing the coefficient matrix \( A \) into the sum of an upper triangular and a lower triangular matrices. The solution is a trivial one and is an \( O(N) \) operation, where \( N \) is the order of the matrix.

The case of homogeneous fluid above a rigid porous solid is particularly simple. In this case the total transformed field is simply the sum of direct and reflected fields. In the wavenumber domain the result for a vertical dipole is:

\[
\varphi_v = (-i) S_d \left\{ \text{sign}(h-z) e^{ik_z|z-h|} + \frac{k_z - k_0 \beta}{k_z + k_0 \beta} \times e^{ik_z(z+h)} \right\}
\]

(5.13)

and for the horizontal component:

\[
\varphi_h = (-i) S_d \left\{ \frac{k_z}{k_z} e^{ik_z|z-h|} + \frac{k_z - k_0 \beta}{k_z + k_0 \beta} \times \frac{k_r}{k_z} e^{ik_z(z+h)} \right\}
\]

(5.14)

The Hankel Transform is carried out either as before by the FFP method (i.e. approximating it by a Discrete Fourier Transform) or by direct numerical integration. Here I use the FFP technique. The total amplitude vector is then:

\[
\phi = \cos \mu \phi_v + \sin \mu \cos (\epsilon - \epsilon_r) \phi_h
\]

(5.15)
where $\epsilon_r$ is the receiver azimuth. Thus calculating the components separately permits us -by varying $\epsilon_r$ - to build up a 2-D contour diagram of an arbitrarily oriented dipole field with only two evaluations of the field potentials. Figure 5.1 is the contour plot of pressure due to an inclined dipole above a porous ground. The parameters used in the calculations are listed in [Table 3.1].

![Contour plot](image)

**Figure 5.1** Pressure (dB) contour plot of a horizontal dipole above a porous elastic ground with azimuthal angle of 45° at $f=200$ Hz. The source and receiver heights are 5.0 and 1.0m respectively.

### 5.3 Approximate asymptotic forms for the dipole field

In this section we endeavour to obtain asymptotic expressions of the integral representation of the dipole field to provide an alternative to numerical evaluation. The aim is to find approximate forms corresponding to the similar forms developed for monopole point sources\textsuperscript{[56,57]}. An intuitive
answer is to use the spherical reflection coefficient for monopole point source for dipoles as well. While this approach has some success in case of the horizontal dipole, the same cannot be said of the vertical dipole. It seems that the ground wave term of the dipole field is more significant. Subsequently, I will concentrate this section on deriving approximate solutions for a vertical dipole only.

5.3.1 Existing solutions

Generalov[54] has developed an expression for the sound field of a multipole above a locally reacting impedance boundary. He represents a multipole of order N as an Nth derivative (wrt spatial co-ordinates) of a monopole field. He forms the Hankel-transformed integral representation of the multipole fields and proceeds to find an approximate, asymptotic form for the integrals by a modified double saddle-point method. He arrives at the following expression for sound field due to a vertical dipole (eqn. 6 of ref [54]):

\[ \phi_d = \phi_z - ik_0 \frac{dR_z}{dz} \frac{e^{ik_0 R_z}}{4\pi R_z} \times \left\{ A(\tau) b_{01} + \frac{2\beta}{ik_0 R_z} \left( L - \frac{\sin^2 \theta}{\cos \theta} \right) \right\} \]

(5.16)

where

\[ \phi_z = ik_0 \frac{dR_z}{dz} \frac{e^{ik_0 R_z}}{4\pi R_z} \left( 1 - \frac{1}{\lambda_o} \right) + ik_0 \frac{dR_z}{dz} \frac{e^{ik_0 R_z}}{4\pi R_z} \left( 1 - \frac{1}{\lambda_o} \right) \]

(5.17)

\[ R_1 = \left( r^2 + (z - h)^2 \right)^{1/2} \quad \text{and} \quad R_2 = \left( r^2 + (z + h)^2 \right)^{1/2} \]

(5.18)

\[ \tau = \left[ -ik_0 \left( 1 + \beta \cos \theta - \sqrt{1 - \beta^2 \sin^2 \theta} \right) \right]^{1/2} \]

(5.19)
\[ A(\tau) = \frac{1}{\tau} \left[ F(\tau) - \frac{1}{2\tau^2} \right] \]  

(5.20)

\[ b_{o1} = \frac{\sqrt{\pi} k_0 R_1 (-\beta)^2 H_0^{(1)}(k_0 r \sqrt{1 - \beta^2}) e^{-ik_0 \sqrt{1 - \beta^2}}}{\cos \theta} \]  

(5.21)

\[ F(\tau) = 1 - \sqrt{\pi \tau e^{z^2}} \text{erfc}(\tau) \]  

(5.22)

\[ L = \frac{(1 + \beta \cos \theta)}{(\cos \theta + \beta)^3} \]  

(5.23)

and

\[ B = \left( \cos \theta + \beta \right)^{-2} \]  

(5.24)

where a typographical error in the expression for \( A(\tau) \) has been corrected.

Here \( F(\tau) \) is the boundary loss factor; \( \beta \) is the admittance of the surface, and \( \tau \) is the numerical distance. Generalov's version of the numerical distance is related to that used by Chien\(^{[57]}\) and Attenborough et al\(^{[56]}\) by \( \tau = -i\omega \).

He then presents a simplified form for the case when \(|\tau| >> 1\):

\[ \phi_z = \phi_2 - \beta \left( L - B \frac{\sin^2 \theta}{\cos \theta} \right) \frac{dR_z}{dz} \frac{e^{ik_0 R_z}}{4\pi R_z^2} + U(- \text{Re}(\tau)) \frac{i k_0^2 R_z^2}{2} \]  

(5.25)

where

\[ \kappa = H_0^{(1)}(k_0 r \sqrt{1 - \beta^2}) e^{-ik_0(z+h)} \]

In conclusion it should be pointed out that that calculation of the complex numerical distance, \( \tau \), requires special care so that the correct phase for it is calculated\(^{[69]}\). The form of \( \tau \) as given in eqn. (5.19) can lead to errors since
almost all complex square root functions available in mathematical packages and FORTRAN compilers at present, invariably produce a positive real value for the square root and this is not always the correct choice. The version of the numerical distance given in references [56] and [57] is more accurate and robust and has been used here.

5.3.2 An accurate Asymptotic form for dipole field.

We start by noting that the free field dipole can also be written as:

\[ \varphi_d = \nabla \cdot \varphi_m \]  

(5.26)

where \( \varphi_m = \frac{e^{ikz-h}}{k_z} \) is the transformed potential for a monopole.

For a source situated above an impedance boundary one could use this differential relationship between the field of a monopole and that of a vertical dipole. In principle, one could differentiate the sound field of a monopole source above a boundary, viz.:

\[ \phi_{\text{monopole}} = \frac{e^{ikR_i}}{R_i} + \left[ R_p + \left( 1 - R_p \right) F(w) \right] \frac{e^{ikR_i}}{R_z} \]  

(5.27)

where the square bracket represents the spherical wave reflection coefficient\(^{[56,57]}\) with \( R_p \) denoting the plane wave reflection coefficient. Other terms have been defined in equations (5.18) and (5.22). However, in deriving the spherical reflection coefficient given in (5.27) several approximations have been made\(^{[56,57]}\), so the exact functional dependence of the reflected term on \( z \) and \( r \) have been lost. This means that differentiating equation (5.27) will not produce an accurate term for the reflected wave. It is
necessary to carry out the differentiation in the wavenumber domain where
the plane wave reflection coefficient is used.

Therefore, the vertical dipole field can be written as:

\[ \phi = \int \frac{d}{dz} \varphi_m J_0(k, r) k,dk, \]  

(5.28)

where \( \varphi_m \) is defined above. I will proceed by rearranging the potential to
some extent so that all the terms are integrable and have known, and tested,
asymptotic values. Then the integration is carried out term by term.

Denoting the direct ray path and by \( R_1 \) and the image or reflected ray path by
\( R_2 \), then the direct field has a known closed form solution:

\[ \int \frac{d}{dz} \left( \frac{e^{ik_z(z-h)}}{k_z} \right) J_0(k, r) k_r,dk_r = \frac{1-ikR_1}{4\pi R^2_1} e^{ik_n} \cos \theta, \]  

(5.29)

where \( \theta_i \) is the angle between direct path and the z-axis. The reflected field
term can be rearranged as follows:

\[ \varphi_r = \frac{k_z - k_0 \beta}{k_z + k_0 \beta} \int \frac{d}{dz} \left( \frac{e^{ik_z(z+h)}}{k_z} \right) = \left( 1 - \frac{-2 \beta}{\cos \theta + \beta} \right) \frac{ik_z}{k_z} e^{ik_z(z+h)} \]  

(5.30)

where \( \cos \theta = \frac{k_z}{k_n} \) is the, possibly complex, angle of incidence.

\[ \varphi_r = \frac{d}{dz} \frac{e^{ik_z(z+h)}}{k_z} + \left( \frac{-2i\beta k_0 \cos \theta}{\cos \theta + \beta} \right) \frac{e^{ik_z(z+h)}}{k_z} \]  

(5.31)

The first term in the above equation can again be evaluated exactly and is
recognised as the 'image' source term:
\[
\int \frac{d}{dz} \left( \frac{e^{ik_z(z+h)}}{k_z} \right) J_\nu(k_z r) k_z dk_z = \frac{1-i k R_z}{4 \pi R_z^2} e^{ik_R z} \cos \theta
\]

(5.32)

The second term needs to be rearranged further:

\[
\left( \frac{-2i k_0 \cos \theta}{\cos \theta + \beta} \right) e^{ik_z(z+h)} = -2i k_0 \left( 1 - \frac{\beta}{\cos \theta + \beta} \right) e^{ik_z(z+h)}
\]

(5.33)

Here the first term is the familiar monopole term with some constant coefficient and the second term has a well known asymptotic form (see [56] and [57]). Combining the last four equations it is possible to arrive at an asymptotic expression for a sound field of a vertical dipole above an impedance boundary:

\[
\phi = \left( \frac{1-i k R_1}{R_1^2} \right) e^{ik_R_1 \cos \theta_1} + \left( \frac{1-i k R_2}{R_2^2} \right) e^{ik_R_2 \cos \theta} + i k_1 \beta \left( \frac{\cos \theta + \beta F(w)}{\cos \theta + \beta} \right) e^{ik_R z} \cos \theta
\]

(5.34)

This equation is the main result of this section. It gives an accurate formula for sound field of a vertical dipole (or the vertical component) above an impedance boundary with normalised admittance \( \beta \). To appreciate the significance of each term, equation (5.34) can be compared with a similar expression for field of a monopole source. After rearranging equation (5.27) one arrives at the following expression:

\[
\phi_{\text{monopole}} = \frac{e^{ik_R_1}}{R_1} + \frac{e^{ik_R_2}}{R_2} + 2 \beta \frac{F(w) - 1}{\cos \theta + \beta} \frac{e^{ik_R z}}{R_z}
\]

(5.35)

Ignoring the direction cosines, the field of a vertical dipole consists of a direct term plus an image source term and a monopole ground wave term.
The third term is somewhat greater for a dipole than it is for a monopole (see Figure 5.10).

By starting from the wave equation for a dipole source and evaluating the resulting Hankel integrals by steepest descent method, Li, Taherzadeh & Attenborough\textsuperscript{[58]} have reproduced the above expression as well as obtaining the corresponding formula for a horizontal dipole. Also they have derived an approximate compact expression for an arbitrarily oriented dipole at high frequencies:

$$\phi = (\hat{\mathbf{i}} \cdot \hat{\mathbf{R}}_1) \left[ \frac{1 - ik_0 R_1}{R_1^2} \right] e^{ik_0 R_1} + (\hat{\mathbf{i}} \cdot \hat{\mathbf{R}}_2) \left[ \frac{1 - ik_0 R_2}{R_2^2} \right] e^{ik_0 R_2}$$

$$+ (\hat{\mathbf{i}} \cdot \hat{\mathbf{R}}_s) (1 - R_p) F(w) \left[ \frac{1 - ik_0 R_s}{R_s^2} \right] e^{ik_0 R_s} \tag{5.36}$$

where $\mathbf{R}_1$ and $\mathbf{R}_2$ are unit vectors pointing from the dipole towards the receiver and image receiver respectively, and, $\mathbf{R}_s$ is the unit vector that characterises the direction of propagation of ground wave term. The approximation is valid as long as the wavenumber, $k_0$, is large compared to the separation of the dipole source and the receiver.
5.4 Validation and analysis

To validate the methods arrived at here, the predicted field from a vertical dipole above a poroelastic ground surface (equation (5.7)) is compared with the field predicted from two vertically separated, out-of-phase, monopoles which can be calculated by FFLAGS (Figure 5.2). This is done in order to check the self-consistency of equation (5.7). Also plotted is the field due to a monopole point source normalised to give the same sound level at the source. The agreement between the two dipole models is good (indeed in the graph the lines representing the two predictions are coincident). The rate of attenuation of a dipole is markedly different from that of a monopole at shorter distances, showing the characteristic decrease of 12 dB per doubling of distance rather than the 6dB per doubling of distance characteristic of a monopole. In the far field, though, both field attenuate at the same rate. On
the other hand, the interference pattern is totally different for a monopole and a vertical dipole. This reflects an extra $\pi$ phase shift of the dipole reflected wave when the receiver is above the source.

The sound field of a dipole above a porous elastic ground is almost indistinguishable to that above an impedance plane for the set of parameters used here and listed in Table 3.1 (as is the field due to a point monopole source). Therefore, from this point on I assume the ground interface to be an impedance plane and utilise an impedance version of the full (poroelastic boundary) dipole FFP (DipoleFFP) where appropriate and compare its predictions with those of the analytical expressions (equations (5.25) and (5.34)) discussed earlier.

Figure 5.3 shows the predictions of the DipoleFFP, equation (5.34), and the Generalov approximation (equation (5.16)), for the sound pressure levels due to a vertical dipole source compared with the field predicted for a monopole source at a frequency of 1 kHz. Figure 5.4 is a plot of a horizontal dipole field above an impedance plane compared to the field of two out-of-phase monopoles separated horizontally, (calculated by adding complex fields of two monopoles) and field of a monopole normalised to the same level. It is clear that, except at short distances compared to source or receiver heights, a horizontal dipole behaves like a monopole. Both asymptotic approximations agree fairly well with predictions of the FFP program at this frequency. However, at a lower frequency, the form derived by Generalov for the vertical dipole does not agree with predictions of DipoleFFP, while
the latter's expressions are indistinguishable from those of the FFP program.

Figures 5.5 and 5.6 are the corresponding plots at a frequency of 100 Hz.

Figure 5.3 Predicted Transmission Loss of a vertical dipole above an impedance plane at 1 kHz. Results of the dipoleFFP and the asymptotic expression (eqn. 5.34) are coincident (solid line), Generalov's solution (dash dot line) and monopole field (dotted line). Hs=2.0m, Hr=1.0m.

Figure 5.4 Predicted Transmission Loss of an horizontal dipole at 1 kHz above an impedance plane. Results of the dipoleFFP and the asymptotic expression are indistinguishable above 1 m. (solid line), monopole field (dotted line). Hs=2.0m, Hr=1.0m.
In the following calculations, involving asymptotic approximations to the dipole field, the ground is characterised by a 2-parameter model\textsuperscript{[59]} for the impedance. This is given by:

\[
Z = 0.436(1 + i) \sqrt{\frac{\sigma_c}{f}} + 19.48i \frac{\alpha_c}{f}
\]  

\textbf{(5.37)}
The parameters used are $\sigma_e=38$ kPa $\text{sm}^{-2}$ and $\alpha_e=15$ m$^{-1}$. We compare predictions of the asymptotic expression for a vertical dipole with those obtained by Generalov and test them against the *impedance version* of the Dipole FFP. Figure 5.3 and Figure 5.5 show predicted SPL of a vertical dipole vs. Range at 1000 and 100 Hz respectively. The source and receiver heights are 2.0 and 1.0 m respectively and the ground is modelled by the 2-parameter model. Results for the lower frequency show that Generalov’s predictions compare poorly with those of the impedance version of the Dipole FFP.

![Figure 5.7 Predicted excess attenuation due to a monopole (solid), horizontal dipole(dotted) and a vertical dipole(dashed line). $l_s=1.0\text{m}$, $h_r=0.5\text{m}$, $R=5.0\text{m}$.

Next, we look at the predictions of the Excess Attenuation spectra for a dipole and compare with that of a monopole source. Figure 5.7 shows excess attenuation of sound (defined as total field divided by direct field)
due to a monopole, a vertical dipole and a horizontal dipole. For this case the source height is 1.0m, the receiver height is 0.5m and the separation is 5.0m. As can be seen, there is little difference between the predictions for a monopole and a horizontal dipole, while the vertical dipole exhibits a greater amplitude of interference pattern. If the height of the vertical dipole is less than that of the receiver (hs<hr) there would be an additional phase difference of $\pi$ between the direct and the reflected terms due to the $\cos\theta_r$ term in the direct wave. This means that the principle of reciprocity, which states that the sound field is the same if source and receiver positions are exchanged, does not hold for a vertical dipole. This is evident also in the form of the dipole potential where the sign of the direct term changes if one exchanges the source and receiver positions. Figure 5.8 shows this effect for a vertical dipole.

![Figure 5.8 Predicted Excess attenuation of sound due to a monopole(solid line), and vertical dipole (dotted line) with hs=0.25m and hr=0.1m. The dashed line represents a vertical dipole.](image-url)
dipole with source and receiver positions reversed. The shift in
ground effect dip is due to the extra phase difference.

The similarity between spectra of monopoles and horizontal dipoles ends
when the separation is comparable to the source or receiver height. In this
the reflected wave can have a large polar angle, thereby reducing its relative
magnitude, and one would predict fewer interference minima. Figure 5.9 is a
comparison of the predicted excess attenuation spectra for a monopole and a
horizontal dipole at a range of 1.0m with source and receiver being (as
before) at 1.0 m and 0.5 m respectively. A practical consequence of this
effect is that extra care must be taken, for example, when using a
loudspeaker as a point source. The geometry of the source-receiver system
must be chosen so that the directionality of the source has little effect of the
measured spectra.

Figure 5.9  As in Figure 5.7 but range=1.0m.
At near-grazing incidence, the surface wave generated by a dipole is predicted to be important. Figure 5.10 shows the predicted excess attenuation of sound due to a monopole (broken line) and a vertical dipole (solid line) at a near grazing angle. The source and receiver heights are 0.1 m and 0.025 m respectively with separation being 2.0 m. Here the predicted enhancement of the dipole sound level at about 1 kHz is due to the ground wave term (dotted line).

Finally, we compare the sensitivity of the field due to a vertical dipole above an impedance plane with that of a monopole source dipole to the assumed impedance parameters. In Figure 5.11 we plot the excess attenuation of sound due to a monopole (plot a) and a vertical dipole (plot b) for $\sigma$ values of 25, 50, 100 and 1000 with $\alpha$ being 20 m$^{-1}$. The source height is 0.4 m, receiver height is 0.1 m and separation is 1.75 m. The Excess Attenuation due to a
vertical dipole is predicted to have greater sensitivity at the higher values of the flow resistivity (between 100 and 1000 kPa s m$^{-2}$) than that due to a monopole, although the shift in frequency with change in flow resistivity from 1000 to 25 units due to a dipole source is same as that of a monopole.

Figure 5.11 Sensitivity of monopole (graph a) and vertical dipole (graph b) fields to effective flow resistivity.
5.5 Measurement of the dipole field above an impedance surface

5.5.1 dipole source

To test the contrasting models described in this section against experimental results, we needed a broadband dipole source. The first choice was an unbaffled loudspeaker. A KEF SP1003 speaker with the housing removed was chosen. The diameter of the speaker was 15 cm. Measurements of sound field spectrum above both rigid and impedance surfaces were carried out in an anechoic chamber using the MLSSA system analyser. The resulting spectra indicated that the source was not a clean source of dipole field, possibly due to the presence of the driving magnet and the steel frame holding the cone (see for example Figure 5.12). An alternative, suggested by David Hothersall, of the university of Bradford, was to use piezoceramic
transducers. These are thin circular plates of piezoceramic material glued to similarly shaped metallic plates. The diameter is between 1 to 3 cm. They produce a resonance frequency. Therefore measurement of excess attenuation spectra was not possible with these devices. Instead the rate of attenuation as a function of range was measured at the resonance frequencies of the device. Two transducers with resonance frequencies of 2930 and 4200 Hz were chosen.

![Transducer disc and supporting rod](image)

**Figure 5.13** drawing of the transducer disc and the supporting rod.

### 5.5.2 Experimental procedure

To model the impedance plane a 1.6 cm thick sheet of wool felt was placed over a rigid hardwood board and secured to it by double sided tape. The impedance of the resulting surface was determined by making separate measurement the Excess Attenuation of sound above the surface using a monopole point source and finding the best fit to the data using a two-parameter impedance model. To achieve this, a Tannoy loudspeaker fitted with a 90 cm long tube of internal diameter 3 cm, was suspended by string above the board. A Bruel & Kjaer type 4311, 1.3 cm-diameter condenser
microphone with a preamplifier suspended above the board by flexible string 1.0 - 2.0m away acted as the receiver. A measurement in the absence of the floor served as the direct, reference field and subsequent characterisation measurements were divided by the reference field to obtain the Excess Attenuation of sound. The measurements and best fit prediction for four sample geometries are shown in Figure 5.15. The best fit parameters were found, by Q. Wang of The Open University, to be 38 kPa s m\(^{-2}\) and 15 m\(^{-1}\).

![Figure 5.14 Directivity pattern of transducer disc (circles) compared with directivity of a dipole (solid line). The disc was positioned perpendicular to the plane of the paper and parallel to 90° axis, as indicated by a thick line.]

These parameters were used in the subsequent predictions of the dipole field and comparisons with the measured attenuation rate.

To support the transducer, a thin steel rod was attached to the transducer disc with the rod in the plane of the disc. This minimised the interference due to the presence of the rod since, theoretically, the field is zero in the plane of a dipole source. The rod need not be physically in contact with the
disc and it was sufficient to secure the wires protruding from the transducer to the rod to ensure stability. This prevented the rod from interfering with the resonance modes of the disc. To test the suitability of the transducer discs as a dipole source, the sound field at resonance frequency, was measured around a circle with a radius of 30 cm, perpendicular to the plane of the disc and is given in Figure 5.14. This was compared with theoretical polar plot of a dipole with levels adjusted to coincide at the dipole axis. The only deviation from the ideal dipole was at zero field plane of dipole where theoretically there is zero field while the transducer produces small but measurable sound field. But since this field is several dB lower than the field along dipole axis (at maximum field), the model is valid. The measurements were carried out with the dipole held in position with the attached rod fastened to a stand and the microphone suspended by flexible string. The MLSSA (Maximum Length Sequence System Analyser) data acquisition and analysis system was used to produce a pseudo-random sequence pulses at the dipole, and capture the microphone signal. The signal was windowed using a half Blackman-Harris window and Fourier Transformed. Then the sound level at the resonance frequency was recorded. The procedure was repeated at several positions along a straight line away from the dipole up to a distance of 1.5m (this was dictated by the limitations in the size of the anechoic chamber, size of the board and desire to avoid the diffraction from the sides of the board). Figure 5.16 shows measured and predicted total fields at 4000 Hz as a function of range with the source disc axis horizontal and at a height of 0.01m above the felt surface. The
agreement between data and both predictions is good indicating that the horizontal dipole field and a monopole field differ little. Figure 5.17 shows measured data and prediction with the transducer disc axis vertical. Again the agreement between data and the approximate expression derived here is good. Also shown are the predictions of the theory due to Generalov and the sound field due to a monopole normalised to the level at the first measurement point. Corresponding results using the 2915 Hz source at a height of 0.035m are shown in Figure 5.18. They predict significant differences between monopole and dipole fields above an impedance plane at short range.

It is also clear that Generalov theory fails to predict the measured interference at short range at either frequency.
Figure 5.16 Measured (crosses) and predicted sound fields from a horizontal dipole at 4096 Hz. Asymptotic form (solid line) and monopole field (dashed line) normalised to coincide at first measured point at 0.1 m in this case. \( h_s=0.03 \text{m}, h_r=0.02 \text{m} \).

Figure 5.17 Measured (crosses) and predicted sound fields from a vertical dipole at 4096 Hz. Asymptotic form (solid line), Generalov's solution (dash-dot line) and monopole field (dashed line) normalised to coincide at first measured point at 0.0 m in this case. \( h_s=0.03 \text{m}, h_r=0.02 \text{m} \).
5.6 Conclusion

In this chapter the sound field due to a dipole source situated near a plane porous surface has been investigated. In the first section the sound field potential from an arbitrarily oriented dipole above a porous elastic solid was derived in the wavenumber domain and the field in spatial domain was calculated using an FFP method. It was shown that the dipole field is made up of two components: a vertical dipole component and a radial component. It was shown also that sound field of a radial (or horizontal) dipole behaves like that of a monopole point source except at distances comparable with the source or receiver heights while a dipole with its axis perpendicular to the plane has a different attenuation rate. In the subsequent section an approximate Weyl-Van Der Pol type expression for the sound field of a
vertical dipole above an impedance plane was derived and its Excess Attenuation spectra predictions were contrasted to those predicted for a monopole source. To test the derived formulae, measurements of the sound field due to a vertical dipole (modelled by a piezoceramic transducer) above an impedance surface (modelled by a sheet of felt stretched over a flat hardboard) have been taken as function of distance. It was shown that the predictions based on eqn. (5.34) (or eqn. (5.36)) agree with the data very well while an earlier theory developed by Generalov (54) fits the data less well.

**Appendix: List of symbols in Chapter 5**

- \( F(w), F(\tau) \) boundary loss factor (see eqn. (5.22))
- \( h, h_s \) source height
- \( h_r \) receiver height
- \( k_r \) horizontal component of the wavenumber
- \( k_z \) vertical component of the wavenumber
- \( R_1 \) direct ray path
- \( R_2 \) image ray path
- \( r \) horizontal distance
- \( \rho \) plane wave reflection coefficient
- \( w \) numerical distance
- \( \beta \) the acoustic surface admittance
- \( \mu \) polar angle of the dipole axis
- \( \varepsilon \) azimuth angle the dipole axis makes with the receiver
- \( \theta \) angle of incidence (equal to the polar angle of the image ray)
- \( \theta_i \) polar angle of the direct ray
- \( \tau \) numerical distance in Generalov's notation
Chapter 6

Sound Field of a Quadrupole Above an Impedance Boundary

6.1 Introduction

In the last chapter the sound field from a dipole source near a solid boundary was discussed and approximate asymptotic expressions for the resulting field above a rigid porous ground were derived. In this chapter similar methods will be employed to discuss the sound field of an arbitrarily-orientated quadrupole. This problem is of some importance since the noises generated by subsonic jet flow and general unsteady fluid flow are quadrupole in nature. Almost all the discussions in the literature have concentrated on the quadrupole field in an unbounded medium or near a rigid solid surface. The one exception is the work of Generalov who derived expressions for a multipole field above a locally reacting rigid porous surface. As discussed in chapter 5 and cited references therein, his formulation for a vertical dipole field does not agree very well with either the measured dipole field or the result of the integral representation of the dipole sound field. His expressions for a quadrupole will be discussed later in section 6.3. Recently, Hu and Bolton have developed integral representation of a longitudinal quadrupole field, as well as that of a dipole.

This chapter is concerned with the sound field of an arbitrarily-orientated quadrupole above a rigid porous solid surface as well as that above an

† An earlier version of this chapter was presented in Inter-Noise 96
impedance boundary. First I derive the Hankel Transform representation of an arbitrarily-oriented quadrupole and state the boundary condition equations at fluid-fluid and fluid-poroelastic solid at the presence of a quadrupole. Then I form the global matrix equation to solve for the wave amplitudes in a stratified layered fluid medium. I show that the quadrupole field can be divided into three components to enable calculation of sound pressure contours of a quadrupole field very efficiently. By differentiating the integral form of the quadrupole field above an impedance boundary, I derive an asymptotic expression for the field due to a quadrupole above a rigid porous ground. The two models will be compared with the result of the superposition of two out of phase dipoles. Finally, as an example application of the quadrupole model, I study the sound field of a jet engine source.

6.2 Hankel Transform function representation

The sound pressure due to a point monopole source can be written in an integral form as:

\[ P(z, r) = \int \varphi_m(z, k_r) J_0(rk_r) k_r dk_r \]

(6.1)

where \( J_0() \) is the Bessel function of zero order and \( \varphi_m \) is the range-independent Green’s function. The form of the Green’s function for a monopole source in a homogeneous fluid above a boundary with an admittance of \( \beta \) is very well known:

\[ \varphi_m = \frac{1}{k_z} e^{ik_z |z-h|} + \frac{k_z - k_0 \beta}{k_z + k_0 \beta} \times \frac{1}{k_z} e^{ik_z (z+h)} \]

(6.2)
Here the first term represents the direct wave and the second term the reflected wave which is the product of the image source term and the plane wave reflection coefficient. The source height is given by $h$ and $k_z = +\sqrt{k_0^2 - k_r^2}$. The Hankel transform for dipole and quadrupole fields are defined as

$$P_{dipole} = (\mathbf{d} \cdot \nabla)P_{\text{monopole}}$$

(6.3)

$$P_{\text{quad.}} = (\mathbf{m} \cdot \nabla)(\mathbf{d} \cdot \nabla)P_{\text{monopole}}$$

(6.4)

where $\mathbf{d}$ and $\mathbf{m}$, called amplitude vectors, are direction cosines of the dipole and the quadrupole axes respectively, so that we have:

$$\mathbf{d} = (\sin \mu \cos \varepsilon, \sin \mu \sin \varepsilon, \cos \mu)$$

(6.5)

and,

$$\mathbf{m} = (\sin \lambda \cos \gamma, \sin \lambda \sin \gamma, \cos \lambda)$$

(6.6)

A dipole can be thought of as superposition of two closely-spaced out-of-phase monopoles. Similarly, a quadrupole is a superposition of two out-of-phase dipoles. Depending on whether or not the axes of the two constituent dipoles are co-linear, the quadrupole can be a longitudinal or a lateral (transverse) one. The axis of a quadrupole is defined to be parallel to the direction of its constituent dipoles for a longitudinal quadrupole and normal (but co-planar) to the dipole axes for transverse (or lateral) quadrupole. This is illustrated in Figure 6.1.
Performing the differentiation on the direct wave term of the dipole field, the kernel function of a dipole source ($\varphi_{d1}$) is given in eqn. (6.7)

$$\varphi_{d1} = (-i) \frac{e^{i|z-h|}}{k_z} \{ \text{sign}(h-z)k_z \cos \mu + k_r \sin \mu \cos \varepsilon \}$$

(6.7)

where $\mu$ and $\varepsilon$ are the polar and azimuthal angles defining the amplitude vector, $d$. The field of a quadrupole is then defined as

$$\varphi_{q1} = (m \cdot \nabla)\varphi_{d1}$$

(6.8)

The explicit expression for $\varphi_q$ in cylindrical co-ordinates is:

$$\varphi_{d1} = \frac{e^{ik_z|z-h|}}{k_z} \{ \text{[sign}(h-z)k_z \cos \mu + k_r \sin \mu \cos \varepsilon]k_z \cos \lambda \\
- \text{[sign}(h-z)k_z \cos \mu + k_r \sin \mu \cos \varepsilon]k_r \sin \lambda \cos \gamma \}$$

(6.9)
with \( \lambda \) and \( \gamma \) being the polar and azimuthal angles of the quadrupole axis. Simplifying it, results in:

\[
\varphi_q = e^{ik_x z - k_y h} \left( \frac{k_x^2 \cos \mu \cos \lambda - k_r^2 \sin \mu \cos \varepsilon \sin \lambda \cos \gamma}{k_z} + \text{sign}(h-z)k_r k_z \left[ \cos \mu \sin \lambda \cos \gamma - \sin \mu \cos \varepsilon \cos \lambda \right] \right)
\]  \( (6.10) \)

The quadrupole field can be divided into three components. The first term in curly brackets represents a component corresponding to a longitudinal vertical quadrupole (orientated along the z-axis), the second term represents a component corresponding to a longitudinal horizontal (i.e. radial) quadrupole and the third term represents a transverse quadrupole component. Therefore an arbitrary quadrupole is the sum of these three components with appropriate direction vector weights. At first sight eqn. \((6.10)\) indicates that there are four angles that determine an arbitrary quadrupole. In fact the definition of quadrupole direction vector in the previous paragraph fixes one of the angles by the following equation:

\[
\sin \mu \sin \lambda \left[ \cos(\varepsilon - \gamma) \right] + \cos \mu \cos \lambda = 0 \text{ or } 1,
\]  \( (6.11) \)

so in general only three angles (or parameters) are required to completely specify a quadrupole.

For a longitudinal quadrupole this reduces further to two angles or parameters. The boundary conditions at the fluid-fluid and solid-fluid interfaces are similar to those considered in chapters 3 and 6 but the source terms on the right hand side of the global matrix equation are formed from the above quadrupole field. The three components can be calculated
separately, as in the case of dipole, to facilitate the production of sound pressure contours.

The reflected wave term is evaluated in a similar fashion resulting in an image source term multiplied by the plane wave reflection coefficient. Once the Hankel Transform is calculated for a given geometry, the integration can be carried out by Fast Field Program method in the far field.

### 6.3 Asymptotic expressions for quadrupole field above a rigid porous ground

Normally the integral representation of a quadrupole field derived in section 6.2 is evaluated numerically by Fourier Transform approximation or other methods. Generalov \(^{54}\) seems to have been the first worker to obtain asymptotic expressions for the multipole field near an impedance boundary. He derived the following approximate expression for the quadrupole field:

\[
\begin{align*}
P_{\text{vertical}} &= \varphi_{zz} + k_0^2 \left( \frac{dR_0}{dz} \right)^2 e^{ik_0 R_2} \frac{dz}{4\pi R_2} \times \left[ A(\tau)b_{02} + \frac{2\beta}{ik_0 R_2} \left( L - 2B\frac{\sin^2 \theta}{\cos \theta} \right) \right] \\
&= \left( \frac{dR_0}{dz} \right)^2 e^{ik_0 R_2} \frac{dz}{4\pi R_2} \times \\
&\quad \left[ A(\tau)b_{02} - \frac{b_{10}}{k R_2} \left( \frac{dR_0}{dz} \right)^2 - 1 \right] + \frac{2\beta}{ik_0 R_2} \left( L + 2B \cos \theta \right) \\
\end{align*}
\]

and

\[
\begin{align*}
P_{\text{transverse}} &= \varphi_{rz} + k_0^2 \left( \frac{dR_0}{dz} \right) e^{ik_0 R_2} \frac{dz}{4\pi R_2} \times \left[ A(\tau)b_{11} + \frac{2\beta}{ik_0 R_2} \left( L - 2B\frac{\sin^2 \theta}{\cos \theta} \right) \right] \\
&= \left( \frac{dR_0}{dz} \right) e^{ik_0 R_2} \frac{dz}{4\pi R_2} \times \\
&\quad \left[ A(\tau)b_{11} + \frac{2\beta}{ik_0 R_2} \left( L - 2B\frac{\sin^2 \theta}{\cos \theta} \right) \right]
\end{align*}
\]

where
\[ \varphi_{uv} = -k_0^2 \left( \frac{dR_1}{du} \frac{dR_1}{dv} \right) \frac{e^{ik_0 R_1}}{R_1} \times \left[ 1 - \frac{3}{ik_0 R_1} \left( \frac{dR_1}{du} \right)^2 \left( \frac{dR_1}{dv} \right)^2 \right] \left( \frac{1}{(ik_0 R_1)} - \frac{1}{(ik_0 R_1)^2} \right) \]

\[ -k_0^2 R_p \left( \frac{dR_2}{du} \frac{dR_2}{dv} \right) \frac{e^{ik_0 R_2}}{R_2} \times \left[ 1 - \frac{3}{ik_0 R_2} \left( \frac{dR_2}{du} \right)^2 \left( \frac{dR_2}{dv} \right)^2 \right] \left( \frac{1}{(ik_0 R_2)} - \frac{1}{(ik_0 R_2)^2} \right) \]

(6.15)

In the above expression:

\[ b_{nm} = \frac{(i)^n k_0 \sqrt{\pi} R_2 (\beta)^{m+1} (1 - \beta^2)^{\frac{m}{2}} H_n^{(1)}(k_0 r \sqrt{1 - \beta^2}) e^{ik_0 r \sqrt{1 - \beta^2}}}{\sin^n \theta \cos^m \theta} \]

(6.16)

and \( \delta_{uv} \) is the Kronecker delta symbol. Other symbols have been defined in chapter 5. He then provides simplified expressions for the case of \( |r| \gg 1 \).

Unlike his formulae for a dipole, the above expressions for quadrupole field agree well with Hankel Transform representation of the same. In part this may be due to the fact that the ground wave term in the field due to a quadrupole is not as significant as that in the dipole field.

Now I try to derive a familiar asymptotic form for the quadrupole field above a rigid porous ground by differentiating the corresponding field for a dipole and using known approximate forms for the resulting terms. First I consider the evaluation of the first component in the quadrupole Hankel Transform function which is the vertical longitudinal term. The vertical axial component of a quadrupole, ignoring the direction scaling factors, is:

\[ P_{v,quad} = \int \left[ \varphi_{qi} - \frac{k_z - k_0 \beta}{k_z + k_0 \beta} \times k_z^2 \times e^{ik_z(z+h)} \right] J_0(r_k) k_r dk_r \]

(6.17)
where $q_{ql}$ is the direct field component. The reflected term can be rearranged to give

$$P_{v,\text{quad}} = \left\{ -k_z^2 \frac{e^{ik_z(z-h)}}{k_z} - k_z^2 \frac{e^{ik_z(z+h)}}{k_z} + \right\}$$

$$\left[ -2ik_0\beta (ik_0 \cos \theta - 2k_0^2 \beta^2 + k_0^2 \beta^2 \frac{2\beta}{\cos \theta + \beta} \right] \frac{e^{ik_z(z+h)}}{k_z} \right\} J_0(\ell k_z) \ell k_z$$

(6.18)

where $\cos \theta = \frac{k_z}{k_0}$ is the, possibly complex, angle of incidence. It is seen that the integrand consists of five terms. The first and second terms have well known closed form solutions and are identified as the direct and image quadrupole source terms. The third term represents an image dipole source (see chapter 5) multiplied by a constant coefficient $(2ik_0\beta)$. The fourth term is an image monopole source term with a coefficient $(2k_0^2 \beta^2)$. Finally, the last term represents the boundary wave term which has a closed-form approximate solution. The resulting expression for the sound field of an axial quadrupole positioned vertically above an impedance boundary is:

$$P_{v,\text{quad}} = \left\{ 3(1 - ik_0 R_1) - k_0^2 R_1^2 \right\} \cos^2 \theta_1 - \left[ 1 - ik_0 R_1 \right] \frac{e^{ik_0 R_1}}{R_1} +$$

$$\left\{ 3(1 - ik_0 R_2) - k_0^2 R_2^2 \right\} \cos^2 \theta_2 - \left[ 1 - ik_0 R_2 \right] \frac{e^{ik_0 R_2}}{R_2} +$$

$$2ik_0\beta (1 - ik_0 R_2) \frac{e^{ik_0 R_2}}{R_2^2} \times \cos \theta_2 - 2k_0^2 \beta^2 \left[ \cos \theta_2 + \beta F(w) \right] \frac{e^{ik_0 R_2}}{R_2}$$

(6.19)

where $R_1$ and $R_2$ are the distance from the source and the image source to the receiver respectively, $\theta_1$ & $\theta_2$ are the polar angles of the lines joining the source and the image to the receiver. Care should be taken in calculating the numerical distance to ensure that the correct phase has been used. The form
for \( w \) given above with the positive real root of the square root ensures that this is always the case. The other two components of the quadrupole can be evaluated in a similar fashion. For a horizontal quadrupole we have:

\[
P_{h,quad} = \int \left[ \varphi_{\text{direct}} - \frac{k_z - k_0 \beta}{k_z + k_0 \beta} \times k_r^2 \times \frac{e^{i k_z (r+h)}}{k_z} \right] J_0(rk_r) k_z dk_r
\]

(6.20)

Li & Taherzadeh \[661\] give a more rigorous derivation of the quadrupole field by starting from the wave equation with a quadrupole source term and arrive at the following compact approximate form for the sound field of an arbitrarily-oriented quadrupole:

\[
P_{\text{quad}} = \frac{-k_0^2}{4\pi} \left( \hat{d} \cdot \hat{R}_1 \right) \left( \hat{m} \cdot \hat{R}_1 \right) e^{ik_0 R_1} \\
+ \frac{-k_0^2}{4\pi} \left\{ \left[ \left( \hat{d} \cdot \hat{R}_2 \right) \left( \hat{m} \cdot \hat{R}_2 \right) R_p + \left( \hat{d} \cdot \hat{R}_s \right) \left( \hat{m} \cdot \hat{R}_s \right) \left( 1 - R_p \right) F(w) \right] e^{ik_0 R_2} \right\} \\
+ \frac{1}{4\pi} \left[ \left( \hat{d} \cdot \hat{m} \right) - 3 \left( \hat{d} \cdot \hat{R}_1 \right) \left( \hat{m} \cdot \hat{R}_1 \right) \right] \times \frac{ik_0 R_1 - 1}{R_1} \times \frac{e^{ik_0 R_1}}{R_1} \\
+ \frac{1}{4\pi} \left[ \left( \hat{d} \cdot \hat{m} \right) - 3 \left( \hat{d} \cdot \hat{R}_2 \right) \left( \hat{m} \cdot \hat{R}_2 \right) \right] \times \frac{ik_0 R_2 - 1}{R_2} \times \frac{e^{ik_0 R_2}}{R_2}
\]

(6.21)

where \( R_p = \frac{\cos \theta_2 - \beta}{\cos \theta_2 + \beta} \) is the plane wave reflection coefficient, and the vector \( \hat{R}_s = (\sin \mu_s \cos \varepsilon, \sin \mu_s \sin \varepsilon, \cos \mu_s) \), may be regarded as the unit vector that characterises the direction of the ground wave term. The complex angle \( \mu_s \) is determined by \( \cos \mu_s + \beta = 0 \).

The vector quantities \( R_1 \) and \( R_2 \) represent unit vectors pointing radially outward from the quadrupole source and image source to the receiver. This is a far-field approximation to the quadrupole field.
6.4 Validation and numerical examples

In what follows, the admittance of the rigid porous ground is modelled by a two-parameter impedance model\(^{[69]}\) given by:

\[
\frac{1}{\beta} = 0.436\left(1 + \frac{1}{i}\right)\sqrt{\frac{\alpha_e}{f}} + 19.48\left(\frac{\alpha_e}{f}\right)
\]  

(6.22)

where \(\alpha_e\) and \(\sigma_e\) are the effective flow resistivity and effective rate of change of porosity with depth respectively. The values for these parameters, used in the following calculations, are 40 kPa s m\(^{-2}\) and 20 m\(^{-1}\) respectively.

![Figure 6.2 transmission Loss of a vertical quadrupole (solid line), vertical dipole (dotted line) and a monopole (dashed line). Frequency is 500 Hz, and source and receiver heights are 1.5 and 1.0m respectively.](image)

The sound field from a longitudinal quadrupole positioned horizontally above an impedance plane is similar to that of a monopole source and attenuates at the same rate, except at short distances comparable to the source and receiver height above the surface. On the other hand, the same quadrupole placed vertically behaves differently. Figure 6.2 shows plot of
Transmission Loss of a vertical longitudinal quadrupole field. The source and receiver heights are 1.5 and 1.0m respectively and the frequency is 500 Hz. Also plotted are the Transmission Loss of dipole and monopole sources. It indicates that a quadruple field attenuates very fast while the dipole source attenuation rate tends to that of a monopole at long range. The Transmission Loss values are calculated with reference to field at 1.m range.

Figure 6.3 shows prediction of excess attenuation spectra of longitudinal quadrupole (solid line), vertical dipole (dotted line) and monopole (dashed line) sources and receiver close to a porous ground 2.0m away. The source and receiver heights are 0.1 and 0.025m respectively.

It can be seen, in this geometry, that the predicted ground wave contribution due to a longitudinal quadrupole is lower than that due to a vertical dipole but it is still higher than that of a monopole.
Comparison of the predictions of the asymptotic expression derived here with that of Generalov and numerical results of QuadFFP for a vertical longitudinal quadrupole and a lateral quadrupole are shown in Figure 6.4 for 1000 Hz and in Figure 6.5 for 100 Hz. All three solutions agree within 0.5 dB. Also plotted are predictions of a monopole (dashed line) and a vertical dipole (dotted line). Figure 6.6 and Figure 6.7 show predictions of Excess Attenuation spectra for a vertical longitudinal quadrupole (plots 6.6a & 6.7a) and a lateral quadrupole (plots 6.6b & 6.7b) for two different geometries. In Figure 6.6 the source and receiver heights are 5.0m and 1.0m respectively and the separation is 5.0m. While in Figure 6.7 the geometry is same as in Figure 6.3.

![Figure 6.4](image)

**Figure 6.4**  a) Sound field, as a function of range, of a longitudinal quadrupole (solid line), a vertical dipole (dotted line), and a monopole (dashed line). Source and receiver heights are 1.0m and 2.0m respectively and frequency= 1 kHz. b) same as (a) above but for a lateral quadrupole (solid line) the dotted line and dashed line represent the vertical dipole and monopole respectively.
Figure 6.5 Same as Figure 6.4 but for $f=100$ Hz.

Figure 6.6a) The excess attenuation spectra for a longitudinal quadrupole (solid line), a vertical dipole (dotted line) and a monopole (dashed line). Source and receiver heights are 1.0m and 0.5m respectively and the separation is 5.0m. 
b) Same as (a) above but for a lateral quadrupole.
In Figure 6.8 polar plots of the far field sound pressure due to a vertical longitudinal quad. (A) and a lateral quad. (B) at f=1000 Hz above an impedance boundary (solid lines) and a rigid plane (broken lines). It indicates that the directivity pattern for an elevated quadrupole source will depend on the impedance of the plane and some minima seen in the pattern in the vicinity of a rigid surface are smoothed out.

Figure 6.7 Same as Figure 6.6 but for a source height of 0.1m, receiver height 0.025m and separation of 2.0m.
6.5 Noise from a subsonic jet near an impedance boundary: an application of quadrupole source

In this section we discuss an application of the approximate theory for the field due to a quadrupole noise source near an impedance boundary to the calculation of the far field propagation of noise from an ensemble of quadrupole sources. It has been shown that a subsonic jet flow in vicinity of a boundary acts as quadrupole source. Powell[67] has argued, using his 'three-sound-pressures theorem' and 'image argument', that noise from a low Mach-number jet engine is reducible to a set of lateral quadrupoles. A simple theoretical model of subsonic jet noise can also be based on the fact that at any one moment the jet noise source is a randomly orientated
quadrupole. Therefore, the average intensity field is defined by an ensemble average of intensity of randomly oriented quadrupoles\[68\].

In all the above references the noise source is assumed to be close to a rigid boundary. It is argued in this section that substitution of an impedance boundary alters the directivity patterns of the ensemble substantially.

For longitudinal quadrupoles this ensemble average is given by:

\[
\bar{I} = \frac{1}{2\pi^2} \int_{0}^{\pi} \int_{0}^{2\pi} \frac{|P|^2}{\rho c} \, d\lambda \, d\gamma
\]

(6.23)

and that of lateral quadrupoles is

\[
\bar{I} = \frac{1}{4\pi^3} \int_{0}^{\phi} \int_{0}^{\pi} \int_{-\pi/2}^{\pi/2} \frac{|P|^2}{\rho c} \, d\lambda \, d\gamma \, d\epsilon
\]

(6.24)

Smith & Carpenter \[68\] state the total far field pressure of a longitudinal quadrupole at an arbitrary orientation as:

\[
P_{\text{tot}} = -2k_0^2 \left[ \sin^2 \lambda \sin^2 \theta \cos^2 (\gamma - \phi) + \cos^2 \lambda \cos^2 \theta \right] \cos(k_0 h \cos \theta) - 2i \sin \lambda \cos \lambda \sin \theta \cos \theta \cos(\gamma - \phi) \sin(k_0 h \cos \theta) \left( \frac{e^{ik_0 r}}{r} \right)
\]

(6.25)

where \( r \) is the distance of the receiver from the origin, \( h \) is the source height, and \( (\theta, \phi) \) are the polar and azimuth angles of the receiver. The integral in eqn. (6.23) can be evaluated to give:

\[
\bar{I} = \left[ 1 - \frac{8}{3} \sin^2 \theta \cos^2 \theta - \frac{1}{4} \sin^4 \theta + \cos^4 \theta \right] \cos^2(k_0 h \cos \theta) + \frac{1}{3} \sin^2 \theta \cos^2 \theta
\]

(6.26)

where we have discarded with a constant term since we are interested in the directional pattern of the source. In the above expression an error in eqn. (6)
of reference [68] has been corrected and can be verified by numerical integration. But both these approaches have assumed the source to be near an acoustically hard surface. To predict the effect of an impedance plane on the directivity patterns we must utilise the formulae developed in the last section. The corresponding intensity pattern above an impedance boundary can be calculated by substituting the expressions for a longitudinal quadrupole into integral (6.23) and performing the integration numerically. In this case a 10 point Gaussian quadrature was sufficient for accurate evaluation of the integral.

According to Smith & Carpenter\textsuperscript{[68]}, the pressure field of a lateral quadrupole above a rigid plane is given by:

\begin{equation}
P_{tot} = -k_0^2 \left\{ \sin \lambda \cos \lambda \cos \mu \left[ \sin^2 \theta \cos(\gamma - \phi) - \cos^2 \theta \right] - \sin \lambda \sin \mu \sin^2 \theta \right. \\
\times \left[ \sin \gamma \cos \gamma \cos 2\phi - \cos \phi \sin \phi \cos 2\gamma \right] \cos(k_0 h \cos \theta) - i \{ \sin \theta \cos \theta \right. \\
\times \left[ \cos \mu \cos 2\lambda \cos(\gamma - \phi) - \sin \mu \cos \lambda \sin(\gamma - \phi) \} \sin(k_0 h \cos \theta) \right) \times \left( \frac{e^{ik_0 r}}{r} \right)
\end{equation}

The above form is substituted in integral (6.24) and evaluated to give (ignoring a constant term):

\begin{equation}
\bar{I} = \left[ 1 - 6 \sin^2 \theta \cos^2 \theta \right] \cos^2(k_0 h \cos \theta) + 3 \sin^2 \theta \cos^2 \theta
\end{equation}

For an ensemble of lateral quadrupoles above an impedance plane, the eqn. (6.28) can be substituted in integral (6.24) and evaluated numerically. The results indicate certain important differences in the directivity patterns for ensemble of quadrupole sources above a hard and an impedance surface. Figure 6.9 shows the ensemble intensities due to longitudinal and lateral
quadrupoles. The most noticeable is the cancellation of sound intensity field along the impedance plane. Another important difference is the fact that the pattern for an impedance surface depends explicitly on $k_0$ and $h$ rather than their product as is the case for a hard surface.

Figure 6.9 The polar plot in the $x$-$z$ plane of an ensemble average of far field intensity of an arbitrarily-oriented quadrupole above a rigid boundary (dashed line) and above an impedance surface (solid line). The frequency is 1 kHz and the source height is 0.11m. a) Ensemble average of longitudinal quadrupole; and, b) ensemble average of lateral quadrupoles.

6.6 Conclusion

In this chapter the sound field due to a quadrupole has been investigated. First the field potential for an arbitrarily oriented quadrupole has been derived in the wavenumber domain and the Hankel transform performed by FFP method thus arriving at an FFP-type algorithm for calculating the SPL due to a quadrupole. In the following section an approximate expression for
the sound field of a longitudinal quadrupole with its axis perpendicular to an impedance plane has been derived in a similar fashion to that used in chapter 5. The differences in the Excess Attenuation spectra and Transmission Loss of quadrupoles and dipoles have been investigated. Finally, as a case study, the modelling of a subsonic jet near a rigid plane as an ensemble of arbitrarily-oriented quadrupoles was extended to include an impedance plane. It has been predicted that the presence of an impedance plane modifies the directivity pattern of such ensembles. However, no measurements have been offered to test the theoretical conclusions of this chapter.
Conclusion and Suggestions for further work

This thesis has been concerned with the general theme of sound propagation in inhomogeneous media.

In particular, three aspects of sound propagation outdoors have been investigated: propagation in a refracting atmosphere, propagation in, and above, a porous elastic medium, and propagation from dipole and quadrupole sources near to porous and elastic surfaces.

Outdoor propagation factors that have been ignored here include atmospheric turbulence and range-dependent inhomogeneities.

In chapter 1, the existing theoretical models for the sound field in a fluid where the speed of sound varies with height above an impedance plane were reviewed. The general form of the sound field under these conditions, based on an earlier work by Pekeris, was derived. A new analytical expression for the sound field in a fluid with a linear sound speed profile was also offered.

In chapter 2, we reviewed Biot's theory of wave propagation in a poroelastic medium and also looked at an alternative theory proposed by de la Cruz and Spanos. It was shown that there are important differences between the two theories in the way the boundary conditions are expressed. In the following chapter the influence of ground porosity on propagation in a porous elastic medium - and in the fluid above such a solid - was investigated and the influence porosity on various propagation parameters was shown. It was
concluded that the porosity of the solid medium plays an important role on wave propagation in both air-filled soils and water-saturated sea bottom sediments.

Chapter 4 was concerned with the coupling of air-borne sound with ground excitations. It was shown that Biot’s poroelastic theory offers a better agreement with measured acoustically induced seismic excitations than could be obtained from a treatment based on (non-porous) elastic layers (see table 4.2). Further measurements performed by the author and others (see footnote at page 88) indicate that there is considerable uncertainty in the measurement of the acoustic-to-seismic coupling spectra so that statements about the preference of one theory to another have to be treated with caution.

Chapters 5 and 6 were concerned with directional sources above poroelastic and impedance planes. In chapter 5 I derived the general form of the Green’s function due to an arbitrarily-oriented dipole situated above a poroelastic solid. We derived also an approximate analytical expression for a field of a vertical dipole above an impedance plane and compared it, together with an existing solution by Generalov, with measurements of dipole field carried out in an anechoic chamber. It was shown that the expressions offered in this thesis provide a better agreement with data than previous models. The vertical dipole was shown to produce a relatively large ground wave term compared to a monopole for certain configurations and hence to offer the potential for determination of ground parameters in a more sensitive manner than possible with a monopole source. In chapter 6 the sound field of a
quadrupole was studied in a similar fashion and predictions made about the apparent directivity of jet noise sources near to an impedance plane although no experimental data was offered.

Most of the work reported in this thesis could be extended and advanced. In particular:

- In the study of propagation in porous elastic media, the new theory by de la Cruz & Spanos needs to be investigated further and resulting predictions of sound propagation above and close to a porous elastic ground contrasted with predictions from established theory by Biot.

- The Fast Field Program is an extremely useful numerical tool in predicting sound propagation in fluids. However, a 2D version of FFP which can be computed efficiently and includes the effects of turbulence is still to be developed.

- At the moment, reflection coefficients from a porous elastic boundary must be computed numerically. Approximate closed-form expressions for these are needed.

- The usefulness of a dipole source, compared to a monopole source, in characterising the ground impedance parameters needs to be investigated.

- The influence of pore size distribution has been demonstrated. However, only log-normal distribution functions have been considered so far. Other distribution functions for the pore sizes should also be considered and investigated.
• The program FFLAGS has been applied successfully to prediction of the sound field above underwater sediments. This application should be pursued.

• Directivity patterns other than those due to multipoles above poroelastic boundaries should be investigated, in particular that due to a sound source with an arbitrary directivity.
References


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