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MAXIMAL BUTTONINGS OF TREES

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Abstract

A buttoning of a tree that has vertices $v_1, v_2, \ldots, v_n$ is a closed walk that
starts at $v_1$ and travels along the shortest path in the tree to $v_2$, and then
along the shortest path to $v_3$, and so forth, finishing with the shortest path
from $v_n$ to $v_1$. Inspired by a problem about buttoning a shirt inefficiently,
we determine the maximum length of buttonings in trees.

Keywords: Centroid, graph metric, tree, walk, Wiener distance.

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Secondary: 05C85.

1. Introduction

At the retirement meeting of Jenny Piggott as director of the mathematics edu-
cation project NRICH, Bernard Murphy proposed the following problem (para-
phrased).

Problem 1. My shirt has eight buttons in a vertical line with a spacing of one
unit between each adjacent pair. Usually I button them from top to bottom,
so that my hands move a distance of seven units. Suppose I button them in a
different order; what is the maximum number of units my hands may travel?

In this partly expository note we address the more general problem of iden-
tifying, for each finite tree $T$ with graph metric $d$, the maximum value of the sum

$$d(v_1, v_2) + d(v_2, v_3) + \cdots + d(v_{n-1}, v_n) + d(v_n, v_1) \quad (1)$$

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among all lists $v_1, v_2, \ldots, v_n$ of the vertices of $T$. Problem 1 is a particular case of this more general problem when $T$ is the linear graph of order 8. (To be precise, we must remove the final term $d(v_n, v_1)$ from (1) to recover Problem 1, but we shall see that this is an insignificant complication.) Our problem is itself a special case of the maximum travelling salesman problem. To see this, observe that the sum (1) is the length of a Hamilton cycle in the weighted complete graph that has vertices $v_1, v_2, \ldots, v_n$ and has, for each distinct pair $i$ and $j$, an edge of weight $d(v_i, v_j)$ between $v_i$ and $v_j$.

All trees throughout the paper are finite. Further, $T$ will always denote a tree with graph metric $d$. We denote by $V_T$ the vertex set of $T$. Let $[u, v]$ denote the unique shortest path from one vertex $u$ to another vertex $v$ in $T$. A buttoning of $T$ is a closed walk in $T$ consisting of $n$ paths $[v_1, v_2], [v_2, v_3], \ldots, [v_{n-1}, v_n], [v_n, v_1]$, where $v_1, v_2, \ldots, v_n$ are the vertices of $T$. The length of this buttoning is the sum (1). A centroid of $T$ is a vertex $v$ such that the sum $\sum_{u \in V_T} d(v, u)$ is minimized. Each tree has either one centroid or two adjacent centroids. Given a centroid $v$ we define

$$\Phi(T) = 2 \sum_{u \in V_T} d(v, u).$$

The theory of centroids is covered briefly in [1, Section 1] and [2, Section 3]. The authors of [1] emphasise the importance of centroids in distance calculations, and our work supports this assertion. We can now state our main theorem.

**Theorem 2.** Given a tree $T$ with vertices $v_1, v_2, \ldots, v_n$ we have

$$2n - 2 \leq d(v_1, v_2) + d(v_2, v_3) + \cdots + d(v_{n-1}, v_n) + d(v_n, v_1) \leq \Phi(T),$$

(2)

and the upper and lower bounds are each attained by the lengths of certain buttonings of $T$.

The lower inequality in (2) has been proven already, in [4, Theorem 1] (including proof that the lower bound is attainable). There are results of a similar nature to Theorem 2 in [3].

A maximal buttoning of a tree $T$ is a buttoning of maximum length $\Phi(T)$. When $T$ is the linear tree of order 8, the two middlemost vertices of $T$ are both centroids, and one can check that $\Phi(T) = 32$. We show in Lemma 5 that you can choose $d(v_n, v_1) = 1$ in a maximal buttoning of such a tree, and so the solution to Problem 1 is 31.

The quantity $\Phi(T)$ is closely related to the Wiener distance $W(T)$, which is given by $W(T) = \sum_{a, b \in V_T} d(a, b)$. It is known (see, for example, [2]) that, among trees of order $n$, $W(T)$ is minimized when $T$ is the star with $n$ vertices and $W(T)$ is maximized when $T$ is the linear graph with $n$ vertices. The same is true of $\Phi(T)$, and we state this as a theorem (which is easily proven). Let $\lfloor x \rfloor$ denote the integer part of a real number $x$. 

Theorem 3. If $T$ is a tree of order $n$ then

$$2n - 2 \leq \Phi(T) \leq \left\lfloor \frac{1}{2} n^2 \right\rfloor.$$  

Furthermore, the lower bound is attained when $T$ is a star and the upper bound is attained when $T$ is a linear graph.

2. Proof of Theorem 2

Theorem 2 concerns the maximum and minimum lengths of buttonings of a tree $T$ of order $n$. Let us briefly summarize the proof from [4, Theorem 1] of the lower bound in (2). Because a buttoning is a closed walk that visits every vertex, each edge must be traversed at least twice, and this proves that each buttoning has length at least $2n - 2$. To see that this lower bound can be attained, between any two adjacent vertices in $T$ introduce a new edge. By ‘opening out’ the resulting graph to form a cycle it is straightforward to construct a buttoning of $T$ of length $2n - 2$. The remainder of this section concerns the upper bound of (2).

Lemma 4. Let $[v_1, v_2], [v_2, v_3], \ldots, [v_{n-1}, v_n], [v_n, v_1]$ be a buttoning of a tree $T$. Then

$$d(v_1, v_2) + d(v_2, v_3) + \cdots + d(v_{n-1}, v_n) + d(v_n, v_1) \leq \Phi(T),$$

with equality if and only if each centroid of $T$ is contained in every path $[v_i, v_{i+1}]$ (including $[v_n, v_1]$).

Proof. Let $v$ be a centroid of $T$ and let $v_{n+1} = v_1$. Then the triangle inequality gives

$$\sum_{i=1}^{n} d(v_i, v_{i+1}) \leq \sum_{i=1}^{n} (d(v_i, v) + d(v, v_{i+1})) = \Phi(T).$$

Equality is attained in this inequality if and only if $d(v_i, v_{i+1}) = d(v_i, v) + d(v, v_{i+1})$ for $i = 1, 2, \ldots, n$. This occurs if and only if $v$ is contained in each path $[v_i, v_{i+1}]$.

We must now prove that the upper bound $\Phi(T)$ in (2) can always be attained. We deal separately with trees that contain two centroids and trees that contain just one centroid. It is an old result of C. Jordan (see [2, Theorem 1]) that a tree with two centroids $u$ and $v$ has even order $2k$, and there is an edge connecting $u$ and $v$ which, once removed, leaves two disconnected subtrees $U$ and $V$ each of order $k$, where $u$ is a leaf of $U$ and $v$ is a leaf of $V$. We use this notation in the next lemma.
Lemma 5. Suppose that a tree $T$ has two centroids $u$ and $v$ and corresponding subtrees $U = \{u_1, u_2, \ldots, u_k\}$ and $V = \{v_1, v_2, \ldots, v_k\}$. Then the buttoning $[u_1, v_1], [v_1, u_2], [u_2, v_2], \ldots, [v_k, u_1]$ of $T$ is a maximal buttoning, and all maximal buttonings arise in this fashion.

Proof. By Lemma 4, each buttoning $[u_1, v_1], [v_1, u_2], [u_2, v_2], \ldots, [v_k, u_1]$ is a maximal buttoning because the paths $[u_i, v_i]$ and $[u_i, u_{i+1}]$ all contain $u$ and $v$. Furthermore, in any buttoning $[w_1, w_2], [w_2, w_3], \ldots, [w_{2k-1}, w_{2k}], [w_{2k}, w_1]$ not of this form there must be two consecutive vertices $w_i$ and $w_{i+1}$ that both lie in $U$, in which case $[w_i, w_{i+1}]$ does not contain $v$, and so, by Lemma 4, the buttoning is not maximal.

All the maximal buttonings of $T$ are described explicitly in Lemma 5, so we have the following corollary.

Corollary 6. A tree $T$ that has two centroids and is of order $2k$ has $2(k!)^2$ maximal buttonings.

Next we turn to trees with a single centroid. A preliminary lemma is needed.

Lemma 7. Let $X_1, X_2, \ldots, X_m$, where $m \geq 2$, be a collection of disjoint finite sets such that $\sum_{i \neq j} |X_i| \geq |X_j|$ for each $j$. Then we can list the elements $v_1, v_2, \ldots, v_n$ of $X_1 \cup X_2 \cup \cdots \cup X_m$ in such a way that no two consecutive terms $v_i$ and $v_{i+1}$ both lie in the same set $X_j$.

Sketch of proof. Remove the elements of $X_1 \cup X_2 \cup \cdots \cup X_m$ one by one and place them in the sequence $v_1, v_2, \ldots, v_n$, each time choosing the element $v_i$ from a set $X_j$ of largest current size (excluding the set $X_k$ from which $v_{i-1}$ was chosen). When $m = 2$, this strategy clearly gives a suitable list. When $m > 2$, the strategy preserves the inequality $\sum_{i \neq j} |X_i| \geq |X_j|$ (until only two elements, in two distinct sets $X_j$, remain), and hence eventually exhausts the sets $X_j$.

If a tree $T$ has a single centroid $v$, then removing $v$ from $T$, and removing all edges connected to $v$, leaves a number of disconnected subtrees of $T$, say $X_1, X_2, \ldots, X_m$. Again, it was proven by C. Jordan (see [2, Theorem 1]) that no one of these subtrees has order larger than the sum of the orders of all the others; in other words $\sum_{i \neq j} |X_i| \geq |X_j|$ for each $j$. We use this notation in the next lemma.

Lemma 8. Suppose that a tree $T$ has a single centroid $v_0$, and removing $v_0$ and its edges from $T$ leaves disconnected subtrees $X_1, X_2, \ldots, X_m$. Then we can label the vertices of $T \setminus \{v_0\}$ as $v_1, v_2, \ldots, v_n$ in such a way that no pair $v_i$ and $v_{i+1}$ both lie in the same set $X_j$, and $[v_0, v_1], [v_1, v_2], \ldots, [v_{n-1}, v_n], [v_n, v_0]$ is a maximal buttoning of $T$. 
Proof. Lemma 7 shows that it is possible to choose the vertices $v_1, v_2, \ldots, v_n$ in the described fashion, and, because each path $[v_i, v_{i+1}]$ passes through $v_0$, we see from Lemma 4 that the resulting buttoning is maximal.

In fact, Lemma 4 shows that all maximal buttonings of $T$ are of the form described in Lemma 8, up to cyclic permutations of the paths $[v_i, v_{i+1}]$ in the buttoning $[v_0, v_1], [v_1, v_2], \ldots, [v_{n-1}, v_n], [v_n, v_0]$. In contrast to Corollary 6, however, there does not appear to be a simple general formula for the number of maximal buttonings.

We proved in Lemma 4 that the length of a buttoning of a tree $T$ is less than or equal to $\Phi(T)$, and Lemmas 5 and 8 show that this bound can always be attained. This completes the proof of Theorem 2.

3. Concluding remarks

The concept of a buttoning extends to all finite connected graphs, and we finish with brief remarks about extremal buttoning lengths in this more general context.

From (2), a buttoning of a tree of order $n$ has length at least $2n - 2$. For more general connected graphs of order $n$, however, the lower bound for buttoning lengths is $n$, rather than $2n - 2$. This is because every buttoning has $n$ constituent paths each of length at least 1, which implies that the total length is at least $n$. Furthermore, the lower bound of length $n$ is achieved by any buttoning of the complete graph of order $n$.

On the other hand, by (3), a buttoning of a tree of order $n$ has length at most $\left\lfloor \frac{1}{2} n^2 \right\rfloor$, and this is also an upper bound for the length of a buttoning of a graph of order $n$. This is because the length of a buttoning of a graph is less than or equal to the length of the same buttoning on a spanning tree of the graph. It follows that among connected graphs of order $n$, the linear graph has the largest maximal buttoning length. In particular, the maximal buttoning length in Problem 1 remains 31 even when we rearrange the eight buttons to form a more general connected graph.

References

