Boundaries of univalent Baker domains

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BOUNDARIES OF UNIVALENT BAKER DOMAINS

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Abstract. Let f be a transcendental entire function and let U be a univalent Baker domain of f. We prove a new result about the boundary behaviour of conformal maps and use this to show that the non-escaping boundary points of U form a set of harmonic measure zero with respect to U. This leads to a new sufficient condition for the escaping set of f to be connected, and also a new general result on Eremenko's conjecture.

1. Introduction

Let f be a transcendental entire function and denote by f^n, $n = 0, 1, 2, \ldots$, the nth iterate of f. The *Fatou set* $F(f)$ is defined to be the set of points $z \in \mathbb{C}$ such that $(f^n)_{n\in\mathbb{N}}$ forms a normal family in some neighborhood of z. The components of $F(f)$ are called *Fatou components*. The complement of $F(f)$ is called the *Julia set* $J(f)$. An introduction to the properties of these sets can be found in [4].

The set $F(f)$ is completely invariant, so for any component U of $F(f)$ there exists, for each $n = 0, 1, 2, \ldots$, a component of $F(f)$, which we call U_n, such that $f^n(U) \subset U_n$. If, for some $p \geq 1$, we have $U_p = U_0 = U$, then we say that U is a *periodic component* of *period* p, assuming p to be minimal. There are then four possible types of periodic components; see [4, Theorem 6] for a classification.

The *escaping set*

$$I(f) = \{z : f^n(z) \to \infty \text{ as } n \to \infty\}$$

was first studied in detail by Eremenko [7] who showed that $I(f) \neq \emptyset$ and indeed that $I(f) \cap J(f) \neq \emptyset$, and also made what is known as 'Eremenko's conjecture' which states that all the components of $I(f)$ are unbounded.

Any Fatou component that meets $I(f)$ must lie in $I(f)$ by normality. A periodic Fatou component in $I(f)$ is called a *Baker domain*; see [11], for example, for the properties of this type of Fatou component, in particular that a Baker domain of a transcendental entire function must be unbounded and simply connected. For the function $f(z) = z + 1 + e^{-z}$, studied by Fatou in [8], the set $F(f)$ is a completely invariant Baker domain, whose boundary is $J(f)$. In this case the Baker domain has many boundary points in $I(f)$ and many that are *not* in $I(f)$.

It is natural to ask whether every Baker domain of a transcendental entire function must have at least one boundary point in $I(f)$. In [15, Remark following the proof of Theorem 1.1] we showed that if U is an invariant Baker domain in which there is an orbit $z_n = f^n(z_0)$, $n \in \mathbb{N}$, such that, for some $k > 1$,

$$|z_{n+1}| \geq k|z_n|, \quad \text{for } n \in \mathbb{N}, \tag{1.1}$$

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then $\partial U \cap I(f)^c$ has harmonic measure zero relative to U. Since a Baker domain of f of period p is an invariant Baker domain of f^p and f maps any boundary point of a Fatou component to a boundary point of a Fatou component, the analogous result holds for periodic Baker domains.

In [2] Barański and Fagella studied transcendental entire functions with univalent Baker domains. A Baker domain U of f of period p is said to be *univalent* if f^p is univalent in U. In this paper we prove the following result about such Baker domains.

Theorem 1.1. *Let f be a transcendental entire function and let U be a univalent Baker domain of f. Then $\partial U \cap I(f)^c$ has harmonic measure zero relative to U. More precisely, if ϕ is a conformal map of the open unit disc $\mathbb{D}$ onto U, then for all $\zeta \in \partial\mathbb{D}$ apart from a set of capacity zero the angular limit $\phi(\zeta)$ exists and lies in $\partial U \cap I(f)$.*

The first statement of Theorem 1.1 follows from the second because the set of angular limits of the conformal map ϕ forms the set of accessible boundary points of U, which has full harmonic measure in U, and any set of boundary points of $\partial\mathbb{D}$ of linear measure zero (in particular, of capacity zero) gives rise to a set of accessible boundary points of U of harmonic measure zero; see [9, page 206].

The paper [2] gives several examples of univalent Baker domains: some with connected boundaries and some with disconnected boundaries; some for which (1.1) holds and some for which it does not hold. Baker and Domínguez [1, Corollary 1.3] showed that the boundary of any non-univalent Baker domain is disconnected, and indeed has uncountably many components, so we have the following immediate corollary of Theorem 1.1.

Corollary 1.1. *Let f be a transcendental entire function and let U be a Baker domain of f whose boundary is connected. Then $\partial U \cap I(f)^c$ has harmonic measure zero relative to U.*

It remains an intriguing open question whether $\partial U \cap I(f) \neq \emptyset$ whenever U is a Baker domain. Note that in [15] we showed that if U is any wandering domain in $I(f)$, then almost all points of ∂U, with respect to harmonic measure in U, are escaping.

We prove Theorem 1.1 using a new result on the boundary behaviour of conformal maps, which we state and prove in Section 3.

Using Theorem 1.1 together with [15, Theorem 5.1], we can give a new sufficient condition for $I(f)$ to be connected, and so satisfy Eremenko's conjecture in a particularly strong way.

Theorem 1.2. *Let f be a transcendental entire function and let E be a set such that $E \subset I(f)$ and $J(f) \subset \overline{E}$. Either $I(f)$ is connected or it has infinitely many components that meet E; in particular, if E is connected, then $I(f)$ is connected.*

Several subsets of $I(f)$ have been studied, involving different rates of escape, including: the *fast escaping set* $A(f)$ (see [6] and [14]), the *slow escaping set* $L(f)$ and *moderately slow escaping set* $M(f)$ (see [13]), the *quite fast escaping set* $Q(f)$ (see [16]), $Z(f)$ (see [12]) and $I'(f)$ (see [5]). Each of these sets contains at least three points and is backwards invariant, so its closure contains $J(f)$ by Montel's theorem. Thus we obtain the following corollary of Theorem 1.2.

Corollary 1.2. *Let f be a transcendental entire function. If one of the sets $A(f)$, $L(f)$, $M(f)$, $Q(f)$, $Z(f)$ or $I'(f)$ is connected, then I(f) is connected.*

The fast escaping set $A(f)$ has the property that all its components are unbounded [14, Theorem 1.1]. Therefore, if we apply Theorem 1.2 in the case when the set E is $A(f)$, then we obtain the following result, which seems to be the strongest general result so far on Eremenko's conjecture. This result can also be deduced directly from [15, Theorem 5.1] in a different way.

Theorem 1.3. *Let f be a transcendental entire function. Either $I(f)$ is connected or it has infinitely many unbounded components.*

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2. Background material

We require several fundamental results from complex analysis, all of which can be found in [10], which we state here for the reader's convenience. The first two results concern the boundary behaviour of a conformal map f of the unit disc $\mathbb{D} = \{z : |z| < 1\}$ into $\mathbb{C}$. Here cap denotes logarithmic capacity and Λ denotes linear measure, both of which are defined for Borel sets.

Theorem 2.1. *Suppose that $f : \mathbb{D} \to \mathbb{C}$ is a conformal map. Then for all $\zeta \in \partial\mathbb{D}$ apart from a set of capacity 0 the angular limit $f(\zeta)$ exists and is finite.*

Theorem 2.1 is a classical result of Beurling [10, Theorem 9.19]. Throughout the paper we use the notation $f(\zeta)$, where $\zeta \in \partial\mathbb{D}$, for the angular limit at ζ of the conformal map f, whenever this exists.

The second result on conformal maps [10, Theorem 9.24] is a quantitative version of the fact that those boundary points of $f(\mathbb{D})$ that can only be reached along relatively long paths in $f(\mathbb{D})$ form a small subset of $\partial f(\mathbb{D})$ in some sense.

Theorem 2.2. *Suppose that $f : \mathbb{D} \to \mathbb{C}$ is a conformal map, $V \subset f(\mathbb{D})$ is open, $E \subset \partial\mathbb{D}$ is a Borel set, and $\alpha \in (0, 1]$. If*

- $\operatorname{dist}(f(0), V) \geq \alpha |f'(0)|$,
- $\Lambda(f(C) \cap V) \geq \beta > 0$, *for all curves C in $\mathbb{D}$ that connect 0 to E,*

then

$$\Lambda(E) \leq 2\pi \operatorname{cap} E < \frac{15}{\sqrt{\alpha}} \exp\left(-\frac{\pi\beta^2}{\operatorname{area} V}\right).$$

We also need various basic results on logarithmic capacity, which can be found in [10, pages 204, 208 and 209].

Theorem 2.3. *Let E and E_n, $n \geq 1$, be Borel subsets of $\mathbb{C}$.*

(a) *If $E_1 \subset E_2$, then $\operatorname{cap} E_1 \leq \operatorname{cap} E_2$.*
(b) *If $\phi(z) = az + b$, then $\operatorname{cap} \phi(E) = |a| \operatorname{cap} E$.*
(c) *If ϕ is a Lipschitz map with constant $M > 0$, then $\operatorname{cap} \phi(E) \leq M \operatorname{cap} E$.*

(d) *If* $E = \bigcup_{n=1}^{\infty} E_n$ *and* $\operatorname{diam} E \le d$, *then*

$$\frac{1}{\log(d/\operatorname{cap} E)} \le \sum_{n=1}^{\infty} \frac{1}{\log(d/\operatorname{cap} E_n)}.$$

(e) *The union of countably many sets of capacity zero has capacity zero.*

(f) *If* ϕ *is a Möbius transformation and* E *has capacity zero, then* $\phi(E)$ *has capacity zero.*

Finally, we need Bagemihl's ambiguous point theorem; see [10, Corollary 2.20]. Let f be a complex-valued function defined in $\mathbb{D}$. For $\zeta \in \partial\mathbb{D}$ and a path $\gamma \subset \mathbb{D}$, we define the *cluster set* of f at ζ along γ as follows:

$$C_\gamma(f,\zeta) = \{w : \lim_{n\to\infty} f(z_n) = w, \text{ for some sequence } z_n \in \gamma \text{ such that } \lim_{n\to\infty} z_n = \zeta\}.$$

A point $\zeta \in \partial\mathbb{D}$ is said to be an *ambiguous point* of f if there exist two paths γ and γ' in $\mathbb{D}$ each ending at ζ such that $C_\gamma(f,\zeta) \cap C_{\gamma'}(f,\zeta) = \emptyset$.

Bagemihl's theorem is as follows — note that there are almost no hypotheses here about the function f; it need not be continuous even.

Theorem 2.4. *Let* f *be a complex-valued function with domain* $\mathbb{D}$. *Then* f *has at most countably many ambiguous points.*

In fact we shall use the obvious adaptation of Theorem 2.4 from $\mathbb{D}$ to the upper half-plane $\mathbb{H} = \{z : \Im z > 0\}$.

3. A result on conformal maps

To prove Theorem 1.1 we require two results on the boundary behaviour of a conformal map, each of which states, roughly speaking, that if the map behaves in a certain way near a boundary point, then its boundary values behave in a similar way nearby. In [10, page 220, Exercise 2] a result of this type is derived from Theorem 2.2, but here we give more precise results of this type, again using Theorem 2.2. For simplicity we state these results in the upper half-plane.

Theorem 3.1. *Let* $\phi : \mathbb{H} \to \mathbb{C}$ *be a conformal map, let* $w_0 \in \mathbb{C} \setminus \phi(\mathbb{H})$, *and let* $\lambda > 1$ *and* $\varepsilon > 0$. *Also, for* $n \ge 0$, *put*

$$I_n = [\lambda^{n-1/2}, \lambda^{n+1/2}] \quad \text{and} \quad E_n = \{t \in I_n : |\phi(t) - w_0| \ge \varepsilon\},$$

$$J_n = [n - \tfrac{1}{2}, n + \tfrac{1}{2}] \quad \text{and} \quad F_n = \{t \in J_n : |\phi(t) - w_0| \ge \varepsilon\}.$$

(a) *If* $\phi(\lambda^n i) \to w_0$ *as* $n \to \infty$, *then*

$$\sum_{n=0}^{\infty} \frac{1}{\log(\lambda^n/\operatorname{cap} E_n)} < \infty. \tag{3.1}$$

(b) *If* $\phi(n+i) \to w_0$ *as* $n \to \infty$, *then*

$$\sum_{n=0}^{\infty} \frac{1}{\log(1/\operatorname{cap} F_n)} < \infty. \tag{3.2}$$

Proof. We first prove part (a). For $n \ge 0$, define

$$z_n = \lambda^n i \quad \text{and} \quad S_n = \{z \in \mathbb{H} : \lambda^{n-1} < |z| < \lambda^{n+1}\}, \tag{3.3}$$

so

$$I_n = [\lambda^{n-1/2}, \lambda^{n+1/2}] \subset \partial S_n \cap \partial \mathbb{H}.$$

Then, for $n \ge 0$, let ψ_n denote a conformal map of $\mathbb{D}$ onto the semi-annulus S_n such that $\psi_n(0) = z_n$. We can choose these maps so that $\psi_n = \lambda^n \psi_0$ for $n \ge 0$. Then, by Carathéodory's theorem [10, page 24], each ψ_n extends to a continuous one-one map between the boundaries of $\partial\mathbb{D}$ and ∂S_n. By the reflection principle, this extension of ψ_n is analytic with finite non-zero derivative on $\partial\mathbb{D}$, except at the four preimages of the vertices of ∂S_n.

Now let $A(\varepsilon) = \{w : \frac{1}{2}\varepsilon < |w - w_0| < \varepsilon\}$ and, for $n \ge 0$, define

- $V_n = \phi(S_n) \cap A(\varepsilon)$,
- $E'_n = \{t \in E_n : t \text{ is not an ambiguous point of } \phi\}$.

Then let the integer N be chosen so large that

$$|\phi(z_n) - w_0| = |\phi(\lambda^n i) - w_0| \le \tfrac{1}{10}\varepsilon, \quad \text{for } n \ge N. \tag{3.4}$$

Note that if $n \ge N$ and $E_n \ne \emptyset$, then V_n is non-empty.

We shall apply Theorem 2.2 to the conformal map $\Psi_n = \phi \circ \psi_n$, the non-empty open subset V_n of $\Psi_n(\mathbb{D})$, and the Borel subset $\psi_n^{-1}(E'_n)$ of $\partial\mathbb{D}$, where $n \ge N$. We first show that

$$\Lambda(\Psi_n(C) \cap V_n) \ge \tfrac{1}{2}\varepsilon, \text{ for all curves } C \text{ in } \mathbb{D} \text{ that connect } 0 \text{ to } \psi_n^{-1}(E'_n). \tag{3.5}$$

Indeed,

$$|\Psi_n(0) - w_0| = |\phi(z_n) - w_0| \le \tfrac{1}{10}\varepsilon,$$

by (3.4), and if a curve C in $\mathbb{D}$ connects 0 to a point $\zeta \in \psi_n^{-1}(E'_n)$, then since Ψ_n has an angular limit at ζ such that $|\Psi_n(\zeta) - w_0| \ge \varepsilon$ and ζ is not an ambiguous point of Ψ_n, we have

$$\limsup_{s \to \zeta,\, s \in C} |\Psi_n(s) - w_0| \ge \varepsilon.$$

Hence $\Psi_n(C)$ must cross the annulus $A(\varepsilon)$, passing through V_n, so (3.5) holds.

Next, for $n \ge N$, by the definition of V_n and (3.4),

$$\operatorname{dist}(\Psi_n(0), V_n) = \operatorname{dist}(\phi(z_n), V_n) \ge \tfrac{1}{2}\varepsilon - \tfrac{1}{10}\varepsilon = \tfrac{2}{5}\varepsilon.$$

Also, for $n \ge N$, we deduce by Koebe's theorem [10, Corollary 1.4] and (3.4) that

$$\begin{aligned} |\Psi'_n(0)| &\le 4 \operatorname{dist}(\Psi_n(0), \partial\Psi_n(\mathbb{D})) \\ &= 4 \operatorname{dist}(\phi(z_n), \partial\phi(S_n)) \\ &\le 4|\phi(z_n) - w_0| \quad (\text{since } w_0 \notin \phi(S_n)) \\ &\le \tfrac{2}{5}\varepsilon. \end{aligned}$$

Therefore, for $n \ge N$ we can indeed apply Theorem 2.2 to Ψ_n, V_n and $\psi_n^{-1}(E'_n)$, with $\alpha = 1$ and $\beta = \frac{1}{2}\varepsilon$ to give

$$2\pi \operatorname{cap} \psi_n^{-1}(E'_n) \le 15 \exp\left(-\frac{\frac{1}{4}\pi\varepsilon^2}{\operatorname{area} V_n}\right). \tag{3.6}$$

As noted earlier, each ψ_n extends analytically to most points of $\partial\mathbb{D}$ and in particular to the interior of the arc $\alpha_n = \partial\mathbb{D} \cap \psi_n^{-1}([\lambda^{n-1}, \lambda^{n+1}])$ with non-zero derivative there. Also, $\psi_n = \lambda^n \psi_0$, for $n \geq 0$. Thus, by Theorem 2.3, part (c), and (3.6), there exists an absolute constant $C > 0$ such that, for $n \geq N$,

$$\begin{aligned} \operatorname{cap} E'_n &\leq \max\{|\psi'_n(\zeta)| : \zeta \in \alpha_n\} \operatorname{cap} \psi_n^{-1}(E'_n) \\ &\leq C\lambda^n \exp\left(-\frac{\frac{1}{4}\pi\varepsilon^2}{\text{area } V_n}\right). \end{aligned} \tag{3.7}$$

Now E_n differs from E'_n by at most a countable set, by Theorem 2.4, so

$$\frac{1}{\log(C\lambda^n/\operatorname{cap} E_n)} \leq \frac{\text{area } V_n}{\frac{1}{4}\pi\varepsilon^2}. \tag{3.8}$$

Since V_n, $n \geq 0$, are disjoint subsets of $A(\varepsilon)$ we have $\sum_{n\geq 0} \operatorname{area} V_n \leq \operatorname{area} A(\varepsilon)$, so (3.1) follows immediately.

The proof of part (b) proceeds in a similar way, on replacing (3.3) by

$$z_n = n + i \quad \text{and} \quad S_n = \{z : |\Re z - n| < 1, 0 < \Im z < 2\}, \quad n \geq 0,$$

and using the fact that $\phi(z_n) \to w_0$ as $n \to \infty$. □

Remark Note that it follows from (3.8) that the conclusion (3.1) can be slightly strengthened to

$$\sum_{n=0}^{\infty} \frac{1}{\log(C\lambda^n/\operatorname{cap} E_n)} \leq 3,$$

where C is a certain positive absolute constant, and similarly (3.2) can be strengthened to

$$\sum_{n=0}^{\infty} \frac{1}{\log(C/\operatorname{cap} F_n)} \leq 3.$$

4. Proof of Theorem 1.1

Without loss of generality we can assume that U is an invariant univalent Baker domain of f. Then U is simply connected, so there is a Riemann map ϕ from the upper half-plane $\mathbb{H}$ to U. Thus

$$g = \phi^{-1} \circ f \circ \phi$$

is a conformal map of $\mathbb{H}$ onto $\mathbb{H}$ and hence a Möbius map with no fixed points in $\mathbb{H}$. Therefore (see [3, page 4], for example) we can choose ϕ in such a way that g is one of the following types:

$$g(w) = \begin{cases} \lambda w, & \text{where } \lambda > 1, \\ w + 1. \end{cases}$$

In the first case the Baker domain is said to be *hyperbolic* and in the second case it is said to be *parabolic*; see [2]. In either case we have $g^n(w) \to \infty$ as $n \to \infty$ whenever $w \in \mathbb{H}$.

The idea of the proof is to show that for all $t \in \partial\mathbb{H}$, apart from a set of capacity zero, we have

$$\phi \text{ has a finite angular limit at } g^n(t) \text{ for all } n \geq 0, \tag{4.1}$$

and

$$\phi(g^n(t)) \to \infty \text{ as } n \to \infty. \tag{4.2}$$

The fact that (4.1) holds follows immediately from Theorem 2.1 and Theorem 2.3, part (f). Once (4.2) has been established we can deduce that for all $t \in \partial\mathbb{H}$, apart from a set of capacity zero, we have

$$\begin{aligned}\phi(g^n(t)) &= \lim_{g^n(w)\to g^n(t)} \phi(g^n(w)) \\ &= \lim_{w\to t} f^n(\phi(w)) \\ &= f^n(\phi(t)),\end{aligned}$$

where the two limits are angular limits at boundary points of $\mathbb{H}$, and hence $\phi(t) \in I(f)$. By Theorem 2.3, part (f), this is sufficient to prove the second statement of Theorem 1.1.

Note that if the boundary of U is a Jordan curve through ∞ (see [2] for examples of this), then both (4.1) and (4.2) hold for all $t \in \partial\mathbb{H}$ with at most one exception, by Carathéodory's theorem, so all points of ∂U are escaping with at most one exception.

We consider first the case when $g(w) = \lambda w$. Let $w_0 = i$ and $z_0 = \phi(w_0) \in U$. Then, for $n \geq 0$, define

$$w_n = g^n(w_0) = \lambda^n i.$$

In order to be able to apply Theorem 3.1, we let $\Phi(w) = 1/(\phi(w) - c)$, where $c \notin U$. Then Φ is a conformal map of $\mathbb{H}$ into $\mathbb{C} \setminus \{0\}$. Since U is a Baker domain,

$$\phi(w_n) = f^n(z_0) \to \infty \ \text{ as } n \to \infty, \quad \text{so} \quad \Phi(w_n) \to 0 \ \text{ as } n \to \infty. \tag{4.3}$$

Thus we can apply Theorem 3.1, part (a), with $w_0 = 0$,

$$I_n = [\lambda^{n-1/2}, \lambda^{n+1/2}] \quad \text{and} \quad E_n = \{t \in I_n : |\Phi(t)| \geq \varepsilon\}, \quad \text{for } n \geq 0,$$

where $\varepsilon > 0$ is arbitrary, to deduce that

$$\sum_{n=0}^{\infty} \frac{1}{\log(\lambda^n/\operatorname{cap} E_n)} < \infty. \tag{4.4}$$

For $n \geq 0$, let $\tilde{E}_n = E_n/\lambda^n \subset I_0$, so $\operatorname{cap} \tilde{E}_n = \operatorname{cap} E_n/\lambda^n$, by Theorem 2.3, part (b). We deduce that if

$$K_m = \bigcup_{n\geq m} \tilde{E}_n, \quad \text{for } m \geq N,$$

then, by Theorem 2.3, part (d), and (4.4), together with the fact that the interval I_0 has length less than $\sqrt{\lambda}$,

$$\begin{aligned}\frac{1}{\log(\sqrt{\lambda}/\operatorname{cap} K_m)} &\leq \sum_{n\geq m} \frac{1}{\log(\sqrt{\lambda}/\operatorname{cap} \tilde{E}_n)} \\ &= \sum_{n\geq m} \frac{1}{\log(\lambda^{n+1/2}/\operatorname{cap} E_n)} < \infty.\end{aligned}$$

It follows that $\operatorname{cap} K_m \to 0$ as $m \to \infty$, so the set

$$K = \bigcap_{m\geq N} K_m = \bigcap_{m\geq N} \bigcup_{n\geq m} \tilde{E}_n \ \text{ has capacity } 0. \tag{4.5}$$

Now K consists of those points $t \in I_0$ such that for infinitely many n the angular limit of Φ at $g^n(t) = \lambda^n t$ exists and lies in $\{z : |z| \geq \varepsilon\}$. Since the set of points where Φ has no angular limit is of capacity 0, by Theorem 2.1, it follows that, for all t in I_0 apart from a set of capacity zero, we have

$$|\Phi(g^n(t))| < \varepsilon, \text{ for all sufficiently large } n.$$

Since $\varepsilon > 0$ is arbitrary and a countable union of sets of capacity zero has capacity zero, we deduce that, for all t in I_0 apart from a set of capacity zero, we have

$$\Phi(g^n(t)) \to 0 \text{ as } n \to \infty$$

and hence

$$\phi(g^n(t)) \to \infty \text{ as } n \to \infty.$$

Since $g(w) = \lambda w$, it follows readily that this property holds for all $t \in \partial\mathbb{H}$ apart from an exceptional set of capacity zero, which proves (4.2).

The proof of (4.2) in the case $g(w) = w + 1$ proceeds in a similar manner with $w_0 = i$, $z_0 = \phi(w_0) \in U$ and

$$w_n = g^n(w_0) = n + i, \quad \text{so} \quad \phi(w_n) = f^n(z_0) \to \infty \text{ as } n \to \infty,$$

by using Theorem 3.1, part (b). We omit the details.

5. Proof of Theorem 1.2

Suppose that f is a transcendental entire function, that E is a subset of $I(f)$ such that $J(f) \subset \overline{E}$, and that E is contained in the union of finitely many components of $I(f)$. Under these hypotheses we proved in [15, Theorem 5.1] that

(a) $I(f) \cap J(f)$ is contained in one component, I_1 say, of $I(f)$;
(b) all the components of $I(f)$ are unbounded, and they consist of
  (i) I_1, which also contains any escaping wandering domains and any Baker domains of f with at least one boundary point in $I(f)$,
  (ii) any Baker domains of f with no boundary points in $I(f)$ and the infinitely many preimage components of such Baker domains.

We shall show that under these hypotheses Baker domains whose boundaries do not meet $I(f)$ cannot occur, so the result from [15] quoted above implies that $I(f)$ has just one component. Therefore, if $I(f)$ is disconnected, then E meets infinitely many components of $I(f)$, as required.

Suppose then that U is a Baker domain of f whose boundary does not meet $I(f)$. By Corollary 1.1, the boundary of U is disconnected. Then U has more than one complementary component, each of which is closed and unbounded, and meets $J(f)$.

We now note that $J(f) = \overline{I(f) \cap J(f)}$, which follows by the blowing up property of $J(f)$ and the fact that $I(f) \cap J(f) \neq \emptyset$; see [7]. All the points of $I(f) \cap J(f)$ lie in complementary components of U, and $I(f) \cap J(f)$ cannot be contained in a single complementary component of U because $J(f) = \overline{I(f) \cap J(f)}$. Hence the component I_1 of $I(f)$ in part (a) meets at least two complementary components of U and so it must meet the boundaries of these two complementary components, which are subsets of ∂U. This contradicts the fact that $\partial U \cap I(f) = \emptyset$. Hence such a Baker domain cannot exist in this case. This completes the proof of Theorem 1.2.

References

[1] I. N. Baker and P. Domínguez, Boundaries of unbounded Fatou components of entire functions, *Ann. Acad. Sci. Fenn. Math.*, 24 (1999), 437–464.

[2] K. Barański and N. Fagella, Univalent Baker domains, *Nonlinearity*, 14 (2001), 411-429.

[3] A. F. Beardon, *Iteration of rational functions*, Springer, 1991.

[4] W. Bergweiler, Iteration of meromorphic functions, *Bull. Amer. Math. Soc.*, 29 (1993), 151–188.

[5] W. Bergweiler, On the Julia set of analytic self-maps of the punctured plane, *Analysis*, 15 (1995), 251–256.

[6] W. Bergweiler and A. Hinkkanen, On semiconjugation of entire functions, *Math. Proc. Cambridge Philos. Soc.*, 126 (1999), 565–574.

[7] A. E. Eremenko, On the iteration of entire functions, *Dynamical systems and ergodic theory*, Banach Center Publications, 23, Polish Scientific Publishers, Warsaw, 339–345, 1989.

[8] P. Fatou, Sur l'itération des fonctions transcendantes entières, *Acta Math.*, 47 (1926), 337–360.

[9] J. B. Garnett and D. E. Marshall, *Harmonic measure*, Cambridge University Press, 2005.

[10] Ch. Pommerenke, *Boundary behaviour of conformal maps*, Springer, 1992.

[11] P. J. Rippon, Baker domains, *Transcendental dynamics and complex analysis*, Cambridge University Press, 371–395, 2008.

[12] P. J. Rippon and G. M. Stallard, On sets where iterates of a meromorphic function zip towards infinity, *Bull. London Math. Soc.*, 32 (2000), 528–536.

[13] P. J. Rippon and G. M. Stallard, Slow escaping points of meromorphic functions, *Trans. Amer. Math. Soc.*, 363 (2011), 4171–4201.

[14] P. J. Rippon and G. M. Stallard, Fast escaping points of entire functions, *Proc. London Math. Soc.*, 105 (2012), 787-820.

[15] P. J. Rippon and G. M. Stallard, Boundaries of escaping Fatou components, *Proc. Amer. Math. Soc.*, 139 (2011), 2807–2820.

[16] P. J. Rippon and G. M. Stallard, Regularity and fast escaping points of entire functions, *Int. Math. Res. Not.*, doi:10.1093/imrn/rnt111.

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