Sets of points determining only acute angles and some related colouring problems

How to cite:


For guidance on citations see FAQs

Copyright and Moral Rights for the articles on this site are retained by the individual authors and/or other copyright owners. For more information on Open Research Online’s data policy on reuse of materials please consult the policies page.
Sets of Points Determining Only Acute Angles and Some Related Colouring Problems

David Bevan
Fernwood, Leaford Crescent,
Watford, Herts. WD24 5TW England
dbevan@emtex.com

Submitted: Jan 20, 2004; Accepted: Feb 7, 2006; Published: Feb 15, 2006
Mathematics Subject Classifications: 05D40, 51M16

Abstract

We present both probabilistic and constructive lower bounds on the maximum size of a set of points \( S \subseteq \mathbb{R}^d \) such that every angle determined by three points in \( S \) is acute, considering especially the case \( S \subseteq \{0,1\}^d \). These results improve upon a probabilistic lower bound of Erdős and Füredi. We also present lower bounds for some generalisations of the acute angles problem, considering especially some problems concerning colourings of sets of integers.

1 Introduction

Let us say that a set of points \( S \subseteq \mathbb{R}^d \) is an acute \( d \)-set if every angle determined by a triple of \( S \) is acute \((< \frac{\pi}{2})\). Let us also say that \( S \) is a cubic acute \( d \)-set if \( S \) is an acute \( d \)-set and is also a subset of the unit \( d \)-cube (i.e. \( S \subseteq \{0,1\}^d \)).

Let us further say that a triple \( u, v, w \in \mathbb{R}^d \) is an acute triple, a right triple, or an obtuse triple, if the angle determined by the triple with apex \( v \) is less than \( \frac{\pi}{2} \), equal to \( \frac{\pi}{2} \), or greater than \( \frac{\pi}{2} \), respectively. Note that we consider the triples \( u, v, w \) and \( w, v, u \) to be the same.

We will denote by \( \alpha(d) \) the size of a largest possible acute \( d \)-set. Similarly, we will denote by \( \kappa(d) \) the size of a largest possible cubic acute \( d \)-set. Clearly \( \kappa(d) \leq \alpha(d) \), \( \kappa(d) \leq \kappa(d+1) \) and \( \alpha(d) \leq \alpha(d+1) \) for all \( d \).
In [EF], Paul Erdős and Zoltán Füredi gave a probabilistic proof that \( \kappa(d) \geq \left[ \frac{1}{2} \left( \frac{2^d}{\sqrt{3}} \right) \right] \) (see also [AZ2]). This disproved an earlier conjecture of Ludwig Danzer and Branko Grünbaum [DG] that \( \alpha(d) = 2d - 1 \).

In the following two sections we give improved probabilistic lower bounds for \( \kappa(d) \) and \( \alpha(d) \). In section 4 we present a construction that gives further improved lower bounds for \( \kappa(d) \) for small \( d \). In section 5, we tabulate the best lower bounds known for \( \kappa(d) \) and \( \alpha(d) \) for small \( d \). Finally, in sections 6–9, we give probabilistic and constructive lower bounds for some generalisations of \( \kappa(d) \), considering especially some problems concerning colourings of sets of integers.

2 A probabilistic lower bound for \( \kappa(d) \)

**Theorem 2.1**

\[
\kappa(d) \geq 2 \left[ \frac{\sqrt{6}}{9} \left( \frac{2}{\sqrt{3}} \right)^d \right] \approx 0.544 \times 1.155^d.
\]

For large \( d \), this improves upon the result of Erdős and Füredi by a factor of \( \frac{4\sqrt{6}}{9} \approx 1.089 \). This is achieved by a slight improvement in the choice of parameters. This proof can also be found in [AZ3].

**Proof:** Let \( m = \left[ \frac{\sqrt{6}}{9} \left( \frac{2}{\sqrt{3}} \right)^d \right] \) and randomly pick a set \( S \) of \( 3m \) point vectors from the vertices of the \( d \)-dimensional unit cube \( \{0, 1\}^d \), choosing the coordinates independently with probability \( \Pr[v_i = 0] = \Pr[v_i = 1] = \frac{1}{2} \), \( 1 \leq i \leq d \), for every \( v = (v_1, v_2, \ldots, v_d) \in S \).

Now every angle determined by a triple of points from \( S \) is non-obtuse (\( \leq \frac{\pi}{2} \)), and a triple of vectors \( u, v, w \) from \( S \) is a right triple iff the scalar product \( \langle u - v, w - v \rangle \) vanishes, i.e. iff either \( u_i - v_i = 0 \) or \( w_i - v_i = 0 \) for each \( i, 1 \leq i \leq d \).

Thus \( u, v, w \) is a right triple iff \( u_i, v_i, w_i \) is neither 0, 1, 0 nor 1, 0, 1 for any \( i, 1 \leq i \leq d \). Since \( u_i, v_i, w_i \) can take eight different values, this occurs independently with probability \( \frac{3}{4} \) for each \( i \), so the probability that a triple of \( S \) is a right triple is \( \left( \frac{3}{4} \right)^d \).

Hence, the expected number of right triples in a set of \( 3m \) vectors is \( 3 \binom{3m}{3} \left( \frac{3}{4} \right)^d \). Thus there is some set \( S \) of \( 3m \) vectors with no more than \( 3 \binom{3m}{3} \left( \frac{3}{4} \right)^d \) right triples, where

\[
3 \binom{3m}{3} \left( \frac{3}{4} \right)^d \leq m \left( \frac{9m}{\sqrt{6}} \right) \left( \frac{3}{4} \right)^d \leq m
\]

by the choice of \( m \).
If we remove one point of each right triple from $S$, the remaining set is a cubic acute $d$-set of cardinality at least $3m - m = 2m$.

3 A probabilistic lower bound for $\alpha(d)$

We can improve the lower bound in theorem 2.1 for non-cubic acute $d$-sets by a factor of $\sqrt{2}$ by slightly perturbing the points chosen away from the vertices of the unit cube. The intuition behind this is that a small random symmetrical perturbation of the points in a right triple is more likely than not to produce an acute triple, as the following diagram suggests.

Theorem 3.1

$$\alpha(d) \geq 2 \left\lfloor \frac{1}{3} \left( \frac{2}{\sqrt{3}} \right)^{d+1} \right\rfloor \approx 0.770 \times 1.155^d.$$ 

Before we can prove this theorem, we need some results concerning continuous random variables.

Definition 3.2 If $F(x) = \Pr[X \leq x]$ is the cumulative distribution function of a continuous random variable $X$, let $\overline{F}(x)$ denote $\Pr[X \geq x] = 1 - F(x)$.

Definition 3.3 Let us say that a continuous random variable $X$ has positive bias if, for all $t$, $\Pr[X \geq t] \geq \Pr[X \leq -t]$, i.e. $\overline{F}(t) \geq F(-t)$.

Property 3.3.1 If a continuous random variable $X$ has positive bias, it follows that $\Pr[X > 0] \geq \frac{1}{2}$.

Property 3.3.2 To show that a continuous random variable $X$ has positive bias, it suffices to demonstrate that the condition $\overline{F}(t) \geq F(-t)$ holds for all positive $t$. 

THE ELECTRONIC JOURNAL OF COMBINATORICS 13 (2006), #R12
Lemma 3.4 If $X$ and $Y$ are independent continuous random variables with positive bias, then $X + Y$ also has positive bias.

Proof: Let $f$, $g$ and $h$ be the probability density functions, and $F$, $G$ and $H$ the cumulative distribution functions, for $X$, $Y$ and $X + Y$ respectively. Then,

\[ H(t) - H(-t) = \int \int_{x+y \geq t} f(x)g(y) \, dy \, dx - \int \int_{x+y \leq -t} f(x)g(y) \, dy \, dx \]

\[ = \int \int_{x+y \geq t} f(x)g(y) \, dy \, dx - \int \int_{y-x \geq t} f(x)g(y) \, dy \, dx \]

\[ + \int \int_{y-x \geq t} f(x)g(y) \, dy \, dx - \int \int_{x+y \leq -t} f(x)g(y) \, dy \, dx \]

\[ = \int_{-\infty}^{\infty} g(y) \left[ F(t-y) - F(y-t) \right] \, dy \]

\[ + \int_{-\infty}^{\infty} f(x) \left[ G(x+t) - G(-x-t) \right] \, dx \]

which is non-negative because $f(t)$, $g(t)$, $F(t) - F(-t)$ and $G(t) - G(-t)$ are all non-negative for all $t$. \qed

Definition 3.5 Let us say that a continuous random variable $X$ is $\epsilon$-uniformly distributed for some $\epsilon > 0$ if $X$ is uniformly distributed between $-\epsilon$ and $\epsilon$.

Let us denote by $j$, the probability density function of an $\epsilon$-uniformly distributed random variable:

\[ j(x) = \left\{ \begin{array}{ll}
\frac{1}{2\epsilon} & \text{if } -\epsilon \leq x \leq \epsilon \\
0 & \text{otherwise}
\end{array} \right. \]

and by $J$, its cumulative distribution function:

\[ J(x) = \left\{ \begin{array}{ll}
0 & \text{if } x < -\epsilon \\
\frac{1}{2} + \frac{x}{2\epsilon} & \text{if } -\epsilon \leq x \leq \epsilon \\
1 & \text{if } x > \epsilon
\end{array} \right. \]

Property 3.5.1 If $X$ is an $\epsilon$-uniformly distributed random variable, then so is $-X$. 

Lemma 3.6 If $X$, $Y$ and $Z$ are independent $\epsilon$-uniformly distributed random variables for some $\epsilon < \frac{1}{2}$, then $U = (Y - X)(1 + Z - X)$ has positive bias.

Proof: Let $G$ be the cumulative distribution function of $U$. By 3.3.2, it suffices to show that $G(u) - G(-u) \geq 0$ for all positive $u$.

Let $u$ be positive. Because $1 + Z - X$ is always positive, $U \geq u$ iff $Y > X$ and $Z \geq -1 + X + \frac{u}{y-x}$. Similarly, $U \leq -u$ iff $X > Y$ and $Z \geq -1 + X + \frac{u}{x-y}$. So,

$$
G(u) - G(-u) = \int \int j(x)j(y)J(-1 + x + \frac{u}{y-x}) dy \, dx
- \int \int j(x)j(y)J(-1 + x + \frac{u}{x-y}) dy \, dx
= \int \int j(x)j(y) \left[ J(1 - x - \frac{u}{y-x}) - J(1 - y - \frac{u}{y-x}) \right] dy \, dx
$$

(because $J(x) = J(-x)$, and by variable renaming)

which is non-negative because $j$ is non-negative and $J$ is non-decreasing (so the expression in square brackets is non-negative over the domain of integration). \qed

Corollary 3.6.1 If $X$, $Y$ and $Z$ are independent $\epsilon$-uniformly distributed random variables for some $\epsilon < \frac{1}{2}$, then $(Y - X)(Z - X - 1)$ has positive bias.

Proof: $(Y - X)(Z - X - 1) = ((-Y) - (-X))(1 + (-Z) - (-X))$. The result follows from 3.5.1 and lemma 3.6. \qed

Lemma 3.7 If $X$, $Y$ and $Z$ are independent $\epsilon$-uniformly distributed random variables, then $V = (Y - X)(Z - X)$ has positive bias.

Proof: Let $H$ be the cumulative distribution function of $V$. By 3.3.2, it suffices to show that $H(v) - H(-v) \geq 0$ for all positive $v$. \qed
Let \( v \) be positive. \( V \ge v \) iff \( Y > X \) and \( Z \ge X + \frac{v}{\sqrt{X}} \) or \( Y < X \) and \( Z \le X + \frac{v}{\sqrt{X}} \). Similarly, \( V \le -v \) iff \( Y > X \) and \( Z \le X - \frac{v}{\sqrt{X}} \) or \( Y < X \) and \( Z \ge X - \frac{v}{\sqrt{X}} \). So,

\[
\overline{H}(v) - H(-v) = \int_{y>x} j(x)j(y) \mathcal{J}(x + \frac{v}{y-x}) \, dy \, dx + \int_{y<x} j(x)j(y) \mathcal{J}(x + \frac{v}{y-x}) \, dy \, dx - \int_{y>x} j(x)j(y) \mathcal{J}(x - \frac{v}{y-x}) \, dy \, dx - \int_{y<x} j(x)j(y) \mathcal{J}(x - \frac{v}{y-x}) \, dy \, dx
\]

which is non-negative because \( j \) is non-negative and \( J \) is non-decreasing (so the expressions in square brackets are non-negative over the domains of integration). \( \square \)

We are now in a position to prove the theorem.

**Proof of theorem 3.1**

Let \( m = \left\lfloor \frac{1}{3} \left( \frac{2}{\sqrt{3}} \right)^{d+1} \right\rfloor \), and randomly pick a set \( S \) of 3m point vectors, \( v_1, v_2, \ldots, v_{3m} \), from the vertices of the \( d \)-dimensional unit cube \( \{0,1\}^d \), choosing the coordinates independently with probability \( \Pr[v_{ki} = 0] = \Pr[v_{ki} = 1] = \frac{1}{2} \) for every \( v_k = (v_{k1}, v_{k2}, \ldots, v_{kd}) \), \( 1 \le k \le 3m, 1 \le i \le d \).

Now for some \( \epsilon, 0 < \epsilon < \frac{1}{2(d+1)} \), randomly pick 3m vectors, \( \delta_1, \delta_2, \ldots, \delta_{3m} \), from the \( d \)-dimensional cube \( [-\epsilon, \epsilon]^d \) of side 2\( \epsilon \) centred on the origin, choosing the coordinates \( \delta_{ki} \), \( 1 \le k \le 3m, 1 \le i \le d \), independently so that they are \( \epsilon \)-uniformly distributed, and let \( S' = \{v'_1, v'_2, \ldots, v'_{3m}\} \) where \( v'_k = v_k + \delta_k \) for each \( k, 1 \le k \le 3m \).

**Case 1: Acute triples in \( S \)**

Because \( \epsilon < \frac{1}{2(d+1)} \), if \( v_j, v_k, v_l \) is an acute triple in \( S \), the scalar product \( \langle v'_j - v'_k, v'_l - v'_k \rangle > \frac{1}{(d+1)^2} \), so \( v'_j, v'_k, v'_l \) is also an acute triple in \( S' \).

**Case 2: Right triples in \( S \)**

If, \( v_j, v_k, v_l \) is a right triple in \( S \) then the scalar product \( \langle v_j - v_k, v_l - v_k \rangle \) vanishes, i.e. either \( v_{ji} - v_{ki} = 0 \) or \( v_{li} - v_{ki} = 0 \) for each \( i, 1 \le i \le d \). There are six possibilities for each triple of coordinates:
Now, the values of the $\delta_{ki}$ are independent and $\varepsilon$-uniformly distributed, so by lemmas 3.7 and 3.6 and corollary 3.6.1, the distribution of the $(v_j' - v_k')(v_i' - v_k')$ has positive bias, and by repeated application of lemma 3.4, the distribution of the scalar product $\langle v_j' - v_k', v_i' - v_k' \rangle = \sum_{i=1}^d (v_j'_{\delta_{ii}} - v_k'_{\delta_{ki}})(v_i'_{\delta_{li}} - v_k'_{\delta_{ki}})$ also has positive bias.

Thus, if $v_j, v_k, v_l$ is a right triple in $S$, then, by 3.3.1,

$$\Pr \left[ \langle v_j' - v_k', v_i' - v_k' \rangle > 0 \right] \geq \frac{1}{2},$$

so the probability that the triple $v_j', v_k', v_l'$ is an acute triple in $S'$ is at least $\frac{1}{2}$.

As in the proof of theorem 2.1, the expected number of right triples in $S$ is $3 \binom{3m}{3} \left( \frac{3}{4} \right)^d$, so the expected number of non-acute triples in $S'$ is no more than half this value. Thus there is some set $S'$ of $3m$ vectors with no more than $\frac{3}{2} \binom{3m}{3} \left( \frac{3}{4} \right)^d$ non-acute triples, where

$$\frac{3}{2} \binom{3m}{3} \left( \frac{3}{4} \right)^d < \frac{3}{2} \binom{3m}{3} \left( \frac{3}{4} \right)^d = m(3m)^2 \left( \frac{3}{4} \right)^{d+1} \leq m$$

by the choice of $m$.

If we remove one point of each non-acute triple from $S'$, the remaining set is an acute $d$-set of cardinality at least $3m - m = 2m$. \qed

### 4 Constructive lower bounds for $\kappa(d)$

In the following proofs, for clarity of exposition, we will represent point vectors in $\{0,1\}^d$ as binary words of length $d$, e.g. $S_3 = \{000, 011, 101, 110\}$ represents a cubic acute 3-set.
Concatenation of words (vectors) \( v \) and \( v' \) will be written \( vv' \).

We begin with a simple construction that enables us to extend a cubic acute \( d \)-set of cardinality \( n \) to a cubic acute \( (d + 2) \)-set of cardinality \( n + 1 \).

**Theorem 4.1**

\[ \kappa(d + 2) \geq \kappa(d) + 1 \]

**Proof:** Let \( S = \{v_0, v_1, \ldots, v_{n-1}\} \) be a cubic acute \( d \)-set of cardinality \( n = \kappa(d) \). Now let \( S' = \{v'_0, v'_1, \ldots, v'_{n-1}\} \subseteq\{0,1\}^{d+2} \) where \( v'_i = v_{i00} \) for \( 0 \leq i \leq n - 2 \), \( v'_{n-1} = v_{n-110} \) and \( v'_n = v_{n-101} \).

If \( v'_i, v'_j, v'_k \) is a triple of distinct points in \( S' \) with no more than one of \( i, j \) and \( k \) greater than \( n - 2 \), then \( v'_i, v'_j, v'_k \) is an acute triple, because \( S \) is an acute \( d \)-set. Also, any triple \( v'_k, v'_{n-1}, v'_i \) or \( v'_k, v'_i, v'_{n-1} \) is an acute triple, because its \((d+1)\)th or \((d+2)\)th coordinates (respectively) are 0,1,0. Finally, for any triple \( v'_{n-1}, v'_k, v'_r \), if \( v_k \) and \( v_{n-1} \) differ in the \( r \)th coordinate, then the \( r \)th coordinates of \( v'_{n-1}, v'_k, v'_r \) are 0,1,0 or 1,0,1. Thus, \( S' \) is a cubic acute \((d+2)\)-set of cardinality \( n + 1 \).

Our second construction combines cubic acute \( d \)-sets of cardinality \( n \) to make a cubic acute \( 3d \)-set of cardinality \( n^2 \).

**Theorem 4.2**

\[ \kappa(3d) \geq \kappa(d)^2 \]

**Proof:** Let \( S = \{v_0, v_1, \ldots, v_{n-1}\} \) be a cubic acute \( d \)-set of cardinality \( n = \kappa(d) \), and let

\[ T = \{w_{ij} = v_i v_j v_{j-i \mod n} : 0 \leq i, j \leq n - 1\}, \]

each \( w_{ij} \) being made by concatenating three of the \( v_i \).

Let \( w_{ps}, w_{qt}, w_{ru} \) be any triple of distinct points in \( T \). They constitute an acute triple iff the scalar product \( \langle w_{ps} - w_{qt}, w_{ru} - w_{qt} \rangle \) does not vanish (is positive). Now,

\[ \langle w_{ps} - w_{qt}, w_{ru} - w_{qt} \rangle = \langle v_p v_s v_{s-p} - v_q v_t v_{t-q}, v_i v_j v_{j-i} v_{s-p} - v_q v_t v_{t-q} \rangle \]
\[ = \langle v_p - v_q, v_r - v_q \rangle \]
\[ + \langle v_s - v_t, v_u - v_t \rangle \]
\[ + \langle v_{s-p} - v_{t-q}, v_{u-r} - v_{t-q} \rangle \]

with all the index arithmetic modulo \( n \).

If both \( p \neq q \) and \( q \neq r \), then the first component of this sum is positive, because \( S \) is an acute \( d \)-set. Similarly, if both \( s \neq t \) and \( t \neq u \), then the second component is positive. Finally, if \( p = q \) and \( u = t \), then \( q \neq r \) and \( s \neq t \) or else the points would not be distinct, so the third component, \( \langle v_{s-p} - v_{t-q}, v_{u-r} - v_{t-q} \rangle \) is positive. Similarly if \( q = r \) and \( s = t \).

Thus, all triples in \( T \) are acute triples, so \( T \) is a cubic acute \( 3d \)-set of cardinality \( n^2 \).  

---

*The Electronic Journal of Combinatorics* 13 (2006), #R12  

8
**Corollary 4.2.1** \( \kappa(3^d) \geq 2^{2^d} \).

**Proof:** By repeated application of theorem 4.2 starting with \( S_3 \), a cubic acute 3-set of cardinality 4. \( \square \)

**Corollary 4.2.2** If \( d \geq 3 \),

\[
\kappa(d) \geq 10^{\frac{(d+1)^\mu}{4}} \approx 1.778^{(d+1)^{0.631}} \quad \text{where } \mu = \frac{\log 2}{\log 3}.
\]

For small \( d \), this is a tighter bound than theorem 2.1.

**Proof:** By induction on \( d \). For \( 3 \leq d \leq 8 \), we have the following cubic acute \( d \)-sets \((S_3, \ldots, S_8)\) that satisfy this lower bound for \( \kappa(d) \) (with equality for \( d = 8 \)):

<table>
<thead>
<tr>
<th>( S_3 ): ( \kappa(3) \geq 4 )</th>
<th>( S_4 ): ( \kappa(4) \geq 5 )</th>
<th>( S_5 ): ( \kappa(5) \geq 6 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>000</td>
<td>0000</td>
<td>00000</td>
</tr>
<tr>
<td>011</td>
<td>0011</td>
<td>00011</td>
</tr>
<tr>
<td>101</td>
<td>0101</td>
<td>01001</td>
</tr>
<tr>
<td>110</td>
<td>1001</td>
<td>1110</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( S_6 ): ( \kappa(6) \geq 8 )</th>
<th>( S_7 ): ( \kappa(7) \geq 9 )</th>
<th>( S_8 ): ( \kappa(8) \geq 10 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>000000</td>
<td>0000000</td>
<td>00000000</td>
</tr>
<tr>
<td>000111</td>
<td>0000011</td>
<td>00000011</td>
</tr>
<tr>
<td>011001</td>
<td>0001101</td>
<td>00001001</td>
</tr>
<tr>
<td>011110</td>
<td>0110001</td>
<td>00011001</td>
</tr>
<tr>
<td>101010</td>
<td>0111110</td>
<td>01100001</td>
</tr>
<tr>
<td>101101</td>
<td>1010101</td>
<td>01111110</td>
</tr>
<tr>
<td>110011</td>
<td>1011010</td>
<td>10101001</td>
</tr>
<tr>
<td>110100</td>
<td>1100110</td>
<td>10110110</td>
</tr>
</tbody>
</table>

If \( \kappa(d) \geq 10^{\frac{(d+1)^\mu}{4}} \), then \( \kappa(3d) \geq \kappa(d)^2 \) by theorem 4.2

\[
\geq 10^{\frac{2(d+1)^\mu}{4}} \quad \text{by the induction hypothesis}
\]

\[
= 10^{\frac{(3d+3)^\mu}{4}} \quad \text{because } 3^\mu = 2.
\]

So, since \( \kappa(3d + 2) \geq \kappa(3d + 1) \geq \kappa(3d) \), if the lower bound is satisfied for \( d \), it is also satisfied for \( 3d, 3d + 1 \) and \( 3d + 2 \). \( \square \)
Theorem 4.3 If, for each $r$, $1 \leq r \leq m$, we have a cubic acute $d_r$-set of cardinality $n_r$, where $n_1$ is the least of the $n_r$, and if, for some dimension $d_Z$, we have a cubic acute $d_Z$-set of cardinality $n_Z$, where

$$n_Z \geq \prod_{r=2}^{m} n_r,$$

then a cubic acute $D$-set of cardinality $N$ can be constructed, where

$$D = \sum_{r=1}^{m} d_r + d_Z \quad \text{and} \quad N = \prod_{r=1}^{m} n_r.$$

This result generalises theorem 4.2, but before we can prove it, we first need some preliminary results.

Definition 4.4 If $n_1 \leq n_2 \leq \ldots \leq n_m$ and $0 \leq k_r < n_r$, for each $r$, $1 \leq r \leq m$, then let us denote by $\langle k_1 k_2 \ldots k_m \rangle_{n_1 n_2 \ldots n_m}$, the number

$$\langle k_1 k_2 \ldots k_m \rangle_{n_1 n_2 \ldots n_m} = \sum_{r=2}^{m} (k_{r-1} - k_r \mod n_r) \prod_{s=r+1}^{m} n_s.$$

Where the $n_r$ can be inferred from the context, $\langle k_1 k_2 \ldots k_m \rangle$ may be used instead of $\langle k_1 k_2 \ldots k_m \rangle_{n_1 n_2 \ldots n_m}$.

The expression $\langle k_1 k_2 \ldots k_m \rangle_{n_1 n_2 \ldots n_m}$ can be understood as representing a number in a number system where the radix for each digit is a different $n_r$ − like the old British monetary system of pounds, shillings and pence − and the digits are the difference of two adjacent $k_r \pmod{n_r}$. For example,

$$\langle 2053 \rangle_{4668} = [2 - 0][0 - 5][5 - 3] = 2 \times 6 \times 8 + 1 \times 8 + 2 = 106,$$

where $[a_2]_{n_2} \ldots [a_m]_{n_m}$ is place notation with the $n_r$ the radix for each place.

By construction, we have the following results:

Property 4.4.1

$$\langle k_1 k_2 \ldots k_m \rangle_{n_1 n_2 \ldots n_m} < \prod_{r=2}^{m} n_r.$$

Property 4.4.2 If $2 \leq t \leq m$ and $j_{t-1} - j_t \neq k_{t-1} - k_t \pmod{n_t}$, then

$$\langle j_1 j_2 \ldots j_m \rangle_{n_1 n_2 \ldots n_m} \neq \langle k_1 k_2 \ldots k_m \rangle_{n_1 n_2 \ldots n_m}.$$

THE ELECTRONIC JOURNAL OF COMBINATORICS 13 (2006), #R12

10
Lemma 4.5  If $n_1 \leq n_2 \leq \ldots \leq n_m$ and $0 \leq j_r, k_r < n_r$, for each $r$, $1 \leq r \leq m$, and the sequences of $j_r$ and $k_r$ are neither identical nor everywhere different (i.e. there exist both $t$ and $u$ such that $j_t = k_t$ and $j_u \neq k_u$), then

$$\langle j_1j_2 \ldots j_m \rangle_{n_1n_2 \ldots n_m} \neq \langle k_1k_2 \ldots k_m \rangle_{n_1n_2 \ldots n_m}.$$ 

Proof: Let $u$ be the greatest integer, $1 \leq u < m$, such that $j_u - j_{u+1} \neq k_u - k_{u+1}$ (mod $n_{u+1}$). (If $j_m = k_m$, then $u$ is the greatest integer such that $j_u \neq k_u$. If $j_m \neq k_m$, then $u$ is at least as great as the greatest integer $t$ such that $j_t = k_t$.) The result now follows from 4.4.2. 

We are now in a position to prove the theorem.

Proof of Theorem 4.3

Let $n_1 \leq n_2 \leq \ldots \leq n_m$, and, for each $r$, $1 \leq r \leq m$, let $S_r = \{v_r^0, v_r^1, \ldots, v_r^{n_r-1}\}$ be a cubic acute $d_r$-set of cardinality $n_r$. Let $Z = \{z_0, z_1, \ldots, z_{n_Z-1}\}$ be a cubic acute $d_Z$-set of cardinality $n_Z$, where

$$n_Z \geq \prod_{r=2}^{m} n_r,$$

and let

$$D = \sum_{r=1}^{m} d_r + d_Z \quad \text{and} \quad N = \prod_{r=1}^{m} n_r.$$

Now let

$$T = \{w_{k_1k_2 \ldots k_m} = v_k^1 v_k^2 \ldots v_k^{n_k} z_k : 0 \leq k_r < n_r, 1 \leq r \leq m\},$$

where $k_Z = \langle k_1k_2 \ldots k_m \rangle_{n_1n_2 \ldots n_m}$, be a point set of dimension $D$ and cardinality $N$, each element of $T$ being made by concatenating one vector from each of the $S_r$ together with a vector from $Z$. (In section 5, we will denote this construction by $d_1 \circ \cdots \circ d_m \circ d_Z$.)

By 4.4.1, we know that $k_Z < \prod_{r=2}^{m} n_r \leq n_Z$, so $k_Z$ is a valid index into $Z$.

Let $w_{i_1i_2 \ldots i_m}, w_{j_1j_2 \ldots j_m}, w_{k_1k_2 \ldots k_m}$ be any triple of distinct points in $T$. They constitute an acute triple iff the scalar product $q = \langle w_{i_1i_2 \ldots i_m} - w_{j_1j_2 \ldots j_m}, w_{k_1k_2 \ldots k_m} - w_{j_1j_2 \ldots j_m} \rangle$ does not vanish (is positive). Now,

$$q = \langle v^{i_1}_{i_1} v^{i_2}_{i_2} \ldots v^{i_m}_{i_m} z_{i_Z} - v^{j_1}_{j_1} v^{j_2}_{j_2} \ldots v^{j_m}_{j_m} z_{j_Z}, v^{k_1}_{k_1} v^{k_2}_{k_2} \ldots v^{k_m}_{k_m} z_{k_Z} - v^{j_1}_{j_1} v^{j_2}_{j_2} \ldots v^{j_m}_{j_m} z_{j_Z} \rangle$$

$$= \sum_{r=1}^{m} \langle v^{i_r}_{i_r} - v^{j_r}_{j_r}, v^{k_r}_{k_r} - v^{j_r}_{j_r} \rangle + \langle z_{i_Z} - z_{j_Z}, z_{k_Z} - z_{j_Z} \rangle.$$ 

If, for some $r$, both $i_r \neq j_r$ and $j_r \neq k_r$, then the first component of this sum is positive, because $S_r$ is an acute set.

If, however, there is no $r$ such that both $i_r \neq j_r$ and $j_r \neq k_r$, then there must be some $t$ for which $i_t \neq j_t$ (or else $w_{i_1i_2 \ldots i_m}$ and $w_{j_1j_2 \ldots j_m}$ would not be distinct) and $j_t = k_t$, and
also some \( u \) for which \( j_u \neq k_u \) (or else \( w_{j_1,j_2\ldots j_m} \) and \( w_{k_1,k_2\ldots k_m} \) would not be distinct) and \( i_u = j_u \). So, by lemma 4.5, \( i_Z \neq j_Z \) and \( j_Z \neq k_Z \), so the second component of the sum for the scalar product is positive, because \( Z \) is an acute set.

Thus, all triples in \( T \) are acute triples, so \( T \) is a cubic acute \( D \)-set of cardinality \( N \).

\[ \boxed{ } \]

**Corollary 4.5.1**

\[ \text{If } d_1 \leq d_2 \leq \ldots \leq d_m, \text{ then } \kappa \left( \sum_{r=1}^{m} rd_r \right) \geq \prod_{r=1}^{m} \kappa(d_r). \]

**Proof:** By induction on \( m \). The bound is trivially true for \( m = 1 \).

Assume the bound holds for \( m - 1 \), and for each \( r, 1 \leq r \leq m \), let \( S_r \) be a cubic acute \( d_r \)-set of cardinality \( n_r = \kappa(d_r) \), with \( d_1 \leq d_2 \leq \ldots \leq d_m \) and thus \( n_1 \leq n_2 \leq \ldots \leq n_m \). By the induction hypothesis, there exists a cubic acute \( d_Z \)-set \( Z \) of cardinality \( n_Z \), where

\[ d_Z = \sum_{r=2}^{m} (r-1)d_r \quad \text{and} \quad n_Z \geq \prod_{r=2}^{m} \kappa(d_r) = \prod_{r=2}^{m} n_r. \]

Thus, by theorem 4.3, there exists a cubic acute \( D \)-set of cardinality \( N \), where

\[ D = \sum_{r=1}^{m} d_r + d_Z = \sum_{r=1}^{m} d_r + \sum_{r=2}^{m} (r-1)d_r = \sum_{r=1}^{m} rd_r, \]

and

\[ N = \prod_{r=1}^{m} n_r = \prod_{r=1}^{m} \kappa(d_r). \]

\[ \boxed{ } \]

**5 Lower bounds for \( \kappa(d) \) and \( \alpha(d) \) for small \( d \)**

The following table lists the best lower bounds known for \( \kappa(d) \), \( 0 \leq d \leq 69 \). For \( 3 \leq d \leq 9 \), an exhaustive computer search shows that \( S_3, \ldots, S_8 \) (corollary 4.2.2), are optimal and also that \( \kappa(9) = 16 \). For other small values of \( d \), the construction used in theorem 4.3 provides the largest known cubic acute \( d \)-set. In the table, these constructions are denoted by \( d_1 \otimes d_2 \otimes d_Z \) or \( d_1 \otimes d_2 \otimes d_3 \otimes d_Z \). For \( 39 \leq d \leq 48 \), the results of a computer program, based on the ‘probabilistic construction’ of theorem 2.1, provide the largest known cubic acute \( d \)-sets. Finally, for \( d \geq 67 \), theorem 2.1 provides the best (probabilistic) lower bound. \( \kappa(d) \) is sequence A089676 in Sloane [S].
Best Lower Bounds Known for $\kappa(d)$

<table>
<thead>
<tr>
<th>$d$</th>
<th>$\kappa(d)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>= 1</td>
</tr>
<tr>
<td>1</td>
<td>= 2</td>
</tr>
<tr>
<td>2</td>
<td>= 2</td>
</tr>
<tr>
<td>3</td>
<td>= 4</td>
</tr>
<tr>
<td>4</td>
<td>= 5</td>
</tr>
<tr>
<td>5</td>
<td>= 6</td>
</tr>
<tr>
<td>6</td>
<td>= 8</td>
</tr>
<tr>
<td>7</td>
<td>= 9</td>
</tr>
<tr>
<td>8</td>
<td>= 10</td>
</tr>
<tr>
<td>9</td>
<td>= 16</td>
</tr>
<tr>
<td>10</td>
<td>$\geq 16$</td>
</tr>
<tr>
<td>11</td>
<td>$\geq 20$</td>
</tr>
<tr>
<td>12</td>
<td>$\geq 25$</td>
</tr>
<tr>
<td>13</td>
<td>$\geq 25$</td>
</tr>
<tr>
<td>14</td>
<td>$\geq 30$</td>
</tr>
<tr>
<td>15</td>
<td>$\geq 36$</td>
</tr>
<tr>
<td>16</td>
<td>$\geq 40$</td>
</tr>
<tr>
<td>17</td>
<td>$\geq 48$</td>
</tr>
<tr>
<td>18</td>
<td>$\geq 64$</td>
</tr>
<tr>
<td>19</td>
<td>$\geq 64$</td>
</tr>
<tr>
<td>20</td>
<td>$\geq 72$</td>
</tr>
<tr>
<td>21</td>
<td>$\geq 81$</td>
</tr>
<tr>
<td>22</td>
<td>$\geq 81$</td>
</tr>
<tr>
<td>23</td>
<td>$\geq 100$</td>
</tr>
<tr>
<td>24</td>
<td>$\geq 125$</td>
</tr>
<tr>
<td>25</td>
<td>$\geq 144$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$d$</th>
<th>$\kappa(d)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>26</td>
<td>$\geq 160$</td>
</tr>
<tr>
<td>27</td>
<td>$\geq 256$</td>
</tr>
<tr>
<td>28</td>
<td>$\geq 256$</td>
</tr>
<tr>
<td>29</td>
<td>$\geq 257$</td>
</tr>
<tr>
<td>30</td>
<td>$\geq 257$</td>
</tr>
<tr>
<td>31</td>
<td>$\geq 320$</td>
</tr>
<tr>
<td>32</td>
<td>$\geq 320$</td>
</tr>
<tr>
<td>33</td>
<td>$\geq 400$</td>
</tr>
<tr>
<td>34</td>
<td>$\geq 400$</td>
</tr>
<tr>
<td>35</td>
<td>$\geq 500$</td>
</tr>
<tr>
<td>36</td>
<td>$\geq 625$</td>
</tr>
<tr>
<td>37</td>
<td>$\geq 625$</td>
</tr>
<tr>
<td>38</td>
<td>$\geq 626$</td>
</tr>
<tr>
<td>39</td>
<td>$\geq 678$</td>
</tr>
<tr>
<td>40</td>
<td>$\geq 762$</td>
</tr>
<tr>
<td>41</td>
<td>$\geq 871$</td>
</tr>
<tr>
<td>42</td>
<td>$\geq 976$</td>
</tr>
<tr>
<td>43</td>
<td>$\geq 1086$</td>
</tr>
<tr>
<td>44</td>
<td>$\geq 1246$</td>
</tr>
<tr>
<td>45</td>
<td>$\geq 1420$</td>
</tr>
<tr>
<td>46</td>
<td>$\geq 1630$</td>
</tr>
<tr>
<td>47</td>
<td>$\geq 1808$</td>
</tr>
<tr>
<td>48</td>
<td>$\geq 2036$</td>
</tr>
<tr>
<td>49</td>
<td>$\geq 2036$</td>
</tr>
<tr>
<td>50</td>
<td>$\geq 2037$</td>
</tr>
<tr>
<td>51</td>
<td>$\geq 2304$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$d$</th>
<th>$\kappa(d)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>52</td>
<td>$\geq 2560$</td>
</tr>
<tr>
<td>53</td>
<td>$\geq 3072$</td>
</tr>
<tr>
<td>54</td>
<td>$\geq 4096$</td>
</tr>
<tr>
<td>55</td>
<td>$\geq 4096$</td>
</tr>
<tr>
<td>56</td>
<td>$\geq 4097$</td>
</tr>
<tr>
<td>57</td>
<td>$\geq 4097$</td>
</tr>
<tr>
<td>58</td>
<td>$\geq 4608$</td>
</tr>
<tr>
<td>59</td>
<td>$\geq 4608$</td>
</tr>
<tr>
<td>60</td>
<td>$\geq 5184$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$d$</th>
<th>$\kappa(d)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>61</td>
<td>$\geq 5184$</td>
</tr>
<tr>
<td>62</td>
<td>$\geq 5832$</td>
</tr>
<tr>
<td>63</td>
<td>$\geq 6561$</td>
</tr>
<tr>
<td>64</td>
<td>$\geq 6561$</td>
</tr>
<tr>
<td>65</td>
<td>$\geq 6562$</td>
</tr>
<tr>
<td>66</td>
<td>$\geq 8000$</td>
</tr>
<tr>
<td>67</td>
<td>$\geq 8342$</td>
</tr>
<tr>
<td>68</td>
<td>$\geq 9632$</td>
</tr>
<tr>
<td>69</td>
<td>$\geq 11122$</td>
</tr>
</tbody>
</table>
The following tables summarise the best lower bounds known for $\alpha(d)$. For $3 \leq d \leq 6$, the best lower bound is Danzer and Grünbaum’s $2d - 1$ [DG]. For $7 \leq d \leq 26$, the results of a computer program, based on the ‘probabilistic construction’ but using sets of points close to the surface of the $d$-sphere, provide the largest known acute $d$-sets. An acute 7-set of cardinality 14 and an acute 8-set of cardinality 16 are displayed. For $27 \leq d \leq 63$, the largest known acute $d$-set is cubic. Finally, for $d \geq 63$, theorem 3.1 provides the best (probabilistic) lower bound.

**Best Lower Bounds Known for $\alpha(d)$**

<table>
<thead>
<tr>
<th>$d$</th>
<th>$\alpha(d)$</th>
<th>$d$</th>
<th>$\alpha(d)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>= 1</td>
<td>16</td>
<td>$\geq 54$ computer</td>
</tr>
<tr>
<td>1</td>
<td>= 2</td>
<td>17</td>
<td>$\geq 63$ computer</td>
</tr>
<tr>
<td>2</td>
<td>= 3</td>
<td>18</td>
<td>$\geq 71$ computer</td>
</tr>
<tr>
<td>3</td>
<td>= 5 [DG]</td>
<td>19</td>
<td>$\geq 76$ computer</td>
</tr>
<tr>
<td>4-6</td>
<td>$\geq 2d - 1$ [DG]</td>
<td>20</td>
<td>$\geq 90$ computer</td>
</tr>
<tr>
<td>7</td>
<td>$\geq 14$ computer</td>
<td>21</td>
<td>$\geq 103$ computer</td>
</tr>
<tr>
<td>8</td>
<td>$\geq 16$ computer</td>
<td>22</td>
<td>$\geq 118$ computer</td>
</tr>
<tr>
<td>9</td>
<td>$\geq 19$ computer</td>
<td>23</td>
<td>$\geq 121$ computer</td>
</tr>
<tr>
<td>10</td>
<td>$\geq 23$ computer</td>
<td>24</td>
<td>$\geq 144$ computer</td>
</tr>
<tr>
<td>11</td>
<td>$\geq 26$ computer</td>
<td>25</td>
<td>$\geq 155$ computer</td>
</tr>
<tr>
<td>12</td>
<td>$\geq 30$ computer</td>
<td>26</td>
<td>$\geq 184$ computer</td>
</tr>
<tr>
<td>13</td>
<td>$\geq 36$ computer</td>
<td>27-62</td>
<td>$\geq \kappa(d)$</td>
</tr>
<tr>
<td>14</td>
<td>$\geq 42$ computer</td>
<td>63</td>
<td>$\geq 6636$ theorem 3.1</td>
</tr>
<tr>
<td>15</td>
<td>$\geq 47$ computer</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Best Lower Bounds Known for $\alpha(7)$** (16)

(62, 1, 9, 10, 17, 38, 46)
(38, 54, 0, 19, 38, 14, 25)
(60, 33, 42, 9, 48, 3, 12)
(62, 35, 41, 16, 39, 44)
(62, 34, 7, 45, 48, 37, 12)
(28, 33, 42, 8, 49, 39, 45)
(40, 16, 22, 12, 0, 0, 25)
(45, 17, 26, 67, 25, 20, 29)
(38, 6, 35, 0, 32, 18, 0)
(62, 0, 42, 45, 49, 3, 48)
(30, 0, 9, 44, 49, 37, 48)
(0, 20, 31, 27, 34, 21, 28)
(48, 19, 24, 22, 33, 20, 73)
(43, 17, 25, 27, 32, 64, 19)

**Best Lower Bounds Known for $\alpha(8)$** (16)

(34, 49, 14, 51, 0, 36, 46, 0)
(31, 17, 14, 51, 1, 5, 44, 31)
(33, 50, 48, 20, 34, 35, 15, 0)
(0, 16, 16, 52, 32, 36, 45, 0)
(37, 31, 46, 52, 13, 0, 0, 22)
(2, 50, 13, 52, 3, 3, 46, 0)
(1, 50, 48, 51, 1, 5, 46, 31)
(24, 0, 43, 2, 17, 20, 32, 16)
(11, 49, 0, 11, 19, 8, 32, 19)
(0, 48, 48, 52, 1, 34, 12, 2)
(0, 48, 47, 51, 34, 37, 47, 32)
(34, 49, 14, 51, 34, 36, 13, 34)
(0, 46, 31, 0, 0, 23, 29, 29)
(16, 40, 29, 23, 54, 3, 17, 16)
(2, 15, 14, 50, 2, 36, 15, 33)
(12, 36, 28, 30, 3, 45, 48, 45)
6 Generalising $\kappa(d)$

We can understand $\kappa(d)$ to be the size of the largest possible set $S$ of binary words such that, for any ordered triple of words $(u, v, w)$ in $S$, there exists an index $i$ for which $(u_i, v_i, w_i) = (0, 1, 0)$ or $(u_i, v_i, w_i) = (1, 0, 1)$. We can generalise this in the following way:

**Definition 6.1** If $T_1, \ldots, T_m$ are ordered $k$-tuples from $\{0, \ldots, r-1\}^k$ (which we will refer to as the matching $k$-tuples), then let us define $\kappa[r, k, T_1, \ldots, T_m](d)$ to be the size of the largest possible set $S$ of $r$-ary words of length $d$ such that, for any ordered $k$-tuple of words $(w_1, \ldots, w_k)$ in $S$, there exist $i$ and $j$, $1 \leq i \leq d$, $1 \leq j \leq m$, for which $(w_{ki}, \ldots, w_{kj}) = T_j$.

Thus we have $\kappa(d) = \kappa[2, 3, (0, 1, 0), (1, 0, 1)](d)$. If the set of matching $k$-tuples is closed under permutation, we will abbreviate by writing a list of matching multisets of cardinality $k$, rather than ordered tuples. For example, instead of $\kappa[2, 3, (0, 0, 1), (0, 1, 0), (1, 0, 0)](d)$, we write $\kappa[2, 3, (0, 0, 1)](d)$.

We can find probabilistic and, in some cases, constructive lower bounds for general $\kappa[r, k, T_1, \ldots, T_m](d)$ using the approaches we used for cubic acute $d$-sets. To illustrate this, in the remainder of this paper, we will consider the set of problems in which it is simply required that at some index the $k$-tuple of words be all different (pairwise distinct). First, we express this in a slightly different form.

Let us say that an $r$-ary $d$-colouring is some colouring of the integers $1, \ldots, d$ using $r$ colours. Let us also also say that a set $R$ of $r$-ary $d$-colourings is a $k$-rainbow set, for some $k \leq r$ if for any set $\{c_1, \ldots, c_k\}$ of $k$ colourings in $R$, there exists some integer $t$, $1 \leq t \leq d$, for which the colours $c_1(t), \ldots, c_k(t)$ are all different, i.e. $c_i(t) \neq c_j(t)$ for any $i$ and $j$, $1 \leq i, j \leq k$, $i \neq j$. For conciseness, we will denote “a $k$-rainbow set of $r$-ary $d$-colourings” by “a $RSC[k, r, d]$”.

Let us further say that a set $\{c_1, \ldots, c_k\}$ of $d$ colourings is a good $k$-set if there exists some integer $t$, $1 \leq t \leq d$, for which the colours $c_1(t), \ldots, c_k(t)$ are all different, and a bad $k$-set if there exists no such $t$.

We will denote by $\rho_{r,k}(d)$ the size of the largest possible $RSC[k, r, d]$, abbreviating $\rho_{k,k}(d)$ by $\rho_k(d)$. Now, $\rho_k(d) = \kappa[k, k, \{0, 1, \ldots, k-1\}] (d)$ and

$$\rho_{r,k}(d) = \kappa[r, k, \{0, 1, \ldots, k-1\}, \ldots, \{r-k, \ldots, r-1\}] (d),$$

where the matching multisets are those of cardinality $k$ with $k$ distinct members.

Clearly, $\rho_{r,k}(d) \leq \rho_{r,k}(d + 1)$, $\rho_{r,k}(d) \leq \rho_{r+1,k}(d)$ and $\rho_{r,k}(d) \geq \rho_{r,k+1}(d)$. Also, $\rho_{r,1}(d)$ is undefined because any set of colourings is a 1-rainbow, $\rho_{r,1}(1) = r$ if $k > 1$, and $\rho_{r,2}(d) = r^d$ because any two distinct $r$-ary $d$-colourings (or $r$-ary words of length $d$) differ somewhere.
In the next two sections we will give a number of probabilistic and constructive lower bounds for \(\rho_{r,k}(d)\), for various \(r\) and \(k\).

7 A probabilistic lower bound for \(\rho_{r,k}(d)\)

Theorem 7.1

\[
\rho_{r,k}(d) \geq (k - 1)m \quad \text{where} \quad m = \left\lfloor k^{-1} \sqrt{k! \left( k^{-1} \sqrt{(r - k)!} r^k - r! \right)} \right\rfloor^d.
\]

Proof: This proof is similar that of theorem 2.1.

Randomly pick a set \(\mathcal{R}\) of \(km\) \(r\)-ary \(d\)-colourings, choosing the colours from \(\{\chi_0, \ldots, \chi_{r-1}\}\) independently with probability \(\Pr[c(i) = \chi_j] = 1/r\), \(1 \leq i \leq d\), \(0 \leq j < r\) for every \(c \in \mathcal{R}\).

Now the probability that a set of \(k\) colourings from \(\mathcal{R}\) is a bad \(k\)-set is

\[
(1 - p)^d \quad \text{where} \quad p = \frac{r!/(r - k)!}{r^k}.
\]

Hence, the expected number of bad \(k\)-sets in a set of \(km\) \(d\)-colourings is \((\frac{km}{k})(1 - p)^d\). Thus there is some set \(\mathcal{R}\) of \(km\) \(d\)-colourings with no more than \((\frac{km}{k})(1 - p)^d\) bad \(k\)-sets, where

\[
\left(\frac{km}{k}\right)(1 - p)^d < \frac{km}{k!} k! (1 - p)^d = m \frac{k^k}{k!} m^{k-1}(1 - p)^d \leq m
\]

by the choice of \(m\).

If we remove one colouring of each bad \(k\)-set from \(\mathcal{R}\), the remaining set is a \(\mathcal{RSC}[k, r, d]\) of cardinality at least \(km - m = (k - 1)m\).

The following results follow directly:

\[
\rho_3(d) \geq 2 \left\lfloor \frac{\sqrt{2}}{3} \left( \frac{3}{\sqrt{7}} \right)^d \right\rfloor \approx 0.943 \times 1.134^d.
\]

\[
\rho_{4,3}(d) \geq 2 \left\lfloor \frac{\sqrt{2}}{3} \left( \frac{4}{\sqrt{10}} \right)^d \right\rfloor \approx 0.943 \times 1.265^d.
\]

\[
\rho_4(d) \geq 3 \left\lfloor \frac{\sqrt{3}}{32} \sqrt[3]{\frac{32}{29}}^d \right\rfloor \approx 1.363 \times 1.033^d.
\]
8 Constructive lower bounds for $\rho_{r,k}(d)$

In the following proofs, for clarity of exposition, we will represent $r$-ary $d$-colourings as $r$-ary words of length $d$, e.g. $R_{2,3,3} = \{000, 011, 102, 121, 212, 220\}$ represents a 3-rainbow set of ternary 3 Colourings (using the colours $\chi_0$, $\chi_1$ and $\chi_2$). Concatenation of words (colourings) $c$ and $c'$ will be written $c.c'$.

We begin with a construction that enables us to extend a $RSC[k, r, d]$ of cardinality $n$ to one of cardinality $n + 1$ or greater.

**Theorem 8.1** If for some $r \geq k \geq 3$, and some $d$, we have a $RSC[k, r, d]$ of cardinality $n$, and for some $r'$, $k - 2 \leq r' \leq r - 2$, and $d'$, we have a $RSC[k - 2, r', d']$ of cardinality at least $n - 1$, then we can construct a $RSC[k, r, d + d']$ of cardinality $N = n - 1 + r - r'$.

**Proof:** Let $\mathcal{R} = \{c_0, c_1, \ldots, c_{n-1}\}$ be a $RSC[k, r, d]$ of cardinality $n$ (using colours $\chi_0, \ldots, \chi_{r-1}$) and $\mathcal{R}' = \{c'_0, c'_1, \ldots, c'_{n-1}\}$ be a $RSC[k - 2, r', d']$ of cardinality $n' \geq n - 1$ (using colours $\chi_0, \ldots, \chi_{r'-1}$).

Now let $\mathcal{Q} = \{q_0, q_1, \ldots, q_{N-1}\}$ be a set of $r$-ary $(d + d')$-colourings where $q_i = c_i.c'_i$ for $0 \leq i \leq n - 2$, and $q_{n-1+j} = c_{n-1}.(r' + j)d'$ for $0 \leq j < r - r'$, each element of $\mathcal{Q}$ being made by concatenating two component colourings, the first from $\mathcal{R}$ and the second being either from $\mathcal{R}'$ or a monochrome colouring.

If $\{q_i, \ldots, q_k\}$ is a set of colourings in $\mathcal{Q}$ with no more than one of the $i_m$ greater than $n - 2$, then it is a good $k$-set because of the first components, since $\mathcal{R}$ is a $k$-rainbow set.

On the other hand, if $\{q_i, \ldots, q_k\}$ is a set of colourings in $\mathcal{Q}$ with no more than $k - 2$ of the $i_m$ less than $n - 1$, then it too is a good $k$-set because of the second components, since $\mathcal{R}'$ is a $(k - 2)$-rainbow set using colours $\chi_0, \ldots, \chi_{r'-1}$ and the second components of the colourings with indices greater than $n - 2$ are each monochrome of a different colour, drawn from $\chi_{r'}, \ldots, \chi_{r-1}$.

Thus $\mathcal{Q}$ is a $RSC[k, r, d + d']$ of cardinality $N$. \hfill $\square$

**Corollary 8.1.1** $\rho_{r,3}(d +1) \geq \rho_{r,3}(d) + r - 2$.

**Proof:** This follows from the theorem due to the fact that there is a 1-rainbow set of 1-ary 1-colourings of any cardinality. \hfill $\square$

**Corollary 8.1.2** $\rho_{r,4}(d + \lceil \log_2(\rho_{r,4}(d) - 1) \rceil) \geq \rho_{r,4}(d) + r - 3$.

**Proof:** Since $\rho_{r,2}(d) = r^d$, we have $\rho_{2,2}(d') \geq \rho_{r,4}(d) - 1$ iff $d' \geq \log_2(\rho_{r,4}(d) - 1)$. \hfill $\square$
Theorem 8.2 If, for each \( s \), \( 1 \leq s \leq m \), we have a \( RSC[3, r, d_s] \) of cardinality \( n_s \), where \( n_1 \) is the least of the \( n_s \), and if, for some \( d_Z \), we have a \( RSC[3, r, d_Z] \) of cardinality \( n_Z \), where
\[
n_Z \geq \prod_{s=2}^{m} \left( 1 + 2 \left\lfloor \frac{n_s}{2} \right\rfloor \right),
\]
then a \( RSC[3, r, D] \) of cardinality \( N \) can be constructed, where
\[
D = \sum_{s=1}^{m} d_s + 2d_Z \quad \text{and} \quad N = \prod_{s=1}^{m} n_s.
\]

This result for 3-rainbow sets corresponds to theorem 4.3 for cubic acute \( d \)-sets. Before we can prove it, we need some further preliminary results.

Definition 8.3 If \( n_1 \leq n_2 \leq \ldots \leq n_m \) and \( 0 \leq k_r < n_r \), for each \( r \), \( 1 \leq r \leq m \), then let us denote by \( \langle \langle k_1 k_2 \ldots k_m \rangle \rangle_{n_1 n_2 \ldots n_m} \), the number
\[
\langle \langle k_1 k_2 \ldots k_m \rangle \rangle_{n_1 n_2 \ldots n_m} = \sum_{r=2}^{m} \left( (k_{r-1} + k_r \mod n_r) \prod_{s=r+1}^{m} n_s \right).
\]

The definition of \( \langle \langle k_1 k_2 \ldots k_m \rangle \rangle_{n_1 n_2 \ldots n_m} \) is the same as that for \( \langle \langle k_1 k_2 \ldots k_m \rangle \rangle_{n_1 n_2 \ldots n_m} \) (see 4.4), but with addition replacing subtraction. By construction, we have
\[
\langle \langle k_1 k_2 \ldots k_m \rangle \rangle_{n_1 n_2 \ldots n_m} < \prod_{r=2}^{m} n_r,
\]
and, if \( 2 \leq t \leq m \) and \( j_{t-1} + j_t \neq k_{t-1} + k_t \) (mod \( n_t \)), then
\[
\langle \langle j_1 j_2 \ldots j_m \rangle \rangle_{n_1 n_2 \ldots n_m} \neq \langle \langle k_1 k_2 \ldots k_m \rangle \rangle_{n_1 n_2 \ldots n_m}.
\]

Lemma 8.4 If \( n_1 \leq n_2 \leq \ldots \leq n_m \), with all the \( n_r \) odd except perhaps \( n_1 \), and \( 0 \leq j_r, k_r, l_r < n_r \), for each \( r \), \( 1 \leq r \leq m \), and the sequences of \( j_r \), \( k_r \) and \( l_r \) are neither pairwise identical nor anywhere pairwise distinct, i.e. there is some \( u, v \) and \( w \) such that \( j_u \neq k_u, k_v \neq l_v \) and \( l_w \neq j_w \) but no \( t \) such that \( j_t \neq k_t, k_t \neq l_t \) and \( l_t \neq j_t \), then either
\[
\langle \langle j_1 \ldots j_m \rangle \rangle_{n_1 \ldots n_m}, \langle \langle k_1 \ldots k_m \rangle \rangle_{n_1 \ldots n_m}, \langle \langle l_1 \ldots l_m \rangle \rangle_{n_1 \ldots n_m} \text{ are pairwise distinct}
\]
or
\[
\langle \langle j_1 \ldots j_m \rangle \rangle_{n_1 \ldots n_m}^+, \langle \langle k_1 \ldots k_m \rangle \rangle_{n_1 \ldots n_m}^+, \langle \langle l_1 \ldots l_m \rangle \rangle_{n_1 \ldots n_m}^+ \text{ are pairwise distinct}.
\]

The Electronic Journal of Combinatorics 13 (2006), #R12
Proof: Without loss of generality, we can assume that we have $j_t \neq k_t$, and that $k_t = l_t$. We will consider two cases:

Case 1: $k_{t-1} \neq l_{t-1}$
Since $j_{t-1} = k_{t-1} \neq l_{t-1}$ and $j_t \neq k_t = l_t$, we have $j_{t-1} - j_t \neq k_{t-1} - k_t$ and $k_{t-1} - k_t \neq l_{t-1} - l_t$, and so $\langle j_1 \ldots j_m \rangle \neq \langle k_1 \ldots k_m \rangle$ and $\langle k_1 \ldots k_m \rangle \neq \langle l_1 \ldots l_m \rangle$. Similarly, $j_{t-1} + j_t \neq k_{t-1} + k_t$ and $k_{t-1} + k_t \neq l_{t-1} + l_t$, and so $\langle j_1 \ldots j_m \rangle^+ \neq \langle k_1 \ldots k_m \rangle^+$ and $\langle k_1 \ldots k_m \rangle^+ \neq \langle l_1 \ldots l_m \rangle^+$.

If $j_{t-1} - j_t \neq l_{t-1} - l_t$, then $\langle j_1 \ldots j_m \rangle \neq \langle l_1 \ldots l_m \rangle$. If $j_{t-1} - j_t = l_{t-1} - l_t$, then $(j_{t-1} - j_t) - (l_{t-1} - l_t) = (j_t - j_t) - (l_t - l_t) = 2(j_t - l_t) \neq 0$ (mod $n_t$) because $j_t \neq l_t$ and $n_t$ is odd, so $j_{t-1} + j_t \neq l_{t-1} + l_t$ and $\langle j_1 \ldots j_m \rangle^+ \neq \langle l_1 \ldots l_m \rangle^+$.

Case 2: $k_{t-1} = l_{t-1}$
Since $j_{t-1} = k_{t-1} = l_{t-1}$ and $j_t \neq k_t = l_t$, we have $j_{t-1} - j_t \neq k_{t-1} - k_t$ and $j_{t-1} - j_t \neq l_{t-1} - l_t$, and so $\langle j_1 \ldots j_m \rangle \neq \langle k_1 \ldots k_m \rangle$ and $\langle j_1 \ldots j_m \rangle \neq \langle l_1 \ldots l_m \rangle$.

If $k_1 = l_1$, let $u$ be the least integer such that $k_u \neq l_u$. Since $k_{u-1} = l_{u-1}$, we have $k_{u-1} - k_u \neq l_{u-1} - l_u$. If $k_1 \neq l_1$, let $u$ be the least integer such that $k_u = l_u$. Since $k_{u-1} \neq l_{u-1}$, we still have $k_{u-1} - k_u \neq l_{u-1} - l_u$. Thus, $\langle k_1 \ldots k_m \rangle \neq \langle l_1 \ldots l_m \rangle$.

\[ \square \]

Proof of Theorem 8.2
Let $n_1 \leq n_2 \leq \ldots \leq n_m$, and, for each $s$, $1 \leq s \leq m$, let $R_s = \{ c_s^0, c_s^1, \ldots, c_s^s \}$ be a RSC[3, r, d_s] of cardinality $n_s$, and let $n'_s = 1 + 2 \lfloor n_s/2 \rfloor$ be the least odd integer not less than $n_s$. Let $Z = \{ z_0, z_1, \ldots, z_{n_Z-1} \}$ be a RSC[3, r, d_Z] of cardinality $n_Z$, where

\[ n_Z \geq \prod_{s=2}^{m} n'_s, \]

and let

\[ D = \sum_{s=1}^{m} d_s + 2d_Z \quad \text{and} \quad N = \prod_{s=1}^{m} n_s. \]

Now let

\[ Q = \{ c_{k_1}^1, c_{k_2}^2 \ldots c_{k_m}^m, z_{k_Z} \cdot z_{k_Z}^+ : 0 \leq k_s < n_s, 1 \leq s \leq m \}, \]

where $k_Z = \langle k_1 k_2 \ldots k_m \rangle_{n'_1 n'_2 \ldots n'_m}$ and $k_Z^+ = \langle k_1 k_2 \ldots k_m \rangle_{n'_1 n'_2 \ldots n'_m}^+$ be a set of $D$-colourings of cardinality $N$, each element of $Q$ being made by concatenating one colouring from each of the $R_s$ together with two colourings from $Z$. (Below, we will denote this construction by $d_1 \otimes \cdots \otimes d_m \otimes d_Z \otimes d_Z$.)

Let $c_{i_1}^1, c_{i_2}^2 \ldots c_{i_m}^m, z_{i_Z} \cdot z_{i_Z}^+$, $c_{j_1}^1, c_{j_2}^2 \ldots c_{j_m}^m, z_{j_Z} \cdot z_{j_Z}^+$ and $c_{k_1}^1, c_{k_2}^2 \ldots c_{k_m}^m, z_{k_Z} \cdot z_{k_Z}^+$ be any three distinct colourings in $Q$. If, for some $s$, $i_s \neq j_s$, $j_s \neq k_s$ and $k_s \neq i_s$, then these three colourings comprise a good 3-set because $R_s$ is a 3-rainbow set.
If, however, there is no $s$ such that $i_s$, $j_s$, and $k_s$ are all different, then the condition of lemma 8.4 holds, and so either $i_Z$, $j_Z$, and $k_Z$ are all different, or $i_Z^+$, $j_Z^+$, and $k_Z^+$ are all different, and the three colourings comprise a good 3-set because $Z$ is a 3-rainbow set.

Thus, any three colourings in $Q$ comprise a good 3-set, so $Q$ is a $\mathcal{RSC}[3, r, D]$ of cardinality $N$.

**Corollary 8.4.1** If $\rho_{r,3}(d)$ is odd, then $\rho_{r,3}(4d) \geq \rho_{r,3}(d)^2$.

**Proof:** By theorem 8.2 using the construction $d \sigma d \sigma d \sigma$.

**Corollary 8.4.2** $\rho_{r,3}(4d + 2) \geq \rho_{r,3}(d)^2$.

**Proof:** By 8.1.1, if $n = \rho_{r,3}(d)$, we can construct a $\mathcal{RSC}[3, r, d + 1]$ of cardinality $n + 1 \geq 1 + 2 \lfloor n/2 \rfloor$. By theorem 8.2, we can then construct a $\mathcal{RSC}[3, r, 4d + 2]$ of cardinality $n^2$ using the construction $d \delta d \delta (d + 1) \delta (d + 1)$.

**Corollary 8.4.3** $\rho_3(4d) \geq 3^{2^d}$.

**Proof:** By repeated application of 8.4.1 starting with $\rho_{3,3}(1) = 3$.

Our final construction enables us to combine $k$-rainbow sets of $r$-ary $d$-colourings for arbitrary $k$.

**Theorem 8.5** If we have a $\mathcal{RSC}[k, r, d_1]$ of cardinality $n_1$, a $\mathcal{RSC}[k, r, d_2]$ of cardinality $n_2 \geq n_1$, and a $\mathcal{RSC}[k, r, d_2]$ of cardinality $n_Z \geq n_2$, with $n_Z$ coprime to each integer in the range $[2, \ldots, h]$ where $h = \binom{k}{2} - 1$, then a $\mathcal{RSC}[k, r, D]$ of cardinality $N$ can be constructed, where $D = d_1 + d_2 + hd_Z$ and $N = n_1n_2$.

As before, we first need a preliminary result:

**Lemma 8.6** Given distinct pairs of integers $(a, b)$ and $(c, d)$ with $0 \leq a, b, c, d < n$ for some $n$, and given a positive integer $h$ such that $n$ is coprime to each integer in the range $[2, \ldots, h]$, then if we let $b_{-1} = a$ and $d_{-1} = c$, and $b_r = b + ra \pmod{n}$ and $d_r = d + rc \pmod{n}$ for $0 \leq r \leq h$, then if $b_i = d_i$ for some $i$, $-1 \leq i \leq h$, we have $b_j \neq d_j$ for all $j \neq i$. 

Proof: We consider two cases:

Case 1: $i = -1$

Since $a = c$, $(b + ja) - (d + jc) = b - d \neq 0 \pmod{n}$ since $(a, b)$ and $(c, d)$ are distinct, and $b$ and $d$ both less than $n$.

Case 2: $i \neq -1$

By the reversing the argument in case 1, $a \neq c$, i.e. $b_{-1} \neq d_{-1}$. For $j \geq 0$, since $b + ia = d + ic$, we have $(b + ja) - (d + jc) = (j - i)a - (j - i)c = (j - i)(a - c) \neq 0 \pmod{n}$ since $a \neq c$ and $|j - i| \leq h$ so $j - i$ is coprime to $n$. □

Proof of Theorem 8.5

Let $R_1 = \{c_0^1, \ldots, c_{n_1-1}^1\}$, $R_2 = \{c_0^2, \ldots, c_{n_2-1}^2\}$ and $Z = \{z_0, \ldots, z_{n_Z-1}\}$ be $k$-rainbow sets of $r$-ary $d_1$-, $d_2$- and $d_Z$-colourings of cardinality $n_1$, $n_2$ and $n_Z$, respectively.

Now let

$$Q = \{c_i^1.c_i^2.z_{j+i}.z_{j+2i} \cdots z_{j+hi}, \, 0 \leq i < n_1, 0 \leq j < n_2\},$$

where $h = \binom{k}{2} - 1$ and the subscript arithmetic is modulo $n_Z$, be a set of $D$-colourings of cardinality $N$, each element of $Q$ being made by concatenating $h+2$ component colourings: one from $R_1$, one from $R_2$, and $h$ from $Z$.

Let

$$S = \{c_i^1.c_i^2.z_{j_1+i_1} \cdots z_{j_1+i_1}, c_i^2.c_i^2.z_{j_2+i_2} \cdots z_{j_2+i_2}, \ldots, c_i^1.c_i^2.z_{j_k+i_k} \cdots z_{j_k+i_k}\}$$

be any set of $k$ distinct colourings in $Q$, and let $b_{s,-1} = i_s$ and $b_{s,t} = j_s + ti_s \pmod{n_Z}$, for each $s$ and $t$, $1 \leq s \leq k$, $0 \leq i \leq h$, so the $s^{th}$ colouring in $S$ is $c_{b_{s,-1},c_{b_{s,0}}.z_{b_{s,1}} \cdots z_{b_{s,h}}}$. Now, for any $s$, $s'$ and $t$, $1 \leq s, s' \leq k$, $-1 \leq t \leq h$, if $b_{s,t} = b_{s',t}$, then by lemma 8.6 we know that for all $u \neq t$, $b_{s,u} \neq b_{s',u}$. So for each pair $\{s, s'\}$, $b_{s,t} = b_{s',t}$ for no more than one value of $t$. Now there are $h + 2$ possible values of $t$, but only $\binom{k}{2} = h + 1$ different pairs $\{s, s'\}$, so there is some $t$ for which $b_{s,t} \neq b_{s',t}$ for all pairs $\{s, s'\}$ and the $(t + 2)^{th}$ component colourings of the elements in $S$ are all different. Since $R_1$, $R_2$ and $Z$ are all $k$-rainbow sets, we know that $S$ is a good $k$-set.

Thus, any $k$ colourings from $Q$ comprise a good $k$-set, so $Q$ is a $\mathcal{RSC}[k, r, D]$ of cardinality $N$. □

Corollary 8.6.1 $\rho_4(6.7^d) \geq 7^2d$.

Proof: The following 4-rainbow set of 4-ary 6-colourings of cardinality 8 — a version of $\mathcal{R}_{4,4,6}$ (see below) displayed with different symbols for each colour — shows that $\rho_4(6) \geq 7$. 

---

THE ELECTRONIC JOURNAL OF COMBINATORICS 13 (2006), #R12

21
The result follows by repeated application of theorem 8.5, noting that 7 is coprime to 2, 3, 4 and $5 = \binom{4}{2} - 1$. \hfill \square

9 Lower bounds for $\rho_{r,k}(d)$ for small $r$, $k$ and $d$

We conclude with tables of the best lower bounds known for $\rho_3(d)$, $\rho_{4,3}(d)$ and $\rho_4(d)$ for small $d$. For very small $d$, exhaustive computer searches have determined the values of $\rho_{r,k}(d)$. For other small values of $d$, the constructions used in theorems 8.2 and 8.5 provide the largest known rainbow sets. In the tables, these constructions are denoted $d_1$\$d_2$\$d_3$\$d_4$\$d_5$, etc., with superscript minus signs ($d^-$) to denote the removal of a single colouring from a largest rainbow set of $d$-colourings (to satisfy the requirement that the cardinality be odd). For $\rho_3(d)$, the probabilistic lower bound of theorem 7.1 is better than the constructions for $d \geq 71$; for $\rho_{4,3}(d)$, this is the case for $d \geq 26$. 
Some $k$-rainbow sets of $r$-ary $d$-colourings, for small $k$, $r$ and $d$

$$
\begin{array}{|c|c|c|c|c|}
\hline
\mathcal{R}_{3,3,3} & \mathcal{R}_{3,3,6} & \mathcal{R}_{4,3,3} & \mathcal{R}_{4,3,4} & \mathcal{R}_{4,4,6} \\
\rho_3(3) \geq 6 & \rho_3(6) \geq 13 & \rho_{4,3}(3) \geq 9 & \rho_{4,3}(4) \geq 16 & \rho_4(6) \geq 8 \\
000 & 000000 & 000 & 0000 & 000000 \\
011 & 000111 & 011 & 0011 & 011111 \\
102 & 000222 & 022 & 0102 & 101222 \\
121 & 011012 & 103 & 0220 & 112033 \\
212 & 022120 & 131 & 1013 & 220312 \\
220 & 101120 & 213 & 1212 & 233103 \\
112021 & 232 & 1230 & 323230 & 323230 \\
112102 & 323 & 1302 & & 3322 \\
112210 & 330 & 2031 & & 3333 \\
120012 & & 2103 & & \\
20012 & & 2121 & & \\
210120 & & 2320 & & \\
221201 & & 3113 & & \\
& & 3231 & & \\
& & 3322 & & \\
& & 3333 & & \\
\hline
\end{array}
$$

Best Lower Bounds Known for $\rho_3(d)$ and $\rho_{4,3}(d)$

$$
\begin{array}{|c|c|}
\hline
\text{d} & \rho_3(d) \\
\hline
1 & 3 \text{ computer, 8.1.1} \\
2 & 4 \text{ computer, } \mathcal{R}_{3,3,3} \\
3 & 9 \text{ computer, } 1 \epsilon 1 \epsilon 1 \epsilon 1 \\
4 & 10 \text{ computer, 8.1.1} \\
5 & 13 \text{ computer, } \mathcal{R}_{3,3,6} \\
6 & 14 \geq 8.1.1 \\
7 & 15 \geq 8.1.1 \\
8 & 16 \geq 8.1.1 \\
9 & 17 \geq 8.1.1 \\
10 & 27 \text{ computer, } 1 \epsilon 1 \epsilon 1 \epsilon 4 \epsilon 4 \\
11 & 28 \geq 8.1.1 \\
12 & 29 \geq 8.1.1 \\
13 & 36 \geq 2 \epsilon 4 \epsilon 4 \epsilon 4 \\
14 & 54 \geq 3 \epsilon 4 \epsilon 4 \epsilon 4 \\
15 & 81 \geq 4 \epsilon 4 \epsilon 4 \epsilon 4 \\
\ldots \ldots \\
70 & 6723 \text{ computer, 8.1.1} \\
71 & 7064 \text{ theorem 7.1} \\
\hline
\end{array}
$$

$$
\begin{array}{|c|c|}
\hline
\text{d} & \rho_{4,3}(d) \\
\hline
1 & 4 \text{ computer, 8.1.1} \\
2 & 6 \text{ computer, } \mathcal{R}_{4,3,3} \\
3 & 9 \text{ computer, } \mathcal{R}_{4,3,4} \\
4 & 16 \text{ computer, 8.1.1} \\
5 & 18 \geq 8.1.1 \\
6 & 20 \geq 8.1.1 \\
7 & 22 \geq 8.1.1 \\
8 & 25 \geq 2^6 \text{ or } 2^6 \text{ or } 2^6 \\
9 & 27 \geq 8.1.1 \\
10 & 36 \text{ computer, } 2 \epsilon 2 \epsilon 3 \epsilon 3 \text{ or } 2 \epsilon 2 \epsilon 3 \epsilon 3 \\
11 & 54 \geq 2 \epsilon 3 \epsilon 3 \epsilon 3 \\
12 & 81 \geq 3 \epsilon 3 \epsilon 3 \epsilon 3 \\
13 & 83 \geq 8.1.1 \\
14 & 90 \geq 2 \epsilon 4 \epsilon 4 \epsilon 4 \\
15 & 135 \geq 3 \epsilon 4 \epsilon 4 \epsilon 4 \\
16 & 225 \geq 4 \epsilon 4 \epsilon 4 \epsilon 4 \\
\ldots \ldots \\
25 & 363 \geq 8.1.1 \\
26 & 424 \text{ theorem 7.1} \\
\hline
\end{array}
$$
Best Lower Bounds Known for $\rho_4(d)$

<table>
<thead>
<tr>
<th>$d$</th>
<th>$\rho_4(d)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$= 4$</td>
</tr>
<tr>
<td>2</td>
<td>$= 4$ computer</td>
</tr>
<tr>
<td>3</td>
<td>$= 5$ computer, 8.1.2</td>
</tr>
<tr>
<td>4</td>
<td>$= 5$ computer</td>
</tr>
<tr>
<td>5</td>
<td>$= 6$ computer, 8.1.2</td>
</tr>
<tr>
<td>6</td>
<td>$= 8$ computer, $R_{4,4,6}$</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>42</td>
<td>$\geq 49$ $6 \leq 6 \leq 6 \leq 6 \leq 6 \leq 6 \leq 6$</td>
</tr>
</tbody>
</table>

Acknowledgements

The author would like to thank Günter Ziegler for his encouragement and helpful comments on earlier drafts of this paper.

References


