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# Colouring of cubic graphs by Steiner triple systems

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## Abstract

Let  $\mathcal{S}$  be a Steiner triple system and  $G$  a cubic graph. We say that  $G$  is  $\mathcal{S}$ -colourable if its edges can be coloured so that at each vertex the incident colours form a triple of  $\mathcal{S}$ . We show that if  $\mathcal{S}$  is a projective system  $PG(n, 2)$ ,  $n \geq 2$ , then  $G$  is  $\mathcal{S}$ -colourable if and only if it is bridgeless, and that every bridgeless cubic graph has an  $\mathcal{S}$ -colouring for every Steiner triple system of order greater than 3. We establish a condition on a cubic graph with a bridge which ensures that it fails to have an  $\mathcal{S}$ -colouring if  $\mathcal{S}$  is an affine system, and we conjecture that this is the only obstruction to colouring any cubic graph with any non-projective system of order greater than 3.

# 1 Introduction

The classical definition of a graph colouring requires adjacent elements – vertices or edges – to carry distinct colours; the aim is to determine the minimum colours needed to colour the graph. If this simple minimality requirement is relaxed, it is natural to impose an additional structure into a colouring and ask whether a colouring with the prescribed property is possible. Among examples of the latter approach so-called list colourings are the most prominent.

The present paper deals with a different, yet very natural generalization of the usual edge colourings of cubic graphs.

As is well known, the edges of any cubic graph (that is, regular of valency 3) can be properly coloured by three or by four colours. The characteristic property of an edge-colouring by three colours – commonly known as a *Tait colouring* – is that the colours of any two adjacent edges determine the colour of the third edge incident with their common vertex. Thus we may regard the colours as the points of a Steiner triple system and the set of colours at each vertex as a triple within the system. In a Tait colouring all the vertices use the same triple of colours, but it is equally reasonable to consider more general edge-colourings of cubic graphs, employing arbitrary Steiner triple systems, where the edges at different vertices are allowed to be coloured by different triples. Such colourings are the main object of study in the present paper.

Recall that a *Steiner triple system*  $\mathcal{S} = (P, B)$  of order  $n$  is a collection  $B$  of three-element subsets (called *triples* or *blocks*) of a set  $P$  of  $n$  points such that each pair of points is together present in exactly one triple.

The smallest Steiner triple system is the *trivial* system which has three points and a single triple. In general, a Steiner triple system of order  $n$  exists if and only if  $n \equiv 1$  or  $3 \pmod{6}$ .

Two infinite classes of Steiner triple systems are of particular interest. The *projective* Steiner triple system  $PG(n, 2)$ ,  $n \geq 2$ , has  $\mathbb{Z}_2^{n+1} - \{0\}$  as its point set, the blocks of the system being the triples  $\{x, y, z\}$  of points such that  $x + y + z = 0$ . The *affine* Steiner triple system  $AG(n, 3)$  has point set  $\mathbb{Z}_3^n$ , the triples of the system being again the triples with zero sum. The first of these classes includes the smallest non-trivial Steiner triple system  $PG(2, 2)$ , the *Fano plane* of order 7.

Given a Steiner triple system  $\mathcal{S}$ , an  $\mathcal{S}$ -*colouring* of a cubic graph  $G$  is a colouring of the edges of  $G$  by points of  $\mathcal{S}$  such that the colours of any

three edges meeting at a vertex form a triple of  $\mathcal{S}$ . If  $G$  admits such a colouring, then we say that it is  $\mathcal{S}$ -colourable. Obviously, a cubic graph is Tait-colourable precisely when it has such a colouring by the trivial Steiner triple system.

The study edge-colourings by Steiner triple systems has been proposed by Archdeacon [A]. The general question reads as follows: *Which cubic graphs can be coloured by which Steiner triple systems?*

The first result along this direction seems to be due to Fu [F] who identified two classes of cubic graphs which can be coloured by the Fano plane. One of these classes consists of all bridgeless cubic graphs of order at most 189, and the second one comprises all such graphs of genus at most 24.

This paper yields a substantial improvement of Fu's results. The following two theorems are our main results.

**Theorem 1.1** *Let  $\mathcal{S}$  be a projective Steiner triple system. Then a cubic graph is  $\mathcal{S}$ -colourable if and only if it is bridgeless.*

**Theorem 1.2** *Every bridgeless cubic graph has an  $\mathcal{S}$ -colouring for every non-trivial Steiner triple system  $\mathcal{S}$ .*

For non-projective Steiner triple systems, the situation concerning cubic graphs with bridges appears to be more complicated. While cubic graphs with bridges cannot be coloured by projective Steiner triple systems, the example in Fig. 1 shows that they can sometimes be coloured with non-projective systems. The colouring uses the affine system  $AG(2, 3)$ , so graph in Fig. 1 is colourable by any affine system. Nevertheless, affine systems do not colour all cubic graphs with bridges.

Figure 1: An  $AG(2, 3)$ -colouring of a cubic graph with a bridge

Consider a cubic graph  $G$  with a bridge  $e = vw$ . Remove the bridge and in the resulting components suppress the 2-valent vertices  $v$  and  $w$ , thereby obtaining cubic graphs  $G_v$  and  $G_w$  at either end of  $e$ . If either  $G_v$  or  $G_w$  is bipartite, we say that  $e$  has a *bipartite end*. As we shall see, bridges with bipartite ends are an obstruction to affine colourings.

**Theorem 1.3** *Let  $\mathcal{S}$  be an affine Steiner triple system, and let  $G$  be a cubic graph containing a bridge with a bipartite end. Then  $G$  cannot be  $\mathcal{S}$ -coloured.*

We have no general positive result concerning cubic graphs with bridges, but offer the following conjecture.

**Conjecture 1.4** *If  $G$  is a simple cubic graph and  $\mathcal{S}$  a non-projective Steiner triple system, then  $G$  fails to have an  $\mathcal{S}$ -colouring only if  $\mathcal{S}$  is affine and  $G$  has a bridge with a bipartite end.*

In the remainder of the paper, graphs are allowed to have multiple edges and loops. Note however that no bridgeless cubic graph can have loops.

## 2 Projective colourings and flows

For any graph  $G$  let  $D(G)$  denote the set which is obtained by replacing each edge of  $G$  with a pair of oppositely directed *darts*; we call  $D(G)$  the *dart-set* of  $G$ . Each dart  $x$ , including those on loops, has its *inverse* dart  $x^{-1} \neq x$  which is incident with the same vertices but has opposite direction. For an arbitrary vertex  $v$ , we let  $D(v)$  be the set of darts emanating from  $v$ . Clearly, these sets partition the whole dart-set.

Let  $A$  be any Abelian group, written additively. We define an  $A$ -flow on  $G$  to be a function  $\phi : D(G) \rightarrow A$  satisfying the following two conditions:

$$(F1) \quad \phi(x^{-1}) = -\phi(x), \quad \text{for each dart } x \in D(G),$$

$$(F2) \quad \sum_{x \in D(v)} \phi(x) = 0, \quad \text{for each vertex } v \in V(G).$$

Such a flow is said to be *nowhere-zero* if  $\phi(x) \neq 0$  for each dart  $x \in D(G)$ .

Observe that if every element of  $A$  is self-opposite, then  $\phi(x) = \phi(x^{-1})$  for each dart  $x$ , and we may simply view an  $A$ -flow on  $G$  to as a function defined on the edges of  $G$  rather than on darts. Note that in this case the group  $A$  will be isomorphic to a direct product of copies of  $\mathbb{Z}_2$ .

A  $\mathbb{Z}$ -flow  $\phi$  such that  $|\phi(x)| < k$  for some integer  $k \geq 2$  is called a  $k$ -flow. Clearly, any  $k$ -flow is at the same time an  $m$ -flow for each  $m \geq k$ . Moreover, it is well known (see, for example, [J] or [D], 6.3–6.4) that a graph admits a nowhere-zero  $k$ -flow if and only if it admits a nowhere-zero  $\mathbb{Z}_k$ -flow, and in turn a nowhere-zero  $A$ -flow in any Abelian group  $A$  of order  $k$ .

**Proof of Theorem 1.1** Let  $\mathcal{S} = PG(n, 2)$  be a projective Steiner triple system,  $n \geq 2$ , and let  $G$  be a bridgeless cubic graph. It follows immediately from the definition that an  $\mathcal{S}$ -colouring of  $G$  is just a nowhere-zero  $\mathbb{Z}_2^{n+1}$ -flow on  $G$ . By Seymour's 6-Flow Theorem (see [D], Theorem ??, or [J], Theorem 4.5), every bridgeless graph has a nowhere-zero 6-flow. It follows from the well-known results about flows that  $G$  also has a nowhere-zero  $2^{n+1}$ -flow, and hence a nowhere-zero  $\mathbb{Z}_2^{n+1}$ -flow. Thus  $G$  is  $\mathcal{S}$ -colourable.

Conversely, note that by a simple counting argument on the ends of edges, an arbitrary  $A$ -flow takes value 0 on any bridge. Therefore a graph with bridges does not admit any nowhere-zero flow. In turn, no cubic graph with bridges admits an  $PG(n, 2)$ -colouring.  $\square$

*Remark.* In the case  $n = 2$ , the projective system becomes the well-known Fano plane. It is tempting to ask whether every bridgeless cubic graph  $G$  has a Fano colouring that avoids some point. The answer is *no*. If  $\phi$  is a Fano colouring of  $G$  that avoids (say)  $(1, 1, 1)$ , then by mapping  $(0, 0, 1)$  and  $(1, 1, 0)$  to 1,  $(0, 1, 0)$  and  $(1, 0, 1)$  to 2, and  $(1, 0, 0)$  and  $(0, 1, 1)$  to 3, we obtain a proper 3-edge-colouring of  $G$ . However, there are cubic graphs which have no 3-edge-colouring, such as the Petersen graph and other snarks. Therefore the required colouring by the Fano plane does not exist in general.

### 3 Factorisable colourings

Let  $\mathcal{S}$  be a Steiner triple system, let  $G$  be a bridgeless cubic graph, and let  $F$  be a 1-factor of  $G$ . We say that an  $\mathcal{S}$ -colouring  $\phi$  of  $G$  is *factorisable over  $F$*  if there is a block  $b$  such that  $\phi(x) \in b$  if and only if  $x \in F$ .

In the following theorem we show that any cyclically 4-edge-connected graph has a factorisable projective colouring. A graph is said to be *cyclically  $k$ -edge-connected* if no edge-cut involving fewer than  $k$  edges leaves two or more components containing circuits.

**Theorem 3.1** *Let  $\mathcal{S}$  be a projective Steiner triple system, and let  $G$  be a cyclically 4-edge-connected cubic graph. Then, for an arbitrarily chosen 1-factor  $F$  in  $G$ , the graph  $G$  has an  $\mathcal{S}$ -colouring that is factorisable over  $F$ .*

*Proof.* Let  $E_1$  be a 1-factor in  $G$ . (Note that by Petersens's theorem ([D], Corollary 2.2.2)  $G$  indeed contains a 1-factor.) Contract the edges of

the complementary 2-factor  $E_2 = E(G) - E_1$  so that each circuit of  $E_2$  is contracted into a single vertex thereby obtaining a graph  $G'$ . In general,  $G'$  may contain multiple edges and loops even when  $G$  was simple.

Theorem 4.7 of [J] shows that every bridgeless graph without edge 3-cuts has a nowhere-zero  $(\mathbb{Z}_2 \times \mathbb{Z}_2)$ -flow. Since  $G$  is cyclically 4-edge-connected,  $G'$  has no 3-cuts and is bridgeless. Thus  $G'$  has a nowhere-zero  $(\mathbb{Z}_2 \times \mathbb{Z}_2)$ -flow, say  $\theta$ . Let  $\lambda : \mathbb{Z}_2 \times \mathbb{Z}_2 \rightarrow \mathbb{Z}_2^{n+1}$  be the mapping which embeds  $\mathbb{Z}_2 \times \mathbb{Z}_2$  into  $\mathbb{Z}_2^{n+1}$  by sending an arbitrary element  $(x, y)$  of  $\mathbb{Z}_2 \times \mathbb{Z}_2$  to  $(x, y, 0, \dots, 0)$ . Then  $\lambda\theta$  is a nowhere-zero  $\mathbb{Z}_2^{n+1}$ -flow on  $G'$ . Expand the vertices of  $G'$  back to the original circuits of  $G$ , so that the edges of  $G'$  become edges of  $G$ , with the flow values determined by  $\lambda\theta$ . Now let  $C$  be any circuit of  $E_2$  formed by the edges  $x_1, x_2, \dots, x_q$  in cyclic order. For  $i = 1, \dots, q - 1$  let  $y_i$  be the edge of  $E_1$  incident with  $x_i$  and  $x_{i+1}$ , and let  $y_q$  be the edge of  $E_1$  incident with  $x_q$  and  $x_1$ . (Any chords of  $C$  will have two labels.) Define

$$\lambda\theta(x_1) = (1, 0, \dots, 0),$$

$$\lambda\theta(x_r) = (1, 0, \dots, 0) + \sum_{i=1}^r \lambda\theta(y_i) \quad (r = 1, \dots, q - 1).$$

When this construction is applied to each circuit of  $E_2$ ,  $\lambda\theta$  is a an  $\mathcal{S}$ -colouring of  $G$  that is factorisable over  $E_1$ .  $\square$

## 4 Affine colourings

It is convenient to address Theorem 1.3 before Theorem 1.2, by considering affine Steiner triple systems in the context of cubic graphs with bridges.

The smallest affine Steiner triple system is  $AG(2, 3)$  of order 9, and we have seen an example of a simple cubic graph with a bridge that has an  $AG(2, 3)$ -colouring. However, not every such graph has an  $AG(2, 3)$ -colouring or, indeed, a colouring with any affine Steiner triple system.

Define a *weak 3-colouring* of a cubic graph  $G$  to be an edge-colouring  $\phi$  (possibly non-proper) with three colours such that, at each vertex, the colours are either all distinct or all equal, and let the *weakness set* of  $\phi$  be the set of vertices of  $G$  at which the colours are all equal.

Then it is straightforward to see that an  $AG(2, 3)$ -colouring of a cubic graph is a colouring  $\phi(x) = (\phi_1(x), \phi_2(x))$ ,  $(x \in E(G))$ , with elements of  $\mathbb{Z}_3 \times \mathbb{Z}_3$  such that  $\phi_1$  and  $\phi_2$  are weak 3-colourings with disjoint weakness

sets. Thus a cubic graph is  $AG(2, 3)$ -colourable if and only if it has two weak 3-colourings with disjoint weakness sets. More generally, for any affine Steiner triple system  $\mathcal{T} = AG(n, 3)$ ,  $n \geq 2$ , a  $\mathcal{T}$ -colouring of  $G$  is a colouring  $\phi(x) = (\phi_1(x), \phi_2(x), \dots, \phi_n(x))$  such that all coordinate mappings  $\phi_i$  are weak 3-colourings with the property that no vertex of  $G$  belongs to all the weakness sets.

We are now in a position to prove Theorem 1.3.

**Proof of Theorem 1.3** Let  $G$  be a cubic graph with a bridge  $b = uv$  where the graph  $G_v$  at one end of  $b$  is bipartite, and let  $\{X, Y\}$  be the bipartition of the vertices of  $G_v$ . If  $G'_v$  is the component of  $G - b$  containing  $v$ , then we can think of  $G'_v$  as the result of reinserting the vertex  $v$  into an edge  $xy$  of  $G_v$ , where  $x \in X$  and  $y \in Y$ .

In order to prove that  $G$  has no affine colouring we shall show that  $v$  belongs to the weakness set of every weak 3-colouring of  $G$ .

Take an arbitrary weak 3-colouring of  $G$ , and let  $\omega : E(G'_v) \rightarrow \mathbb{Z}_3$  be its restriction on  $G'_v$ . Direct the edges of  $G'_v$  from  $X$  to  $Y$  so that  $xv$  will be directed to  $v$  and  $vy$  will be directed from  $v$ . Now extend the mapping  $\omega : E(G'_v) \rightarrow \mathbb{Z}_3$  to a mapping  $\bar{\omega} : D(G'_v) \rightarrow \mathbb{Z}_3$  in such a way that the directions from  $X$  to  $Y$  inherit the original value of  $\omega$  whereas the inverse directions receive the corresponding opposite value. With this definition,  $\bar{\omega}$  fulfils the flow condition (F1). Moreover, since  $\omega$  is a weak 3-colouring,  $\bar{\omega}$  satisfies (F2) at every vertex of  $X \cup Y$ . However, (F2) must then hold at the vertex  $v$  as well, which immediately implies that  $\omega(xv) = \omega(vy)$ . The latter is only possible when  $v$  is a weakness vertex of the original weak 3-colouring of  $G$ . Therefore  $G$  cannot have an  $AG(n, 3)$ -colouring for any  $n \geq 2$ .  $\square$

## 5 General Steiner colourings

Section 2 resolves the question of colourings by projective Steiner triple systems. In order to investigate the more general case, we need to establish the existence of certain configurations in non-projective Steiner triple systems. Grannell, Griggs and Mendelsohn [G] define a  $C_{14}$ -configuration in an Steiner triple system to be a configuration (that is, a collection of blocks) of seven distinct points  $a, b, c, d, e, f, g$  and four blocks  $\{a, b, f\}$ ,  $\{a, c, e\}$ ,  $\{b, c, d\}$  and  $\{e, f, g\}$ . Theorem 3.2 of [S], together with the formula for the number



of  $C_{14}$ -configurations in a Steiner triple system given in [G], shows that every non-projective Steiner triple system contains a  $C_{14}$ -configuration.

For the following discussion, if  $x$  and  $y$  are any two distinct points of  $\mathcal{S}$ , we denote by  $xy$  the third point of the block containing them. Now consider the points  $eb$  and  $cf$ . By uniqueness, neither of the blocks  $\{b, e, eb\}$ ,  $\{c, f, cf\}$  is equal to any of the blocks of the  $C_{14}$ , and so neither  $h = eb$  nor  $i = cf$  can coincide with any of  $a, \dots, g$ , though it is possible that  $h = i$ . A  $D$ -configuration is such a configuration in  $\mathcal{S}$ , consisting of eight or nine points  $a, b, \dots, h, i$  (where possibly  $h = i$ ), and the six blocks  $\{a, b, f\}$ ,  $\{a, c, e\}$ ,  $\{b, c, d\}$ ,  $\{e, f, g\}$ ,  $\{b, e, h\}$ ,  $\{c, f, i\}$ . Then a  $D$ -colouring of a cubic graph  $G$  is an edge-colouring such that the colours at each vertex form one of the triples of some  $D$ -configuration of a given Steiner triple system. The configuration is further described as a  $D_8$  or a  $D_9$  according as the number of points is 8 or 9.

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Figure 2

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In a  $D$ -configuration, a vertex belonging to exactly two triples of the configuration is said to be an *apex*. Thus, a  $D_8$  contains two apexes ( $a$  and  $h$  in Figure 2) while a  $D_9$  contains one ( $a$  in Figure 2).

**Lemma 5.1** *Let  $\mathcal{S}$  be any non-trivial Steiner triple system that does not contain the Fano plane as a configuration, and let  $\{a, b, f\}$  be a block of  $\mathcal{S}$ . Then  $\mathcal{S}$  contains a  $D$ -configuration of which  $a$  is an apex and  $\{a, b, f\}$  is a block.*

*Proof.* Since  $\mathcal{S}$  is non-trivial, there is a second block  $\{a, c, e\}$  where  $b, f, c$  and  $e$  are distinct.

If either  $ef \neq bc$  or  $eb \neq cf$ , then the points  $a, b, c, e, f, ef, bc, eb, cf$  and the blocks  $\{a, b, f\}$ ,  $\{a, c, e\}$ ,  $\{b, c, bc\}$ ,  $\{e, f, ef\}$ ,  $\{b, e, eb\}$ ,  $\{c, f, cf\}$  form a  $D$ -configuration with  $a$  as an apex.

If, on the other hand, both  $ef = bc$  and  $be = cf$ , then denote these points by  $d$  and  $g$  respectively. By uniqueness of triples containing pairs,  $d$  and  $g$  must be distinct from each other and from  $a, b, c, e$  and  $f$ .

Let  $ag = h$ . Now  $\{a, d, g\}$  cannot be a block, since otherwise the seven points and seven blocks would form a Fano plane. Thus,  $h \neq d$ , and clearly  $h$  cannot coincide with any of  $b, f, c$  or  $e$ . Now consider the configuration

whose points are  $a, b, c, e, f, g, h, bh$  and  $fh$  and whose blocks are  $\{a, b, f\}$ ,  $\{a, g, h\}$ ,  $\{b, e, g\}$ ,  $\{f, h, fh\}$ ,  $\{f, g, c\}$ ,  $\{b, h, bh\}$ . Since each of  $bh$  and  $fh$  must be distinct from  $a, b, c, e, f$  and  $h$  and from each other, the configuration is a  $D_9$  with  $a$  the apex.  $\square$

**Lemma 5.2** *Let  $G$  be a cyclically 4-edge-connected cubic graph, and let  $\mathcal{S}$  be a non-projective Steiner triple system containing a  $D$ -configuration. Then  $G$  has a  $D$ -colouring. Moreover, if  $a$  is an apex of  $D$ ,  $\{a, b, f\}$  is a block and  $v \in V(G)$ , then the colouring may be chosen so that the colours at  $v$  are  $a, b, f$  in any chosen order.*

*Proof.* By Petersen's Theorem ([D], Corollary 2.2.2),  $G$  can be decomposed into a 1-factor  $E_1$  and a 2-factor  $E_2$ . Let  $G'$  be the graph resulting from contracting each circuit of  $E_2$  into a single vertex; let us denote by  $v'$  the vertex of  $G'$  corresponding to the circuit of  $G$  that contains  $v$ . Then  $G'$  is bridgeless and has no edge 3-cut. Thus by Theorem 4.7 of [J],  $G'$  has a nowhere-zero  $(\mathbb{Z}_2 \times \mathbb{Z}_2)$ -flow, say  $\theta$ . For any non-zero element  $h$  of  $\mathbb{Z}_2 \times \mathbb{Z}_2$  let  $G'_h$  be the spanning subgraph of  $G'$  formed by all edges carrying the flow value  $h$ . If we identify the elements  $(0, 1)$ ,  $(1, 0)$  and  $(1, 1)$  with 1, 2 and 3, respectively, we obtain a partition of  $G'$  into  $G'_1, G'_2$  and  $G'_3$  with the property that for each vertex  $w$  of  $G'$  and for each  $i \in \{1, 2, 3\}$ , the number of edges of  $G'_i$  incident with  $w$  has the same parity as the valency of  $w$ . Moreover, we may choose  $\theta$  so that  $v' \in G'_3$ .

Let us repeatedly remove circuits from  $G'_1$  and from  $G'_2$  until we arrive at a spanning subgraph  $H' \subseteq G'$  such that the corresponding subgraphs  $H'_1$  and  $H'_2$  are acyclic. Clearly,  $\theta|_{H'}$  is a nowhere-zero  $(\mathbb{Z}_2 \times \mathbb{Z}_2)$ -flow on  $H'$ , so the partition of  $H'$  into  $H'_1, H'_2$  and  $H'_3$  has the same parity property as the corresponding partition of  $G'$ . We now use  $H'_1$  and  $H'_2$  to construct two weak 3-colourings  $\phi_1$  and  $\phi_2$  of  $G$  with disjoint weakness sets.

In order to construct  $\phi_1$ , let us order the vertices of  $G'$  as  $w'_1, w'_2, \dots, w'_z$  in such a way that each  $w'_j$  is adjacent in  $H'_1$  to at most one of its predecessors. Such an ordering is possible due to the fact that  $H'_1$  is a spanning forest of  $G'$ . We give the circuits of  $E_2$  (which are circuits in  $G$ ) the corresponding ordering,  $C_1, C_2, \dots, C_z$ , and let  $T_1$  be the subset of  $E_1$  corresponding to the edge-set of  $H'_1$ . For each  $e \in E_1 - T_1$  define  $\phi_1(e) = 0$ . Next, for each  $j = 1, 2, \dots, z$  label the edges of  $C_1$  by  $f_{11}, f_{12}, \dots, f_{1m}$  in cyclic order (where  $m$  is the length of  $C_1$ ). Define  $\phi_1(f_{11}) = 1$ , and proceed in cyclic order, so

that  $\phi_1(f_{1k}) = 1$  or  $2$  ( $k = 2, \dots, c_j$ ) where  $\phi_1(f_{1k}) = \phi_1(f_{1k+1})$  if and only if the edge  $e$  in  $E_1$  that is incident with  $f_{1k}$  and  $f_{1k+1}$  belongs to  $T_1$ . Where this is so, define also  $\phi_1(e) = \phi_1(f_{1k})$ . Due to the parity-respecting nature of  $T_1$ , we also have  $f_{1m} = f_{11}$  if and only if the appropriate edge in  $E_1$  belongs to  $T_1$ .

Process the circuits in order. In each case, if an edge  $t$  in  $T_1$  is adjacent both to  $C_j$  and to some predecessor, start the cycle from an edge of  $C_j$  adjacent to  $t$  and colour appropriately. Since there is at most one such adjacency with a predecessor circuit, the result is a weak 3-edge-colouring  $\phi_1$  of  $G$  whose weakness set consists of all vertices of  $G$  that are incident with an edge in  $T_1$ .

Now repeat the process with  $H'_2$  in place of  $H'_1$ , to obtain a second weak colouring  $\phi_2$ . Its weakness vertices are disjoint from those of  $\phi_1$ , since  $T_1$  and  $T_2$  are disjoint subsets of a 1-factor. Define the colouring  $\phi : E(G) \rightarrow \mathbb{Z}_3 \times \mathbb{Z}_3$  by setting

$$\phi(e) = (\phi_1(e), \phi_2(e))$$

for each edge  $e$  of  $G$ .

We further show that the affine colouring  $\phi$  which we have constructed can be modified into a  $D$ -colouring having the colours  $a, b, f$  incident with  $v$ .

By the method of construction of  $\phi$ , there are at most six triples of colours at the vertices of  $G$ . Indeed, at a weakness vertex of  $\phi_1$ , the triple is  $\{(1, 0), (1, 1), (1, 2)\}$  or  $\{(2, 0), (2, 1), (2, 2)\}$ ; at a weakness vertex of  $\phi_2$ , it is  $\{(0, 1), (1, 1), (2, 1)\}$  or  $\{(0, 2), (1, 2), (2, 2)\}$ ; and at any other vertex, it is  $\{(0, 0), (1, 1), (2, 2)\}$  or  $\{(0, 0), (1, 2), (2, 1)\}$ . Since  $v$  is not a weakness vertex, the triple at  $v$  is one of the last two. Thus, by composing each  $\phi_i$  ( $i = 1, 2$ ) with a suitable permutation of  $\mathbb{Z}_3$  if necessary, we may arrange that the triple at  $v$  is  $\{(0, 0), (1, 1), (2, 2)\}$  in any chosen order.

Now let  $a$  be an apex of  $D$  and let  $b, c, d, e, f, g, h, i$  be the other points of  $D$  (where, if  $D$  is a  $D_8$ , then  $h = i$ ), chosen such that the blocks are  $\{a, b, f\}$ ,  $\{a, c, e\}$ ,  $\{b, c, d\}$ ,  $\{e, f, g\}$ ,  $\{b, e, h\}$ , and  $\{c, f, i\}$ . Define the mapping  $\sigma : \mathbb{Z}_3 \times \mathbb{Z}_3 \rightarrow D$  as follows:

$$\begin{aligned} \sigma(0, 0) &= a; \sigma(1, 1) = b; \sigma(2, 2) = f; \\ \sigma(1, 2) &= c; \sigma(1, 0) = d; \sigma(2, 1) = e; \\ \sigma(2, 0) &= g; \sigma(0, 1) = h; \sigma(0, 2) = i. \end{aligned}$$

Thus, if the configuration is a  $D_8$ , then  $\sigma(0, 1) = \sigma(0, 2)$ . Otherwise, all  $\sigma$ -images are distinct.

Finally, define  $\Phi = \sigma\phi$ . Then  $\Phi$  is a  $D$ -colouring of  $G$  with colours  $a, b, f$  at  $v$  in any desired order.  $\square$

**Proof of Theorem 1.2.** Let  $\mathcal{S}$  be any non-trivial Steiner triple system, and let  $G$  be a bridgeless cubic graph. We want to show that  $G$  is  $\mathcal{S}$ -colourable. If  $\mathcal{S}$  contains the Fano plane, then the result follows from Theorem 1.1. Thus we may assume that  $\mathcal{S}$  does not contain the Fano plane as a configuration.

Assume the theorem to be false, and let  $G$  be a bridgeless cubic graph of minimum order that does not have an  $\mathcal{S}$ -colouring. By Lemma 5.2,  $G$  contains a cycle-separating edge-cut of size 2 or 3. At least one of these cuts must be such that one of the resulting components, say  $P$ , does not itself have a cycle-separating edge-cut of size smaller than 4. Take both the cut and  $P$  to be of minimum size. Denote the cut by  $S$ , and let  $Q$  be the other component of  $G - S$ .

*Case 1.* First assume that  $|S|=3$ , where  $S = \{r, s, t\}$ . Let  $G/P$  and  $G/Q$  be the graphs formed from  $G$  by contracting  $P$  or  $Q$ , respectively, into a single vertex (denoted by  $p$  and  $q$  respectively). By the minimality assumption,  $G/P$  has an  $\mathcal{S}$ -colouring, say  $\phi_P$ .

Let  $\phi_P(r) = a$ ,  $\phi_P(s) = b$ ,  $\phi_P(t) = f$ . By Lemma 5.1, a  $D$ -configuration may be chosen in  $\mathcal{S}$  such that  $\{a, b, f\}$  is a block and  $a$  an apex. Now by Lemma 5.2,  $G/Q$  has a  $D$ -colouring  $\phi_Q$  such that  $\phi_P(z) = \phi_Q(z)$  for any  $z \in S$ . Hence there is an  $\mathcal{S}$ -colouring of  $G$ .

*Case 2.* Now assume that  $|S|=2$ , where  $S = \{r, s\}$ . Construct  $G/P$  and  $G/Q$  as in Case 1, but now suppress the 2-valent vertices  $p, q$ , resulting in edges  $e_p$  and  $e_q$  respectively. Let  $\phi_P(e_p) = a$ . By Lemma 5.1,  $D$  may be chosen so that  $a$  is an apex. Thus we may choose the colouring of  $G/Q$  so that  $\phi_Q(e_q) = a$ . Hence, once again there is an  $\mathcal{S}$ -colouring of  $G$ , and the result is proved.  $\square$

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