Sets of points determining only acute angles and some related colouring problems

Sets of Points Determining Only Acute Angles and Some Related Colouring Problems

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Abstract

We present both probabilistic and constructive lower bounds on the maximum size of a set of points $\mathcal{S} \subseteq \mathbb{R}^d$ such that every angle determined by three points in $\mathcal{S}$ is acute, considering especially the case $\mathcal{S} \subseteq \{0,1\}^d$. These results improve upon a probabilistic lower bound of Erdős and Füredi. We also present lower bounds for some generalisations of the acute angles problem, considering especially some problems concerning colourings of sets of integers.

1 Introduction

Let us say that a set of points $\mathcal{S} \subseteq \mathbb{R}^d$ is an **acute *d*-set** if every angle determined by a triple of $\mathcal{S}$ is acute ($< \frac{\pi}{2}$). Let us also say that $\mathcal{S}$ is a **cubic acute *d*-set** if $\mathcal{S}$ is an acute d-set and is also a subset of the unit d-cube (i.e. $\mathcal{S} \subseteq \{0,1\}^d$).

Let us further say that a triple $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in \mathbb{R}^d$ is an **acute triple**, a **right triple**, or an **obtuse triple**, if the angle determined by the triple with apex $\boldsymbol{v}$ is less than $\frac{\pi}{2}$, equal to $\frac{\pi}{2}$, or greater than $\frac{\pi}{2}$, respectively. Note that we consider the triples $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ and $\boldsymbol{w}, \boldsymbol{v}, \boldsymbol{u}$ to be the same.

We will denote by $\alpha(d)$ the size of a largest possible acute d-set. Similarly, we will denote by $\kappa(d)$ the size of a largest possible cubic acute d-set. Clearly $\kappa(d) \leq \alpha(d)$, $\kappa(d) \leq \kappa(d+1)$ and $\alpha(d) \leq \alpha(d+1)$ for all d.

In [EF], Paul Erdős and Zoltán Füredi gave a probabilistic proof that $\kappa(d) \geq \left\lfloor \frac{1}{2}\left(\frac{2}{\sqrt{3}}\right)^d \right\rfloor$ (see also [AZ2]). This disproved an earlier conjecture of Ludwig Danzer and Branko Grünbaum [DG] that $\alpha(d) = 2d - 1$.

In the following two sections we give improved probabilistic lower bounds for $\kappa(d)$ and $\alpha(d)$. In section 4 we present a construction that gives further improved lower bounds for $\kappa(d)$ for small d. In section 5, we tabulate the best lower bounds known for $\kappa(d)$ and $\alpha(d)$ for small d. Finally, in sections 6–9, we give probabilistic and constructive lower bounds for some generalisations of $\kappa(d)$, considering especially some problems concerning colourings of sets of integers.

2 A probabilistic lower bound for $\kappa(d)$

Theorem 2.1

$$\kappa(d) \;\geq\; 2\left\lfloor \frac{\sqrt{6}}{9}\left(\frac{2}{\sqrt{3}}\right)^d \right\rfloor \;\approx\; 0.544 \times 1.155^d.$$

For large d, this improves upon the result of Erdős and Füredi by a factor of $\frac{4\sqrt{6}}{9} \approx 1.089$. This is achieved by a slight improvement in the choice of parameters. This proof can also be found in [AZ3].

Proof: Let $m = \left\lfloor \frac{\sqrt{6}}{9}\left(\frac{2}{\sqrt{3}}\right)^d \right\rfloor$ and randomly pick a set $\mathcal{S}$ of $3m$ point vectors from the vertices of the d-dimensional unit cube $\{0,1\}^d$, choosing the coordinates independently with probability $\Pr[\boldsymbol{v}_i = 0] = \Pr[\boldsymbol{v}_i = 1] = \frac{1}{2}$, $1 \leq i \leq d$, for every $\boldsymbol{v} = (\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_d) \in \mathcal{S}$.

Now every angle determined by a triple of points from $\mathcal{S}$ is non-obtuse ($\leq \frac{\pi}{2}$), and a triple of vectors $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ from $\mathcal{S}$ is a right triple iff the scalar product $\langle \boldsymbol{u} - \boldsymbol{v}, \boldsymbol{w} - \boldsymbol{v} \rangle$ vanishes, i.e. iff either $\boldsymbol{u}_i - \boldsymbol{v}_i = 0$ or $\boldsymbol{w}_i - \boldsymbol{v}_i = 0$ for each i, $1 \leq i \leq d$.

Thus $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ is a right triple iff $\boldsymbol{u}_i, \boldsymbol{v}_i, \boldsymbol{w}_i$ is neither $0,1,0$ nor $1,0,1$ for any i, $1 \leq i \leq d$. Since $\boldsymbol{u}_i, \boldsymbol{v}_i, \boldsymbol{w}_i$ can take eight different values, this occurs independently with probability $\frac{3}{4}$ for each i, so the probability that a triple of $\mathcal{S}$ is a right triple is $\left(\frac{3}{4}\right)^d$.

Hence, the expected number of right triples in a set of $3m$ vectors is $3\binom{3m}{3}\left(\frac{3}{4}\right)^d$. Thus there is *some* set $\mathcal{S}$ of $3m$ vectors with no more than $3\binom{3m}{3}\left(\frac{3}{4}\right)^d$ right triples, where

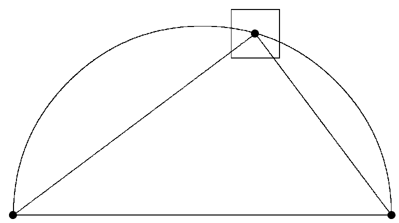
$$3\binom{3m}{3}\left(\frac{3}{4}\right)^d \;<\; 3\frac{(3m)^3}{6}\left(\frac{3}{4}\right)^d \;=\; m\left(\frac{9m}{\sqrt{6}}\right)^2\left(\frac{3}{4}\right)^d \;\leq\; m$$

by the choice of m.

If we remove one point of each right triple from $\mathcal{S}$, the remaining set is a cubic acute d-set of cardinality at least $3m - m = 2m$. $\square$

3 A probabilistic lower bound for $\alpha(d)$

We can improve the lower bound in theorem 2.1 for non-cubic acute d-sets by a factor of $\sqrt{2}$ by slightly perturbing the points chosen away from the vertices of the unit cube. The intuition behind this is that a small random symmetrical perturbation of the points in a right triple is more likely than not to produce an acute triple, as the following diagram suggests.



Theorem 3.1

$$\alpha(d) \;\geq\; 2\left\lfloor \frac{1}{3}\left(\frac{2}{\sqrt{3}}\right)^{d+1} \right\rfloor \;\approx\; 0.770 \times 1.155^d.$$

Before we can prove this theorem, we need some results concerning continuous random variables.

Definition 3.2 *If $F(x) = \Pr[X \leq x]$ is the cumulative distribution function of a continuous random variable X, let $\overline{F}(x)$ denote $\Pr[X \geq x] \;=\; 1 - F(x)$.*

Definition 3.3 *Let us say that a continuous random variable X has* ***positive bias*** *if, for all t, $\Pr[X \geq t] \;\geq\; \Pr[X \leq -t]$, i.e. $\overline{F}(t) \geq F(-t)$.*

Property 3.3.1 *If a continuous random variable X has positive bias, it follows that $\Pr[X > 0] \geq \frac{1}{2}$.*

Property 3.3.2 *To show that a continuous random variable X has positive bias, it suffices to demonstrate that the condition $\overline{F}(t) \geq F(-t)$ holds for all* ***positive*** *t.*

Lemma 3.4 *If X and Y are independent continuous random variables with positive bias, then $X+Y$ also has positive bias.*

Proof: Let f, g and h be the probability density functions, and F, G and H the cumulative distribution functions, for X, Y and $X+Y$ respectively. Then,

$$\begin{aligned}
\overline{H}(t) - H(-t) &= \iint_{x+y\,\geq\,t} f(x)g(y)\,\mathrm{d}y\,\mathrm{d}x \;-\; \iint_{x+y\,\leq\,-t} f(x)g(y)\,\mathrm{d}y\,\mathrm{d}x \\
&= \quad \iint_{x+y\,\geq\,t} f(x)g(y)\,\mathrm{d}y\,\mathrm{d}x \;-\; \iint_{y-x\,\geq\,t} f(x)g(y)\,\mathrm{d}y\,\mathrm{d}x \\
&\quad + \iint_{y-x\,\geq\,t} f(x)g(y)\,\mathrm{d}y\,\mathrm{d}x \;-\; \iint_{x+y\,\leq\,-t} f(x)g(y)\,\mathrm{d}y\,\mathrm{d}x \\
&= \quad \int_{-\infty}^{\infty} g(y)\left[\overline{F}(t-y) - F(y-t)\right]\,\mathrm{d}y \\
&\quad + \int_{-\infty}^{\infty} f(x)\left[\overline{G}(x+t) - G(-x-t)\right]\,\mathrm{d}x
\end{aligned}$$

which is non-negative because $f(t)$, $g(t)$, $\overline{F}(t) - F(-t)$ and $\overline{G}(t) - G(-t)$ are all non-negative for all t. $\square$

Definition 3.5 *Let us say that a continuous random variable X is **ϵ-uniformly distributed** for some $\epsilon > 0$ if X is uniformly distributed between $-\epsilon$ and ϵ.*

Let us denote by j, the probability density function of an ϵ-uniformly distributed random variable:

$$j(x) \;=\; \begin{cases} \frac{1}{2\epsilon} & -\epsilon \leq x \leq \epsilon \\ 0 & \text{otherwise} \end{cases}$$

and by J, its cumulative distribution function:

$$J(x) \;=\; \begin{cases} 0 & x < -\epsilon \\ \frac{1}{2} + \frac{x}{2\epsilon} & -\epsilon \leq x \leq \epsilon \\ 1 & x > \epsilon \end{cases}$$

Property 3.5.1 *If X is an ϵ-uniformly distributed random variable, then so is $-X$.*

Lemma 3.6 *If X, Y and Z are independent ϵ-uniformly distributed random variables for some $\epsilon < \frac{1}{2}$, then $U = (Y - X)(1 + Z - X)$ has positive bias.*

Proof: Let G be the cumulative distribution function of U. By 3.3.2, it suffices to show that $\overline{G}(u) - G(-u) \geq 0$ for all positive u.

Let u be positive. Because $1 + Z - X$ is always positive, $U \geq u$ iff $Y > X$ and $Z \geq -1 + X + \frac{u}{Y-X}$. Similarly, $U \leq -u$ iff $X > Y$ and $Z \geq -1 + X + \frac{u}{X-Y}$. So,

$$\begin{aligned}
\overline{G}(u) - G(-u) &= \iint_{y>x} j(x)j(y)\overline{J}(-1 + x + \frac{u}{y-x})\,\mathrm{d}y\,\mathrm{d}x \\
&\quad - \iint_{x>y} j(x)j(y)\overline{J}(-1 + x + \frac{u}{x-y})\,\mathrm{d}y\,\mathrm{d}x \\
&= \iint_{y>x} j(x)j(y)\left[J(1 - x - \frac{u}{y-x}) - J(1 - y - \frac{u}{y-x})\right]\,\mathrm{d}y\,\mathrm{d}x \\
&\quad \text{(because } \overline{J}(x) = J(-x)\text{, and by variable renaming)}
\end{aligned}$$

which is non-negative because j is non-negative and J is non-decreasing (so the expression in square brackets is non-negative over the domain of integration). □

Corollary 3.6.1 *If X, Y and Z are independent ϵ-uniformly distributed random variables for some $\epsilon < \frac{1}{2}$, then $(Y - X)(Z - X - 1)$ has positive bias.*

Proof: $(Y - X)(Z - X - 1) = ((-Y) - (-X))(1 + (-Z) - (-X))$. The result follows from 3.5.1 and lemma 3.6. □

Lemma 3.7 *If X, Y and Z are independent ϵ-uniformly distributed random variables, then $V = (Y - X)(Z - X)$ has positive bias.*

Proof: Let H be the cumulative distribution function of V. By 3.3.2, it suffices to show that $\overline{H}(v) - H(-v) \geq 0$ for all positive v.

Let v be positive. $V \geq v$ iff $Y > X$ and $Z \geq X + \frac{v}{Y-X}$ or $Y < X$ and $Z \leq X + \frac{v}{Y-X}$. Similarly, $V \leq -v$ iff $Y > X$ and $Z \leq X - \frac{v}{Y-X}$ or $Y < X$ and $Z \geq X - \frac{v}{Y-X}$. So,

$$\begin{aligned}
\overline{H}(v) - H(-v) \quad = \quad & \iint_{y>x} j(x)j(y)\overline{J}(x + \frac{v}{y-x})\,\mathrm{d}y\,\mathrm{d}x \\
& + \iint_{y<x} j(x)j(y)J(x + \frac{v}{y-x})\,\mathrm{d}y\,\mathrm{d}x \\
& - \iint_{y>x} j(x)j(y)J(x - \frac{v}{y-x})\,\mathrm{d}y\,\mathrm{d}x \\
& - \iint_{y<x} j(x)j(y)\overline{J}(x - \frac{v}{y-x})\,\mathrm{d}y\,\mathrm{d}x \\
= \quad & \iint_{y>x} j(x)j(y)\left[J(-x - \frac{v}{y-x}) - J(-y - \frac{v}{y-x})\right]\mathrm{d}y\,\mathrm{d}x \\
& + \iint_{y<x} j(x)j(y)\left[J(x + \frac{v}{y-x}) - J(y + \frac{v}{y-x})\right]\mathrm{d}y\,\mathrm{d}x \\
& \text{(because } \overline{J}(x) \ = \ J(-x)\text{, and by variable renaming)}
\end{aligned}$$

which is non-negative because j is non-negative and J is non-decreasing (so the expressions in square brackets are non-negative over the domains of integration). $\square$

We are now in a position to prove the theorem.

Proof of theorem 3.1

Let $m = \left\lfloor \frac{1}{3}\left(\frac{2}{\sqrt{3}}\right)^{d+1} \right\rfloor$, and randomly pick a set $\mathcal{S}$ of $3m$ point vectors, $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_{3m}$, from the vertices of the d-dimensional unit cube $\{0,1\}^d$, choosing the coordinates independently with probability $\Pr[\boldsymbol{v}_{ki} = 0] = \Pr[\boldsymbol{v}_{ki} = 1] = \frac{1}{2}$ for every $\boldsymbol{v}_k = (\boldsymbol{v}_{k1}, \boldsymbol{v}_{k2}, \ldots, \boldsymbol{v}_{kd})$, $1 \leq k \leq 3m$, $1 \leq i \leq d$.

Now for some ϵ, $0 < \epsilon < \frac{1}{2(d+1)}$, randomly pick $3m$ vectors, $\boldsymbol{\delta}_1, \boldsymbol{\delta}_2, \ldots, \boldsymbol{\delta}_{3m}$, from the d-dimensional cube $[-\epsilon, \epsilon]^d$ of side 2ϵ centred on the origin, choosing the coordinates $\boldsymbol{\delta}_{ki}$, $1 \leq k \leq 3m$, $1 \leq i \leq d$, independently so that they are ϵ-uniformly distributed, and let $\mathcal{S}' = \{\boldsymbol{v}'_1, \boldsymbol{v}'_2, \ldots, \boldsymbol{v}'_{3m}\}$ where $\boldsymbol{v}'_k = \boldsymbol{v}_k + \boldsymbol{\delta}_k$ for each k, $1 \leq k \leq 3m$.

Case 1: Acute triples in $\mathcal{S}$

Because $\epsilon < \frac{1}{2(d+1)}$, if $\boldsymbol{v}_j, \boldsymbol{v}_k, \boldsymbol{v}_l$ is an acute triple in $\mathcal{S}$, the scalar product $\langle \boldsymbol{v}'_j - \boldsymbol{v}'_k, \boldsymbol{v}'_l - \boldsymbol{v}'_k \rangle > \frac{1}{(d+1)^2}$, so $\boldsymbol{v}'_j, \boldsymbol{v}'_k, \boldsymbol{v}'_l$ is also an acute triple in $\mathcal{S}'$.

Case 2: Right triples in $\mathcal{S}$

If, $\boldsymbol{v}_j, \boldsymbol{v}_k, \boldsymbol{v}_l$ is a right triple in $\mathcal{S}$ then the scalar product $\langle \boldsymbol{v}_j - \boldsymbol{v}_k, \boldsymbol{v}_l - \boldsymbol{v}_k \rangle$ vanishes, i.e. either $\boldsymbol{v}_{j_i} - \boldsymbol{v}_{ki} = 0$ or $\boldsymbol{v}_{li} - \boldsymbol{v}_{ki} = 0$ for each i, $1 \leq i \leq d$. There are six possibilities for each triple of coordinates:

$\boldsymbol{v}_{j_i}, \boldsymbol{v}_{ki}, \boldsymbol{v}_{li}$	$(\boldsymbol{v}'_{j_i} - \boldsymbol{v}'_{ki})(\boldsymbol{v}'_{li} - \boldsymbol{v}'_{ki})$
0, 0, 0	$(\boldsymbol{\delta}_{j_i} - \boldsymbol{\delta}_{ki})(\boldsymbol{\delta}_{li} - \boldsymbol{\delta}_{ki})$
1, 1, 1	$(\boldsymbol{\delta}_{j_i} - \boldsymbol{\delta}_{ki})(\boldsymbol{\delta}_{li} - \boldsymbol{\delta}_{ki})$
0, 0, 1	$(\boldsymbol{\delta}_{j_i} - \boldsymbol{\delta}_{ki})(1 + \boldsymbol{\delta}_{li} - \boldsymbol{\delta}_{ki})$
1, 0, 0	$(\boldsymbol{\delta}_{li} - \boldsymbol{\delta}_{ki})(1 + \boldsymbol{\delta}_{j_i} - \boldsymbol{\delta}_{ki})$
0, 1, 1	$(\boldsymbol{\delta}_{li} - \boldsymbol{\delta}_{ki})(\boldsymbol{\delta}_{j_i} - \boldsymbol{\delta}_{ki} - 1)$
1, 1, 0	$(\boldsymbol{\delta}_{j_i} - \boldsymbol{\delta}_{ki})(\boldsymbol{\delta}_{li} - \boldsymbol{\delta}_{ki} - 1)$

Now, the values of the $\boldsymbol{\delta}_{ki}$ are independent and ϵ-uniformly distributed, so by lemmas 3.7 and 3.6 and corollary 3.6.1, the distribution of the $(\boldsymbol{v}'_{j_i} - \boldsymbol{v}'_{ki})(\boldsymbol{v}'_{li} - \boldsymbol{v}'_{ki})$ has positive bias, and by repeated application of lemma 3.4, the distribution of the scalar product $\langle \boldsymbol{v}'_j - \boldsymbol{v}'_k, \boldsymbol{v}'_l - \boldsymbol{v}'_k \rangle = \sum_{i=1}^{d} (\boldsymbol{v}'_{j_i} - \boldsymbol{v}'_{ki})(\boldsymbol{v}'_{li} - \boldsymbol{v}'_{ki})$ also has positive bias.

Thus, if $\boldsymbol{v}_j, \boldsymbol{v}_k, \boldsymbol{v}_l$ is a right triple in $\mathcal{S}$, then, by 3.3.1,

$$\Pr\left[\langle \boldsymbol{v}'_j - \boldsymbol{v}'_k, \boldsymbol{v}'_l - \boldsymbol{v}'_k \rangle > 0\right] \;\geq\; \frac{1}{2},$$

so the probability that the triple $\boldsymbol{v}'_j, \boldsymbol{v}'_k, \boldsymbol{v}'_l$ is an acute triple in $\mathcal{S}'$ is at least $\frac{1}{2}$.

As in the proof of theorem 2.1, the expected number of right triples in $\mathcal{S}$ is $3\binom{3m}{3}\left(\frac{3}{4}\right)^d$, so the expected number of non-acute triples in $\mathcal{S}'$ is no more than half this value. Thus there is *some* set $\mathcal{S}'$ of $3m$ vectors with no more than $\frac{3}{2}\binom{3m}{3}\left(\frac{3}{4}\right)^d$ non-acute triples, where

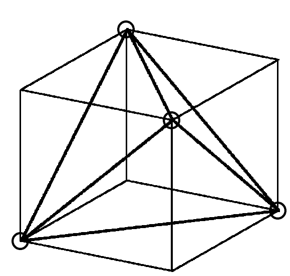
$$\frac{3}{2}\binom{3m}{3}\left(\frac{3}{4}\right)^d \;<\; \frac{3}{2}\frac{(3m)^3}{6}\left(\frac{3}{4}\right)^d \;=\; m(3m)^2\left(\frac{3}{4}\right)^{d+1} \;\leq\; m$$

by the choice of m.

If we remove one point of each non-acute triple from $\mathcal{S}'$, the remaining set is an acute d-set of cardinality at least $3m - m = 2m$. □

4 Constructive lower bounds for $\kappa(d)$

In the following proofs, for clarity of exposition, we will represent point vectors in $\{0,1\}^d$ as binary words of length d, e.g. $\mathcal{S}_3 = \{000, 011, 101, 110\}$ represents a cubic acute 3-set.



Concatenation of words (vectors) $\boldsymbol{v}$ and $\boldsymbol{v}'$ will be written $\boldsymbol{v}\boldsymbol{v}'$.

We begin with a simple construction that enables us to extend a cubic acute d-set of cardinality n to a cubic acute $(d+2)$-set of cardinality $n+1$.

Theorem 4.1

$$\kappa(d+2) \;\geq\; \kappa(d)+1$$

Proof: Let $\mathcal{S} = \{\boldsymbol{v}_0, \boldsymbol{v}_1, \ldots, \boldsymbol{v}_{n-1}\}$ be a cubic acute d-set of cardinality $n = \kappa(d)$. Now let $\mathcal{S}' = \{\boldsymbol{v}'_0, \boldsymbol{v}'_1, \ldots, \boldsymbol{v}'_n\} \subseteq \{0,1\}^{d+2}$ where $\boldsymbol{v}'_i = \boldsymbol{v}_i 00$ for $0 \leq i \leq n-2$, $\boldsymbol{v}'_{n-1} = \boldsymbol{v}_{n-1}10$ and $\boldsymbol{v}'_n = \boldsymbol{v}_{n-1}01$.

If $\boldsymbol{v}'_i, \boldsymbol{v}'_j, \boldsymbol{v}'_k$ is a triple of distinct points in $\mathcal{S}'$ with no more than one of i, j and k greater than $n-2$, then $\boldsymbol{v}'_i, \boldsymbol{v}'_j, \boldsymbol{v}'_k$ is an acute triple, because $\mathcal{S}$ is an acute d-set. Also, any triple $\boldsymbol{v}'_k, \boldsymbol{v}'_{n-1}, \boldsymbol{v}'_n$ or $\boldsymbol{v}'_k, \boldsymbol{v}'_n, \boldsymbol{v}'_{n-1}$ is an acute triple, because its $(d+1)$th or $(d+2)$th coordinates (respectively) are $0,1,0$. Finally, for any triple $\boldsymbol{v}'_{n-1}, \boldsymbol{v}'_k, \boldsymbol{v}'_n$, if $\boldsymbol{v}_k$ and $\boldsymbol{v}_{n-1}$ differ in the rth coordinate, then the rth coordinates of $\boldsymbol{v}'_{n-1}, \boldsymbol{v}'_k, \boldsymbol{v}'_n$ are $0,1,0$ or $1,0,1$. Thus, $\mathcal{S}'$ is a cubic acute $(d+2)$-set of cardinality $n+1$. $\square$

Our second construction combines cubic acute d-sets of cardinality n to make a cubic acute $3d$-set of cardinality n^2.

Theorem 4.2

$$\kappa(3d) \;\geq\; \kappa(d)^2.$$

Proof: Let $\mathcal{S} = \{\boldsymbol{v}_0, \boldsymbol{v}_1, \ldots, \boldsymbol{v}_{n-1}\}$ be a cubic acute d-set of cardinality $n = \kappa(d)$, and let

$$\mathcal{T} = \{\boldsymbol{w}_{ij} = \boldsymbol{v}_i \boldsymbol{v}_j \boldsymbol{v}_{j-i \bmod n} \;:\; 0 \leq i, j \leq n-1\},$$

each $\boldsymbol{w}_{ij}$ being made by concatenating three of the $\boldsymbol{v}_i$.

Let $\boldsymbol{w}_{ps}, \boldsymbol{w}_{qt}, \boldsymbol{w}_{ru}$ be any triple of distinct points in $\mathcal{T}$. They constitute an acute triple iff the scalar product $\langle \boldsymbol{w}_{ps} - \boldsymbol{w}_{qt}, \boldsymbol{w}_{ru} - \boldsymbol{w}_{qt} \rangle$ does not vanish (is positive). Now,

$$\begin{aligned}
\langle \boldsymbol{w}_{ps} - \boldsymbol{w}_{qt}, \boldsymbol{w}_{ru} - \boldsymbol{w}_{qt} \rangle &= \langle \boldsymbol{v}_p \boldsymbol{v}_s \boldsymbol{v}_{s-p} - \boldsymbol{v}_q \boldsymbol{v}_t \boldsymbol{v}_{t-q}, \boldsymbol{v}_r \boldsymbol{v}_u \boldsymbol{v}_{u-r} - \boldsymbol{v}_q \boldsymbol{v}_t \boldsymbol{v}_{t-q} \rangle \\
&= \quad \langle \boldsymbol{v}_p - \boldsymbol{v}_q, \boldsymbol{v}_r - \boldsymbol{v}_q \rangle \\
&\quad + \langle \boldsymbol{v}_s - \boldsymbol{v}_t, \boldsymbol{v}_u - \boldsymbol{v}_t \rangle \\
&\quad + \langle \boldsymbol{v}_{s-p} - \boldsymbol{v}_{t-q}, \boldsymbol{v}_{u-r} - \boldsymbol{v}_{t-q} \rangle
\end{aligned}$$

with all the index arithmetic modulo n.

If both $p \neq q$ and $q \neq r$, then the first component of this sum is positive, because $\mathcal{S}$ is an acute d-set. Similarly, if both $s \neq t$ and $t \neq u$, then the second component is positive. Finally, if $p = q$ and $t = u$, then $q \neq r$ and $s \neq t$ or else the points would not be distinct, so the third component, $\langle \boldsymbol{v}_{s-p} - \boldsymbol{v}_{t-q}, \boldsymbol{v}_{u-r} - \boldsymbol{v}_{t-q} \rangle$ is positive. Similarly if $q = r$ and $s = t$.

Thus, all triples in $\mathcal{T}$ are acute triples, so $\mathcal{T}$ is a cubic acute $3d$-set of cardinality n^2. $\square$

Corollary 4.2.1 $\kappa(3^d) \geq 2^{2^d}$.

Proof: By repeated application of theorem 4.2 starting with $\mathcal{S}_3$, a cubic acute 3-set of cardinality 4. $\square$

Corollary 4.2.2 *If* $d \geq 3$,

$$\kappa(d) \geq 10^{\frac{(d+1)^\mu}{4}} \approx 1.778^{(d+1)^{0.631}} \qquad \textit{where } \mu = \frac{\log 2}{\log 3}.$$

For small d, this is a tighter bound than theorem 2.1.

Proof: By induction on d. For $3 \leq d \leq 8$, we have the following cubic acute d-sets $(\mathcal{S}_3, \ldots, \mathcal{S}_8)$ that satisfy this lower bound for $\kappa(d)$ (with equality for $d = 8$):

$\mathcal{S}_3 : \kappa(3) \geq 4$
000
011
101
110

$\mathcal{S}_4 : \kappa(4) \geq 5$
0000
0011
0101
1001
1110

$\mathcal{S}_5 : \kappa(5) \geq 6$
00000
00011
00101
01001
10001
11110

$\mathcal{S}_6 : \kappa(6) \geq 8$
000000
000111
011001
011110
101010
101101
110011
110100

$\mathcal{S}_7 : \kappa(7) \geq 9$
0000000
0000011
0001101
0110001
0111110
1010101
1011010
1100110
1101001

$\mathcal{S}_8 : \kappa(8) \geq 10$
00000000
00000011
00000101
00011001
01100001
01111110
10101001
10110110
11001110
11010001

If $\kappa(d) \geq 10^{\frac{(d+1)^\mu}{4}}$, then

$$\begin{aligned} \kappa(3d) &\geq \kappa(d)^2 && \text{by theorem 4.2} \\ &\geq 10^{\frac{2(d+1)^\mu}{4}} && \text{by the induction hypothesis} \\ &= 10^{\frac{(3d+3)^\mu}{4}} && \text{because } 3^\mu = 2. \end{aligned}$$

So, since $\kappa(3d+2) \geq \kappa(3d+1) \geq \kappa(3d)$, if the lower bound is satisfied for d, it is also satisfied for $3d$, $3d+1$ and $3d+2$. $\square$

Theorem 4.3 *If, for each r, $1 \le r \le m$, we have a cubic acute d_r-set of cardinality n_r, where n_1 is the least of the n_r, and if, for some dimension d_Z, we have a cubic acute d_Z-set of cardinality n_Z, where*

$$n_Z \ge \prod_{r=2}^{m} n_r,$$

then a cubic acute D-set of cardinality N can be constructed, where

$$D = \sum_{r=1}^{m} d_r + d_Z \quad \textit{and} \quad N = \prod_{r=1}^{m} n_r.$$

This result generalises theorem 4.2, but before we can prove it, we first need some preliminary results.

Definition 4.4 *If $n_1 \le n_2 \le \ldots \le n_m$ and $0 \le k_r < n_r$, for each r, $1 \le r \le m$, then let us denote by $\langle\!\langle\, k_1 k_2 \ldots k_m \,\rangle\!\rangle_{n_1 n_2 \ldots n_m}$, the number*

$$\langle\!\langle\, k_1 k_2 \ldots k_m \,\rangle\!\rangle_{n_1 n_2 \ldots n_m} \;=\; \sum_{r=2}^{m} \Big((k_{r-1} - k_r \bmod n_r) \prod_{s=r+1}^{m} n_s \Big).$$

Where the n_r can be inferred from the context, $\langle\!\langle\, k_1 k_2 \ldots k_m \,\rangle\!\rangle$ may be used instead of $\langle\!\langle\, k_1 k_2 \ldots k_m \,\rangle\!\rangle_{n_1 n_2 \ldots n_m}$.

The expression $\langle\!\langle\, k_1 k_2 \ldots k_m \,\rangle\!\rangle_{n_1 n_2 \ldots n_m}$ can be understood as representing a number in a number system where the radix for each digit is a different n_r — like the old British monetary system of pounds, shillings and pennies — and the digits are the difference of two adjacent k_r (mod n_r). For example,

$$\langle\!\langle\, 2053 \,\rangle\!\rangle_{4668} \;=\; [2-0]_6[0-5]_6[5-3]_8 \;=\; 2 \times 6 \times 8 \;+\; 1 \times 8 \;+\; 2 \;=\; 106,$$

where $[a_2]_{n_2} \ldots [a_m]_{n_m}$ is place notation with the n_r the radix for each place.

By construction, we have the following results:

Property 4.4.1

$$\langle\!\langle\, k_1 k_2 \ldots k_m \,\rangle\!\rangle_{n_1 n_2 \ldots n_m} \;<\; \prod_{r=2}^{m} n_r.$$

Property 4.4.2 *If $2 \le t \le m$ and $j_{t-1} - j_t \ne k_{t-1} - k_t \pmod{n_t}$, then*

$$\langle\!\langle\, j_1 j_2 \ldots j_m \,\rangle\!\rangle_{n_1 n_2 \ldots n_m} \;\ne\; \langle\!\langle\, k_1 k_2 \ldots k_m \,\rangle\!\rangle_{n_1 n_2 \ldots n_m}.$$

Lemma 4.5 *If $n_1 \le n_2 \le \ldots \le n_m$ and $0 \le j_r, k_r < n_r$, for each r, $1 \le r \le m$, and the sequences of j_r and k_r are neither identical nor everywhere different (i.e. there exist both t and u such that $j_t = k_t$ and $j_u \ne k_u$), then*

$$\langle\!\langle\, j_1 j_2 \ldots j_m \,\rangle\!\rangle_{n_1 n_2 \ldots n_m} \ \ne\ \langle\!\langle\, k_1 k_2 \ldots k_m \,\rangle\!\rangle_{n_1 n_2 \ldots n_m}.$$

Proof: Let u be the greatest integer, $1 \le u < m$, such that $j_u - j_{u+1} \ne k_u - k_{u+1} \pmod{n_{u+1}}$. (If $j_m = k_m$, then u is the greatest integer such that $j_u \ne k_u$. If $j_m \ne k_m$, then u is at least as great as the greatest integer t such that $j_t = k_t$.) The result now follows from 4.4.2. □

We are now in a position to prove the theorem.

Proof of Theorem 4.3

Let $n_1 \le n_2 \le \ldots \le n_m$, and, for each r, $1 \le r \le m$, let $\mathcal{S}_r = \{\boldsymbol{v}^r_0, \boldsymbol{v}^r_1, \ldots, \boldsymbol{v}^r_{n_r-1}\}$ be a cubic acute d_r-set of cardinality n_r. Let $\mathcal{Z} = \{\boldsymbol{z}_0, \boldsymbol{z}_1, \ldots, \boldsymbol{z}_{n_Z-1}\}$ be a cubic acute d_Z-set of cardinality n_Z, where

$$n_Z \ge \prod_{r=2}^{m} n_r,$$

and let

$$D = \sum_{r=1}^{m} d_r + d_Z \quad \text{and} \quad N = \prod_{r=1}^{m} n_r.$$

Now let

$$\mathcal{T} = \{\boldsymbol{w}_{k_1 k_2 \ldots k_m} = \boldsymbol{v}^1_{k_1} \boldsymbol{v}^2_{k_2} \ldots \boldsymbol{v}^m_{k_m} \boldsymbol{z}_{k_Z} \ : \ 0 \le k_r < n_r, 1 \le r \le m\},$$

where $k_Z = \langle\!\langle\, k_1 k_2 \ldots k_m \,\rangle\!\rangle_{n_1 n_2 \ldots n_m}$, be a point set of dimension D and cardinality N, each element of $\mathcal{T}$ being made by concatenating one vector from each of the $\mathcal{S}_r$ together with a vector from $\mathcal{Z}$. (In section 5, we will denote this construction by $d_1 \otimes \cdots \otimes d_m \oplus d_Z$.)

By 4.4.1, we know that $k_Z < \prod_{r=2}^{m} n_r \le n_Z$, so k_Z is a valid index into $\mathcal{Z}$.

Let $\boldsymbol{w}_{i_1 i_2 \ldots i_m}, \boldsymbol{w}_{j_1 j_2 \ldots j_m}, \boldsymbol{w}_{k_1 k_2 \ldots k_m}$ be any triple of distinct points in $\mathcal{T}$. They constitute an acute triple iff the scalar product $q = \langle \boldsymbol{w}_{i_1 i_2 \ldots i_m} - \boldsymbol{w}_{j_1 j_2 \ldots j_m}, \boldsymbol{w}_{k_1 k_2 \ldots k_m} - \boldsymbol{w}_{j_1 j_2 \ldots j_m} \rangle$ does not vanish (is positive). Now,

$$\begin{aligned} q &= \langle \boldsymbol{v}^1_{i_1} \boldsymbol{v}^2_{i_2} \ldots \boldsymbol{v}^m_{i_m} \boldsymbol{z}_{i_Z} - \boldsymbol{v}^1_{j_1} \boldsymbol{v}^2_{j_2} \ldots \boldsymbol{v}^m_{j_m} \boldsymbol{z}_{j_Z},\ \boldsymbol{v}^1_{k_1} \boldsymbol{v}^2_{k_2} \ldots \boldsymbol{v}^m_{k_m} \boldsymbol{z}_{k_Z} - \boldsymbol{v}^1_{j_1} \boldsymbol{v}^2_{j_2} \ldots \boldsymbol{v}^m_{j_m} \boldsymbol{z}_{j_Z} \rangle \\ &= \sum_{r=1}^{m} \langle \boldsymbol{v}^r_{i_r} - \boldsymbol{v}^r_{j_r}, \boldsymbol{v}^r_{k_r} - \boldsymbol{v}^r_{j_r} \rangle \ + \ \langle \boldsymbol{z}_{i_Z} - \boldsymbol{z}_{j_Z}, \boldsymbol{z}_{k_Z} - \boldsymbol{z}_{j_Z} \rangle. \end{aligned}$$

If, for some r, both $i_r \ne j_r$ and $j_r \ne k_r$, then the first component of this sum is positive, because $\mathcal{S}_r$ is an acute set.

If, however, there is no r such that both $i_r \ne j_r$ and $j_r \ne k_r$, then there must be some t for which $i_t \ne j_t$ (or else $\boldsymbol{w}_{i_1 i_2 \ldots i_m}$ and $\boldsymbol{w}_{j_1 j_2 \ldots j_m}$ would not be distinct) and $j_t = k_t$, and also some u for which $j_u \neq k_u$ (or else $\boldsymbol{w}_{j_1 j_2 \ldots j_m}$ and $\boldsymbol{w}_{k_1 k_2 \ldots k_m}$ would not be distinct) and $i_u = j_u$. So, by lemma 4.5, $i_Z \neq j_Z$ and $j_Z \neq k_Z$, so the second component of the sum for the scalar product is positive, because $\mathcal{Z}$ is an acute set.

Thus, all triples in $\mathcal{T}$ are acute triples, so $\mathcal{T}$ is a cubic acute D-set of cardinality N. $\square$

Corollary 4.5.1

$$\textit{If } d_1 \leq d_2 \leq \ldots \leq d_m, \textit{ then } \kappa\Big(\sum_{r=1}^{m} r d_r\Big) \geq \prod_{r=1}^{m} \kappa(d_r).$$

Proof: By induction on m. The bound is trivially true for $m = 1$.

Assume the bound holds for $m-1$, and for each r, $1 \leq r \leq m$, let $\mathcal{S}_r$ be a cubic acute d_r-set of cardinality $n_r = \kappa(d_r)$, with $d_1 \leq d_2 \leq \ldots \leq d_m$ and thus $n_1 \leq n_2 \leq \ldots \leq n_m$. By the induction hypothesis, there exists a cubic acute d_Z-set $\mathcal{Z}$ of cardinality n_Z, where

$$d_Z = \sum_{r=2}^{m} (r-1) d_r \quad \text{and} \quad n_Z \geq \prod_{r=2}^{m} \kappa(d_r) = \prod_{r=2}^{m} n_r.$$

Thus, by theorem 4.3, there exists a cubic acute D-set of cardinality N, where

$$D = \sum_{r=1}^{m} d_r + d_Z = \sum_{r=1}^{m} d_r + \sum_{r=2}^{m} (r-1) d_r = \sum_{r=1}^{m} r d_r,$$

and

$$N = \prod_{r=1}^{m} n_r = \prod_{r=1}^{m} \kappa(d_r).$$

$\square$

5 Lower bounds for $\kappa(d)$ and $\alpha(d)$ for small d

The following table lists the best lower bounds known for $\kappa(d)$, $0 \leq d \leq 69$. For $3 \leq d \leq 9$, an exhaustive computer search shows that $\mathcal{S}_3, \ldots, \mathcal{S}_8$ (corollary 4.2.2), are optimal and also that $\kappa(9) = 16$. For other small values of d, the construction used in theorem 4.3 provides the largest known cubic acute d-set. In the table, these constructions are denoted by $d_1 \otimes d_2 \oplus d_Z$ or $d_1 \otimes d_2 \otimes d_3 \oplus d_Z$. For $39 \leq d \leq 48$, the results of a computer program, based on the 'probabilistic construction' of theorem 2.1, provide the largest known cubic acute d-sets. Finally, for $d \geq 67$, theorem 2.1 provides the best (probabilistic) lower bound. $\kappa(d)$ is sequence A089676 in Sloane [S].

Best Lower Bounds Known for $\kappa(d)$

d	$\kappa(d)$	
0	$= 1$	
1	$= 2$	
2	$= 2$	
3	$= 4$	*computer*, $\mathcal{S}_3$
4	$= 5$	*computer*, $\mathcal{S}_4$
5	$= 6$	*computer*, $\mathcal{S}_5$
6	$= 8$	*computer*, $\mathcal{S}_6$
7	$= 9$	*computer*, $\mathcal{S}_7$
8	$= 10$	*computer*, $\mathcal{S}_8$
9	$= 16$	*computer*, $3 \otimes 3 \oplus 3$
10	≥ 16	
11	≥ 20	$3 \otimes 4 \oplus 4$
12	≥ 25	$4 \otimes 4 \oplus 4$
13	≥ 25	
14	≥ 30	$4 \otimes 5 \oplus 5$
15	≥ 36	$5 \otimes 5 \oplus 5$
16	≥ 40	$4 \otimes 6 \oplus 6$
17	≥ 48	$5 \otimes 6 \oplus 6$
18	≥ 64	$6 \otimes 6 \oplus 6$ *or* $3 \otimes 3 \otimes 3 \oplus 9$
19	≥ 64	
20	≥ 72	$6 \otimes 7 \oplus 7$
21	≥ 81	$7 \otimes 7 \oplus 7$
22	≥ 81	
23	≥ 100	$3 \otimes 4 \otimes 4 \oplus 12$
24	≥ 125	$4 \otimes 4 \otimes 4 \oplus 12$
25	≥ 144	$7 \otimes 9 \oplus 9$

d	$\kappa(d)$	
26	≥ 160	$8 \otimes 9 \oplus 9$
27	≥ 256	$9 \otimes 9 \oplus 9$
28	≥ 256	
29	≥ 257	*theorem* 4.1
30	≥ 257	
31	≥ 320	$9 \otimes 11 \oplus 11$
32	≥ 320	
33	≥ 400	$11 \otimes 11 \oplus 11$
34	≥ 400	
35	≥ 500	$11 \otimes 12 \oplus 12$
36	≥ 625	$12 \otimes 12 \oplus 12$
37	≥ 625	
38	≥ 626	*theorem* 4.1
39	≥ 678	*computer*
40	≥ 762	*computer*
41	≥ 871	*computer*
42	≥ 976	*computer*
43	≥ 1086	*computer*
44	≥ 1246	*computer*
45	≥ 1420	*computer*
46	≥ 1630	*computer*
47	≥ 1808	*computer*
48	≥ 2036	*computer*
49	≥ 2036	
50	≥ 2037	*theorem* 4.1
51	≥ 2304	$17 \otimes 17 \oplus 17$

d	$\kappa(d)$	
52	≥ 2560	$16 \otimes 18 \oplus 18$
53	≥ 3072	$17 \otimes 18 \oplus 18$
54	≥ 4096	$18 \otimes 18 \oplus 18$ *or* $9 \otimes 9 \otimes 9 \oplus 27$
55	≥ 4096	
56	≥ 4097	*theorem* 4.1
57	≥ 4097	
58	≥ 4608	$18 \otimes 20 \oplus 20$
59	≥ 4608	
60	≥ 5184	$20 \otimes 20 \oplus 20$

d	$\kappa(d)$	
61	≥ 5184	
62	≥ 5832	$20 \otimes 21 \oplus 21$
63	≥ 6561	$21 \otimes 21 \oplus 21$
64	≥ 6561	
65	≥ 6562	*theorem* 4.1
66	≥ 8000	$11 \otimes 11 \otimes 11 \oplus 33$
67	≥ 8342	*theorem* 2.1
68	≥ 9632	*theorem* 2.1
69	≥ 11122	*theorem* 2.1

The following tables summarise the best lower bounds known for $\alpha(d)$. For $3 \le d \le 6$, the best lower bound is Danzer and Grünbaum's $2d-1$ [DG]. For $7 \le d \le 26$, the results of a computer program, based on the 'probabilistic construction' but using sets of points close to the surface of the d-sphere, provide the largest known acute d-sets. An acute 7-set of cardinality 14 and an acute 8-set of cardinality 16 are displayed. For $27 \le d \le 62$, the largest known acute d-set is cubic. Finally, for $d \ge 63$, theorem 3.1 provides the best (probabilistic) lower bound.

Best Lower Bounds Known for $\alpha(d)$

d	$\alpha(d)$	
0	$= 1$	
1	$= 2$	
2	$= 3$	
3	$= 5$	[DG]
4–6	$\ge 2d-1$	[DG]
7	≥ 14	*computer*
8	≥ 16	*computer*
9	≥ 19	*computer*
10	≥ 23	*computer*
11	≥ 26	*computer*
12	≥ 30	*computer*
13	≥ 36	*computer*
14	≥ 42	*computer*
15	≥ 47	*computer*

d	$\alpha(d)$	
16	≥ 54	*computer*
17	≥ 63	*computer*
18	≥ 71	*computer*
19	≥ 76	*computer*
20	≥ 90	*computer*
21	≥ 103	*computer*
22	≥ 118	*computer*
23	≥ 121	*computer*
24	≥ 144	*computer*
25	≥ 155	*computer*
26	≥ 184	*computer*
27–62	$\ge \kappa(d)$	
63	≥ 6636	*theorem* 3.1

$\alpha(7) \ge 14$
$(62, 1, 9, 10, 17, 38, 46)$
$(38, 54, 0, 19, 38, 14, 25)$
$(60, 33, 42, 9, 48, 3, 12)$
$(62, 35, 41, 44, 16, 39, 44)$
$(62, 34, 7, 45, 48, 37, 12)$
$(28, 33, 42, 8, 49, 39, 45)$
$(40, 16, 22, 12, 0, 0, 25)$
$(45, 17, 26, 67, 25, 20, 29)$
$(38, 6, 35, 0, 32, 18, 0)$
$(62, 0, 42, 45, 49, 3, 48)$
$(30, 0, 9, 44, 49, 37, 48)$
$(0, 20, 31, 27, 34, 21, 28)$
$(48, 19, 24, 22, 33, 20, 73)$
$(43, 17, 25, 27, 32, 64, 19)$

$\alpha(8) \ge 16$
$(34, 49, 14, 51, 0, 36, 46, 0)$
$(31, 17, 14, 51, 1, 5, 44, 31)$
$(33, 50, 48, 20, 34, 35, 15, 0)$
$(0, 16, 16, 52, 32, 36, 45, 0)$
$(37, 31, 46, 52, 13, 0, 0, 22)$
$(2, 50, 13, 52, 3, 3, 46, 0)$
$(1, 50, 48, 51, 1, 5, 46, 31)$
$(24, 0, 43, 2, 17, 20, 32, 16)$
$(11, 49, 0, 11, 19, 8, 32, 19)$
$(0, 48, 48, 52, 1, 34, 12, 2)$
$(0, 48, 47, 51, 34, 37, 47, 32)$
$(34, 49, 14, 51, 34, 36, 13, 34)$
$(0, 46, 31, 0, 0, 23, 29, 29)$
$(16, 40, 29, 23, 54, 3, 17, 16)$
$(2, 15, 14, 50, 2, 36, 15, 33)$
$(12, 36, 28, 30, 3, 45, 48, 45)$

6 Generalising $\kappa(d)$

We can understand $\kappa(d)$ to be the size of the largest possible set $\mathcal{S}$ of binary words such that, for any ordered triple of words $(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w})$ in $\mathcal{S}$, there exists an index i for which $(\boldsymbol{u}_i, \boldsymbol{v}_i, \boldsymbol{w}_i) = (0, 1, 0)$ or $(\boldsymbol{u}_i, \boldsymbol{v}_i, \boldsymbol{w}_i) = (1, 0, 1)$. We can generalise this in the following way:

Definition 6.1 *If $T_1, \ldots, T_m$ are ordered k-tuples from $\{0, \ldots, r-1\}^k$ (which we will refer to as the matching k-tuples), then let us define $\kappa[\![r, k, T_1, \ldots, T_m]\!](d)$ to be the size of the largest possible set $\mathcal{S}$ of r-ary words of length d such that, for any ordered k-tuple of words $(\boldsymbol{w}_1, \ldots, \boldsymbol{w}_k)$ in $\mathcal{S}$, there exist i and j, $1 \leq i \leq d$, $1 \leq j \leq m$, for which $(\boldsymbol{w}_{1i}, \ldots, \boldsymbol{w}_{ki}) = T_j$.*

Thus we have $\kappa(d) = \kappa[\![2, 3, (0, 1, 0), (1, 0, 1)]\!](d)$. If the set of matching k-tuples is closed under permutation, we will abbreviate by writing a list of matching *multisets* of cardinality k, rather than ordered tuples. For example, instead of $\kappa[\![2, 3, (0, 0, 1), (0, 1, 0), (1, 0, 0)]\!](d)$, we write $\kappa[\![2, 3, \{0, 0, 1\}]\!](d)$.

We can find probabilistic and, in some cases, constructive lower bounds for general $\kappa[\![r, k, T_1, \ldots, T_m]\!](d)$ using the approaches we used for cubic acute d-sets. To illustrate this, in the remainder of this paper, we will consider the set of problems in which it is simply required that at some index the k-tuple of words be all different (pairwise distinct). First, we express this in a slightly different form.

Let us say that an ***r*-ary *d*-colouring** is some colouring of the integers $1, \ldots, d$ using r colours. Let us also also say that a set $\mathcal{R}$ of r-ary d-colourings is a ***k*-rainbow set**, for some $k \leq r$ if for any set $\{c_1, \ldots, c_k\}$ of k colourings in $\mathcal{R}$, there exists some integer t, $1 \leq t \leq d$, for which the colours $c_1(t), \ldots, c_k(t)$ are all different, i.e. $c_i(t) \neq c_j(t)$ for any i and j, $1 \leq i, j \leq k$, $i \neq j$. For conciseness, we will denote "a k-rainbow set of r-ary d-colourings" by "a $\mathcal{RSC}[k, r, d]$".

Let us further say that a set $\{c_1, \ldots, c_k\}$ of k d-colourings is a **good *k*-set** if there exists some integer t, $1 \leq t \leq d$, for which the colours $c_1(t), \ldots, c_k(t)$ are all different, and a **bad *k*-set** if there exists no such t.

We will denote by $\rho_{r,k}(d)$ the size of the largest possible $\mathcal{RSC}[k, r, d]$, abbreviating $\rho_{k,k}(d)$ by $\rho_k(d)$. Now, $\rho_k(d) = \kappa[\![k, k, \{0, 1, \ldots, k-1\}]\!](d)$ and

$$\rho_{r,k}(d) = \kappa[\![r, k, \{0, \ldots, k-1\}, \ldots, \{r-k, \ldots, r-1\}]\!](d),$$

where the matching multisets are those of cardinality k with k distinct members.

Clearly, $\rho_{r,k}(d) \leq \rho_{r,k}(d+1)$, $\rho_{r,k}(d) \leq \rho_{r+1,k}(d)$ and $\rho_{r,k}(d) \geq \rho_{r,k+1}(d)$. Also, $\rho_{r,1}(d)$ is undefined because any set of colourings is a 1-rainbow, $\rho_{r,k}(1) = r$ if $k > 1$, and $\rho_{r,2}(d) = r^d$ because any two distinct r-ary d-colourings (or r-ary words of length d) differ somewhere.

In the next two sections we will give a number of probabilistic and constructive lower bounds for $\rho_{r,k}(d)$, for various r and k.

7 A probabilistic lower bound for $\rho_{r,k}(d)$

Theorem 7.1

$$\rho_{r,k}(d) \;\geq\; (k-1)m \quad \textit{where} \quad m = \left\lfloor \sqrt[k-1]{\frac{k!}{k^k}} \left(\sqrt[k-1]{\frac{(r-k)!\,r^k}{(r-k)!\,r^k - r!}} \right)^d \right\rfloor .$$

Proof: This proof is similar that of theorem 2.1.

Randomly pick a set $\mathcal{R}$ of km r-ary d-colourings, choosing the colours from $\{\chi_0, \dots, \chi_{r-1}\}$ independently with probability $\Pr[c(i) = \chi_j] = 1/r$, $1 \leq i \leq d$, $0 \leq j < r$ for every $c \in \mathcal{R}$.

Now the probability that a set of k colourings from $\mathcal{R}$ is a bad k-set is

$$(1-p)^d \quad \text{where} \quad p \;=\; \frac{r!/(r-k)!}{r^k}.$$

Hence, the expected number of bad k-sets in a set of km d-colourings is $\binom{km}{k}(1-p)^d$. Thus there is *some* set $\mathcal{R}$ of km d-colourings with no more than $\binom{km}{k}(1-p)^d$ bad k-sets, where

$$\binom{km}{k}(1-p)^d \;<\; \frac{(km)^k}{k!}(1-p)^d \;=\; m\,\frac{k^k}{k!}\,m^{k-1}(1-p)^d \;\leq\; m$$

by the choice of m.

If we remove one colouring of each bad k-set from $\mathcal{R}$, the remaining set is a $\mathcal{RSC}[k,r,d]$ of cardinality at least $km - m = (k-1)m$. $\square$

The following results follow directly:

$$\rho_3(d) \;\geq\; 2 \left\lfloor \frac{\sqrt{2}}{3} \left(\frac{3}{\sqrt{7}} \right)^d \right\rfloor \;\approx\; 0.943 \times 1.134^d.$$

$$\rho_{4,3}(d) \;\geq\; 2 \left\lfloor \frac{\sqrt{2}}{3} \left(\frac{4}{\sqrt{10}} \right)^d \right\rfloor \;\approx\; 0.943 \times 1.265^d.$$

$$\rho_4(d) \;\geq\; 3 \left\lfloor \sqrt[3]{\frac{3}{32}}\, \sqrt[3]{\frac{32}{29}}^{\,d} \right\rfloor \;\approx\; 1.363 \times 1.033^d.$$

8 Constructive lower bounds for $\rho_{r,k}(d)$

In the following proofs, for clarity of exposition, we will represent r-ary d-colourings as r-ary words of length d, e.g. $\mathcal{R}_{3,3,3} = \{000, 011, 102, 121, 212, 220\}$ represents a 3-rainbow set of ternary 3-colourings (using the colours χ_0, χ_1 and χ_2). Concatenation of words (colourings) c and c' will be written $c.c'$.

We begin with a construction that enables us to extend a $\mathcal{RSC}[k, r, d]$ of cardinality n to one of cardinality $n + 1$ or greater.

Theorem 8.1 *If for some $r \geq k \geq 3$, and some d, we have a $\mathcal{RSC}[k, r, d]$ of cardinality n, and for some r', $k - 2 \leq r' \leq r - 2$, and d', we have a $\mathcal{RSC}[k - 2, r', d']$ of cardinality at least $n - 1$, then we can construct a $\mathcal{RSC}[k, r, d + d']$ of cardinality $N = n - 1 + r - r'$.*

Proof: Let $\mathcal{R} = \{c_0, c_1, \ldots, c_{n-1}\}$ be a $\mathcal{RSC}[k, r, d]$ of cardinality n (using colours $\chi_0, \ldots, \chi_{r-1}$) and $\mathcal{R}' = \{c'_0, c'_1, \ldots, c'_{n'-1}\}$ be a $\mathcal{RSC}[k - 2, r', d']$ of cardinality $n' \geq n - 1$ (using colours $\chi_0, \ldots, \chi_{r'-1}$).

Now let $\mathcal{Q} = \{q_0, q_1, \ldots, q_{N-1}\}$ be a set of r-ary $(d + d')$-colourings where $q_i = c_i.c'_i$ for $0 \leq i \leq n - 2$, and $q_{n-1+j} = c_{n-1}.(r' + j)^{d'}$ for $0 \leq j < r - r'$, each element of $\mathcal{Q}$ being made by concatenating two component colourings, the first from $\mathcal{R}$ and the second being either from $\mathcal{R}'$ or a monochrome colouring.

If $\{q_{i_1}, \ldots, q_{i_k}\}$ is a set of colourings in $\mathcal{Q}$ with no more than one of the i_m greater than $n - 2$, then it is a good k-set because of the first components, since $\mathcal{R}$ is a k-rainbow set.

On the other hand, if $\{q_{i_1}, \ldots, q_{i_k}\}$ is a set of colourings in $\mathcal{Q}$ with no more than $k - 2$ of the i_m less than $n - 1$, then it too is a good k-set because of the second components, since $\mathcal{R}'$ is a $(k-2)$-rainbow set using colours $\chi_0, \ldots, \chi_{r'-1}$ and the second components of the colourings with indices greater than $n - 2$ are each monochrome of a different colour, drawn from $\chi_{r'}, \ldots, \chi_{r-1}$.

Thus $\mathcal{Q}$ is a $\mathcal{RSC}[k, r, d + d']$ of cardinality N. □

Corollary 8.1.1 $\rho_{r,3}(d + 1) \;\geq\; \rho_{r,3}(d) + r - 2$.

Proof: This follows from the theorem due to the fact that there is a 1-rainbow set of 1-ary 1-colourings of any cardinality. □

Corollary 8.1.2 $\rho_{r,4}(d + \lceil \log_2(\rho_{r,4}(d) - 1) \rceil) \;\geq\; \rho_{r,4}(d) + r - 3$.

Proof: Since $\rho_{r,2}(d) = r^d$, we have $\rho_{2,2}(d') \geq \rho_{r,4}(d) - 1$ iff $d' \geq \log_2(\rho_{r,4}(d) - 1)$. □

Theorem 8.2 *If, for each s, $1 \leq s \leq m$, we have a $\mathcal{RSC}[3, r, d_s]$ of cardinality n_s, where n_1 is the least of the n_s, and if, for some d_Z, we have a $\mathcal{RSC}[3, r, d_Z]$ of cardinality n_Z, where*

$$n_Z \geq \prod_{s=2}^{m} (1 + 2 \left\lfloor \frac{n_s}{2} \right\rfloor),$$

then a $\mathcal{RSC}[3, r, D]$ of cardinality N can be constructed, where

$$D = \sum_{s=1}^{m} d_s + 2d_Z \quad \textit{and} \quad N = \prod_{s=1}^{m} n_s.$$

This result for 3-rainbow sets corresponds to theorem 4.3 for cubic acute d-sets. Before we can prove it, we need some further preliminary results.

Definition 8.3 *If $n_1 \leq n_2 \leq \ldots \leq n_m$ and $0 \leq k_r < n_r$, for each r, $1 \leq r \leq m$, then let us denote by $\langle\!\langle\, k_1 k_2 \ldots k_m \,\rangle\!\rangle^{+}_{n_1 n_2 \ldots n_m}$, the number*

$$\langle\!\langle\, k_1 k_2 \ldots k_m \,\rangle\!\rangle^{+}_{n_1 n_2 \ldots n_m} \;=\; \sum_{r=2}^{m} \Big((k_{r-1} + k_r \bmod n_r) \prod_{s=r+1}^{m} n_s \Big).$$

The definition of $\langle\!\langle\, k_1 k_2 \ldots k_m \,\rangle\!\rangle^{+}_{n_1 n_2 \ldots n_m}$ is the same as that for $\langle\!\langle\, k_1 k_2 \ldots k_m \,\rangle\!\rangle_{n_1 n_2 \ldots n_m}$ (see 4.4), but with addition replacing subtraction. By construction, we have

$$\langle\!\langle\, k_1 k_2 \ldots k_m \,\rangle\!\rangle^{+}_{n_1 n_2 \ldots n_m} \;<\; \prod_{r=2}^{m} n_r,$$

and, if $2 \leq t \leq m$ and $j_{t-1} + j_t \not\equiv k_{t-1} + k_t \pmod{n_t}$, then

$$\langle\!\langle\, j_1 j_2 \ldots j_m \,\rangle\!\rangle^{+}_{n_1 n_2 \ldots n_m} \;\neq\; \langle\!\langle\, k_1 k_2 \ldots k_m \,\rangle\!\rangle^{+}_{n_1 n_2 \ldots n_m}.$$

Lemma 8.4 *If $n_1 \leq n_2 \leq \ldots \leq n_m$, with all the n_r odd except perhaps n_1, and $0 \leq j_r, k_r, l_r < n_r$, for each r, $1 \leq r \leq m$, and the sequences of j_r, k_r and l_r are neither pairwise identical nor anywhere pairwise distinct, i.e. there is some u, v and w such that $j_u \neq k_u$, $k_v \neq l_v$ and $l_w \neq j_w$ but no t such that $j_t \neq k_t$, $k_t \neq l_t$ and $l_t \neq j_t$, then either*

$$\langle\!\langle\, j_1 \ldots j_m \,\rangle\!\rangle_{n_1 \ldots n_m},\ \langle\!\langle\, k_1 \ldots k_m \,\rangle\!\rangle_{n_1 \ldots n_m},\ \langle\!\langle\, l_1 \ldots l_m \,\rangle\!\rangle_{n_1 \ldots n_m} \textit{ are pairwise distinct}$$

or

$$\langle\!\langle\, j_1 \ldots j_m \,\rangle\!\rangle^{+}_{n_1 \ldots n_m},\ \langle\!\langle\, k_1 \ldots k_m \,\rangle\!\rangle^{+}_{n_1 \ldots n_m},\ \langle\!\langle\, l_1 \ldots l_m \,\rangle\!\rangle^{+}_{n_1 \ldots n_m} \textit{ are pairwise distinct.}$$

Proof: Without loss of generality, we can assume that we have $j_1 = k_1$, that $t > 1$ is the least integer for which $j_t \neq k_t$, and that $k_t = l_t$. We will consider two cases:

Case 1: $k_{t-1} \neq l_{t-1}$

Since $j_{t-1} = k_{t-1} \neq l_{t-1}$ and $j_t \neq k_t = l_t$, we have $j_{t-1} - j_t \neq k_{t-1} - k_t$ and $k_{t-1} - k_t \neq l_{t-1} - l_t$, and so $\langle\!\langle\, j_1 \dots j_m \,\rangle\!\rangle \neq \langle\!\langle\, k_1 \dots k_m \,\rangle\!\rangle$ and $\langle\!\langle\, k_1 \dots k_m \,\rangle\!\rangle \neq \langle\!\langle\, l_1 \dots l_m \,\rangle\!\rangle$. Similarly, $j_{t-1} + j_t \neq k_{t-1} + k_t$ and $k_{t-1} + k_t \neq l_{t-1} + l_t$, and so $\langle\!\langle\, j_1 \dots j_m \,\rangle\!\rangle^+ \neq \langle\!\langle\, k_1 \dots k_m \,\rangle\!\rangle^+$ and $\langle\!\langle\, k_1 \dots k_m \,\rangle\!\rangle^+ \neq \langle\!\langle\, l_1 \dots l_m \,\rangle\!\rangle^+$.

If $j_{t-1} - j_t \neq l_{t-1} - l_t$, then $\langle\!\langle\, j_1 \dots j_m \,\rangle\!\rangle \neq \langle\!\langle\, l_1 \dots l_m \,\rangle\!\rangle$. If $j_{t-1} - j_t = l_{t-1} - l_t$ then $(j_{t-1} + j_t) - (l_{t-1} + l_t) = (j_{t-1} - j_t + 2j_t) - (l_{t-1} - l_t + 2l_t) = 2(j_t - l_t) \neq 0 \pmod{n_t}$ because $j_t \neq l_t$ and n_t is odd, so $j_{t-1} + j_t \neq l_{t-1} + l_t$ and $\langle\!\langle\, j_1 \dots j_m \,\rangle\!\rangle^+ \neq \langle\!\langle\, l_1 \dots l_m \,\rangle\!\rangle^+$.

Case 2: $k_{t-1} = l_{t-1}$

Since $j_{t-1} = k_{t-1} = l_{t-1}$ and $j_t \neq k_t = l_t$, we have $j_{t-1} - j_t \neq k_{t-1} - k_t$ and $j_{t-1} - j_t \neq l_{t-1} - l_t$, and so $\langle\!\langle\, j_1 \dots j_m \,\rangle\!\rangle \neq \langle\!\langle\, k_1 \dots k_m \,\rangle\!\rangle$ and $\langle\!\langle\, j_1 \dots j_m \,\rangle\!\rangle \neq \langle\!\langle\, l_1 \dots l_m \,\rangle\!\rangle$.

If $k_1 = l_1$, let u be the least integer such that $k_u \neq l_u$. Since $k_{u-1} = l_{u-1}$, we have $k_{u-1} - k_u \neq l_{u-1} - l_u$. If $k_1 \neq l_1$, let u be the least integer such that $k_u = l_u$. Since $k_{u-1} \neq l_{u-1}$, we still have $k_{u-1} - k_u \neq l_{u-1} - l_u$. Thus, $\langle\!\langle\, k_1 \dots k_m \,\rangle\!\rangle \neq \langle\!\langle\, l_1 \dots l_m \,\rangle\!\rangle$. $\square$

Proof of Theorem 8.2

Let $n_1 \leq n_2 \leq \ldots \leq n_m$, and, for each s, $1 \leq s \leq m$, let $\mathcal{R}_s = \{c^s_0, c^s_1, \ldots, c^s_{n_s-1}\}$ be a $\mathcal{RSC}[3, r, d_s]$ of cardinality n_s, and let $n'_s = 1 + 2\lfloor n_s/2 \rfloor$ be the least odd integer not less than n_s. Let $\mathcal{Z} = \{z_0, z_1, \ldots, z_{n_Z-1}\}$ be a $\mathcal{RSC}[3, r, d_Z]$ of cardinality n_Z, where

$$n_Z \geq \prod_{s=2}^{m} n'_s,$$

and let

$$D = \sum_{s=1}^{m} d_s + 2d_Z \quad \text{and} \quad N = \prod_{s=1}^{m} n_s.$$

Now let

$$\mathcal{Q} = \{c^1_{k_1}.c^2_{k_2} \ldots c^m_{k_m}.z_{k_Z}.z_{k_Z^+} \;:\; 0 \leq k_s < n_s, 1 \leq s \leq m\},$$

where $k_Z = \langle\!\langle\, k_1 k_2 \ldots k_m \,\rangle\!\rangle_{n'_1 n'_2 \ldots n'_m}$ and $k_Z^+ = \langle\!\langle\, k_1 k_2 \ldots k_m \,\rangle\!\rangle^+_{n'_1 n'_2 \ldots n'_m}$ be a set of D-colourings of cardinality N, each element of $\mathcal{Q}$ being made by concatenating one colouring from each of the $\mathcal{R}_s$ together with two colourings from $\mathcal{Z}$. (Below, we will denote this construction by $d_1 \otimes \cdots \otimes d_m \oplus d_Z \oplus d_Z$.)

Let $c^1_{i_1}.c^2_{i_2} \ldots c^m_{i_m}.z_{i_Z}.z_{i_Z^+}$, $c^1_{j_1}.c^2_{j_2} \ldots c^m_{j_m}.z_{j_Z}.z_{j_Z^+}$ and $c^1_{k_1}.c^2_{k_2} \ldots c^m_{k_m}.z_{k_Z}.z_{k_Z^+}$ be any three distinct colourings in $\mathcal{Q}$. If, for some s, $i_s \neq j_s$, $j_s \neq k_s$ and $k_s \neq i_s$, then these three colourings comprise a good 3-set because $\mathcal{R}_s$ is a 3-rainbow set.

If, however, there is no s such that i_s, j_s and k_s are all different, then the condition of lemma 8.4 holds, and so either i_Z, j_Z and k_Z are all different, or i_Z^+, j_Z^+ and k_Z^+ are all different, and the three colourings comprise a good 3-set because $\mathcal{Z}$ is a 3-rainbow set.

Thus, any three colourings in $\mathcal{Q}$ comprise a good 3-set, so $\mathcal{Q}$ is a $\mathcal{RSC}[3, r, D]$ of cardinality N. $\square$

Corollary 8.4.1 *If $\rho_{r,3}(d)$ is odd, then $\rho_{r,3}(4d) \geq \rho_{r,3}(d)^2$.*

Proof: By theorem 8.2 using the construction $d \otimes d \oplus d \oplus d$. $\square$

Corollary 8.4.2 $\rho_{r,3}(4d+2) \;\geq\; \rho_{r,3}(d)^2$.

Proof: By 8.1.1, if $n = \rho_{r,3}(d)$, we can construct a $\mathcal{RSC}[3, r, d+1]$ of cardinality $n+1 \geq 1 + 2\lfloor n/2 \rfloor$. By theorem 8.2, we can then construct a $\mathcal{RSC}[3, r, 4d+2]$ of cardinality n^2 using the construction $d \otimes d \oplus (d+1) \oplus (d+1)$. $\square$

Corollary 8.4.3 $\rho_3(4^d) \;\geq\; 3^{2^d}$.

Proof: By repeated application of 8.4.1 starting with $\rho_{3,3}(1) = 3$. $\square$

Our final construction enables us to combine k-rainbow sets of r-ary d-colourings for arbitrary k.

Theorem 8.5 *If we have a $\mathcal{RSC}[k, r, d_1]$ of cardinality n_1, a $\mathcal{RSC}[k, r, d_2]$ of cardinality $n_2 \geq n_1$, and a $\mathcal{RSC}[k, r, d_Z]$ of cardinality $n_Z \geq n_2$, with n_Z coprime to each integer in the range $[2, \ldots, h]$ where $h = \binom{k}{2} - 1$, then a $\mathcal{RSC}[k, r, D]$ of cardinality N can be constructed, where $D = d_1 + d_2 + hd_Z$ and $N = n_1 n_2$.*

As before, we first need a preliminary result:

Lemma 8.6 *Given distinct pairs of integers (a, b) and (c, d) with $0 \leq a, b, c, d < n$ for some n, and given a positive integer h such that n is coprime to each integer in the range $[2, \ldots, h]$, then if we let $b_{-1} = a$ and $d_{-1} = c$, and $b_r = b + ra \pmod{n}$ and $d_r = d + rc \pmod{n}$ for $0 \leq r \leq h$, then if $b_i = d_i$ for some i, $-1 \leq i \leq h$, we have $b_j \neq d_j$ for all $j \neq i$.*

Proof: We consider two cases:

Case 1: $i = -1$
Since $a = c$, $(b + ja) - (d + jc) = b - d \neq 0 \pmod n$ since (a, b) and (c, d) are distinct, and b and d both less than n.

Case 2: $i \neq -1$
By the reversing the argument in case 1, $a \neq c$, i.e. $b_{-1} \neq d_{-1}$. For $j \geq 0$, since $b + ia = d + ic$, we have $(b + ja) - (d + jc) = (j - i)a - (j - i)c = (j - i)(a - c) \neq 0 \pmod n$ since $a \neq c$ and $|j - i| \leq h$ so $j - i$ is coprime to n. □

Proof of Theorem 8.5

Let $\mathcal{R}_1 = \{c^1_0, \ldots, c^1_{n_1-1}\}$, $\mathcal{R}_2 = \{c^2_0, \ldots, c^2_{n_2-1}\}$ and $\mathcal{Z} = \{z_0, \ldots, z_{n_Z-1}\}$ be k-rainbow sets of r-ary d_1-, d_2- and d_Z-colourings of cardinality n_1, n_2 and n_Z, respectively.

Now let

$$\mathcal{Q} = \{c^1_i.c^2_j.z_{j+i}.z_{j+2i} \ldots z_{j+hi} \;:\; 0 \leq i < n_1, 0 \leq j < n_2\},$$

where $h = \binom{k}{2} - 1$ and the subscript arithmetic is modulo n_Z, be a set of D-colourings of cardinality N, each element of $\mathcal{Q}$ being made by concatenating $h+2$ component colourings: one from $\mathcal{R}_1$, one from $\mathcal{R}_2$, and h from $\mathcal{Z}$.

Let

$$\mathcal{S} = \{c^1_{i_1}.c^2_{j_1}.z_{j_1+i_1} \ldots z_{j_1+hi_1},\; c^1_{i_2}.c^2_{j_2}.z_{j_2+i_2} \ldots z_{j_2+hi_2},\; \ldots,\; c^1_{i_k}.c^2_{j_k}.z_{j_k+i_k} \ldots z_{j_k+hi_k}\}$$

be any set of k distinct colourings in $\mathcal{Q}$, and let $b_{s,-1} = i_s$ and $b_{s,t} = j_s + ti_s \pmod{n_Z}$, for each s and t, $1 \leq s \leq k$, $0 \leq t \leq h$, so the s^{th} colouring in $\mathcal{S}$ is $c^1_{b_{s,-1}}.c^2_{b_{s,0}}.z_{b_{s,1}} \ldots z_{b_{s,h}}$.

Now, for any s, s' and t, $1 \leq s, s' \leq k$, $-1 \leq t \leq h$, if $b_{s,t} = b_{s',t}$, then by lemma 8.6 we know that for all $u \neq t$, $b_{s,u} \neq b_{s',u}$. So for each pair $\{s, s'\}$, $b_{s,t} = b_{s',t}$ for no more than one value of t. Now there are $h + 2$ possible values of t, but only $\binom{k}{2} = h + 1$ different pairs $\{s, s'\}$, so there is *some* t for which $b_{s,t} \neq b_{s',t}$ for all pairs $\{s, s'\}$ and the $(t + 2)^{\text{th}}$ component colourings of the elements in $\mathcal{S}$ are all different. Since $\mathcal{R}_1$, $\mathcal{R}_2$ and $\mathcal{Z}$ are all k-rainbow sets, we know that $\mathcal{S}$ is a good k-set.

Thus, any k colourings from $\mathcal{Q}$ comprise a good k-set, so $\mathcal{Q}$ is a $\mathcal{RSC}[k, r, D]$ of cardinality N. □

Corollary 8.6.1 $\rho_4(6.7^d) \;\geq\; 7^{2^d}$.

Proof: The following 4-rainbow set of 4-ary 6-colourings of cardinality 8 — a version of $\mathcal{R}_{4,4,6}$ (see below) displayed with different symbols for each colour — shows that $\rho_4(6) \geq 7$.

$$\boxed{\begin{array}{cccccc}
\spadesuit & \clubsuit & \spadesuit & \diamondsuit & \heartsuit & \clubsuit \\
\spadesuit & \heartsuit & \diamondsuit & \clubsuit & \clubsuit & \diamondsuit \\
\heartsuit & \clubsuit & \diamondsuit & \spadesuit & \diamondsuit & \heartsuit \\
\heartsuit & \heartsuit & \clubsuit & \diamondsuit & \spadesuit & \spadesuit \\
\diamondsuit & \diamondsuit & \spadesuit & \heartsuit & \clubsuit & \heartsuit \\
\diamondsuit & \spadesuit & \heartsuit & \clubsuit & \heartsuit & \spadesuit \\
\clubsuit & \diamondsuit & \heartsuit & \spadesuit & \spadesuit & \clubsuit \\
\clubsuit & \spadesuit & \clubsuit & \heartsuit & \diamondsuit & \diamondsuit
\end{array}}$$

The result follows by repeated application of theorem 8.5, noting that 7 is coprime to 2, 3, 4 and $5 = \binom{4}{2} - 1$. □

9 Lower bounds for $\rho_{r,k}(d)$ for small r, k and d

We conclude with tables of the best lower bounds known for $\rho_3(d)$, $\rho_{4,3}(d)$ and $\rho_4(d)$ for small d. For very small d, exhaustive computer searches have determined the values of $\rho_{r,k}(d)$. For other small values of d, the constructions used in theorems 8.2 and 8.5 provide the largest known rainbow sets. In the tables, these constructions are denoted $d_1 \otimes d_2 \oplus d_Z \oplus d_Z$, etc., with superscript minus signs (d^-) to denote the removal of a single colouring from a largest rainbow set of d-colourings (to satisfy the requirement that the cardinality be odd). For $\rho_3(d)$, the probabilistic lower bound of theorem 7.1 is better than the constructions for $d \geq 71$; for $\rho_{4,3}(d)$, this is the case for $d \geq 26$.

Some k-rainbow sets of r-ary d-colourings, for small k, r and d

$\mathcal{R}_{3,3,3}$ $\rho_3(3) \geq 6$
000
011
102
121
212
220

$\mathcal{R}_{3,3,6}$ $\rho_3(6) \geq 13$
000000
000111
000222
011012
022120
101120
112021
112102
112210
120012
202012
210120
221201

$\mathcal{R}_{4,3,3}$ $\rho_{4,3}(3) \geq 9$
000
011
022
103
131
213
232
323
330

$\mathcal{R}_{4,3,4}$ $\rho_{4,3}(4) \geq 16$
0000
0011
0102
0220
1013
1212
1230
1302
2031
2103
2121
2320
3113
3231
3322
3333

$\mathcal{R}_{4,4,6}$ $\rho_4(6) \geq 8$
000000
011111
101222
112033
220312
233103
323230
332321

Best Lower Bounds Known for $\rho_3(d)$ and $\rho_{4,3}(d)$

d	$\rho_3(d)$	
1	$= 3$	
2	$= 4$	*computer*, 8.1.1
3	$= 6$	*computer*, $\mathcal{R}_{3,3,3}$
4	$= 9$	*computer*, $1 \otimes 1 \oplus 1 \oplus 1$
5	$= 10$	*computer*, 8.1.1
6	$= 13$	*computer*, $\mathcal{R}_{3,3,6}$
7	≥ 14	8.1.1
8	≥ 15	8.1.1
9	≥ 16	8.1.1
10	≥ 17	8.1.1
11	≥ 27	$1 \otimes 1 \otimes 1 \oplus 4 \oplus 4$
12	≥ 28	8.1.1
13	≥ 29	8.1.1
14	≥ 36	$2 \otimes 4 \oplus 4 \oplus 4$
15	≥ 54	$3 \otimes 4 \oplus 4 \oplus 4$
16	≥ 81	$4 \otimes 4 \oplus 4 \oplus 4$
...	...	
70	≥ 6723	$16 \otimes 18 \oplus 18 \oplus 18$
71	≥ 7064	*theorem* 7.1

d	$\rho_{4,3}(d)$	
1	$= 4$	
2	$= 6$	*computer*, 8.1.1
3	$= 9$	*computer*, $\mathcal{R}_{4,3,3}$
4	$= 16$	*computer*, $\mathcal{R}_{4,3,4}$
5	≥ 18	8.1.1
6	≥ 20	8.1.1
7	≥ 22	8.1.1
8	≥ 25	$2^- \otimes 2^- \oplus 2 \oplus 2$
9	≥ 27	8.1.1
10	≥ 36	$1 \otimes 3 \oplus 3 \oplus 3$ *or* $2 \otimes 2 \oplus 3 \oplus 3$
11	≥ 54	$2 \otimes 3 \oplus 3 \oplus 3$
12	≥ 81	$3 \otimes 3 \oplus 3 \oplus 3$
13	≥ 83	8.1.1
14	≥ 90	$2 \otimes 4^- \oplus 4 \oplus 4$
15	≥ 135	$3 \otimes 4^- \oplus 4 \oplus 4$
16	≥ 225	$4^- \otimes 4^- \oplus 4 \oplus 4$
...	...	
25	≥ 363	8.1.1
26	≥ 424	*theorem* 7.1

Best Lower Bounds Known for $\rho_4(d)$

d	$\rho_4(d)$	
1	$= 4$	
2	$= 4$	*computer*
3	$= 5$	*computer*, 8.1.2
4	$= 5$	*computer*
5	$= 6$	*computer*, 8.1.2
6	$= 8$	*computer*, $\mathcal{R}_{4,4,6}$
$\cdots$	$\cdots$	
42	≥ 49	$6^{-}\otimes 6^{-}\oplus 6^{-}\oplus 6^{-}\oplus 6^{-}\oplus 6^{-}\oplus 6^{-}$

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