

# CONFORMAL AUTOMORPHISMS OF COUNTABLY CONNECTED REGIONS

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ABSTRACT. We prove that the conformal automorphism group of a countably connected circular region of connectivity at least three is either a Fuchsian group or a discrete elementary group of Möbius transformations.

## 1. INTRODUCTION

It is a well known result of Heins [3] that the only groups that arise as conformal automorphism groups of plane regions of finite connectivity at least three are the finite groups  $A_4$ ,  $S_4$ ,  $A_5$ ,  $C_n$ , and  $D_n$ , for  $n = 1, 2, \dots$ . This paper is about regions of countable, rather than just finite, connectivity. We define a *circular region* to be a region  $D$  such that each component of the complement of  $D$  is either a single point or a closed spherical disc.

**Theorem 1.1.** *The conformal automorphism group of a countably connected circular region of connectivity at least three is either a Fuchsian group or a discrete elementary group of Möbius transformations. Furthermore, each Fuchsian group and discrete elementary group arises as the conformal automorphism group of a countably connected circular region.*

Koebe proved that any region of finite connectivity is conformally equivalent to a circular region, and He and Schramm [4] proved that this is also true of countably connected regions. (It is unknown whether Koebe's theorem holds for uncountably connected regions.) Using He and Schramm's result, we have the following corollary of Theorem 1.1.

**Corollary 1.2.** *Each countably connected region of connectivity at least three is conformally equivalent to a region whose conformal automorphism group is either a Fuchsian group or a discrete elementary group of Möbius transformations.*

A Fuchsian group is, by definition, a discrete group of Möbius transformations with an invariant disc. After conjugating by a Möbius transformation, we may assume that the invariant disc of a Fuchsian group is the unit disc  $\mathbb{D}$ , in which case the group preserves the hyperbolic metric on  $\mathbb{D}$ . A discrete elementary group is, up to conjugation by a Möbius transformation, either a discrete group of conformal isometries of the complex plane  $\mathbb{C}$  (with the Euclidean metric), a discrete group of conformal isometries of the extended complex plane  $\mathbb{C}_\infty$  (with the spherical metric), or a discrete group of conformal isometries of  $\mathbb{C} \setminus \{0\}$  (with the metric  $|dz|/|z|$ ). Therefore Theorem 1.1 says that the conformal automorphism group of a countably connected circular region of connectivity at least three is conjugate by a Möbius transformation to a discrete subgroup of conformal isometries of either  $\mathbb{D}$ ,  $\mathbb{C}$ ,  $\mathbb{C}_\infty$ , or  $\mathbb{C} \setminus \{0\}$ .

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## 2. CONNECTIVITY LESS THAN THREE

A *region*  $D$  in the extended complex plane  $\mathbb{C}_\infty$  is an open, connected set. The *connectivity* of  $D$  is the number (possibly infinite) of connected components in the complement of  $D$ . We recall that a *circular region* is a region  $D$  such that each component of the complement of  $D$  is either a single point or a closed disc. Here and throughout the paper the term *disc* refers to a proper spherical disc in  $\mathbb{C}_\infty$ ; not a single point, and not  $\mathbb{C}_\infty$  itself.

The *conformal automorphism group* of  $D$ , denoted  $\text{Aut}(D)$ , is the group of all conformal maps of  $D$  to itself. It can be equipped with the compact-open topology. Let  $\mathcal{M}$  denote the group of Möbius transformations. Each element of  $\mathcal{M}$  has the form  $z \mapsto (az + b)/(cz + d)$ , where  $a, b, c$ , and  $d$  are complex numbers, and  $ad - bc \neq 0$ . It can be shown (see, for example, [4, Theorem 0.1]) that the conformal automorphism group of a countably connected circular region consists only of Möbius transformations.

Every region of connectivity less than three is conformally equivalent to either  $\mathbb{C}_\infty, \mathbb{C}, \mathbb{D}, \mathbb{C} \setminus \{0\}, \mathbb{D} \setminus \{0\}$ , or an annulus. In each case, the conformal automorphism group is a Lie group with positive dimension (see [5, Chapter 12]). In contrast, for regions of connectivity greater than two,  $\text{Aut}(D)$  is discrete (see [6, page 496]).

## 3. PUNCTURED SPHERES

A *punctured sphere*  $D$  is a circular region in which each connected component of  $\mathbb{C}_\infty \setminus D$  is a point. For punctured spheres, there is a stronger version of Theorem 1.1.

**Theorem 3.1.** *The conformal automorphism group of a countably connected punctured sphere of connectivity at least three is a discrete elementary group of Möbius transformations.*

Before proving Theorem 3.1 we first review some of the standard theory of Möbius transformations. The Möbius group  $\mathcal{M}$  acts on  $\mathbb{C}_\infty$  and, by the Poincaré extension, it also acts isometrically on three-dimensional hyperbolic upper half-space  $\mathbb{H}^3$ . The sphere  $\mathbb{C}_\infty$  is the ideal boundary of  $\mathbb{H}^3$ . A subgroup  $G$  of  $\mathcal{M}$  is *elementary* if  $G$ , in its action on either  $\mathbb{C}_\infty$  or  $\mathbb{H}^3$ , has a finite orbit.

The *limit set*  $\Lambda_G$  of a discrete subgroup  $G$  of  $\mathcal{M}$  is the set of accumulation points in  $\mathbb{C}_\infty$  of the  $G$ -orbit of a point in  $\mathbb{H}^3$ . Either  $\Lambda_G$  has order 0, 1, or 2, in which case  $G$  is elementary, or else it has uncountable order, in which case  $G$  is non-elementary (see [1, Section 5] for details, including a classification of the discrete elementary groups). If  $G$  is non-elementary then the limit set  $\Lambda_G$  is the smallest non-empty  $G$ -invariant closed subset of  $\mathbb{C}_\infty$  [1, Theorem 5.3.7].

*Proof of Theorem 3.1.* Let  $G$  be the conformal automorphism group of a countably connected punctured sphere  $D$  of connectivity at least three. Then  $G$  is a group of Möbius transformations, and it was observed at the end of the previous section that  $G$  is discrete [6, page 496]. The set  $\mathbb{C}_\infty \setminus D$  is closed and  $G$ -invariant, so if  $G$  is non-elementary then  $\mathbb{C}_\infty \setminus D$  contains the limit set  $\Lambda_G$ , which is uncountable. But this cannot be, because  $\mathbb{C}_\infty \setminus D$  is countable. We conclude, therefore, that  $G$  is elementary after all.  $\square$

Theorem 3.1 fails for uncountably connected punctured spheres. To see this, take any discrete group  $G$  that is not a Fuchsian group and that has an uncountable, totally disconnected limit set ( $G$  may be a Schottky group, for example). Define  $D$  to be the complement of  $\Lambda_G$ . Then  $\text{Aut}(D)$  contains  $G$ , so it is neither Fuchsian nor elementary.

## 4. PROOF OF THEOREM 1.1: PART I

In this section we prove the first part of Theorem 1.1. Before this, however, we review some more well known facts about the Möbius group. Each Möbius transformation can be classified as *elliptic*, *parabolic*, or *loxodromic*. A loxodromic map is *hyperbolic* if it fixes an open disc in  $\mathbb{C}_\infty$  (besides  $\mathbb{C}_\infty$  itself). Otherwise it is said to be *strictly loxodromic*. Recall that a *Fuchsian group* is a discrete subgroup of  $\mathcal{M}$  with an invariant disc. The next lemma is proven in [1, Theorem 5.2.1].

**Lemma 4.1.** *Let  $G$  be a non-elementary subgroup of  $\mathcal{M}$ . There exists a  $G$ -invariant disc if and only if  $G$  contains no strictly loxodromic elements.*

We also use the following pair of simple lemmas.

**Lemma 4.2.** *Suppose that  $D$  is a region of connectivity at least two, and  $p$  is a fixed point of a loxodromic element of  $\text{Aut}(D)$ . Then  $p \notin D$ .*

*Proof.* Choose a loxodromic element  $f$  of  $\text{Aut}(D)$  with attracting fixed point  $p$  and repelling fixed point  $q$ . The sequence  $f^n$  converges locally uniformly on  $\mathbb{C}_\infty \setminus \{q\}$  to  $p$ . Since  $D$  has connectivity at least two, we can choose a component  $K$  of  $\mathbb{C}_\infty \setminus D$  that does not contain  $q$ . Thus  $f^n(K)$  converges to  $p$  as  $n \rightarrow \infty$ . On the other hand,  $f^n$  fixes  $D$  for each  $n$ . Hence  $p \notin D$ .  $\square$

**Lemma 4.3.** *Suppose that  $D$  is a circular region, and that a loxodromic element  $f$  of  $\text{Aut}(D)$  has a fixed point that lies inside one of the spherical disc components of  $\mathbb{C}_\infty \setminus D$ . Then  $f$  is hyperbolic.*

*Proof.* Let  $E$  be the spherical disc component of  $\mathbb{C}_\infty \setminus D$  that contains a fixed point  $p$  of  $f$ . Since  $f$  acts on the whole of  $\mathbb{C}_\infty$  it permutes the components of  $\mathbb{C}_\infty \setminus D$ . But  $f(p) = p$ , and so  $f(E) = E$ . Thus, by definition,  $f$  is hyperbolic.  $\square$

We can now prove the first part of Theorem 1.1.

Let  $G$  be the conformal automorphism group of a countably connected circular region  $D$  of connectivity at least three. Then, as observed at the end of Section 2,  $G$  is a discrete group of conformal Möbius transformations. Suppose that  $G$  is non-elementary. We must show that  $G$  is a Fuchsian group. By Lemma 4.1, it suffices to show that there are no strictly loxodromic elements in  $G$ .

Let  $E_1$  denote the union of all the components of  $\mathbb{C}_\infty \setminus D$  that are closed discs, and let  $E_2$  denote the union of all the components of  $\mathbb{C}_\infty \setminus D$  that are single points. By Lemma 4.2, every fixed point of a loxodromic element of  $G$  lies in  $E_1 \cup E_2$ . Suppose first that all these fixed points lie in  $E_1$ . Then Lemma 4.3 implies that all the loxodromics are hyperbolic, and the result is proven.

Suppose then, in order to reach a contradiction, that there is a loxodromic element  $f$  of  $G$  with attracting fixed point  $p$ , repelling fixed point  $q$ , and  $p \in E_2$ . Since  $G$  is non-elementary, we can choose an element  $g$  of  $G$  such that  $r = g(p)$  is not equal to  $p$  or  $q$ . Now,  $\{r\}$  is a component of  $E_2$ , and  $r$  is a fixed point of the loxodromic  $gfg^{-1}$ . Moreover, for  $n = 1, 2, \dots$ , each singleton  $\{f^n(r)\}$  is a component of  $E_2$ , and  $f^n(r)$  is a fixed point of the loxodromic  $f^n g f g^{-1} f^{-n}$ . The sequence  $f^n(r)$  accumulates at  $p$ . Let  $F_2$  denote the subset of  $E_2$  consisting of those elements of  $E_2$  that are fixed points of loxodromic elements of  $G$ . We have shown that each element of  $F_2$  is an accumulation point of other elements of  $F_2$ .

Let  $U$  be a closed spherical disc centred on  $p$  that is chosen to be sufficiently small that no spherical disc component of  $E_1$  of radius greater than 1 intersects  $U$ . Choose two points  $p_1$  and  $p_2$  from  $F_2$  that lie in the interior of  $U$ . For  $i = 1, 2$ , let  $U_i$  be a closed spherical disc centred on  $p_i$  that lies inside  $U$  and is chosen to

be sufficiently small that no spherical disc component of  $E_1$  of radius greater than  $1/2$  intersects  $U_i$ . Ensure also that  $U_1$  and  $U_2$  are disjoint. Next, for  $i = 1, 2$ , choose two points  $p_{i1}$  and  $p_{i2}$  from  $F_2$  that lie in the interior of  $U_i$ . For  $j = 1, 2$ , let  $U_{ij}$  be a closed spherical disc centred on  $p_{ij}$  that lies inside  $U_i$  and is chosen to be sufficiently small that no spherical disc component of  $E_1$  of radius greater than  $1/2^2$  intersects  $U_{ij}$ . Ensure also that  $U_{i1}$  and  $U_{i2}$  are disjoint. Continuing in this fashion we obtain, for each sequence  $i_1, i_2, \dots$  from  $\{1, 2\}$ , a sequence of nested closed discs

$$(4.1) \quad U \supset U_{i_1} \supset U_{i_1 i_2} \supset U_{i_1 i_2 i_3} \supset \dots$$

and a sequence of points  $p, p_{i_1}, p_{i_1 i_2}, p_{i_1 i_2 i_3}, \dots$ , where  $p_{i_1 i_2 \dots i_k}$  is the centre of  $U_{i_1 i_2 \dots i_k}$  for each  $k$ , and  $U_{i_1 i_2 \dots i_k}$  intersects no spherical disc component of  $E_1$  of radius greater than  $1/2^k$ . What is more, for each  $k$ , the  $2^k$  discs of the form  $U_{i_1 i_2 \dots i_k}$  are pairwise disjoint.

Let  $u$  be an intersection point of one of the nested sequences (4.1). Then  $u$  is a limit of some sequence  $p, p_{i_1}, p_{i_1 i_2}, \dots$  from  $E_2$ , which means that  $u$  belongs to the closed set  $\mathbb{C}_\infty \setminus D$ . By construction,  $u \notin E_1$ , because no disc component from  $E_1$  can intersect all the sets  $U_{i_1 i_2 \dots i_k}$ . Therefore  $u \in E_2$ . However, there are uncountably many such points  $u$ , which implies that  $E_2$  is uncountable. This contradicts one of our hypotheses. We conclude that  $G$  contains no strictly loxodromic elements, and so it is a Fuchsian group.  $\square$

## 5. PROOF OF THEOREM 1.1: PART II

To complete the proof of Theorem 1.1 it remains to construct, for each discrete group  $G$  that is either Fuchsian or elementary, a countably connected circular region  $D$  such that  $\text{Aut}(D) = G$ . We supply an argument only for a general Fuchsian group; the other constructions are similar.

Let  $G$  be a Fuchsian group acting on the unit disc  $\mathbb{D}$ . Because  $G$  is countable,  $\mathbb{D}$  is uncountable, and each non-identity element of  $G$  has only one or two fixed points in  $\mathbb{D}$ , we can choose a point  $p$  in  $\mathbb{D}$  that is fixed only by the identity element of  $G$ . Since  $G$  acts discontinuously on  $\mathbb{D}$ , we may choose an open hyperbolic disc  $\Delta$  centred on  $p$  such that  $g(\Delta) \cap \Delta = \emptyset$  for each non-identity element  $g$  of  $G$ . Let  $U$  be a closed disc in  $\Delta \setminus \{p\}$ . Let  $G(p)$  denote the orbit of  $p$  under  $G$ , and let  $G(U)$  denote the orbit of the disc  $U$  under  $G$ . Define

$$D = \mathbb{D} \setminus (G(p) \cup G(U)).$$

Then  $G \leq \text{Aut}(D)$ . On the other hand, if  $h \in \text{Aut}(D)$  then  $h$  is a Möbius map that fixes  $\mathbb{D}$ , and permutes the orbit  $G(p)$ . Therefore there exists an element  $g$  of  $G$  such that  $h(p) = g(p)$ . Thus  $g^{-1}h$  fixes  $p$ . Since  $g^{-1}h$  permutes the discs  $G(U)$ , and  $U$  is the unique disc in this collection closest in hyperbolic distance to  $p$ ,  $g^{-1}h$  fixes  $U$  also. Thus  $g^{-1}h$  is the identity map, so  $h = g$ . Therefore  $\text{Aut}(D) \leq G$ , and hence  $G = \text{Aut}(D)$ .

## 6. CONCLUDING REMARKS

At the end of Section 3 we gave an example of an uncountably connected circular region whose conformal automorphism group is neither Fuchsian nor discrete elementary. There are, however, many discrete groups that cannot be obtained as conformal automorphism groups. For instance, if  $G$  is a discrete group of the first kind then  $\Lambda_G = \mathbb{C}_\infty$ , so there are no non-trivial open subsets of  $\mathbb{C}_\infty$  that are invariant under  $G$ . There are also discrete groups that are neither Fuchsian nor discrete elementary, but nonetheless arise as conformal automorphism groups

of countably connected (non-circular) regions. Examples can be constructed from quasi-Fuchsian groups.

In higher dimensions it is no longer true that all finitely connected regions are conformally equivalent to circular regions. This is because a theorem of Liouville says that the only conformal maps of domains in  $\mathbb{R}^n$ , for  $n > 2$ , are Möbius transformations. It can easily be shown that each finite group arises as the conformal automorphism group of some finitely connected region in  $\mathbb{R}^n$ , for some  $n$ . Finitely connected regions (of any connectivity) may not have finite conformal automorphism groups though. Consider, for example, the three circles that lie on the unit sphere in  $\mathbb{R}^6$  given by the equations  $x_1^2 + x_2^2 = 1$ ,  $x_3^2 + x_4^2 = 1$ , and  $x_5^2 + x_6^2 = 1$ . If  $D$  is the complement of these three circles in  $\mathbb{R}^6$  then  $\text{Aut}(D)$  contains  $\text{SO}_2(\mathbb{R}) \times \text{SO}_2(\mathbb{R}) \times \text{SO}_2(\mathbb{R})$ , the direct product of three special orthogonal groups.

Each finite group also arises as the conformal automorphism group of a Riemann surface. This was first established by Greenberg [2], who in fact proved that each countable group arises as the conformal automorphism group of a Riemann surface.

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