Multilevel Systems and Policy

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Abstract
A formal theory of multilevel systems is needed for modelling the outcomes of policy. The dynamics of social systems are coupled between levels and models that do not reflect this are incomplete and a poor basis for policymaking. The structures developed here are necessary for the new kinds of models and computer simulations needed to support policy makers as they tackle the problems of an increasingly connected multilevel world. Almost all social systems have many levels of organisation, from micro to macro levels, and multilevel structure is fundamental to their dynamics. Part-whole structures play a major role in multilevel systems, where intermediate wholes may themselves be parts in higher level structures. Taxonomic aggregation plays another major role in multilevel systems, but is very different to part-whole aggregation. Generally part-whole and taxonomic aggregation are interleaved. Both space and time have natural multilevel structure relevant to social organisation. Multilevel structure can be relational or numerical. Generally relations determine organisational structure, with patterns of numbers defined on it. These numbers aggregate and disaggregate with the multilevel part-whole and taxonomic structures. In this chapter the ideas are developed descriptively in the first instance in order to be accessible to everyone. The second part uses a more formal approach that enables the fundamental concepts to be developed with greater precision.

1. Introduction
The social and economic world has many levels of organisations. For example, Barclays Bank is part of the financial system, and any branch is part of Barclays Bank. Parts exist at lower levels of organisation than wholes, and wholes exist at higher levels of organisation than parts. Thus Barclays bank exists at a higher level than its branches and at a lower level than the financial system, so the financial system has at least three levels. Of course, it has many more. For example, Barclays Bank is part of the banking system, and the banking system is part of the financial system. A system with three or more levels of organisation will be called a multilevel system.

Almost all social and economic systems have many levels. For example, the writings of Voltaire are part of French literature, French literature is part of literature, and literature is part of Art. As another example, a candidate is part of a political campaign, and a political campaign is part of democracy. So, both art and democracy are multilevel systems. A moments thought makes it clear that every ministry in every country is responsible for a multilevel system, e.g. health, agriculture, transport, justice, defence, education, and so on.

Why do socio-economic systems suddenly change? In 2015 why are there suddenly so many refugees fleeing Africa for Europe creating an immigration crisis? Why did the global economic system collapse in 2008? Why did the European policy on biofuels cause starvation in Swaziland? In each of these cases abrupt and unpredicted change was due to the dynamics of the micro and macro levels being coupled through intermediate meso levels. In each of these cases this was not factored into the policy makers’ theories, models and action.

A working theory of the micro-meso-macro coupled dynamics of multilevel systems is essential if policy is not to be constantly wrong-footed by unexpected events. This chapter presents a way of representing multilevel systems at all levels providing a necessary intellectual architecture for understanding the policy-relevant coupled dynamics of multilevel systems. This forms a basis for the predictions implicit or explicit in policy to be made well-defined and tested before implementation.

The first part of this chapter, Sections 1 to 13, gives a descriptive introduction to multilevel systems using everyday language and illustrations. The second part, Sections 14 and 15, develops a formal method of representing multilevel systems using a more symbolic approach. The descriptive approach is ideal for giving a broad understanding by explaining and illustrating the main ideas. However multilevel systems and their dynamics can be very complicated making a descriptive approach very verbose with the details hard to understand. In contrast, too formal an approach can make even the simplest ideas seem over-complicated and unintuitive. However when things get complicated the formal approach can give great clarity and generality, and the formal takes over where the descriptive meets its limitations.

2. Social organisation

Organisation requires at least two distinct levels of aggregation: the organisation and the things that are organised. The organisation could be said to exist at a higher level of aggregation than the things that are organised. In social organisations the things that are organised include people.

Human society is characterised by individual people and interactions between individuals to form social organisations. For example, Adam Smith’s pin factory (1776) is a social structure made up of interacting individuals:

“One man draws out the wire, another straightens it, a third cuts it, a fourth points it, a fifth grinds it at the top for receiving the head: to make the head requires two or three distinct operations: to put it on is a particular business, to whiten the pins is another ... and the important business of making a pin is, in this manner, divided into about eighteen distinct operations, which in some manufactories are all performed by distinct hands, though in others the same man will sometime perform two or three of them.”

So, suppose man-1 draws out the wire, man-2 straightens it, man-3 cuts it, man-4 points it, man-5 grinds it to receive the head, man-6 and man-7 make the head, etc. with man-18 performing the last operation. In order to make the pin these men are organised. Let us say there is an organising relation on them. Then the list of men with the organising relation forms the socio-economic structure of the pin factory.

Technically the organising relation is called an 18-ary relation, since it forms 18 people into an organisation. In this case the relationship is rather simple since man-1 hands on partially made pins to man-2 who hands on partially made pins to man-3 and so on until man-18 performs the last task. The men and their organising relation form a whole of which the men are the parts. The whole has an emergent property not possessed by the individual men, namely the ability to make pins efficiently.

Suppose one of the men is injured and cannot complete his task. Then the whole socio-economic structure of the pin factory collapses, and pin production stops. The 18-ary relation that organises the men requires that they are all present performing their functions. Remove one part and the whole collapses. Of course in some circumstances, e.g. a minor injury, the remaining seventeen men might self-organise as a 17-ary relation to create a new structure with a (probably lesser) ability to make pins. However, if the injury were serious the other men would probably self-organise into a new life-saving social structure, with the socio-economic structure for making pins temporarily or permanently ceasing to exist.

Social and economic organisation takes place at higher levels of aggregation. For example, pin factories compete in a more aggregate structure called a market. In this context the factories act as individuals and they too are organised by structural relations. For example, suppose six factories held a monopoly. Then they could mutually agree to abide by a 6-ary relation to fix the price of their goods and not to undercut each other. Again remove just one

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2 Apart from the men, the pin factory may consist of other things such as land and a building, tools and machines, and other things. It would be possible to add all these things, but it would make the development more complicated and add little.
of the factories from the organising relation and the monopoly structure collapses, e.g. if one of the factories suddenly starts selling its goods at a lower price.

Parts and wholes exist at different levels of aggregation. In general it takes more than one part to make a whole. It can be said that the parts exist at a lower level of aggregation to the whole or, equivalently, the whole exists at a high level of aggregation to the parts.

Individual people interact at what will be called the microlevel to form social structures such as factories and businesses. These organisations exist at what will be called a mesolevel. Mesolevel entities also interact to create higher-level dynamics such as competition and price wars. The whole system, such as a market, is said to exist at the macrolevel.

**Figure 1. The pin-market as a 3-level system**

Socio-economic structures are important to policy and government regulatory bodies. They are also important to self-organising businesses. For example, two factories may interact with one being a sole-supplier to the other, with one side benefitting from a better price and the other benefitting from guaranteed orders.

In Figure 1 the pin market is a socio-economic structure that exists at a more aggregate level to the individual factories. This could be called the macrolevel. In Figure 1 the parts are assembled bottom-up to form wholes at the next level. Alternatively the parts of the higher levels can be identified top-down.

The problem with the micro-meso-macro nomenclature is that three levels are not enough. For example, consider the world of literature with words at the microlevel. Let’s call this ‘Level 1’. These are assembled into sentences at ‘Level 2’. These are assembled into paragraphs at ‘Level 3’. These are assembled into chapters at ‘Level 4’, and these are assembled into books at ‘Level 7’. So, it could be said that the symbols exist at the microlevel, the books exist the macrolevel, and everything else exists at the mesolevel. But this would be rather arbitrary.

Generally the terms micro-, meso-, and macro- are not used very precisely when systems have more than three levels. Although these terms are useful in describing the lower, middle and upper levels of systems, for policy it is necessary and possible to be much more precise.

Multilevel systems such as markets, ghettos, terrorism, education, politics, and art can be very complicated and looked at globally can be overwhelming. However, in isolation the part-whole structures of intermediate levels can be much easier to understand. This can lead to ‘local’ disconnected theories (possibly untrue) about isolated parts of systems, especially at micro and macro levels, e.g. “the (macrolevel) welfare bill is too high because of (microlevel) miscreants who are abusing the system”, or “migration is a (macrolevel) problem because too many (microlevel) immigrants are economically motivated rather than fleeing persecution”.

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Certainly microlevel dynamics impact on the macrolevel through the various levels of aggregation. And certainly, macrolevel dynamics impact on the microlevel by the imposition of rules and other constraints. However, exactly how these micro-macro dynamics are coupled though the various meso-level structures is rarely made explicit.

This is one reason for policy failing or having unintended consequences. Policy involves prediction – ‘if we do this then the outcome will be that’. Paradoxically the social world is both very predictable and very unpredictable. On a day-to-day basis our social systems at all levels function more-or-less as normal. Our microlevel interactions are mostly predictable. Companies predictably produce their output that predictably fills the shelves of our shops. Governments and oppositions predictably argue about the best way to do things, and the mass media predictably reports on these things, trying to predict how events will turn out. Until something unusual happens.

A working theory of the micro-meso-macro coupled dynamics of multilevel systems is essential to understanding how social and economic systems change if policy is not to be constantly thwarted by unexpected events and unintended consequences.

3. Systems

Let a system be defined to be a set of parts related in various ways to form the whole. Often the whole will have properties not possessed by the parts. These properties are said to emerge from the relationships and the interactions between the parts.

For example, a health system has many parts including doctors, patients, surgeries, medicines, and much else. These parts interact in many ways, e.g. a receptionist, doctor, patient and pharmacist all interact for the emergent properties of the patient being diagnosed and receiving the appropriate medication. This relatively local interaction could be considered to be part of a medical General Practice subsystem. There are other subsystems such hospitals. The subsystems may interact, for example the doctor in a general practice subsystem may refer a patient to a specialist in a hospital subsystem. At the macrolevel, the National Health Service in the UK employs 1.6 million people, deals with more than a million patients every thirty-six hours, and costs £115 billion per year. It is a very complex multilevel system.

The Health system is just one of many in the UK, including its agricultural system, military system, education system, religious system, financial system, welfare system, transportation system, criminal justice system, housing system, retail system, and so on.

These systems all interact, e.g. the UK health system and welfare system are funded independently leading to an emergent phenomenon of ‘bed blocking’ when a patient has completed their clinical treatment in hospital, but cannot leave because the welfare system cannot provide a suitable place for them. This has knock-on effects for accident and emergency (A&E) units across the country: “Accident and emergency units have missed their waiting time targets for the 80th week in a row as the winter crisis continues to grip the NHS. Damning figures showed 93% of A&E patients were admitted, transferred or discharged within the four-hour limit last week - two per cent below the key 95% target. And bed-blocking has risen to its highest level this winter, with 4,300 hospitals beds unavailable for incoming patients because of delays in transfers, up from 4,200 in the previous week. The knock-on effect filters through the system, leading to delays at struggling casualty departments.”

The pressure on A&E units is exacerbated by unintended consequences of the NHS 111 phoneline which advises “Call 111 if you need medical help fast but it's not a 999 emergency; you think you need to go to A&E or need another NHS urgent care service; you

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3 http://www.nhs.uk/NHSEngland/thenhs/about/Pages/overview.aspx (all websites referenced here were viewed on 8/2/15)
5 http://www.mirror.co.uk/news/uk-news/nhs-crisis-ac-waiting-time-5074164
6 http://www.nhs.uk/NHSEngland/AboutNHSservices/Emergencyandurgentcareservices/Pages/NHS-111.aspx
don't know who to call or you don't have a GP to call; you need health information or reassurance about what to do next.” Dr Cliff Mann, president of the College of Emergency Medicine, told a parliamentary committee “Of the 450,000 extra attendances in the [A&E] system in the last year, 220,000 were advised by NHS 111 to come to the emergency department and another 220,000 had an ambulance dispatched to them by NHS 111.”

The problem of struggling A&E departments is systemic. In any geographical area the health system admits and discharges patients. In the simplest case when all beds are occupied, the arithmetic is that the number of patients admitted to hospital is at best equal to the number patients discharged. The inadequate supply of places for discharged patients in the welfare system reduces the number of patients that can be admitted to a hospital in the health system while the NHS 111 phone system increases the number of people presenting at A&E units.

The systemic nature of this particular policy failure means it might have been anticipated as a possible outcome. However to do this it is necessary to have a 'model' of the system that describes the parts and how they interact to give the behaviour of the whole. However the system has components at different levels of aggregation, and the behaviour of the whole depends on the bottom-up and top-down interactions of the multilevel parts.

4. Systems, models and policy

Policy concerns the working of existing systems, the creation of new systems, and the replacement of old systems with something else. Policy involves arguments such as “if we do this now, then at some future time there will be an outcome that achieves our objectives”. Evidence-based policy attempts to make explicit the chain of argument that concludes the objectives will indeed be achieved.

This involves creating a model of the system. In the first instance the models are formulated in vernacular language such as English, and many policy documents include implicit models formulated in this way. They also include other ways of modelling systems including maps, diagrams and numbers. Numbers are inescapable in policy constrained by costs. Numbers are also essential for counting things such as the 8,000 GP practices in the UK, and making comparisons such as “The UK had 2.8 hospital beds per 1,000 people in 2012, compared to 8.3 in Germany, 6.3 in France, 3.4 in Italy, 3.0 in Spain and 2.8 in New Zealand”. Often models contain formulae that represent relationships between the numbers and in some cases these can be used for forecasts such as the number of people expected to suffer from a given disease and the healthcare provision that needs to be made.

Vernacular policy modelling often takes the form of narratives, or stories about the system and its behaviours. For example, the narrative of the A&E crisis includes the story of bed-blockers and the story that the NHS 111 telephone system is increasing rather than decreasing the number of people seeking professional medical advice because of “non-clinically trained staff who follow a formulaic script rather than using clinical judgment to assess how calls are dealt with” which “had inevitably led call handlers, with limited experience of medicine, to refer patients to the NHS when a trained professional could have encouraged them to effectively self-care”.

The narrative of the UK General Practitioner (GP) system tells various stories of a patient with a health problem wanting to see a doctor. The process begins by contacting the receptionist to request an appointment, which either results in an appointment being made, or the patient being left with their problem unresolved. In the former case the patient has a clinical consultation with the doctor resulting in a diagnosis with or without a prescription for medicinal treatment, or it may result in a referral to a specialist physician.

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8 http://www.nhsconfed.org/resources/key-statistics-on-the-nhs
9 http://www.ft.com/cms/s/0/a6e3d928-abb4-11e4-b05a-00144feab7de.html#axzz3R9X3U00
This verbal description can be converted into a diagram, Figure 2, that shows parts of the system and how they interact. This is more obviously a 'model' of the system than the text above, but both represent the same information about the GP system and they are both models.

![Diagram of GP system](image)

**Figure 2. A systems description of the General Practitioner system in the UK**

### 5. Multilevel Part-Whole Aggregation

Any part-whole system has at least two levels. The whole system is at the highest level and the parts are at the lowest level. For example, Figure 3 shows the GP system drawn as a part-whole cone. At the lowest level are all the parts, and at the highest level the apex represents the whole system. In Figure 3 these are called Level Top, and Level Bottom. The ellipse at the bottom of the cone is called its base. We write base(GP System) = \{ patient, problem, receptionist, doctor, diagnosis, advice, prescription, referral, pharmacist, … \} to show the set of parts. The symbol $R_{GP}$ signifies that the parts work together as shown in Figure 2 or, equivalently, the narrative description that precedes it. Thus $R_{GP}$ maps the base of the cone at one level to its apex at a higher level.

![Diagram of part-whole cone](image)

**Figure 3. The GP system as a part-whole cone.**

However, the GP system is just one subsystem of the whole NHS. There are other subsystems such as the Hospital and Ambulance Systems. These all have to fit together to make the whole, and there will be documents defining this. Let the symbol $R_{NHS}$ represent all the relational information that says how these subsystems work together.

For example, the Ambulance subsystem is connected to the Emergency Phone Line System (not shown) used by members of the public for emergency access the Police, Fire and Ambulance systems. Given the necessary information, an ambulance is dispatched in the Ambulance system and either treats the patient where they are or transports them to a hospital, where the Ambulance subsystem hands over responsibility for the patient to the Hospital System. This is part of the $R_{NHS}$ relationship.
Now, as shown in Figure 4, the representation has three levels: the whole system at the top level, the parts of the system at an intermediate level (which in this case are subsystems), and the parts of the parts at the bottom level.

The part-whole relationship by assembly is particularly important in multilevel systems because constructed wholes are never parts of their parts, e.g. a car is not part of its engine and the NHS is not part of a hospital. This means that the parts are always at a lower level to the whole, and there is an immutable upwards arrow that establishes a difference in levels.

6. Taxonomic Aggregation

Another kind of immutable arrow arises in the use of taxonomies and the words that go with them. For example, the Statistical Classification of Economic Activities in the European Community, known as NACE\textsuperscript{10}, has twenty highest-level classes:

A - Agriculture, forestry and fishing
B - Mining and quarrying
C - Manufacturing
D - Electricity, gas, steam and air conditioning supply
E - Water supply; sewerage; waste management and remediation activities
F - Construction
G - Wholesale and retail trade; repair of motor vehicles and motorcycles
H - Transporting and storage
I - Accommodation and food service activities
J - Information and communication
K - Financial and insurance activities
L - Real estate activities
M - Professional, scientific and technical activities
N - Administrative and support service activities
O - Public administration and defence; compulsory social security
P - Education
Q - Human health and social work activities
R - Arts, entertainment and recreation
S - Other services activities
T - Activities of households as employers; undifferentiated goods - and services - producing activities of households for own use
U - Activities of extraterritorial organisations and bodies

\textsuperscript{10} NACE is an abbreviation of Nomenclature statistique des activités économiques dans la Communauté européenne. The 2008 version can be found as http://ec.europa.eu/eurostat/documents/3859598/5902521/KS-RA-07-015-EN.PDF
Each of the classes has a variety of subclasses, as illustrated for Construction in Table 1. The taxonomic aggregations can be drawn in the form of trees, as illustrated in Figure 5.

Table 1. The NACE classification for construction

Figure 5. Part of the NACE taxonomic aggregation drawn as a tree.
Usually taxonomies are created for a purpose, such as the collection of statistics. For example, Table 2 shows the added value of a company’s activities classified as Construction (C), Wholesale and retail trade; repair of motor vehicles and motorcycles (G), and Professional, scientific and technical activities (M). In principle firms can assign the added value of their activities to the different classes, and this can be summed over all firms to give national statistics of the activities at the various levels of aggregation. Typically such figures are used for economic planning by national government and the European Commission.

<table>
<thead>
<tr>
<th>Section</th>
<th>Division</th>
<th>Group</th>
<th>Class</th>
<th>Description of the class</th>
<th>Share</th>
</tr>
</thead>
<tbody>
<tr>
<td>C</td>
<td>25</td>
<td>25.9</td>
<td>25.91</td>
<td>Manufacture of steel drums and similar containers</td>
<td>10%</td>
</tr>
<tr>
<td></td>
<td>28</td>
<td>28.1</td>
<td>28.11</td>
<td>Manufacture of engines and turbines, except aircraft, vehicle &amp; cycle engines</td>
<td>6%</td>
</tr>
<tr>
<td></td>
<td>28.2</td>
<td>28.24</td>
<td></td>
<td>Manufacture of power-driven hand tools</td>
<td>5%</td>
</tr>
<tr>
<td></td>
<td>28.9</td>
<td>28.93</td>
<td></td>
<td>Manufacture of machinery for food, beverages and tobacco processing</td>
<td>23%</td>
</tr>
<tr>
<td>G</td>
<td>46</td>
<td>46.1</td>
<td>46.14</td>
<td>Agents involved in the sale of machinery, industrial equipment, ships &amp; aircraft</td>
<td>7%</td>
</tr>
<tr>
<td>M</td>
<td>71</td>
<td>71.1</td>
<td>71.12</td>
<td>Engineering activities and related technical consultancy</td>
<td>13%</td>
</tr>
</tbody>
</table>

Table 2. An example of using the NACE classification to record a company’s added value

7. Intension and Extension

There are two ways of defining classes. One way involves listing all the things referred to by a word or phrase. For example, at the time of writing ‘twentieth first century British Prime Ministers’ refers to Blair, Brown and Cameron. This listing is called definition by extension.

In contrast to defining terms by extension, they can be defined by intension, which usually means establishing a rule. For example, according to the Oxford Dictionary, a car is “a road vehicle, typically with four wheels, powered by an internal-combustion engine and able to carry a small number of people”. Thus my desk is not a car, but the machine I drove to work this morning is a car. Intensional definitions can be difficult to apply in practice because they require interpretation. For example, some ‘cars’ can carry nine people, so may not qualify as cars. Also as shown in Figure 6(c), the definition or car needs updating to include things desired in the extension such as being powered by electricity, as with the BMW i3 shown.

When building structured vocabularies inconsistencies can arise between intension and extension. For example, a vocabulary to classify television programmes had a class ‘Sports Not Requiring Equipment’ illustrated extensionally with the examples wrestling and boxing (Figure 5(b)). However boxing requires equipment such as gloves, gum shields, and a ring and is not a sport not requiring equipment. The problem here is that the name of the class itself has meaning, and in this case the intension contained in the name does not match the desired extension.

Note that the taxonomic cones shown are different to part-whole cones. Extension means ‘belongs to this list’, intension means ‘obeys this rule’, but part-whole aggregation also requires a ‘the parts are put together this way’ assembly relation.

![Figure 6. Care is needed when naming classes and defining them by extension and intension](image)
8. The Intermediate Word Problem

For any system, the intermediate word problem asks ‘what are the appropriate words to describe the system between its highest and lowest levels, as shown in Figure 8(a).

When a system is formally analysed for the first time it is usually easy to see many things that clearly belong to it. For example, the people that formulated Standard Industrial Classifications such as NACE would have been aware that industry could include, for example, making footballs, growing rice and making Stilton cheese.

(a) what are the intermediate words between the system and the soup?

(b) the NACE statistical classification is one answer to the intermediate word problem.

Figure 7. The Intermediate Word Problem

Figure 7(b) shows how the NACE classification gives an answer to this intermediate word problem, allowing the levels to be assigned numbers 1 to 5. The original collection of entities is called the hierarchical soup because it contains entities at different levels, e.g. Agriculture
and growing rice, and it contains something that is not an industry, \textit{i.e.} taking a holiday rather than providing a holiday. The soup can contain any word or expression, and the intermediate word problem involves lifting a coherent structured vocabulary out of the soup. It has been called an “arbitrary and unstructured collection of things … a prelogical primordial source containing the building blocks of all subsequent structures” (Gould \textit{et al}, 1984).

The NACE classification is imperfect in a number of ways. For example, \textit{Agriculture} has to be subsumed under \textit{Agriculture, Forestry and Fishing}. This means, for example, that any Level 4 statistics concerning \textit{Agriculture} are ‘contaminated’ with data related to \textit{Forestry and Fishing}. Another problem is that many of the terms are stilted and artificial, often containing peculiar combinations of different things, \textit{e.g.} G47.7.6 - Retail sale of flowers, plants, seeds, fertilisers, pet animals and pet food in specialised stores.

Hierarchical schemes of words are constructed to describe multilevel systems and their dynamics. The success of a scheme depends on its purpose. For example, if it were useful to distinguish \textit{Agriculture} from \textit{Forestry} from \textit{Fishing}, a new \textit{Level 3.5} could be created between \textit{Level 3} and \textit{Level 4} with these words. Of course the levels would be renumbered so that \textit{Level 3.5}, \textit{Level 4} became \textit{Level 5}, and \textit{Level 5} became \textit{Level 6}. This illustrates that the numbering of the levels is not absolute.

Another problem with NACE is that it stops too soon. For example, it might be useful to have ‘Making Footballs’ explicitly in the vocabulary at, say, \textit{Level 0}. Also it would be possible to have ‘Making Blue Cheese’ at \textit{Level 0} and ‘Making Stilton Cheese’ at \textit{Level -1}.

NACE is an example of the definition of taxonomic intermediate words. This is different to the definition of part-whole intermediate words. For this case it is necessary to have words at one level to represent the parts, and a word at the higher level to represent the whole. Furthermore it is necessary to make explicit the relational structure that combines the parts into the whole.

The relative nature of the levels can be stressed by the notation \textit{Level N}, \textit{Level N+1}, \textit{etc.} where \textit{N} is an arbitrary fixed base level. When constructing a vocabulary to represent a system it is common to find a word that lies between two levels. When new levels are introduced notation such as \textit{Level N+2½} could be used, but the simplest thing is to renumber so that all the levels are designated by whole numbers.

Intermediate words vocabulary is usually created to enable systems to be described for explicit purposes. When a system is being analysed for policy purposes the vocabulary needs to include everything relevant to the policy, and need not contain words and phrases that are irrelevant to the policy. The ‘resolution’ of the words and phrases depends on the purpose in hand.

\textbf{9. Grounding and transitivity in taxonomic aggregation}

![Figure 8. The words Einstein, Curie and Lavoisier are grounded elements, not classes.](image-url)
Taxonomic aggregation does not create new entities, it simply provides a way of defining subsets of entities. For example, Figure 8 shows a taxonomy of scientists with a subclass of physicists. The chemist Lavoisier is not included in this (the class of chemists is not shown). All of these scientists existed irrespective of such taxonomies. The taxonomy is just one of many ways these individuals could be grouped. For example, they could have been grouped into women and men. Figure 8 also illustrates how words can be grounded in real things. Here Einstein, Curie and Lavoisier will be called grounded elements, e.g. the name ‘Curie’ here uniquely identifies the real person Marie Curie. In contrast the word ‘physicist’ does not uniquely name any individual, but names a group of individuals. The extension of the word ‘physicist’ uncontroversially includes Einstein and Curie. Whether or not it should include Lavoisier is a matter of debate and agreement between those using the taxonomy.

Taxonomies are transitive. If a grounded element belongs to a lower level class, and that lower level class aggregates into a high level class, then the grounded element also aggregates into the higher level class. This is because the classes of the taxonomy define sets of grounded elements, and a subset of a subset of a set is also a subset of that set (Figure 9).

**Figure 9.** Partitional taxonomies have nested bases
10. Non-partitional lattice aggregation and top-down cones

Taxonomic aggregations are usually tree-like partitions with disjoint classes for any level. This is not essential. For example, consider a classification of sports with the classes ‘ball games’ and ‘water sports’. Then, for example, water polo belongs to both classes, and this leads to a non-tree aggregation as shown in Figure 10(a). This is an example of a lattice taxonomy. Lattice taxonomies do not have nested bases as illustrated in Figure 10(c), and they have top-down cones as shown in Figure 10(b).

![Figure 10. A lattice aggregation and the related top-down cone.](image)

Conventional taxonomies are usually *partitional*, *i.e.* the classes do not intersect, so that, at any level, a grounded object belongs to one and only one class. One reason for this is that aggregation of numbers is a little more complicated for non-partitional schemes. For example, suppose one were counting minutes of programmes broadcast on the sports in Figure 10. At the lowest level suppose the times were football (200), tennis (200), water polo (100), swimming (100) and diving (100). Then the broadcast time for ball games is $200 + 200 + 100 = 500$, and for water sports is $100 + 100 + 100 = 300$. Add these together to get the broadcast time for SPORTS as $500 + 300 = 800$. But there are only $700 (= 200 + 200 + 100 + 100 + 100)$ minutes broadcast altogether! Of course summing between levels double-counts the time for water polo. However this is not necessary. Although the ‘sum between levels’ algorithm gives the wrong results for lattice taxonomies, the algorithm that computes the broadcast time for sport as the sum of the broadcast times for the individual grounded sports works perfectly.

Frequently taxonomic schemes are devised to collect and aggregate statistics, as with NACE. Forcing grounded elements to belong to one class rather than another may distort the statistics and introduce errors rather than statistical consistency. The way the numbers are computed implicitly or explicitly involves the intensional interpretation of the classes in a taxonomy, and arbitrary non-assignment to classes is a poor basis for interpretation.
11. Grounded intermediate words in systems of system of systems

There is subtle distinction to be made between the parts of a system and the parts of the parts. Winston et al (1987) give the following example:

- Simpson’s finger is part of Simpson’s hand
- Simpson’s hand is part of Simpson’s body
- Simpson’s finger is part of Simpson’s body

This suggests that the ‘part of’ relation is transitive, so that a part of a part also a part. However

- Simpson’s finger is part of Simpson
- Simpson is part of the Philosophy Department
- Simpson’s finger is part of the Philosophy Department

shows that the transitivity of the ‘being a part of’ relation can be problematic.

The assembly of Simpson and other academics into the Philosophy Department is determined by the university’s rules and statutes. These establish (i) which people are members and (ii) how those people should act and interact to form the higher level structure. This is very different to the relation that assembles the parts of Simpson and the others into their bodies.

Top-down, it is possible to disassemble the Philosophy department into its unstructured parts, as shown at the top of Figure 11(a). It also possible to take one of the parts of the Philosophy Department such as Simpson and top-down, identify parts of him such as his finger as shown at the bottom of Figure 11(a).

![Diagram](image)

(a) disaggregating to parts of parts  (b) how are the parts of parts assembled?

**Figure 11. The ‘part of’ relation is not transitive for part-whole assemblies**

It is possible to make the set of the parts of the parts of the Philosophy Department as shown at the bottom of Figure 11(b). However, what is the part-whole relation that assembles the set of parts of parts into the whole? Of course it does not exist explicitly. The definition of the Philosophy Department is not framed this way. It is framed in terms of whole people without reference to their parts. In this case the assembly that creates the Philosophy Department is grounded at the intermediate level. Disassembling the parts into parts of parts adds no useful information in this case.

In general let $w$ be a word or phrase that names a part-whole entity in a system. Let base($w$) be the set of all the parts required to make the assembled entity. Let $N+k$ be the lowest level of all parts in base($w$). Then $w$ is said to be grounded at Level $N+k$ and the assembly defining the existence of the entity named by $w$ is said to be information-closed at Level $N+k$. 

12. Aggregation over Space and Time

All human activity is referenced in space and time. This is because every human being exists somewhere on earth, and often it matters where they are at any particular time.

For the purposes of social administration and planning, the surface of the earth is usually divided into zones, almost always areas of contiguous land and/or water. Again the definition of the zones depends on the purpose, and very often zones correspond to politically defined areas such as a country, a region of a country, a city, part of a city, and so on. Invariably the zones do not intersect so that, for example, the sum of the populations in the zones of a city aggregates into the population of the whole city.

Time is also divided up, usually into continuous intervals such as an hour, a day, a week, a month, a year, a decade, a century, and a millennium. Usually the time intervals are selected for a particular purpose, and usually they do not intersect.

13. Interleaved Multilevel Part-Whole and Taxonomic Aggregation

Although they are different, part-whole aggregations and taxonomic aggregations work together in multilevel systems. To illustrate this consider the arch construction in Figure 12.

![Figure 12](image_url)

**Figure 12. Interleaved taxonomic and part-whole aggregations**

In this multilevel system blocks are assembled to form three types of arches. Each individual arch is assembled into a class of arches of that type. At the lowest level, the individual blocks are assembled into sets of their type. Thus the part-whole aggregation is sandwiched between two taxonomic aggregations. At the top level the different classes of arches are brought together under a single taxonomic class of arches, denoted $A$. In this example there are three taxonomic aggregations and one part-whole aggregation.
Figure 13 illustrates see how such a scheme might be used and the subtle distinction between 
an organisational description of the objects in a multilevel system, the objects themselves, and 
the part-whole structure of the objects. On the left is a multilevel description of the arch in 
terms of the types of blocks that are assembled to make it. This is different to an actual 
instance of the block, as shown on the right.

The multilevel description of the arch means that one can work top-down on the left to find 
out the components are necessary to build an instance of the arch. This information enables 
the set of parts to be assembled, as with a furniture flat pack. The relation $R$ then says how 
those parts should be assembled, as with the instructions in a flat pack.

Bottom-up, the assembly of the actual parts into sets of parts of the same kind is called a 
*homogeneous grounded taxonomic aggregation* at the bottom left of Figure 12. This 
aggregation is like putting components of the same kind into a bin container.

The bottom-up assembly of the set of components needed to make the arch is called a 
*heterogeneous grounded taxonomic aggregation* at the bottom right of Figure 13. This is like 
taking the parts from bins of homogeneous components to make the set of heterogeneous 
components, ready for the bottom up part-whole arch assembly. This process creates a real 
object, *grounded* in the real components at the lowest level.

Although this illustrative example involves physical objects, these ideas can equally well be 
applied to social systems. For example when building a team of people, the types of people 
will be specified and so will the way they must work together to achieve the team objective. 
Then individual people will be appointed as a heterogeneous grounded taxonomic 
aggregation. Following this, part-whole aggregation will be needed to meld the individuals 
into a well-working team. This might involve training for individuals and the whole group.

**14. Multilevel Gestalt Perception**

Human perception has the remarkable property that we perceive our environment at all levels 
simultaneously. For example Figure 14 shows the London skyline. At one level this can be 
perceived as whole, but at the same time one sees the individual buildings.
Look around you. Do you see your environment as one ‘picture’. Almost certainly not. On the one hand you perceive the whole, but you will also perceive many of the parts that make up the whole. This may be due to the ways that eyes scan scenes, temporarily fixating on parts to build up an impression of the whole.

Figure 15. A stamp illustrating multilevel perception

To illustrate the idea of multilevel perception, consider the image of a stamp in Figure 15. The whole is clearly a Spanish postage stamp. At a lower level of aggregation are various items of text and a copy of Picasso’s iconic painting Guernica. At a lower level still, one sees the a horse and a bull, and at yet a lower level fixating on the bull one can see its horns, ears, mouth and eyes. I find myself constantly moving between the whole and its multilevel details. For example on the left is a mother and baby, then I see Picassos’ signature, then the value of 200 pesetas, and so on until I am looking again at the whole, and suddenly looking at the serrated edges, before going back to the signature and then further details in the painting. If you look at the stamp I expect you too will find your perception rapidly shifting between the levels.

Figure 16 illustrates the bottom-up and top-down nature of our perception. On the left one can take two Level N rectangles and arrange them to form a cross at Level N+1. Immediately new shapes emerge, including the square where the rectangles intersect. This Level N object can be perceived as being formed from four straight lines at a lower level. These lines can be assembled to make [ and ] shapes at Level N-1 and these can be assembled to make a new rectangle at Level N. This has then been assembled to form an i-shape at Level N+1.

The multilevel way our brains process information is very remarkable. It seems that reduction to the parts and recombination the Gestalt whole is a very effective strategy.

Figure 16. Dynamics bottom-up and top-down perception creates new parts and wholes

15. Towards a formal mathematical theory of multilevel systems.

The great challenge for multilevel systems science is to understand how the dynamics of systems are coupled across and between levels, bottom-up from micro to macro through the meso, and top-down from macro to micro through the meso. Mathematical notation can make the structures and operations involved much clearer, as will be seen.

15.1 Traffic on the multilevel backcloth

In an early attempt to formulate a mathematical theory of multilevel systems, R. H. Atkin made a distinction between relational structure and patterns of numbers distributed over relational objects called simplices. The distinction is clear to see in network theory. Networks are formed from vertices and edges, also called nodes and links in social analysis. Whereas a network such as a road system may not change over a given period of time, the speed and number of vehicles may change considerably during that time. Atkin called the relatively static relational structure the backcloth and he called the relatively dynamic patterns of numbers the traffic.

Backcloth structure at the microlevel could include the relational structure of a family. The traffic on this structure could include the number of family holidays taken and the money spent on them. At higher level, institutions such as businesses form relatively fixed relational structures supporting patterns of numbers such as their costs and incomes. Another example is the relational structure of a hospital supporting a traffic of patients treated.

Here, any part-whole structure is part of the multilevel backcloth, and the numbers associated with part-whole structures forms the multilevel traffic. The mathematical theory to be developed shows how multilevel systems and their multilevel dynamics can be precisely defined.

15.2 The mathematics of parts and wholes

![Part-whole aggregations](image)

**Figure 17. Part-whole aggregations**

Figure 17 shows four part-whole aggregations. Figure 17(a) shows four blocks, a, b, c, and d, aggregated by the 4-ary relation $R_{arch}$ to form and arch. This arch has an emergent property, namely the gap between its sides and the top. None of its parts have this property.

Figure 17(b) shows the same four block arranged by another 4-ary relation, $R_{arrow}$, to form an arrow. This illustrates the possibility of the same set of parts being assembled into different whole. Thus knowing the parts is necessary but not sufficient to define the whole. It is also necessary to know how the parts are to be put together. Different assembly relations can create different wholes.

Figure 17(c) shows how a 3-ary relation $R_{dog}$ on the letters O, D, and G assembles them in to the word $D O G$. Let $R_{dog}$ be given as “take the first letter and put it to the right of the second
letter. Then take the third letter and put to the right of the letter just placed”. When this rule is applied in Figure 17(d) it result in the word \( G O D \) rather than the word \( D O G \). This illustrates that the order of the parts must be specified if the \( n \)-ary relations acting on those parts are to work properly.

These observations make clear that in order to represent the whole in terms of its parts:

(i) the \( n \) parts must be known

(ii) the \( n \)-ary relation that assembles the parts into the whole must be known

(iii) the \( n \) parts must be consistently ordered for the \( n \)-ary relation that assembles the parts into the whole to work properly.

These consideration underlie the following definitions and notation:

**Definition 1.** Let \( p_1, p_2, \ldots, p_n \) be the parts of a whole. Then the expression \( < p_1, p_2, \ldots, p_n > \) is called a *part-list simplex*. The order of the parts is arbitrary but fixed.

For example, \( < p_n, p_{n-1}, \ldots, p_1 > \) is also a part-list simplex, but \( < p_n, p_{n-1}, \ldots, p_1 > \neq < p_1, p_2, \ldots, p_n > \) because the parts are ordered differently.

**Definition 2.** Let \( \sigma_{\text{whole}} \) be an entity with parts given by the part-list simplex \( < p_1, p_2, \ldots, p_n > \). Let \( R \) be the \( n \)-ary relation that assembles the parts into a whole, \( \sigma_{\text{whole}} \). Then the expression \( < p_1, p_2, \ldots, p_n ; R > \) is called the *hypersimplex* that represents the whole. We write:

\[
\sigma_{\text{whole}} = < p_1, p_2, \ldots, p_n ; R >
\]

The part-list simplex and the \( n \)-ary relation must be matched, meaning that the definition of the \( n \)-ary relation assumes the parts are ordered in a particular way. For example, as seen in Figure 17, the 3-ary relation \( R_{\text{dog}} \) assumes that the part-list simplex is \( < O, G, D > \). When \( R_{\text{dog}} \) is applied to the part-list simplex \( < O, G, D > \) it gives the result \( G O D \), which is not what is required. Of course, one can define another 3-ary relation \( R_{\text{god}} \) as “take the first letter of the part-list simplex \( < O, D, G > \) and place it the right of the third. Then take the second letter and place it on the right of the letter just placed”. Then \( R_{\text{dog}} \) and \( R_{\text{god}} \) are different relations and \( < O, D, G ; R_{\text{dog}} > = D O G \) and \( < O, D, G ; R_{\text{god}} > = G O D \) as required.

15.3 Aggregation of numbers: multilevel traffic on the multilevel backcloth.\(^\text{12}\)

For decisionmaking purposes, the patterns of numbers over the multilevel structure are of great importance. In particular it is important to understand how numbers at the lower level aggregate into number at the higher level, and how higher level quantities can be distributed top-down.

In Figure 18 the symbol \( \sigma^{N+1} \) represents a whole entity at Level \( N+1 \). The symbols \( \sigma^N_j \) represent the parts of the whole at Level \( N \). The parts are assembled to the whole by the \( n \)-ary relation \( R^N \), as shown to the left of the diagram.

The Greek symbol \( \phi \) (phi) will be used to represent functions that map the parts and whole to numbers. In Figure 18 these are shown as time series. Thus \( \phi^N_j \) maps \( \sigma^N_j \) to a time series at

---

\(^{12}\)In this section we use the Greek symbols \( \sigma \) (sigma), \( \phi \) (phi), \( \psi \) (psi) and the symbol \( \varphi \) (varphi).
the bottom right, $\phi_2^N$ maps $\sigma_2^N$ to a time series placed above it, and so on with $\phi_n^N$ mapping $\sigma_n^N$ to another time series.

The idea is that these Level $N$ time series may be aggregated to a higher Level $N+1$ time series, $\phi^{N+1}$, associated with the higher level whole, $\sigma^{N+1}$.

The situation is summarised in Figure 19. At Level $N$ the individual mappings $\phi_i^N$ are combined into a single mapping $\phi^N$ that maps the part-list simplex to the list of $n$ numbers (vector) represented as $(\phi^N \sigma_1^N, \phi^N \sigma_2^N, \ldots, \phi^N \sigma_p^N)$. The $n$-ary relation $R^N$ maps the part-list simplex to the whole, $\sigma^{N+1}$. At Level $N+1$ the mapping $\phi^{N+1}$ associates the number $\phi^{N+1} \sigma^{N+1}$ with the hypersimplex $\sigma^{N+1}$.

**Figure 18. Aggregation of numbers over the multilevel backcloth.**

![Diagram](image)

**Figure 19.** $\phi^{N+1}$ is an aggregation of $\phi_i^N$, $i = 1, \ldots, n$ when the diagram commutes

The general question is whether or not the Level $N+1$ mapping $\phi^{N+1}$ can be reconstructed from the Level $N$ mapping $\phi^N$? The answer is that it can if there exists a mapping, here represented by the $\psi^N$ (Greek symbol psi with superscript $N$), where the application of $R^N$ followed by $\phi^{N+1}$ gives the same result as the application of $\phi^N$ followed by $\psi^N$. In this case it is said that the diagram commutes, i.e. starting at the bottom left, going round the diagram either way gives the same result.

With these concepts it is possible to define the aggregation of traffic across the backcloth.

**Definition 4.** The mapping $\phi^{N+1}$ is an aggregation of the mappings $\phi_i^N$, $i = 1, \ldots, n$, if there exists a mapping $\psi^N$ called the aggregator mapping such that the diagram in Figure 19 commutes, i.e.

\[ \phi^{N+1} R^N = \psi^N \phi^N \]
15.4 Example: aggregation of lower levels statistics

Let \( \sigma^{N+1} \) be a city at Level \( N+1 \) made up of four Level \( N \) zones, \( \sigma_1^N, \sigma_2^N, \sigma_3^N, \sigma_4^N \). Then \( \sigma_1^N, \sigma_2^N, \sigma_3^N, \sigma_4^N \) is Level \( N \) part-list simplex. Let \( \phi^N \sigma^N \) be the population of zone \( \sigma_i^N \). Using the populations in Figure 20, \( \psi^N \sigma^N = (6000, 1000, 4500, 2500) \). Let \( \psi^N \) be the mapping that adds together all these population:

\[
\psi^N \phi^N < \sigma_1^N, \sigma_2^N, \sigma_3^N, \sigma_4^N > = \phi_1^N \sigma_1^N + \phi_2^N \sigma_2^N + \phi_3^N \sigma_3^N + \phi_4^N \sigma_4^N = 6000 + 1000 + 4500 + 2500 = 14,000
\]

Let \( \phi^{N+1} \sigma^{N+1} \) be the population of the city. Let this be defined to be the sum of the population in all the zones. Then \( \phi^{N+1} \sigma^{N+1} = 14,000 = \psi^N \phi^N < \sigma_1^N, \sigma_2^N, \sigma_3^N, \sigma_4^N > \), so that \( \phi^{N+1} R^N < \sigma_1^N, \sigma_2^N, \sigma_3^N, \sigma_4^N > = \psi^N \phi^N < \sigma_1^N, \sigma_2^N, \sigma_3^N, \sigma_4^N > \), and \( \phi^{N+1} R^N = \psi^N \phi^N \).

Therefore \( \psi^N \) is an aggregation mapping, or aggregator, of \( \phi^N \) into \( \phi^{N+1} \). In this case \( \phi^{N+1} \) is deliberately constructed from the population counts \( \phi^N \). Although very simple, this kind of additive statistical aggregation is very common in multilevel systems. It will be called a linear aggregation.

15.5 Example: Adam Smith’s pin factory

Let \( \sigma^{N+2} \) be Adam Smith’s pin factory. Let the Level \( N+1 \) part-list simplex \( \sigma^{N+1} = \sigma_1^{N+1} = \text{buildings}, \sigma_2^{N+1} = \text{workers}, \sigma_3^{N+1} = \text{materials} \). Let the worker part-list simplex be \( \sigma_1^N, \sigma_2^N, \ldots, \sigma_{18}^N \). There are many other parts for even such a simple organisation (the details are omitted in Figure 21). Although aggregations are linear, such as summing the wages of the individual workers, the traffic aggregations for even this simple system are quite complicated.

![Figure 20. A city assembled from four zones](image)

![Figure 21. A partial multilevel description of a pin factory](image)
For the owners of the pin factory one of the most important numbers is the profit, which is the income minus the outgoings. The income is related to the number of pins produced per day. Smith report that he observed ten men make about twelve pounds or forty-eight thousand pins a day, four thousand eight hundred each. “But if they had all wrought separately and independently, and without any of them having been educated to this peculiar business, they certainly could not each of them have made twenty”. Smith is saying that the division of labour, the creation of the part-list simplex \(< \alpha_1^N, \alpha_2^N, \ldots, \alpha_p^N >\), and the way the men are organised by the assembly relation \(R^N\), can dramatically improve productively.

### 15.6 Disassembly relations.

Let the assembly relation \(R^N\) be rewritten as \(R^\Phi^N\) to emphasize that that it aggregates parts up into wholes. \(R^\Phi^N: < \alpha_1^N, \alpha_2^N, \ldots, \alpha_p^N > \rightarrow \alpha_{N+1}^N\). Associated with this let us define a disassembly relation \(R^{\Psi^N}: \alpha_{N+1}^N \rightarrow < \alpha_1^N, \alpha_2^N, \ldots, \alpha_p^N >\). This ‘forgets’ the part-whole relational structure, \(R^\Phi\), of \(\alpha_{N+1}^N = < \alpha_1^N, \alpha_2^N, \ldots, \alpha_p^N, R^\Phi^N >\). Figure 22 redraws Figure 19 including \(R^\Phi^N\) and \(R^{\Psi^N}\).

![Figure 22. \(R^{\Psi^+}\) disassembles the hypersimplex \(\alpha^{N+1}\) into its part-list \(< \alpha_1^N, \alpha_2^N, \ldots, \alpha_p^N >\)](image)

**Proposition.** The mapping \(\phi^{N+1}\) is an aggregation of the mappings \(\phi_i^N\), \(i = 1, \ldots, n\), if there exists an aggregation mapping \(\psi^N\) such that

\[
\phi^{N+1} = \psi^N \phi^N R^{\Phi^{N+1}}
\]

This follows from the existence of the aggregation mapping \(\psi^N\). The symbol \(R^{\Phi^N}\) is defined to be the same as \(R^N\), so by Definition 4 we can write \(\phi^{N+1} R^{\Phi^N} = \psi^N \phi^N\) and

\[
\phi^{N+1} R^{\Phi^N} < \alpha_1^N, \alpha_2^N, \ldots, \alpha_p^N > = \psi^N \phi^N < \alpha_1^N, \alpha_2^N, \ldots, \alpha_p^N >
\]

Substituting \(\alpha_{N+1}^N\) for \(R^{\Phi^N} < \alpha_1^N, \alpha_2^N, \ldots, \alpha_p^N >\) on the left, and substituting \(R^{\Psi^N} \alpha_{N+1}^N\) for \(< \alpha_1^N, \alpha_2^N, \ldots, \alpha_p^N >\) on the right gives \(\phi^{N+1} \alpha_{N+1}^N = \psi^N \phi^N R^{\Psi^N} \alpha_{N+1}^N\) so that the mapping \(\phi^{N+1} = \psi^N \phi^N R^{\Psi^N}\) as required.

This formulation makes clearer the nature of aggregator mappings. The disassembler relation \(R^{\Psi^N}\) makes clear that the mapping \(\phi^{N+1}\) is to be constructed bottom-up from lower level information, namely the mappings \(\phi_i^N\).

### 15.7 Information Closure

What is the lowest level necessary to understand the dynamics of multilevel systems? For example, when trying to understand the behaviour of the people in a social system is it necessary to know their genetic makeup? The answer is that it depends on the purpose of the
analysis. Genetic makeup is central in personalized medicine (Royal Society, 2005), but currently it is not considered relevant to the majority of social systems in the public and private sectors.

In a different context and using a different notation, Pfantte et al (2014) suggest the following properties for multilevel systems

**I. Information closure**: The higher level process is informational closed, i.e., there is no information flow from the lower to the higher level. Knowledge of the microstate will not improve predictions of the macrostate.”

**II. Observational commutativity**: It makes no difference, whether we perform the aggregation first, and then observe the upper process, or we observe the process on the microstate level, and then lump together the states.”

![Diagram](23)

Observational commutativity is illustrated in Figure 23. This diagram is commutative because you can go round it either way to get the same result:

<table>
<thead>
<tr>
<th>aggregate</th>
<th>observe</th>
<th>result</th>
</tr>
</thead>
<tbody>
<tr>
<td>lower process</td>
<td>observe</td>
<td>results</td>
</tr>
<tr>
<td>upper process</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

To illustrate how these ideas can be applied to multilevel systems as defined in this chapter, consider the following example.

Let \( \sigma^{N+1} \) = ‘bag of \( n \) potatoes’ and \( \sigma^N_i = \) ‘potato number \( i \)’, for \( i = 1, \ldots, n \). Let \( \sigma^N \) be the bag. Let \( \phi^{N+1} \sigma^{N+1} \) be the weight of the bag of potatoes. Let \( \phi^N \sigma^N_i \) be the weight of potato number \( i \). Let \( \psi^N (\phi^N \sigma^N_1, \ldots, \phi^N \sigma^N_n) \) be the sum of the weights of the individual potatoes.

Let \( R^{N+1} \) be the disaggregation of the bag of potatoes in the list of the individual potatoes. Then \( \phi^{N+1} = R^{N+1} \phi^N \psi^N \). Thus \( \psi^N \) aggregates the weights of the individual potatoes into the weight of the whole bag of potatoes.

Suppose a merchant is selling the potatoes. You bring a bag of potatoes off the shelf, \( \sigma^{N+1} \). The merchant can put the bag of potatoes on the scales and read off the weight of the whole immediately as \( \phi^{N+1} \sigma^{N+1} \). Alternatively the merchant can disassemble the whole via \( R^{N+1} \) and weigh the bag and each potato individually via \( \phi^N \), and sum the weights via \( \psi^N \), to get the weight of the whole.

Weighing the whole bag of potatoes at Level \( N+1 \) or weighing each potato individually and adding the Level \( N \) weights gives the same result. No new information is obtained by going to the lower level.

In this case, not only does going to the lower level give no new information, it also creates extra work.
This discussion motivates the following definitions:

**Definition.** Assume a multilevel system is described as shown in Figure 24. Let $\phi^N$ and $\phi^{N+1}$ be methods of observing the system at Level $N$ and $N+1$.

(i) the system has **observational commutativity** if there exists an aggregator mapping $\psi^N$ and the diagram commutes: $\phi^{N+1} R^N = \psi^N \phi^N$

(ii) Level $N+1$ has **information closure** with respect to Level $N$ if $\phi^{N+1} = \psi^N \phi^N R^{N+1}$ where the process of observing $\phi^{N+1}$ is independent of the process of observing $\phi^N$, i.e. the observation $\phi^{N+1}$ can be made without reference to $\phi^N$.

By these definitions there is observational commutativity in the potato system, and Level $N+1$ is information-closed with respect to Level $N$.

**15.8 Disaggregation of multilevel traffic – synthetic micropopulations.**

In complex systems science and its applications to decisionmaking, computer simulation is a major tool for investigating the possible consequences of policy actions. Moeckel *et al* (2003) write “Microsimulation models require micro data. However, the collection of individual micro data, *i.e.* data that can be associated with single buildings, or the retrieval of individual micro data from administrative registers is neither allowed in most countries nor desirable for privacy reasons. Therefore these models work with synthetic micro data that can be retrieved from general accessible aggregate data. A synthetic population has to be generated that represents individual actors in the form of households and household members. A synthetic population is statistically equivalent to a real population. For each household characteristics such as house- hold size, income, number of cars and address are generated. Each person is described by characteristics such as age, sex, religion, and work location. For creating addresses for the synthetic population, land-use data are disaggregated to raster data by GIS techniques.” The first large-scale applications of synthetic micropopulations for micro simulation was the TRANSIMS systems developed at Los Alamos National Laboratory for road traffic modelling in the nineteen nineties (Barrett *et al*, 1999).

Figure 25 shows the generality of creating synthetic micropopulations. On the left, at Level $N$ each sampled individual has some characteristic given by $\phi^N_i$ (individual person $i$). These values are aggregated by $\psi^N$ to give the population statistics $\phi^{N+1}$ for the real population. Implicitly there is a population to which these statistics can be extrapolated. For example, if data were collected on the incomes of people in various British cities, the extrapolation population could be Britain, but would not include Spain or Germany.
Given $\phi^{N+k}$, the statistical values can be distributed across a new population at Level $N$, i.e. every member of the new population is assigned a value denoted as $\psi^N(j)$ (individual person $j$). This is done by a process represented by the disaggregation symbol, $\mu^{N+k}$, which usually uses Monte Carlo methods to assign values to the members of the new population. Typically there is synthetic data on 100% of the people in the original sample area, or some other area within the extrapolation population, to create a synthetic micropopulation.

Importantly, $\mu^{N+k}$ and $\psi^N$ must have the property that $\psi^N \mu^{N+k} \phi^{N+1} = \psi^N \phi^N = \psi^N \phi^N$ so that $\phi^{N+k} = \phi^{N+k}$ and that the synthetic micropopulation has the same statistical properties as the original data. With or without other synthetic Level $N$ data, the Level $N$ synthetic micropopulation may support aggregations to high levels than the original $N+k$.

15.9 Connectivity and multilevel transmission dynamics

A part-whole hypersimplex $\sigma_{\text{whole}} = \langle p_1, p_2, \ldots, p_n, R \rangle$ at Level $N+1$ has an associated polyhedron at Level $N$ representing its part-list simplex. This is illustrated in Figure 26. A relation between two things gives an edge, as in a network. A relation between three people such as a business deal is represented by a two-dimensional polyhedron, namely a triangle. A relationship between four elements is represented by a 3-dimensional tetrahedron, e.g. Figure 26 shows a relation between the four musicians of a piano quartet. A relation between five people such as being a team is represented by a polyhedron with five vertices in four dimensional space. In general a relation between $p+1$ elements is represented by a polyhedron in $p$-dimensional space.

Figure 26 shows how polyhedra can be connected. The symbols $\sigma_1, \sigma_2, \sigma_3$ at the bottom left represent part-whole subsystems as connected polyhedra at Level $N$. These three polyhedra support various patterns of numbers represented by the three swirling shapes at the lower right of the figure. These dynamics have the possibility of coupled interactions because the polyhedra that support them are connected. At the higher level, $N+1$, the behaviour of the system is represented by a time series.
Figure 27. Coupled interaction dynamics in multilevel systems.

Figure 27 illustrates the possibility of interactions of the dynamics between and across levels. For example, it is possible that the coupled interactions of the dynamic at Level N+1 can only be determined by studying the interactions at Level N. The **Fundamental Challenge of Multilevel Systems** is to unify the dynamics between levels as illustrated in Figure 28. The commutativity of the rectangles in this diagram suggest that this challenge could be addressed through category and topos theory (Goldblatt, 1980).

Figure 28. The **Fundamental Challenge of Multilevel Systems** (Johnson, 2014)
16. Multilevel triangle diagrams and multidimensional structures

16.1 Multilevel triangles

When reasoning about sets, Euler circles in Venn diagrams can be very useful. These diagrams have their limitations, but they are extremely useful for sketching out ideas, as shown in Figure 29(a). Representing multilevel systems can get very complicated as illustrated by the NACE classification published as a 36-page document. Taking inspiration from Euler circles, it is suggested that multilevel triangles provide a simple way to sketch the complexity of multilevel systems, as shown in Figure 29(b).

(a) Euler circles provide a simplified way to sketch ideas about sets
(b) Multilevel triangles give a simplified way to sketch ideas about multilevel systems

Figure 29. Multilevel triangles as a way of sketching ideas about multilevel systems

Continuing the analogy with Venn diagrams, Figure 29 shows the representation of subsystems, intersections, and unions. Although these diagrams are very simple, they provide a way of representing great complexity. For example, the grey triangle in Figure 30(a) shows the Bank of England as a subsystem of the finance system. Figure 30(b) shows the intersection of the banking system and the finance system, with the Bank of England as a subsystem. Figure 30(c) shows the multilevel system A as the ‘union’ of multilevel systems B and C. The caption includes a question mark because the combination of two multilevel systems is likely to involve new structures as the substructures of A and B interact. What the ‘union’ of two multilevel systems might be is a research question stimulated by the multilevel triangle convention.

(a) subsystem
(b) intersection
(c) union ?

Figure 30. Representing subsystems, intersections and unions by multilevel triangles

Consider the multilevel transportation system and the multilevel land use system. They are tightly coupled since land uses create travel demand, and the possibility of travel creates forces for new or different land uses. This could be represented as a link in a multilevel network, as shown in Figure 31(a). This suggests the possibility of networks of multilevel systems, as shown in Figure 31(b).
One of the problems with complex multilevel systems is that almost everything interacts with almost everything else. However some systems are more tightly coupled than others, so networks of multilevel systems may have some kind of multilevel weightings on the links.

Figure 32 shows a way of representing the evolution of multilevel systems as trajectories through time. For example, the thick grey line shows a trajectory from A to C to E to H. In general it is not possible to know for certain that a system will evolve along one trajectory rather another, and the trajectories fan out. As the clock ticks some trajectories cease to be possible, and new triangles will evolve to create a new ‘front’ at time \( t_4 \) after G, H, I and J at time \( t_4 \).

### 16.2 Example: Narratives as multilevel multidimensional structures

One of the great challenges to policy is to understand the dynamics of narratives that can have such immense impact on political and political agenda. Let a narrative be defined to be a structured set of words\(^1\). For example, a narrative could evolve through a sequence of online newspaper articles. Each of these is a structure with substructures the paragraphs, sentences, phrases and multilevel words. There are other more subtle relationships between words with emergent meanings that combine to form parts of narratives as they evolve through publication of the articles through time.

Narratives can include anything. Also, they can have complicated dynamics. For example, a narrative may have elements of business, as in the NACE classification, but also elements of the environment, religion and government. These four multilevel dimensions can each be represented by a multilevel triangle, as shown in Figure 33.

Consider a narrative developing that combines all these four dimensions represented by a tetrahedron (Fig. 33(a)). For example, it could be a story about the Pope commenting on the responsibilities of governments and big business in the context of climate change. Or it could be a story about nuclear proliferation in the Middle East. The particular narrative emerges as lower level details are instantiated in the multilevel triangles (Fig. 33(b)).

\(^{13}\) Pictures such as photographs can be a strong part of a narrative, but for simplicity here the discussion is restricted to words.
When there is interaction across all dimensions under a multilevel 4-ary relation, \( R \), the resulting dynamic system can be represented by the notation < Business, Environment, Government, Religion; \( R \) > and represented graphically by a tetrahedron as shown in Figure 33. This structure is a hypersimplex, and can be very dynamic with the focus rapidly changing from high to low levels in the multilevel schema.

![Tetrahedron](image)

(a) <business, environment, government, religion>  (b) the narrative instantiated at lower levels

**Figure 33. A narrative as multidimensional structures in multilevel systems**

Narratives are much richer and more dynamic than the simple diagrams in Figure 33 suggest. In general there will be more multilevel triangles expressing other dimensions, including space and time. Also there will not be just one hypersimplex but many at many levels forming complex multilevel hypernetwork structures.

### 15.3 The multilevel algebra of narratives

Figure 34 shows two narrative trajectories, \( N_1 \) and \( N_2 \). For example, \( N_1 \) could be the narrative that “because successive governments have neglected house building, houses are in short supply”. House prices in the UK are very high, and for many it is impossible to buy houses in the capital. Let \( N_1 \) be the narrative that “the rich in Russia and Asia, some criminal, find the UK a safer place to invest their money than their own countries, and that the London property market is an ideal and politically safe place to invest large sums of money”.

![Narrative Trajectories](image)

**Figure 34. When narratives crash into each other: \( N_1 \bullet N_2 = \text{what?} \)**

Figure 34 shows these narratives spinning from time \( t_1 \) to time \( t_4 \) until they crash into each other at time \( t_5 \) to create a new narrative \( N_3 \) at time \( t_6 \). “The shortage of houses and Russian and Asian money are making housing in London unaffordable for ordinary people”.

From a mathematical viewpoint this is very interesting because we can write \( N_1 \bullet N_2 = N_3 \) and define \( \bullet \) to be the **multilevel crash operator**. Understanding the properties of the multilevel crash operator is research programme to be addressed.
16. Conclusions

This chapter has outlined a method of representing multilevel systems for policy purposes. Making clear the multilevel structure is particularly important when the impact of policy is to be investigated by computer simulation. However, even for less formal analyses of policy awareness of the subtleties of multilevel systems can add greater precision and make successful policy more likely. The main ideas covered include:

- Socio-economic systems have multilevel structure. This was illustrated by Smith’s pin factory in Section 2, and the UK health system in Sections 3, 4 and 5.

- Part-whole systems play a major role in multilevel systems, where intermediate level wholes may become parts in higher level structure. The idea of part-whole cone aggregation was introduced in Sections 5.

- Taxonomic aggregation plays another major role in multilevel systems but, as shown in Section 6, taxonomic aggregation is very different to part-whole aggregation.

- Multilevel classes can be defined by extension (listing the members) or intension (giving a rule for membership). Section 7 shows that care is required to ensure that class names with intensional meaning are consistent with the intended extensions.

- The Intermediate Word Problem is introduced in Section 8 as a practical challenge for the representation multilevel systems and their dynamics.

- A theory of multilevel systems must be grounded in real things. Section 9 discusses this for taxonomic aggregations where the part-of relation is transitive.

- Section 10 shows that taxonomic aggregations need not force observers to assign things to just one class.

- Confusing anomalies can appear in multilevel system. This is illustrated in Section 11 by the example of Simpson’s finger being part of the Philosophy Department. The problem is created by the way that part-whole aggregations are defined. Objects in multilevel systems can be grounded at intermediate levels, and be information-closed so that analysing them at lower levels gives no extra information.

- Space and time have multilevel structure, as discussed in Section 12.

- Part-whole and taxonomic aggregation are interleaved in multilevel system, as shown in Section 13.

- Section 14 argues that human beings perceive their environment simultaneously at all levels, and that perception creates new parts and wholes at all intermediate levels between micro and macro, i.e. multilevel perception is ‘hard wired’ into our brains.

- Although multilevel structure is built into language and the way we think, a formal approach to multilevel system can be useful for complex systems. Section 15 gives a mathematical formalism that can add precision to multilevel analyses.

- A new graphical way of representing multilevel systems is presented in Section 16. Multilevel triangles are inspired by Euler circles. Although like Venn diagrams they have their limitations, the use of multilevel triangles is illustrated by the need for multilevel algebra to support a dynamic theory of narratives. It is suggested that the ‘crash’ operator of this algebra could be a new mathematical structure that could be investigated by the mathematical theory of categories and topoi.

In an increasingly complex multilevel world the risks inherent in ignoring or not knowing the dependencies between subsystems are increasing. An operational theory of multilevel systems will not prevent the wilful implementation of bad policies, but can support the design and implementation of policies that achieve their objectives without unintended consequences.
Bibliography


