The origins of Euler's early work on continued fractions

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Abstract

In this paper, I examine Euler's early work on the elementary properties of continued fractions in the 1730s, and investigate its possible links to previous writings on continued fractions by authors such as William Brouncker. By analysing the content of Euler's first paper on continued fractions, 'De fractionibus continuis dissertatio' (1737, published 1744) I conclude that, contrary to what one might expect, Euler's work on continued fractions initially arose not from earlier writings on continued fractions, but from a wish to solve the Riccati differential equation.

Keywords: Euler, continued fractions, Riccati equation, 18th-century, Bernoulli, mathematics
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1. Introduction

With the exception of a few isolated results which appeared in the sixteenth and seventeenth centuries, most of the elementary theory of continued fractions was developed in a single paper written in 1737 by Leonhard Euler. In this paper, 'De fractionibus continuis dissertatio' ('Essay on continued fractions',[2] E71 in Gustav Eneström's 1913 index of Euler's works), Euler

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[2]The paper, as with almost all of Euler's published works, is available from http://www.eulerarchive.org/. There also exists a complete English translation of E71: see Wyman and Wyman [1985]. However, the translation given there does contain a few

presented continued fractions as an alternative to infinite series or products for representing irrational and transcendental quantities.[3] He established most of the basic properties of continued fractions, briefly examined certain special cases, and then used the properties of continued fractions to arrive at the result for which the paper is arguably best known: namely, the first known proof that the regular continued fraction expansion of e continues infinitely. It follows from this fact (though Euler does not explicitly say so) that e is irrational.[4]

It is not immediately clear why Euler wrote E71, and existing commentaries on the paper have not fully explored this puzzle [Sandifer, 2007a,b; Baltus, 2007]. It is known that Euler was well-acquainted with John Wallis's 1656 book *Arithmetica infinitorum* [Knobloch, 1989, 279; Calinger, 1996, 124]. The *Arithmetica* contains some of the earliest results in the theory of continued fractions, obtained by William Brouncker and published by Wallis as part of their efforts to find exact expressions for $4/\pi$. At first glance it might, therefore, be tempting to look to the *Arithmetica* as the source of Euler's interest in the subject of continued fractions. However, though Euler clearly had the *Arithmetica infinitorum* in mind when writing E71, the focus of his own paper is completely different. Apart from a brief mention of Wallis and Brouncker at the beginning of the paper, along with a note that Brouncker's derivation of his continued fraction representation of $4/\pi$ remained lost, there are seemingly no connections between the *Arithmetica infinitorum* and E71. Euler did eventually make several attempts to recover Brouncker's missing proof, but the first of these did not occur until 1739, in 'De fractionibus continuis observationes' ('Observations on continued fractions', E123), Euler's second paper on continued fractions.

For this reason, we must look elsewhere for the stimulus or stimuli that drove Euler to write E71. In this paper, I will propose that this stimulus

misprints and so all translations from Latin or French in the present paper are my own.

[3]The notion of an 'irrational' quantity as used by Euler is equivalent to the modern notion: an irrational quantity is one which cannot be written as a quotient of two whole numbers [Euler, 1771, 54–55]. However, his notion of 'transcendental' quantities is somewhat different to the modern sense of the word: see [Petrie, 2012].

[4]As will be discussed below, the denominators of the regular continued fraction expansion of e form an interpolated arithmetic progression. This was known to Roger Cotes in 1714; however, he only observed that the progression seemed to continue indefinitely, and did not prove that it actually did so [Cotes, 1714, 11]. For more details, see Fowler [1999, Chapter 9].

came not from the work of Wallis and Brouncker, but from some early work of Daniel Bernoulli on ordinary differential equations, via some of the letters Euler exchanged with Christian Goldbach in the early 1730s. To do this, I will first outline some seventeenth century results in the basic theory of continued fractions, including work by Rafael Bombelli, Pietro Antonio Cataldi, Daniel Schwenter, John Pell, John Wallis, and William Brouncker. Then, I will briefly explain the content of E71, showing that it has little in common with any of these earlier works. Finally, I will discuss some of the contents of Daniel Bernoulli's 1724 book *Exercitationes quaedam mathematicae*, and a letter which Euler wrote to Christian Goldbach on 25 November 1731, and argue that in fact it was these writings which prompted the main result of E71.

2. Earlier works on continued fractions

2.1. Bombelli's L'algebra *(1572) and Cataldi's* Trattato del modo brevissimo *(1613)*

One of the earliest instances of a continued fraction-related method in western mathematics is in the work of Rafael Bombelli. His *L'algebra parte maggiore dell'aritmetica* was published in Bologna in 1572; a second edition followed in 1579. The text is best known now for its treatment of complex numbers. However, it also includes an approximation method for square roots which produces what we now interpret as a continued fraction.[5] It consists of finding an integer approximation to the square root, and then finding values which alternately over- and under-approximate the non-integer part, which when written out in full form a continued fraction. Though Bombelli did not explicitly write out the continued fraction (and, indeed, he used no symbolic notation at all) it is clear that his method does produce convergents of a periodic continued fraction, and that he was aware that these could be used to approximate the root as closely as was desired.

This method later appeared in Pietro Antonio Cataldi's *Trattato del modo brevissimo di trovare la radice quadra delli numeri*, published in 1613, also in Bologna.[6] It is not known whether Cataldi had read Bombelli's work,

[5] Bombelli's exposition of this method is reproduced in [Smith, 1959, 80–82].

[6] Cataldi's continued fraction method from the *Trattato del modo brevissimo* has been reproduced in English translation in [Smith, 1959, 82–84]; there is a partial facsimile of the relevant page of the *Trattato* in [Fowler, 1994, 734–735].

but one important way in which the two differ is that Cataldi's work contained the first hint of modern continued fraction notation. Having found an approximation to $\sqrt{18}$ as a continued fraction (which he called *rotti di rotti*, fractions of fractions), Cataldi exhibited it using the natural 'diagonal' notation. However, as this was (and still is) difficult to typeset, from that point onward on he wrote

$$4.\&\frac{2}{8}.\&\frac{2}{8}.\&\frac{2}{8}.$$

This is strikingly similar to the shorthand

$$4+\frac{2}{8+}\,\frac{2}{8+}\,\frac{2}{8+}$$

which is still used in many modern texts. However, since his aim here was to approximate square roots, Cataldi was more interested in finding the convergents of the expansion than in the continued fraction itself, and so he did not dwell on the new mathematical object he had created.

2.2. Brouncker's continued fraction for $4/\pi$, 1655

Continued fractions first became an object of study in their own right in work which was completed in 1655 by Viscount William Brouncker and published by his friend John Wallis in his *Arithmetica infinitorum* [1656]. Wallis had set out to find an exact expression for the ratio of the area of a square to that of its inscribed circle. After a long process of interpolating between the rows and columns of a table of figurate numbers, he eventually arrived [Wallis, 1656, 179] at the infinite product

$$\frac{4}{\pi}=\frac{1\times 3\times 3\times 5\times 5\times 7\times\cdots}{2\times 2\times 4\times 4\times 6\times 6\times\cdots}$$

and when Brouncker saw this, he somehow converted it to the form

$$\frac{4}{\pi}=1+\cfrac{1}{2+\cfrac{9}{2+\cfrac{25}{2+\cfrac{49}{2+\ddots}}}}.$$

Unfortunately, Brouncker did not give the method by which he had obtained this result. Wallis was unable to reconstruct it, although his efforts to do so can be found in a Scholium following Proposition 191 of the *Arithmetica infinitorum*.[7] The discussion there concluded with a method for approximating $4/\pi$ using the convergents of the continued fraction. Notably, this discussion contained the first example of a general algebraic continued fraction, and showed that Brouncker and Wallis were aware of several of its basic properties, including the recursion method for finding successive convergents.

2.3. Approximation methods: Schwenter, Pell, Huygens and Wallis

A best rational approximation to a real number x is a rational number p/q, with $q > 0$, such that $|x - p/q| < |x - a/b|$ for any rational number a/b such that $0 < b < q$. It is well known that one may find the best possible rational approximations to x by applying a method derived from the Euclidean algorithm, which produces continued fractions [Fowler, 1999, 304–310]. This method was known to several mathematicians in the seventeenth century, and in E71 Euler attributed it to John Wallis. Such a method can indeed be found in Chapters 10 and 11 of Wallis's *Treatise of algebra* of 1685, which Euler probably knew in the Latin edition of 1693; here, Wallis showed how to find approximations to a rational quantity, and then applied the same method to approximate π using rational numbers.[8] However, he was by no means the first to develop such a method.

Stedall [2002, 143–156] argues that it is highly likely that Wallis learned of this method from John Pell, who used it to approximate π as early as 1636, in an unpublished manuscript whose treatment of the problem bears striking similarities to Wallis's.[9] Furthermore, a very similar method had been used almost twenty years earlier by Daniel Schwenter, a professor of mathematics and Oriental languages at the University of Altdorf. In his *Geometria practica nova, Tractatus II*, a text on surveying published in Nuremberg in 1617, Schwenter found rational approximations in small terms to 177/233, using the same method that was later taken up by Pell and Wallis [Schwenter, 1617, II, 69].[10] It is not known whether Pell was aware of Schwenter's

[7] Brouncker's method certainly proved elusive; see [Euler, 1750] for Euler's early attempts to recover it, and [Stedall, 2000] for a modern treatment of the problem using 17th-century methods.

[8] For a detailed account of Wallis's exposition of the method, see [Fowler, 1991].

[9] British Library Add. Ms. 4416, f. 31.

[10] The first edition of *Geometria practica nova* was published in three parts: Tractatus II is dated 1617, and Tractatus I and III followed in 1618. Later editions also include a

work. A collection of books which belonged to Pell has survived and is now held at the Busby Library of Westminster School, London, and among these there is indeed a copy of the 1623 edition of Schwenter's Tractatus II.[11] However, there is no evidence that Pell read the book or recognised Schwenter's approximation method as his own: unlike most of Pell's books, the copy is entirely free of annotations.[12]

What appears to have gone unnoticed by most of the authors who took up this method is that the method naturally gives rise to a continued fraction, and, moreover, none but Wallis realised that the truncations of that continued fraction are the best possible rational approximations to the value of the original quantity without using larger terms. As an example of this, we follow Pell in finding rational approximations to π by taking $x = 3.141592653589793$. This gives

$$x = 3 + \cfrac{1}{7 + \cfrac{1}{15 + \cfrac{1}{1 + \cfrac{1}{291 + \ddots}}}}$$

and, if we take the first and third convergents, we obtain the approximations 22/7 and 355/113, which are the approximations to π found by Archimedes (c.250BCE) and Metius (1585) respectively. Neither can be improved upon without increasing the denominator.

The first to explicitly relate this method to continued fractions was Christiaan Huygens when, in 1680, he used it to calculate gear ratios for a planetarium [Huygens, 1888, XXI, 587–652]. Indeed, when Huygens read Wallis's *Algebra* a few years later, he recognised the method given there as a more clumsy version of his own [Huygens, 1888, XX, 392–394].

We might wonder why Huygens was not driven to make any study of continued fractions, as he certainly did spot those that arose from his own approximations. I suspect that this was for the same reason that Cataldi neglected to study the properties of his continued fraction for $\sqrt{18}$. Huy-

fourth treatise, which is a German translation of Curzio Casati's 'Geometricum problema' (1602).

[11]Busby Library, Westminster School: pressmark I B 31.

[12]For a detailed discussion of Pell's mathematics, see Malcolm and Stedall [2005, Part II].

gens was interested in the practical problem of best rational approximations. Like Cataldi, his aim was to find the truncated fractions, and the continued fractions themselves were of little or no interest, since they were merely a byproduct of the solution he sought.

3. 'De fractionibus continuis dissertatio', 1737: Commentary

'De fractionibus continuis dissertatio' was first presented to the St Petersburg Academy of Sciences on 7 March 1737. Euler read the first half of it to the Academy on 1 April, before submitting it for publication in the journal *Commentarii academiae scientiarum imperialis petropolitanae*, where it filled 40 pages of the volume for 1737 (which was not published until 1744). Euler's motivation for writing the paper is at first unclear; here, I will analyse it with a view to uncovering why he might have come to approach the topic of continued fractions.

Euler opened the paper by stating that there are various ways by which irrational and transcendental quantities could be represented. He began by noting the two most commonly used types of representation by infinite series: those in which the terms were related by addition and subtraction, and those in which the terms were related by multiplication. When searching for such series, he continued, it was preferable that they should converge quickly and require as few terms as possible to yield a good approximation to the quantity. Then, he suggested a third kind of series for representing irrational and transcendental quantities, in which the terms would be related by *divisio continua*: continuing division. He called such series *fractiones continuae*, perhaps echoing the use of the similar term *fractiones continuae fractae* (continually broken fractions) by John Wallis in the *Arithmetica infinitorum* [Wallis, 1656, 182]. Though *fractiones continuae* (continued fractions) were less often used than the other two types of series, Euler noted that they were very well suited for finding approximations, and lamented the fact that apart from a few special cases, no theory or methods had been established for dealing with them. This, he said, was what he intended to achieve in this paper.

This introduction is the natural place to seek Euler's reasons for writing the paper: in particular, it is tempting to assume that the suitability of continued fractions for finding approximations was the main reason. However, this does not explain why Euler came to consider them in the first place. In Wallis's exposition of the approximation algorithm, the continued fractions produced are not made explicit, and there is no evidence that Euler read any of the works in which continued fractions were made explicit. Thus, I do not believe that the use of continued fractions for producing approximations was the *a priori* motivation for the paper; rather, it was a useful byproduct found in the course of investigating a different problem.

After these introductory remarks, Euler began his investigation by writing out a general continued fraction:

$$a + \cfrac{\alpha}{b + \cfrac{\beta}{c + \cfrac{\gamma}{d + \cfrac{\delta}{e + \cfrac{\varepsilon}{f + \text{etc.}}}}}}$$

where $a, b, c, \ldots, \alpha, \beta, \gamma, \ldots$ are all non-negative.[13] Here, we begin to see the influence of Wallis's *Arithmetica infinitorum*: Euler's use of Roman letters for denominators and Greek letters for numerators is identical to the notation used for the general continued fraction presented there. Indeed, Euler immediately went on to say that to the best of his knowledge, Brouncker's work in Wallis's *Arithmetica* was the first presentation of a continued fraction in this way. It is unclear whether Euler meant he knew of no earlier continued fraction at all than Brouncker's, or simply of no earlier *general* continued fraction.

After a brief summary of Wallis's attempt to reconstruct Brouncker's proof, Euler moved on to the question of how to approximate the quantity represented by a continued fraction. Continuing to assume that all of the numerators and denominators of the general continued fraction were positive, he pointed out that an approximate value for the continued fraction could be obtained by truncation, and that the successive truncations formed a sequence which alternated between over- and under-estimates and approached the true value of the continued fraction as closely as desired.

Next, Euler calculated the values of the first few convergents as simple fractions, and noted the recursion formula for calculating their numerators

[13] Recall that a regular continued fraction has all of its numerators, here represented by Greek letters, equal to 1.

and denominators, as first stated in Wallis's *Arithmetica infinitorum*. By taking differences between successive convergents, he converted the continued fraction into an infinite series, stating that this converged quickly and was of great use for approximating the value of the continued fraction. Indeed, by studying the series representation of the continued fraction, Euler noted that the general continued fraction given above converges fastest when its numerators $\alpha, \beta, \gamma, \ldots$ are small and its denominators $a, b, c, \ldots$ are large. Since he had already found a way to replace fractional numerators or denominators with integers,[14] he was able to assume without loss of generality that all numerators and denominators are non-negative integers. This allowed him to deduce that the fastest rate of convergence was to be obtained by setting all of the numerators equal to 1 and making the denominators as large as possible. After ensuring that the denominators could be required to be whole numbers while the numerators were equal to 1, Euler noted that doing this created an easy way to distinguish between rational and irrational numbers:

> Moreover, every finite fraction whose numerator and denominator are finite whole numbers is transformed in this way into a continued fraction which is somewhere broken off; on the other hand, any fraction whose numerator and denominator are infinitely large numbers, such as those given for irrational and transcendental numbers, will go over to a truly continued fraction that extends to infinity.[15]

In other words, a quantity is rational if and only if its continued fraction has finite length. As we shall see, this result became crucial later on in the paper.

Euler then gave a brief exposition of the now standard algorithm for converting a simple fraction into a continued fraction, noting that it arose as a corollary of the Euclidean algorithm for finding the greatest common divisor of the numerator and denominator. For converting irrational quantities

[14]See [Euler, 1744b, 107].

[15]'Omnis autem fractio finita, cuius numerator et denominator sunt numeri integri finiti in huiusmodi fractionem continuam transformatur, quae alicubi abrumpitur; fractio autem cuius numerator et denominator sunt numeri infinite magni, cuiusmodi dantur pro quantitatibus irrationalibus et transcendentibus, in fractionem vere continuam et in infinitum excurrentem transibit.' [Euler, 1744b, 108]

into continued fractions, he recommended approximating them by rational numbers: for example, by taking a finite decimal fraction. Then, he combined this algorithm with his earlier observation that truncating a continued fraction gives an approximation to its true value, and pointed out that this provided a convenient solution to a problem which had been worked on, and solved, albeit in a laborious way, by John Wallis.[16] He illustrated this result with two examples: first, he applied it to the quantity 355/113, and second, he applied it to the problem of determining how often leap years should occur in order to make the calendar stay aligned with the movements of the planets.

Having determined that 97 years out of every 400 should be declared leap years (as is indeed the case in the Gregorian calendar), Euler changed the subject slightly. He calculated a continued fraction expansion of $\sqrt{2}$, and showed that all of the denominators except the first had the value 2. Upon further investigation, it became apparent to him that similar patterns could be found for other integer square roots, and so he began an investigation of (in modern terminology) periodic continued fractions. He gave a simple method for calculating the exact value of any periodic continued fraction, and proved that every such continued fraction is the root of a quadratic equation.

Euler then took the natural step of looking at continued fractions whose denominators displayed other types of patterns. He opened this investigation

[16] As mentioned above, Wallis's treatment of the problem can be found in Chapters X and XI of his *De algebra tractatus* of 1693, in which the problem is stated as follows: 'Given any Fraction (or Ratio), to exhibit a number as closely as possible equal to it, in numbers no greater than the given one, and in minimal terms.' ('Data Fractione (seu Ratione) quavis, ei quam potest proxime aequalem exhibere, in numeris dato non majoribus, & in minimis terminis.' [Wallis, 1693, 40])

by observing that the quantity e has a continued fraction expansion beginning

$$e = 2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{4 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{6 + \cfrac{1}{1 + \text{etc.}}}}}}}}}}.$$

He noted that the denominators in every third place formed an arithmetic progression which appears to continue infinitely; the infinitude of this progression, he promised, would be proved later on in the paper. For the time being, by investigating other powers of e, he found that their continued fraction representations exhibited similar patterns. With the possible exception of the first few denominators of each continued fraction, their denominators formed arithmetic progressions; for some fractions these progressions were 'interrupted' by constant quantities (as in the expansion of e given above), and for others they were uninterrupted. Through some straightforward algebraic manipulations, Euler found that he could transform interrupted progressions into uninterrupted progressions, and vice versa.

Here we come to what is now considered the most important result of the paper: Euler's derivation of the continued fraction expansions of e and its powers, from which we now deduce the irrationality of those quantities. He did this by starting with an ordinary differential equation in x and y, and reducing it to a separable equation in p and q via a substitution which, he claimed, takes the form of a continued fraction. He used this substitution to write the solution of the separable differential equation as a continued fraction in terms of p, x, and y. When this fraction is allowed to continue infinitely, the terms in x and y vanish, and so he obtained a continued fraction for

$$\frac{e^{\frac{2p}{a}} + 1}{e^{\frac{2p}{a}} - 1},$$

which, after a little rearrangement, could be rearranged to give a continued fraction for $e^{2p/a}$. Euler proceeded as follows.

Acknowledging the 'peculiar' nature of the proof, he opened by stating that it hinged on the reduction of the equation

$$a\,dy + y^2\,dx = x^{\frac{-4n}{2n+1}}\,dx$$

to the separable equation

$$a\,dq + q^2\,dp = dp.$$

Euler claimed that this could be done by setting

$$p = (2n+1)x^{\frac{1}{2n+1}}$$

and then, he said, he had found that taking

$$q = \tfrac{a}{p} + \cfrac{1}{\tfrac{3a}{p} + \cfrac{1}{\tfrac{5a}{p} + \cfrac{1}{\tfrac{7a}{p} + \cfrac{1}{\ddots + \cfrac{1}{\tfrac{(2n-1)a}{p} + \cfrac{1}{x^{\frac{2n}{2n+1}}y}}}}}} \quad .$$

would reduce the equation to the separable case as desired. This statement is not at all trivial, but Euler made no attempt to justify it. We will revisit this omission later in the present paper.

Next, Euler allowed n to become infinite, and hence obtained an infinite continued fraction with denominators in arithmetic progression:

$$q = \frac{a}{p} + \cfrac{1}{\tfrac{3a}{p} + \cfrac{1}{\tfrac{5a}{p} + \cfrac{1}{\tfrac{7a}{p} + \text{etc.}}}}.$$

Moreover, he claimed, this continued fraction satisfies the equation $a\,dq + q^2\,dp = dp$. But Euler could easily find the general solution of this equation by separating variables and integrating, deducing that

$$\frac{a}{2}\log\frac{1+q}{1-q} = p + C$$

where C is a constant of integration which Euler found to be zero by requiring q to become infinite when $p = 0$, as is the case for the continued fraction form of q. Hence, he obtained

$$q = \frac{e^{\frac{2p}{a}}+1}{e^{\frac{2p}{a}}-1}$$

and so

$$e^{\frac{2p}{a}} = 1 + \frac{2}{q-1} = 1 + \cfrac{2}{\frac{a-p}{p} + \cfrac{1}{\frac{3a}{p} + \cfrac{1}{\frac{5a}{p} + \cfrac{1}{\frac{7a}{p} + \cfrac{1}{\frac{9a}{p} + \text{etc.}}}}}}.$$

By setting $a/2p = s$ and applying the results he had obtained earlier in the paper when considering interrupted and uninterrupted progressions of denominators, Euler concluded that

$$e^{\frac{1}{s}} = 1 + \cfrac{1}{s-1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{3s-1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{5s-1+\cfrac{1}{1+\text{etc.}}}}}}}}.$$

Letting $s = 1$ in this expression gives us a regular continued fraction expression for e; as Euler conjectured earlier in the paper, this expression continues infinitely, and though Euler does not say so, by immediate corollary of this fact, we may deduce that e is irrational. It is somewhat strange that Euler did not explicitly point this out; it is possible that he thought the result obvious or unimportant and did not wish to labour the point.

Euler ended the paper by using an intricate sequence of substitutions to give a method for finding the value of any infinite continued fraction whose denominators form an arithmetic progression. It has been suggested by Sandifer [2007b, 190] that this final section, contained in paragraphs 31–35 of Euler's paper, was meant to constitute the promised proof that the continued fraction for e continues infinitely. Sandifer suggested that since Euler did not prove the reduction of the differential equation $a\,dy + y^2\,dx = x^{\frac{-4n}{2n+1}}\,dx$, he did not intend paragraphs 28–30 to be a proof that the expansion for $e^{1/s}$ continues infinitely. However, I find this extremely unlikely. The last few words of paragraph 30 seem unambiguous [Euler, 1744b, 133]:

> From these formulae all those found above, by which we have expressed some powers of e by continued fractions, result; therefore the progression, which was earlier merely observed, is understood to be necessarily true.[17]

In contradiction to Sandifer, I believe that Euler thought the gap in the proof an 'easy exercise' for the reader. In the following section, we shall see that this could indeed have been an easy exercise for a reader familiar with the mathematical literature of the decades preceding the writing of E71.

4. Daniel Bernoulli's *Exercitationes mathematicae* (1724) and the Euler–Goldbach correspondence (1731)

None of the theorems Euler proved on continued fractions in E71 seems to have been the spark that made him write the paper, and none of them seems to relate strongly to earlier work on continued fractions; though Wallis's publications are mentioned in passing, one would think that if Wallis's work was the stimulus for E71, then the latter would surely have featured Brouncker's continued fraction more prominently. We must, therefore, look elsewhere

[17]'Ex his vero formulis fluunt omnes supra inventae, quibus potestates quasdam ipsius e per fractiones continuas expressimus; ex quo necessitas progressionis ante tantum observatae intelligitur.'

for Euler's reason for studying continued fractions. The major achievement of the paper is the derivation of the infinite continued fraction for e; it is natural, therefore, to wonder where this came from, especially in light of the 'peculiar way' in which Euler came to it, via differential equations.

Hence, we must examine work done in the 1720s and 1730s on ordinary differential equations: in particular, we will look at the Riccati equation, of which both of the equations Euler uses in E71 are particular cases.[18] As one of the major areas of research arising from the new calculus, differential equations were an area of great interest for much of the mathematical community in the early eighteenth century. Until around 1730, no systematic methods of solution were known other than separation of variables, and so nearly all studies on differential equations until that time focused on the search for transformations to make equations separable.

In the years before he wrote E71, Euler wrote several papers on the solution of differential equations: 'Nova methodus innumerabiles aequationes differentiales secundi gradus reducendi ad aequationes differentiales primi gradus' ('A new method for reducing innumerable second order differential equations to first order differential equations', E10), presented to the Academy in 1728, was the first of these. It was followed in 1733 by three papers on the construction of particular differential equations:[19] 'Constructio aequationum quarundam differentialium, quae indeterminatarum separationem non admittunt' ('Construction of certain differential equations which do not admit a separation of variables', E11); 'Specimen de constructione aequationum differentialium sine indeterminatarum separatione' ('Example of the construction of differential equations without a separation of variables', E28); and 'Constructio aequationis differentialis $ax^n dx = dy + y^2 dx$' (Construction of the equation $ax^n dx = dy + y^2 dx$, E31). Though these papers contain considerable detail on the general Riccati equation, here Euler was considering the impossibility of separation of variables in the general case, rather than treating a specific separable case as in E71. Thus, it is unlikely that the result from E71 has much to do with the contents of E10, E11, E28,

[18]For detailed accounts of the early history of various forms of the Riccati equation, see [Bittanti, 1989] and [Bottazzini, 1996].

[19]A full discussion of the meaning of 'construction' here is outside the scope of the present paper. Euler uses the word to mean the reduction of the differential equation to a problem of quadrature or rectification [Euler, 1733a, 369]. For some more information on the early history of differential equations, see [Archibald, 2003].

and E31. The Riccati equation also appeared in two more of the papers Euler wrote prior to E71: 'De constructione aequationum ope motus tractorii aliisque ad methodum tangentium inversam pertinentibus' ('On the construction of equations using tractory motion, and of other things pertinent to the inverse tangent method', E51), written in 1735, and 'De constructione aequationum' ('On the construction of equations', E70), written in 1737. Again, there is no hint of continued fractions in either of these; the latter is perhaps notable in the present context because it was presented to the St Petersburg Academy on 7 February 1737, only a month before E71, but it mentions the Riccati equation only very briefly, at its very end.

Instead, we can find a more likely origin for the result on the continued fraction for e by turning to Euler's correspondence: in particular, to some of the earliest of a long sequence of letters he exchanged with Christian Goldbach. Euler and Goldbach had met during the late 1720s when Euler arrived at the St Petersburg Academy as an adjunct in physiology. He became friends with Goldbach, then the recording secretary of the Academy, and they remained friends until Goldbach's death in 1764. The Euler–Goldbach correspondence, which dates from 1729 until shortly before Goldbach's death, comprises around 200 extant letters, is rich in mathematical problems, and is notable particularly for its enthusiastic discussion of number-theoretic problems in a period when number theory was largely regarded as trivial and unimportant. It has been lauded as 'a jewel in the history of science of the eighteenth century' [Fellmann, 2007, 36]; indeed, the editors of Euler's *Opera omnia* deemed it significant enough to merit translation into English in its entirety, an honour not afforded to any other part of Euler's correspondence [Kleinert and Mattmüller, 2007]. One of the earliest letters written by Euler to Goldbach is dated 25 November 1731, and contains the statement, without proof, of a result which undoubtedly influenced the writing of E71 six years later. In what follows, I will refer to this letter as R 729, after the reference number it was allocated by the editors of the Euler correspondence index [Juškevič et al., 1975]. The *Opera omnia* volume of Euler–Goldbach has not yet been published, but most of the text of R 729 can be found in the 1965 edition of the correspondence by A.P. Juškevič and E. Winter, or in P.H. Fuss's *Correspondance mathématique et physique de quelques célèbres géomètres du XVIIIème siècle*, of which the Euler–Goldbach correspondence occupies the whole of the first of two volumes [Juškevič and Winter, 1965, 50–53; Fuss, 1843, I, 56–60].

In R 729, Euler discussed a number of different and mostly unrelated subjects: integrals; composite numbers of the form $2^n - 1$; and the separable Riccati differential equation $a\,dq = q^2\,dp - dp$. I shall focus here on the few paragraphs dealing with the Riccati equation, which are particularly noteworthy. Here, Euler stated:

> Recently, considering separable cases of the Riccati equation, I have uncovered the following universal substitution, by which the equation $a\,dq = q^2\,dp - dp$ may be restored to the form $a\,dy = y^2\,dx - x^{\frac{-4n}{2n+1}}\,dx$.[20]

This remark came seemingly unprompted; there is no hint of the Riccati equation earlier in this letter or in those that precede it. Euler continued by claiming that if one makes the substitution $p = (2n+1)x^{\frac{1}{2n+1}}$, then it is easy to determine that the substitution

$$q = -\tfrac{a}{p} + \cfrac{1}{\frac{-3a}{p} + \cfrac{1}{\frac{-5a}{p} + \cfrac{1}{\frac{-7a}{p} + \cfrac{1}{\ddots + \cfrac{1}{\frac{-(2n-1)a}{p} + \cfrac{1}{x^{\frac{2n}{2n+1}}y}}}}}}$$

will give the desired form of the equation.

Reciprocally, he noted, one may transform the equation

$$a\,dy = y^2\,dx - x^{\frac{-4n}{2n+1}}\,dx$$

back into the equation

$$a\,dq = q^2\,dp - dp$$

by simply reversing these transformations.

[20]'Casus nuper formulae Riccatianae separabiles considerans, sequentem universalem detexi substitutionem, qua aequatio $a\,dq = q^2\,dp - dp$ ad hanc formam $a\,dy = y^2\,dx - x^{\frac{-4n}{2n+1}}\,dx$ reduci potest' [Fuss, 1843, I: 58; Juškevič and Winter, 1965, 52].

This is a non-trivial result: it is far from obvious that q is the correct substitution. In his editorial comments to E71 in Euler's *Opera omnia*, Georg Faber notes that the equation

$$a\frac{dy}{dx} + y^2 = x^{\frac{-4n}{2n+1}},$$

which is the form used in E71, can be transformed to

$$a\frac{dz}{dp} = 1 + \frac{2na}{p}z - z^2$$

by letting $p = (2n+1)x^{\frac{1}{2n+1}}$ (as given by Euler) and taking z to be the expression in the last partial denominator of the continued fraction, so $z = x^{\frac{2n}{2n+1}}y$ [Faber, 1935, IC]. Faber then notes that if one writes z^* for a solution of the same equation but with n replaced by $n-1$, then one will have

$$z^* = \frac{(2n-1)a}{p} + \frac{1}{z}.$$

Composing substitutions of this form will give Euler's continued fraction. However, this is still not at all clear unless one has already seen the continued fraction; Faber's argument does not explain how Euler came to find the continued fraction in the first place.

In the letter R 729, it is also curious that Euler was beginning with an easily manageable separable equation and 'restoring' or 'reducing' it to a more complicated form. Yet even though Euler gave neither a proof nor any further explanation of the claim, Goldbach seemed to understand immediately, and even extended the result a little in his reply (*Opera omnia* number OO730), giving another type of equation that can easily be transformed into either of the forms given by Euler [Fuss, 1843, I: 61; Juškevič and Winter, 1965, 54]. One might wonder if Goldbach was simply reluctant to admit that he did not understand Euler. But this is unlikely: in the same letter, Goldbach willingly confessed that he did not understand another claim that Euler had made later on in the letter, and asked for clarification. Yet, he did not question this one.

I therefore contend that Goldbach did understand why Euler's claim was true, and that his immediate understanding was based on a shared experience: some mutual prior knowledge, a previous discussion of this equation that Euler knew Goldbach would recall. The correspondence between Euler and Goldbach prior to R 729 yields no evidence that the two had ever discussed this problem before. So, the shared knowledge that tacitly drove this conversation must have come from another source.

There are at least two people with whom Goldbach had discussed the Riccati equation prior to 1731: Nicolaus Bernoulli (1695–1726), and his younger brother Daniel (1700–1782), the sons of Euler's former tutor Johann Bernoulli (1667–1748). Goldbach corresponded with Nicolaus Bernoulli during the early 1720s, and the two discussed various forms of the Riccati equation. However, during this correspondence Goldbach's primary concern was to find cases of the equation which admitted an algebraic solution. While Nicolaus Bernoulli did make some investigations into separable cases, notably in his letter to Goldbach of 14 March 1722, the equation he studied has a slightly different form to the one studied by Euler in 1731, and so the expressions he obtained for exponents in separable cases are also different and the connection to Euler's work is not clear [Fuss, 1843, II, 140–144].[21] What I find more plausible as a source of Goldbach's prior knowledge is Daniel Bernoulli's 1724 book *Exercitationes quaedam mathematicae*: this contains a treatment of the Riccati equation which is very clearly connected to Euler's treatment from 1731.

Daniel Bernoulli published his first work on the Riccati equation in the *Actorum eruditorum supplementa* of 1724. In this paper, titled 'Notata in praecedens schediasma Ill. Co. Jacobi Riccati' ('Note on the preceding sketch by the illustrious Count Jacopo Riccati'), Bernoulli took up the challenge issued by Jacopo Riccati in the paper printed immediately before his own: to find infinitely many n for which the equation $ax^n\,dx + uu\,dx = b\,du$ is separable [Bernoulli, 1724b]. He succeeded, but in the 'Notata' he gave the solution only in the form of an obscure anagram, saying he wanted to stake a priority claim for the first solution while still leaving the problem open for others. It was not until he wrote *Exercitationes quaedam mathematicae*, his first major mathematical book, later in the same year, that he revealed his method. The book consists of six chapters on various topics: recurrent series and their application to the card game 'Pharaon'; the behaviour of a fluid flowing from a container; the Riccati equation; and the quadrature of lunes.[22]

[21]Nicolaus Bernoulli studied the equation $ax^m\,dx + by^2x^p\,dx = dy$, and found that if it is separable in the case $m = p$ then it is also separable when $m = -3p - 4$ and when $m = (-p-4)/3$.

[22]For a detailed commentary on *Exercitationes quaedam mathematicae*, see [Bottazzini,

Several of these topics appear in Bernoulli's correspondence with Goldbach in the two years prior to the publication of the *Exercitationes*.[23]

In the chapter on the Riccati equation, titled 'Auctoris explanatio notationum suarum ... una cum ejusdem solutione problematis Riccatiani' ('Explanation by the author of his 'Notata' ... together with a solution of the problem of Riccati'), Bernoulli gave a way of finding infinitely many n for which the equation

$$ax^n\,dx + uu\,dx = b\,du \tag{1}$$

is separable. This method relies upon two lemmas. In the first of these (*Lemma primum*, [Bernoulli, 1724a, 78]) he noted that if one begins with equation (1) and applies the substitutions $u = y^{-1}$ and $x = s^{\frac{1}{n+1}}$, then the resulting equation in y and s is

$$\frac{1}{a}s^{\frac{-n}{n+1}}\,ds + y^2\,ds = \frac{-b(n+1)}{a}\,dy.$$

This has the same form as the original equation (1), but with new constant coefficients, and with the exponent n replaced by $\frac{-n}{n+1}$. Therefore, if equation (1) is separable in the case $n = m$, it is also separable in the case $n = \frac{-m}{m+1}$. The second lemma (*Lemma secundum*, [Bernoulli, 1724a, 78]) gives a similar result: once again beginning with equation (1), one can apply the substitutions $u = -bx^{-1} + x^{-2}y$ and $x = s^{-1}$ to obtain

$$as^{-n-4}\,ds + y^2\,ds = -b\,dy.$$

This equation in y and s has the same form as the original equation (1), with different constant coefficients; this time, the exponent n is replaced by $-n-4$. Hence, if equation (1) is separable in the case $n = m$, then it is also separable in the case $n = -m-4$.

The two lemmas mean that if the differential equation (1) is separable in the case $n = m$, then it is also separable in the cases $n = \frac{-m}{m+1}$ and $n = -m-4$. But since the equation is clearly separable when $n = 0$, Bernoulli could apply the two lemmas alternately to obtain infinitely many more separable cases.[24]

1996].

[23]See the correspondence between Daniel Bernoulli and Christian Goldbach in [Fuss, 1843, II], and commentary on its links with the *Exercitationes* in [Bottazzini, 1996].

[24]Both of the maps $f : m \mapsto -m-4$ and $g : m \mapsto \frac{-m}{m+1}$ are self-inverse, so two consecutive applications of either f or g will give a value of m already obtained. In order to produce a new value of m each time, we must apply f and g alternately.

Indeed, doing this gives the sequence

$$\begin{gathered}
0, \\
-0-4=-4, \\
\frac{-(-4)}{-4+1}=-\frac{4}{3}, \\
-\left(-\frac{4}{3}\right)-4=-\frac{8}{3} \\
\frac{-\left(-\frac{8}{3}\right)}{-\frac{8}{3}+1}=-\frac{8}{5}, \\
-\left(-\frac{8}{5}\right)-4=-\frac{12}{5}, \\
\ldots,
\end{gathered}$$

or $n=\frac{-4m}{2m\pm1}$, where m may be any non-negative integer. These cases are almost exactly those considered by Euler and Goldbach seven years later.

It remains to show how this result is linked to the continued fraction Euler found in R 729. The mathematics underlying this is simple, though the process is somewhat cumbersome. Here, we use subscript notation for added clarity, though this was not yet in common use in the 1720s and 1730s. We begin with the separable equation $a\,dq=q^2\,dp-dp$. This is the form used by Euler in R 729; it is a particular case of equation (1), with $n=0$. As suggested by Bernoulli, we apply the two Lemmas alternately. Applying *Lemma primum*, the substitutions are $q=y^{-1}$ and $p=s$; these give the same equation, but with y instead of q. Applying *Lemma secundum* to the original equation, the relevant substitutions are $q=-ap^{-1}+p^{-2}q_1$ and $p=p_1^{-1}$, and we obtain the equation

$$-a\,dq_1=(q_1^2-p_1^{-4})\,dp_1.$$

If we now apply *Lemma primum* starting with this equation, then the substitutions are $q_1=q_2^{-1}$ and $p_1=p_2^{-\frac{1}{3}}$, and the new equation is

$$3a\,dq_2=(q_2^2-p_2^{-\frac{4}{3}})\,dp_2.$$

Starting from this equation, the substitutions from *Lemma secundum* are $q_2=-3ap_2^{-1}+p_2^{-2}q_3$ and $p_2=p_3^{-1}$, and the new equation is

$$-3a\,dq_3=(q_3^2-p_3^{-\frac{8}{3}})\,dp_3.$$

If we once again apply *Lemma primum*, the substitutions are $q_3 = q_4^{-1}$ and $p_3 = p_4^{-\frac{3}{5}}$, and we obtain the equation

$$5a\,dq_4 = (q_4^2 - p_4^{-\frac{8}{5}})\,dp_4.$$

Applying *Lemma secundum* again, the substitutions are $q_4 = -5ap_4^{-1} + p_4^{-2}q_5$ and $p_4 = p_5^{-1}$; this gives the equation

$$-5a\,dq_5 = (q_5^2 - p_5^{-\frac{12}{5}})\,dp_5.$$

We could continue in this vein. But instead, we merely note that by composing the substitutions we have produced so far, we obtain

$$p = p_1^{-1} = p_2^{\frac{1}{3}} = p_3^{-\frac{1}{3}} = p_4^{\frac{1}{5}} = p_5^{-\frac{1}{5}}$$

and

$$\begin{aligned} q &= -ap^{-1} + p^{-2}q_1 \\ &= -ap^{-1} + p^{-2}q_2^{-1} \\ &= -ap^{-1} + \frac{1}{p^2(-3ap_2^{-1} + p_2^{-2}q_3)} \\ &= -ap^{-1} + \frac{1}{-3ap^2p_2^{-1} + p^2p_2^{-2}q_4^{-1}} \\ &= -ap^{-1} + \cfrac{1}{-3ap^2p_2^{-1} + \cfrac{1}{p^{-2}p_2^2(-5ap_4^{-1} + p_4^{-2}q_5)}}. \end{aligned}$$

Rewriting each of the terms in p_2 and p_4 in terms of the original variable p, we obtain

$$q = -ap^{-1} + \cfrac{1}{-3ap^{-1} + \cfrac{1}{-5ap^{-1} + p^{-6}q_5}}$$

and if we continue in this way, clearly we will obtain the same continued fraction that Euler found in R 729.

If we now re-examine the continued fraction in R 729 in the light of its possible connection to Daniel Bernoulli's *Exercitationes*, it becomes less surprising that Euler was 'restoring' or 'reducing' a separable equation to one that is more difficult to deal with: he was not trying to reduce one equation to another simpler one, but rather, he was rewriting Bernoulli's result on separable forms of the Riccati equation, by composing the substitutions given in the two Lemmas. What is more, there is strong circumstantial evidence that the *Exercitationes* was indeed in the minds of both Euler and Goldbach in 1731. Euler would certainly have known the book, given his close connection with the Bernoulli family: indeed, at the time he wrote the letter he and Daniel Bernoulli were living in St Petersburg Academy accommodation, and the two often worked closely together on mathematical problems. Goldbach had been in regular correspondence with Bernoulli at the time when *Exercitationes quaedam mathematicae* was published; it is clear from these letters that he was well aware of the book's existence and would have been familiar with its contents.[25] It is, therefore, undoubtable that both Euler and Goldbach would have been intimately familiar with the contents of the *Exercitationes*.

Thus, I suggest that in R 729, Euler was tacitly applying the method used by Bernoulli, secure in the knowledge that Goldbach would understand the implicit reference to the *Exercitationes* even though Bernoulli had not written down the composition of the substitutions. It is this result that he returned to in 1737, when he saw that it could be extended to prove that the regular continued fraction for e does indeed continue infinitely. It was this realisation, I suggest, and not any previous writings on continued fractions, that provided the stimulus for Euler to write E71.

5. Conclusions

As we have seen, there are several 17th-century appearances of continued fractions which, given the right circumstances, could have led Euler directly to the study of continued fractions. Instead, it transpires that he came to study continued fractions by an altogether more surprising route: he started from Daniel Bernoulli's work on differential equations, and returned to the topic later via a letter to Goldbach. This is an excellent example both of the startling connections that can be unveiled within mathematics, and of Euler's rare talent for spotting them. Moreover, it raises questions about the role of correspondence in Euler's working practices, and about the ways in which

[25] See, for example, the mentions of the *Exercitationes* in various letters exchanged between Bernoulli and Goldbach during 1724 and 1725 [Fuss, 1843, II, 211–239].

other mathematicians influenced Euler to take up certain topics. For example, it is curious that there was a gap of six years between the first appearance of the continued fraction in R 729, and its reappearance in E71. Several other papers written around the same time as E71 also treat topics that Euler discussed in correspondence with Goldbach in the early 1730s: for example, as mentioned above, 'De constructione aequationum' (E70), presented to the Academy a month before E71, touches upon the Riccati equation, and 'Solutio problematis geometrici circa lunulas a circulis formatas' ('Solution to a geometrical problem about lunes formed from circles', E73) was presented a few months after E71 and deals with geometrical problems similar to some which Euler discussed with Goldbach in 1730. There is, therefore, a small cluster of papers written in 1737 whose topics all appear to have Euler's correspondence with Goldbach as their common origin. Further work is needed to determine whether any of Euler's contact with Goldbach prompted Euler to extend the contents of these letters into full-length papers.[26] It would also be worthwhile to investigate how other correspondence influenced Euler's publishing activities.

Finally, I note that while E71 was not itself motivated by questions relating to the theory of continued fractions, it did open the door for Euler, and later others, to ask more questions concerning continued fractions. He returned to the subject several times during his career: in 1738, a year after he wrote E71, he wrote a paper titled 'De fractionibus continuis observationes' ('Observations on continued fractions', E123) in which he returned to Brouncker's continued fraction for π, which he had mentioned in passing in E71. Much later, in the late 1750s, he explored the connection between continued fractions and the Pell equation in 'De specimen algorithmi singularis' ('On an example of a singular algorithm', E281) and 'De usu novi algorithmi in problemate Pelliano solvendo' ('On the use of a new algorithm in solving Pell's equation', E323). This work was later taken up by Lagrange in his work on the solution of numerical equations in the late 1760s. Thus, by developing the basic theory of continued fractions in E71 and demonstrating their usefulness, Euler allowed them to be put to more significant uses than simply as approximations or as curious representations of irrational numbers,

[26] Goldbach returned to St. Petersburg when the Russian court moved there from Moscow in 1732, and resumed his duties as secretary to the St. Petersburg Academy, so it is possible that such a discussion could have occurred face-to-face.

and turned them into a useful tool in analysis.

Archibald, T., 2003. Differential equations: a historical overview to circa 1900, in: Jahnke, H.N. (Ed.), A history of analysis. American Mathematical Society, pp. 325–354.

Baltus, C., 2007. Notes on Euler's continued fractions (http://www.oswego.edu/documents/mathematics/eulercfproceedings.pdf, accessed 25 january 2013).

Bernoulli, D., 1724a. Exercitationes quaedam mathematicae. Venice.

Bernoulli, D., 1724b. Notata in praecedens schediasma Ill. Co.Jacobi Riccati. Actorum eruditorum supplementa 8, 73–75.

Bittanti, S. (Ed.), 1989. Count Riccati and the early days of the Riccati equation. Pitagora.

Bottazzini, U., 1996. The mathematical writings from Daniel Bernoulli's youth, in: Speiser, D. (Ed.), Die Werke von Daniel Bernoulli. Birkhäuser, Basel. volume I, pp. 129–197.

Calinger, R., 1996. Leonhard Euler: the first St Petersburg years (1727–1741). Historia Mathematica 23, 121–166.

Cataldi, P., 1613. Trattato del modo brevissimmo di trovar la radice quadra delli numeri. Bologna.

Cotes, R., 1714. Logometria. Philosophical Transactions of the Royal Society of London 29, 5–45.

Eneström, G., 1913. Die Schriften Eulers chronologisch nach den Jahren geordnet, in denen sie verfasst worden sind. Jahresbericht der Deutschen Mathematiker-Vereinigung .

Euler, L., 1733a. Constructio aequationum quarundam differentialium, quae indeterminatarum separationem non admittunt [E11]. Nova acta eruditorum , 369–373.

Euler, L., 1733b. Nova methodus innumerabiles aequationes differentiales secundi gradus reducendi ad aequationes differentiales primi gradus [E10]. Commentarii academiae scientiarum imperialis petropolitanae 3, 124–137.

Euler, L., 1738a. Constructio aequationis differentialis $ax^n dx = dy + y^2 dx$ [E31]. Commentarii academiae scientiarum imperialis petropolitanae 6, 231–246.

Euler, L., 1738b. Specimen de constructione aequationum differentialium sine indeterminatarum separatione [E28]. Commentarii academiae scientiarum imperialis petropolitanae 6, 168–174.

Euler, L., 1741. Nova methodus innumerabiles aequationes differentiales secundi gradus reducendi ad aequationes differentiales primi gradus [E51]. Commentarii academiae scientiarum imperialis petropolitanae 8, 66–85.

Euler, L., 1744a. De constructione aequationum [E70]. Commentarii academiae scientiarum imperialis petropolitanae 9, 85–97.

Euler, L., 1744b. De fractionibus continuis dissertatio [E71]. Commentarii academiae scientiarum imperialis petropolitanae 9, 98–137.

Euler, L., 1750. De fractionibus continuis observationes [E123]. Commentarii academiae scientiarum imperialis petropolitanae 11, 32–81.

Euler, L., 1764. Specimen algorithmi singularis [E281]. Novi commentarii academiae scientiarum imperialis petropolitanae 9, 53–69.

Euler, L., 1767. De usu novi algorithmi in problemate pelliano solvendo [E323]. Novi commentarii academiae scientiarum imperialis petropolitanae 11, 29–66.

Euler, L., 1771. Vollständige Anleitung zur Algebra: erster Theil [E387]. St. Petersburg.

Faber, G., 1935. Übersicht über die Bände 14, 15, 16, 16* der ersten Serie, in: Boehm, C. (Ed.), Commentationes analyticae ad theoriam serierum infinitarum pertinentes. Volumen tertium. Sectio altera. Leonhardi Euleri opera omnia I.16.2. Teubner, pp. VII–CXII.

Fellmann, E.A., 2007. Leonhard Euler. Birkhäuser.

Fowler, D.H., 1991. An approximation technique and its use by Wallis and Taylor. Archive for History of Exact Sciences 41, 189–233.

Fowler, D.H., 1994. Continued fractions, in: Grattan-Guinness, I. (Ed.), Companion encyclopedia of the history and philosophy of the mathematical sciences. Routledge, London. volume 1, pp. 730–740.

Fowler, D.H., 1999. The mathematics of Plato's Academy: A new reconstruction. Clarendon Press. 2nd edition.

Fuss, P.H., 1843. Correspondance mathématique et physique de quelques célèbres géomètres du XVIIIème siècle. St. Petersburg.

Huygens, C., 1888. Oeuvres complètes de Christiaan Huygens, publiées par la Société Hollandaise des Sciences. Martinus Nijhoff, The Hague.

Juškevič, A.P., Smirnov, V.I., Habicht, W. (Eds.), 1975. Briefwechsel von Leonhard Euler: Beschreibung, Zusammenfassungen der Briefe und Verzeichnisse. Number 1 in Leonhardi Euleri opera omnia, series 4A: Commercium epistolicum, Birkhäuser, Basel.

Juškevič, A.P., Winter, E. (Eds.), 1965. Leonhard Euler und Christian Goldbach. Briefwechsel 1729–1764. Akademie-Verlag, Berlin.

Kleinert, A., Mattmüller, M., 2007. Leonhardi Euleri Opera Omnia: A centenary project. European Mathematical Society Newsletter , 25–31.

Knobloch, E., 1989. Leonhard Eulers mathematische Notizbücher. Annals of Science 46, 277–302.

Malcolm, N., Stedall, J.A., 2005. John Pell (1611–1685) and his correspondence with Sir Charles Cavendish: The mental world of an early modern mathematician. Oxford University Press.

Petrie, B.J., 2012. Leonhard Euler's use and understanding of mathematical transcendence. Historia Mathematica 39, 280–291.

Sandifer, C.E., 2007a. The early mathematics of Leonhard Euler. Mathematical Association of America.

Sandifer, C.E., 2007b. How Euler did it. Mathematical Association of America.

Schwenter, D., 1617. Geometria practica nova. Nuremberg.

Schwenter, D., 1623. Geometriae practicae novae et auctae. Tractatus II. Nuremberg.

Smith, D.E., 1959. A source book in mathematics. Dover, New York.

Stedall, J.A., 2000. Catching Proteus: the collaborations of Wallis and Brouncker. I: Squaring the circle. Notes and Records of the Royal Society of London 54, 293–316.

Stedall, J.A., 2002. A discourse concerning algebra: English algebra to 1685. Oxford University Press, Oxford.

Wallis, J., 1656. Arithmetica infinitorum. Oxford.

Wallis, J., 1685. A treatise of algebra, both historical and practical. London.

Wallis, J., 1693. De algebra tractatus, historicus et practicus. Oxford.

Wyman, M.F., Wyman, B.F., 1985. An essay on continued fractions. Mathematical systems theory 18, 295–328.