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Sets of Points Determining Only Acute Angles
and Some Related Colouring Problems

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Abstract

We present both probabilistic and constructive lower bounds on the maximum size
of a set of points $S \subseteq \mathbb{R}^d$ such that every angle determined by three points in $S$
is acute, considering especially the case $S \subseteq \{0,1\}^d$. These results improve upon
a probabilistic lower bound of Erdős and Füredi. We also present lower bounds
for some generalisations of the acute angles problem, considering especially some
problems concerning colourings of sets of integers.

1 Introduction

Let us say that a set of points $S \subseteq \mathbb{R}^d$ is an acute $d$-set if every angle determined by a
triple of $S$ is acute ($< \frac{\pi}{2}$). Let us also say that $S$ is a cubic acute $d$-set if $S$ is an acute
d-set and is also a subset of the unit $d$-cube (i.e. $S \subseteq \{0,1\}^d$).

Let us further say that a triple $u, v, w \in \mathbb{R}^d$ is an acute triple, a right triple, or an
obtuse triple, if the angle determined by the triple with apex $v$ is less than $\frac{\pi}{2}$, equal to
$\frac{\pi}{2}$, or greater than $\frac{\pi}{2}$, respectively. Note that we consider the triples $u, v, w$ and $w, v, u$
to be the same.

We will denote by $\alpha(d)$ the size of a largest possible acute $d$-set. Similarly, we will denote
by $\kappa(d)$ the size of a largest possible cubic acute $d$-set. Clearly $\kappa(d) \leq \alpha(d)$, $\kappa(d) \leq \kappa(d+1)$
and $\alpha(d) \leq \alpha(d+1)$ for all $d$. 
In [EF], Paul Erdős and Zoltán Füredi gave a probabilistic proof that \( \kappa(d) \geq \left[ \frac{1}{2} \left( \frac{2}{\sqrt{3}} \right)^d \right] \) (see also [AZ2]). This disproved an earlier conjecture of Ludwig Danzer and Branko Grünbaum [DG] that \( \alpha(d) = 2d - 1 \).

In the following two sections we give improved probabilistic lower bounds for \( \kappa(d) \) and \( \alpha(d) \). In section 4 we present a construction that gives further improved lower bounds for \( \kappa(d) \) for small \( d \). In section 5, we tabulate the best lower bounds known for \( \kappa(d) \) and \( \alpha(d) \) for small \( d \). Finally, in sections 6–9, we give probabilistic and constructive lower bounds for some generalisations of \( \kappa(d) \), considering especially some problems concerning colourings of sets of integers.

## 2 A probabilistic lower bound for \( \kappa(d) \)

**Theorem 2.1**

\[
\kappa(d) \geq 2 \left[ \frac{\sqrt{6}}{9} \left( \frac{2}{\sqrt{3}} \right)^d \right] \approx 0.544 \times 1.155^d.
\]

For large \( d \), this improves upon the result of Erdős and Füredi by a factor of \( \frac{4\sqrt{6}}{9} \approx 1.089 \). This is achieved by a slight improvement in the choice of parameters. This proof can also be found in [AZ3].

**Proof:** Let \( m = \left\lfloor \frac{\sqrt{6}}{9} \left( \frac{2}{\sqrt{3}} \right)^d \right\rfloor \) and randomly pick a set \( S \) of \( 3m \) point vectors from the vertices of the \( d \)-dimensional unit cube \( \{0, 1\}^d \), choosing the coordinates independently with probability \( \Pr[ \mathbf{v}_i = 0 ] = \Pr[ \mathbf{v}_i = 1 ] = \frac{1}{2} \), \( 1 \leq i \leq d \), for every \( \mathbf{v} = ( \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_d ) \in S \).

Now every angle determined by a triple of points from \( S \) is non-obtuse (\( \leq \frac{\pi}{2} \)), and a triple of vectors \( \mathbf{u}, \mathbf{v}, \mathbf{w} \) from \( S \) is a right triple iff the scalar product \( \langle \mathbf{u} - \mathbf{v}, \mathbf{w} - \mathbf{v} \rangle \) vanishes, i.e. iff either \( \mathbf{u}_i - \mathbf{v}_i = 0 \) or \( \mathbf{w}_i - \mathbf{v}_i = 0 \) for each \( i, 1 \leq i \leq d \).

Thus \( \mathbf{u}, \mathbf{v}, \mathbf{w} \) is a right triple iff \( \mathbf{u}_i, \mathbf{v}_i, \mathbf{w}_i \) is neither 0, 1, 0 nor 1, 0, 1 for any \( i, 1 \leq i \leq d \). Since \( \mathbf{u}_i, \mathbf{v}_i, \mathbf{w}_i \) can take eight different values, this occurs independently with probability \( \frac{3}{4} \) for each \( i \), so the probability that a triple of \( S \) is a right triple is \( \left( \frac{3}{4} \right)^d \).

Hence, the expected number of right triples in a set of \( 3m \) vectors is \( 3 \binom{3m}{3} \left( \frac{3}{4} \right)^d \). Thus there is some set \( S \) of \( 3m \) vectors with no more than \( 3 \binom{3m}{3} \left( \frac{3}{4} \right)^d \) right triples, where

\[
3 \binom{3m}{3} \left( \frac{3}{4} \right)^d < 3 \binom{3m}{3} \left( \frac{3}{4} \right)^d = m \left( \frac{9m}{\sqrt{6}} \right)^2 \left( \frac{3}{4} \right)^d \leq m
\]

by the choice of \( m \).
If we remove one point of each right triple from $S$, the remaining set is a cubic acute $d$-set of cardinality at least $3m - m = 2m$.  

\[ \square \]

### 3 A probabilistic lower bound for $\alpha(d)$

We can improve the lower bound in theorem 2.1 for non-cubic acute $d$-sets by a factor of $\sqrt{2}$ by slightly perturbing the points chosen away from the vertices of the unit cube. The intuition behind this is that a small random symmetrical perturbation of the points in a right triple is more likely than not to produce an acute triple, as the following diagram suggests.

![Diagram showing a right triangle with a small perturbation]

**Theorem 3.1**

\[
\alpha(d) \geq 2 \left[ \frac{1}{3} \left( \frac{2}{\sqrt{3}} \right)^{d+1} \right] \approx 0.770 \times 1.155^d.
\]

Before we can prove this theorem, we need some results concerning continuous random variables.

**Definition 3.2** If $F(x) = \Pr[X \leq x]$ is the cumulative distribution function of a continuous random variable $X$, let $\overline{F}(x)$ denote $\Pr[X \geq x] = 1 - F(x)$.

**Definition 3.3** Let us say that a continuous random variable $X$ has **positive bias** if, for all $t$, $\Pr[X \geq t] \geq \Pr[X \leq -t]$, i.e. $\overline{F}(t) \geq F(-t)$.

**Property 3.3.1** If a continuous random variable $X$ has positive bias, it follows that $\Pr[X > 0] \geq \frac{1}{2}$.

**Property 3.3.2** To show that a continuous random variable $X$ has positive bias, it suffices to demonstrate that the condition $\overline{F}(t) \geq F(-t)$ holds for all positive $t$. 

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Lemma 3.4 If $X$ and $Y$ are independent continuous random variables with positive bias, then $X + Y$ also has positive bias.

Proof: Let $f$, $g$ and $h$ be the probability density functions, and $F$, $G$ and $H$ the cumulative distribution functions, for $X$, $Y$ and $X + Y$ respectively. Then,

$$
\overline{H}(t) - H(-t) = \int\int_{x+y \geq t} f(x)g(y) \, dy \, dx - \int\int_{x+y \leq -t} f(x)g(y) \, dy \, dx
$$

$$
= \int\int_{x+y \geq t} f(x)g(y) \, dy \, dx - \int\int_{y-x \geq t} f(x)g(y) \, dy \, dx
\ + \int\int_{y-x \geq t} f(x)g(y) \, dy \, dx - \int\int_{x+y \leq -t} f(x)g(y) \, dy \, dx
$$

$$
= \int_{-\infty}^{\infty} g(y) \left[ F(t-y) - F(y-t) \right] \, dy
\ + \int_{-\infty}^{\infty} f(x) \left[ G(x+t) - G(-x-t) \right] \, dx
$$

which is non-negative because $f(t)$, $g(t)$, $F(t) - F(-t)$ and $G(t) - G(-t)$ are all non-negative for all $t$. \hfill \Box

Definition 3.5 Let us say that a continuous random variable $X$ is $\epsilon$-uniformly distributed for some $\epsilon > 0$ if $X$ is uniformly distributed between $-\epsilon$ and $\epsilon$.

Let us denote by $j$, the probability density function of an $\epsilon$-uniformly distributed random variable:

$$
j(x) = \begin{cases} 
\frac{1}{\epsilon} & -\epsilon \leq x \leq \epsilon \\
0 & \text{otherwise}
\end{cases}
$$

and by $J$, its cumulative distribution function:

$$
J(x) = \begin{cases} 
0 & x < -\epsilon \\
\frac{1}{2} + \frac{x}{2\epsilon} & -\epsilon \leq x \leq \epsilon \\
1 & x > \epsilon
\end{cases}
$$

Property 3.5.1 If $X$ is an $\epsilon$-uniformly distributed random variable, then so is $-X$. 
Lemma 3.6 If $X$, $Y$ and $Z$ are independent $\epsilon$-uniformly distributed random variables for some $\epsilon < \frac{1}{2}$, then $U = (Y - X)(1 + Z - X)$ has positive bias.

Proof: Let $G$ be the cumulative distribution function of $U$. By 3.3.2, it suffices to show that $G(u) - G(-u) \geq 0$ for all positive $u$.

Let $u$ be positive. Because $1 + Z - X$ is always positive, $U \geq u$ iff $Y > X$ and $Z \geq -1 + X + \frac{u}{Y - X}$. Similarly, $U \leq -u$ iff $X > Y$ and $Z \geq -1 + X + \frac{u}{X - Y}$. So,

$$G(u) - G(-u) = \int\int_{y>x} j(x)j(y)J(-1 + x + \frac{u}{y - x}) dy dx - \int\int_{x>y} j(x)j(y)J(-1 + x + \frac{u}{x - y}) dy dx$$

$$= \int\int_{y>x} j(x)j(y) \left[ J(1 - x - \frac{u}{y - x}) - J(1 - y - \frac{u}{y - x}) \right] dy dx$$

(because $J(x) = J(-x)$, and by variable renaming)

which is non-negative because $j$ is non-negative and $J$ is non-decreasing (so the expression in square brackets is non-negative over the domain of integration).

Corollary 3.6.1 If $X$, $Y$ and $Z$ are independent $\epsilon$-uniformly distributed random variables for some $\epsilon < \frac{1}{2}$, then $(Y - X)(Z - X - 1)$ has positive bias.

Proof: $(Y - X)(Z - X - 1) = ((-Y) - (-X))(1 + (-Z) - (-X))$. The result follows from 3.5.1 and lemma 3.6. □

Lemma 3.7 If $X$, $Y$ and $Z$ are independent $\epsilon$-uniformly distributed random variables, then $V = (Y - X)(Z - X)$ has positive bias.

Proof: Let $H$ be the cumulative distribution function of $V$. By 3.3.2, it suffices to show that $H(v) - H(-v) \geq 0$ for all positive $v$. □
Let \( v \) be positive, \( V \geq v \) iff \( Y > X \) and \( Z \geq X + \frac{v}{y-x} \) or \( Y < X \) and \( Z \leq X + \frac{v}{y-x} \). Similarly, \( V \leq -v \) iff \( Y > X \) and \( Z \leq X - \frac{v}{y-x} \) or \( Y < X \) and \( Z \geq X - \frac{v}{y-x} \). So,

\[
\overline{H}(v) - H(-v) = \int_{y>x} \int_{y<x} j(x)j(y)J(x + \frac{v}{y-x}) \, dy \, dx \\
+ \int_{y>x} \int_{y<x} j(x)j(y)J(x - \frac{v}{y-x}) \, dy \, dx \\
- \int_{y>x} \int_{y<x} j(x)j(y)J(x) \, dy \, dx \\
- \int_{y<x} \int_{y>x} j(x)j(y)J(x) \, dy \, dx
\]

which is non-negative because \( j \) is non-negative and \( J \) is non-decreasing (so the expressions in square brackets are non-negative over the domains of integration). \( \square \)

We are now in a position to prove the theorem.

**Proof of theorem 3.1**

Let \( m = \left\lfloor \frac{1}{3} \left( \frac{2}{\sqrt{3}} \right)^{d+1} \right\rfloor \), and randomly pick a set \( S \) of 3m point vectors, \( \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{3m} \), from the vertices of the \( d \)-dimensional unit cube \( \{0, 1\}^d \), choosing the coordinates independently with probability \( \Pr[\mathbf{v}_{ki} = 0] = \Pr[\mathbf{v}_{ki} = 1] = \frac{1}{2} \) for every \( \mathbf{v}_k = (\mathbf{v}_{k1}, \mathbf{v}_{k2}, \ldots, \mathbf{v}_{kd}) \), \( 1 \leq k \leq 3m, 1 \leq i \leq d \).

Now for some \( \epsilon, 0 < \epsilon < \frac{1}{2(d+1)} \), randomly pick 3m vectors, \( \mathbf{\delta}_1, \mathbf{\delta}_2, \ldots, \mathbf{\delta}_{3m} \), from the \( d \)-dimensional cube \( [-\epsilon, \epsilon]^d \) of side 2\( \epsilon \) centred on the origin, choosing the coordinates \( \mathbf{\delta}_{ki} \), \( 1 \leq k \leq 3m, 1 \leq i \leq d \), independently so that they are \( \epsilon \)-uniformly distributed, and let \( S' = \{ \mathbf{v}'_1, \mathbf{v}'_2, \ldots, \mathbf{v}'_{3m} \} \) where \( \mathbf{v}'_k = \mathbf{v}_k + \mathbf{\delta}_k \) for each \( k, 1 \leq k \leq 3m \).

**Case 1: Acute triples in \( S \)**

Because \( \epsilon < \frac{1}{2(d+1)} \), if \( \mathbf{v}_j, \mathbf{v}_k, \mathbf{v}_l \) is an acute triple in \( S \), the scalar product \( \langle \mathbf{v}'_j - \mathbf{v}'_k, \mathbf{v}'_l - \mathbf{v}'_k \rangle > \frac{1}{(d+1)^2} \), so \( \mathbf{v}'_j, \mathbf{v}'_k, \mathbf{v}'_l \) is also an acute triple in \( S' \).

**Case 2: Right triples in \( S \)**

If, \( \mathbf{v}_j, \mathbf{v}_k, \mathbf{v}_l \) is a right triple in \( S \) then the scalar product \( \langle \mathbf{v}_j - \mathbf{v}_k, \mathbf{v}_l - \mathbf{v}_k \rangle \) vanishes, i.e. either \( \mathbf{v}_{j_i} - \mathbf{v}_{k_i} = 0 \) or \( \mathbf{v}_{l_i} - \mathbf{v}_{k_i} = 0 \) for each \( i, 1 \leq i \leq d \). There are six possibilities for each triple of coordinates:
<table>
<thead>
<tr>
<th>$v_{ji}, v_{ki}, v_{li}$</th>
<th>$(v'<em>{ji} - v'</em>{ki})(v'<em>l - v'</em>{ki})$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0, 0, 0</td>
<td>$(\delta_{ji} - \delta_{ki})(\delta_{li} - \delta_{ki})$</td>
</tr>
<tr>
<td>1, 1, 1</td>
<td>$(\delta_{ji} - \delta_{ki})(\delta_{li} - \delta_{ki})$</td>
</tr>
<tr>
<td>0, 0, 1</td>
<td>$(\delta_{ji} - \delta_{ki})(1 + \delta_{li} - \delta_{ki})$</td>
</tr>
<tr>
<td>1, 0, 0</td>
<td>$(\delta_{li} - \delta_{ki})(1 + \delta_{ji} - \delta_{ki})$</td>
</tr>
<tr>
<td>0, 1, 1</td>
<td>$(\delta_{li} - \delta_{ki})(\delta_{ji} - \delta_{ki} - 1)$</td>
</tr>
<tr>
<td>1, 1, 0</td>
<td>$(\delta_{ji} - \delta_{ki})(\delta_{li} - \delta_{ki} - 1)$</td>
</tr>
</tbody>
</table>

Now, the values of the $\delta_{ki}$ are independent and $\epsilon$-uniformly distributed, so by lemmas 3.7 and 3.6 and corollary 3.6.1, the distribution of the $(v'_{ji} - v'_{ki})(v'_l - v'_{ki})$ has positive bias, and by repeated application of lemma 3.4, the distribution of the scalar product $\langle v'_{j} - v'_{k}, v'_l - v'_{k} \rangle = \sum_{i=1}^{d}(v'_{ji} - v'_{ki})(v'_l - v'_{ki})$ also has positive bias.

Thus, if $v_{j}, v_{k}, v_{l}$ is a right triple in $S$, then, by 3.3.1,

$$\Pr[\langle v'_{j} - v'_{k}, v'_l - v'_{k} \rangle > 0] \geq \frac{1}{2},$$

so the probability that the triple $v'_{j}, v'_{k}, v'_{l}$ is an acute triple in $S'$ is at least $\frac{1}{2}$.

As in the proof of theorem 2.1, the expected number of right triples in $S$ is $3m(\frac{3}{3})^d$, so the expected number of non-acute triples in $S'$ is no more than half this value. Thus there is some set $S'$ of $3m$ vectors with no more than $\frac{3}{2} \binom{3m}{3} \left(\frac{3}{4}\right)^d$ non-acute triples, where

$$\frac{3}{2} \binom{3m}{3} \left(\frac{3}{4}\right)^d < \frac{3}{2} \binom{3m}{3} \left(\frac{3}{4}\right)^{d+1} \leq m$$

by the choice of $m$.

If we remove one point of each non-acute triple from $S'$, the remaining set is an acute $d$-set of cardinality at least $3m - m = 2m$. \(\square\)

## 4 Constructive lower bounds for $\kappa(d)$

In the following proofs, for clarity of exposition, we will represent point vectors in $\{0, 1\}^d$ as binary words of length $d$, e.g. $S_3 = \{000, 011, 101, 110\}$ represents a cubic acute 3-set.
Concatenation of words (vectors) $v$ and $v'$ will be written $vv'$.

We begin with a simple construction that enables us to extend a cubic acute $d$-set of cardinality $n$ to a cubic acute $(d+2)$-set of cardinality $n+1$.

**Theorem 4.1**

$$\kappa(d+2) \geq \kappa(d) + 1$$

**Proof:** Let $S = \{v_0, v_1, \ldots, v_{n-1}\}$ be a cubic acute $d$-set of cardinality $n = \kappa(d)$. Now let $S' = \{v'_0, v'_1, \ldots, v'_{n-1}\} \subseteq \{0,1\}^{d+2}$ where $v'_i = v_{i00}$ for $0 \leq i \leq n-2$, $v'_{n-1} = v_{n-100}$ and $v'_{n} = v_{n-101}$.

If $v'_i, v'_j, v'_k$ is a triple of distinct points in $S'$ with no more than one of $i$, $j$ and $k$ greater than $n - 2$, then $v'_i, v'_j, v'_k$ is an acute triple, because $S$ is an acute $d$-set. Also, any triple $v'_k, v'_{n-1}$, $v'_i$, $v'_k$, $v'_0$, $v'_{n-1}$ is an acute triple, because its $(d+1)$th or $(d+2)$th coordinates (respectively) are 0, 1, 0. Finally, for any triple $v'_{n-1}, v'_{k}, v'_{n}$, if $v_k$ and $v_{n-1}$ differ in the $r$th coordinate, then the $r$th coordinates of $v'_{n-1}, v'_{k}, v'_{n}$ are 0, 1, 0 or 1, 0, 1. Thus, $S'$ is a cubic acute $(d+2)$-set of cardinality $n + 1$. \(\square\)

Our second construction combines cubic acute $d$-sets of cardinality $n$ to make a cubic acute $3d$-set of cardinality $n^2$.

**Theorem 4.2**

$$\kappa(3d) \geq \kappa(d)^2.$$ 

**Proof:** Let $S = \{v_0, v_1, \ldots, v_{n-1}\}$ be a cubic acute $d$-set of cardinality $n = \kappa(d)$, and let

$$T = \{w_{ij} = v_iv_jv_{j-i \mod n} : 0 \leq i, j \leq n - 1\},$$

each $w_{ij}$ being made by concatenating three of the $v_i$.

Let $w_{ps}, w_{qt}, w_{ru}$ be any triple of distinct points in $T$. They constitute an acute triple iff the scalar product $\langle w_{ps} - w_{qt}, w_{ru} - w_{qt} \rangle$ does not vanish (is positive). Now,

$$\langle w_{ps} - w_{qt}, w_{ru} - w_{qt} \rangle = \langle v_pv_sv_{s-p}, v_qv_tv_{t-q}, v_iv_uv_{u-r}, v_qv_tv_{t-q} \rangle = \langle v_p - v_q, v_r - v_q \rangle + \langle v_s - v_t, v_u - v_t \rangle + \langle v_{s-p} - v_{t-q}, v_{u-r} - v_{t-q} \rangle$$

with all the index arithmetic modulo $n$.

If both $p \neq q$ and $q \neq r$, then the first component of this sum is positive, because $S$ is an acute $d$-set. Similarly, if both $s \neq t$ and $t \neq u$, then the second component is positive. Finally, if $p = q$ and $t = u$, then $q \neq r$ and $s \neq t$ or else the points would not be distinct, so the third component, $\langle v_{s-p} - v_{t-q}, v_{u-r} - v_{t-q} \rangle$ is positive. Similarly if $q = r$ and $s = t$.

Thus, all triples in $T$ are acute triples, so $T$ is a cubic acute $3d$-set of cardinality $n^2$. \(\square\)
Corollary 4.2.1 \( \kappa(3^d) \geq 2^{2^d} \).

**Proof:** By repeated application of theorem 4.2 starting with \( S_3 \), a cubic acute 3-set of cardinality 4. \( \square \)

Corollary 4.2.2 If \( d \geq 3 \),
\[
\kappa(d) \geq 10^{\frac{(d+1)^\mu}{4}} \approx 1.778^{(d+1)^0.631} \text{ where } \mu = \frac{\log 2}{\log 3}.
\]

For small \( d \), this is a tighter bound than theorem 2.1.

**Proof:** By induction on \( d \). For \( 3 \leq d \leq 8 \), we have the following cubic acute \( d \)-sets \( (S_3, \ldots, S_8) \) that satisfy this lower bound for \( \kappa(d) \) (with equality for \( d = 8 \)):

\[
\begin{array}{|c|c|c|}
\hline
S_3 : \kappa(3) \geq 4 & S_4 : \kappa(4) \geq 5 & S_5 : \kappa(5) \geq 6 \\
000 & 000 & 00000 \\
011 & 0011 & 00011 \\
101 & 0101 & 00101 \\
110 & 1001 & 01001 \\
\hline
S_6 : \kappa(6) \geq 8 & S_7 : \kappa(7) \geq 9 & S_8 : \kappa(8) \geq 10 \\
000000 & 0000000 & 00000000 \\
000111 & 0000011 & 00000011 \\
011001 & 0001101 & 00000101 \\
011110 & 0110001 & 00011001 \\
101010 & 0111110 & 01100001 \\
101101 & 1010101 & 01111110 \\
110011 & 1011010 & 10101001 \\
110100 & 1100110 & 10110110 \\
110100 & 1101001 & 11001110 \\
\hline
\end{array}
\]

If \( \kappa(d) \geq 10^{\frac{(d+1)^\mu}{4}} \), then \( \kappa(3d) \geq \kappa(d)^2 \geq 10^{2\frac{(d+1)^\mu}{4}} \geq 10^{\frac{(3d+3)^\mu}{4}} \) by theorem 4.2, by the induction hypothesis, because \( 3^\mu = 2 \).

So, since \( \kappa(3d + 2) \geq \kappa(3d + 1) \geq \kappa(3d) \), if the lower bound is satisfied for \( d \), it is also satisfied for \( 3d, 3d + 1 \) and \( 3d + 2 \). \( \square \)
Theorem 4.3  If, for each $r$, $1 \leq r \leq m$, we have a cubic acute $d_r$-set of cardinality $n_r$, where $n_1$ is the least of the $n_r$, and if, for some dimension $d_Z$, we have a cubic acute $d_Z$-set of cardinality $n_Z$, where

$$n_Z \geq \prod_{r=2}^{m} n_r,$$

then a cubic acute $D$-set of cardinality $N$ can be constructed, where

$$D = \sum_{r=1}^{m} d_r + d_Z \quad \text{and} \quad N = \prod_{r=1}^{m} n_r.$$

This result generalises theorem 4.2, but before we can prove it, we first need some preliminary results.

Definition 4.4  If $n_1 \leq n_2 \leq \ldots \leq n_m$ and $0 \leq k_r < n_r$, for each $r$, $1 \leq r \leq m$, then let us denote by $\langle\langle k_1 k_2 \ldots k_m \rangle\rangle_{n_1 n_2 \ldots n_m}$, the number

$$\langle\langle k_1 k_2 \ldots k_m \rangle\rangle_{n_1 n_2 \ldots n_m} = \sum_{r=2}^{m} \left( (k_{r-1} - k_r \mod n_r) \prod_{s=r+1}^{m} n_s \right).$$

Where the $n_r$ can be inferred from the context, $\langle\langle k_1 k_2 \ldots k_m \rangle\rangle$ may be used instead of $\langle\langle k_1 k_2 \ldots k_m \rangle\rangle_{n_1 n_2 \ldots n_m}$.

The expression $\langle\langle k_1 k_2 \ldots k_m \rangle\rangle_{n_1 n_2 \ldots n_m}$ can be understood as representing a number in a number system where the radix for each digit is a different $n_r$ — like the old British monetary system of pounds, shillings and pennies — and the digits are the difference of two adjacent $k_r$ (mod $n_r$). For example,

$$\langle\langle 2053 \rangle\rangle_{4668} = [2 - 0]_6[0 - 5]_6[5 - 3]_8 = 2 \times 6 \times 8 + 1 \times 8 + 2 = 106,$$

where $[a_2]_{n_2} \ldots [a_m]_{n_m}$ is place notation with the $n_r$ the radix for each place.

By construction, we have the following results:

Property 4.4.1

$$\langle\langle k_1 k_2 \ldots k_m \rangle\rangle_{n_1 n_2 \ldots n_m} < \prod_{r=2}^{m} n_r.$$

Property 4.4.2  If $2 \leq t \leq m$ and $j_{t-1} - j_i \neq k_{t-1} - k_t$ (mod $n_t$), then

$$\langle\langle j_1 j_2 \ldots j_m \rangle\rangle_{n_1 n_2 \ldots n_m} \neq \langle\langle k_1 k_2 \ldots k_m \rangle\rangle_{n_1 n_2 \ldots n_m}.$$
Lemma 4.5 If \( n_1 \leq n_2 \leq \ldots \leq n_m \) and \( 0 \leq j_r, k_r < n_r \), for each \( r, 1 \leq r \leq m \), and the sequences of \( j_r \) and \( k_r \) are neither identical nor everywhere different (i.e. there exist both \( t \) and \( u \) such that \( j_t = k_t \) and \( j_u \neq k_u \)), then
\[
\langle j_1 j_2 \ldots j_m \rangle_{n_1 n_2 \ldots n_m} \neq \langle k_1 k_2 \ldots k_m \rangle_{n_1 n_2 \ldots n_m}.
\]

**Proof:** Let \( u \) be the greatest integer, \( 1 \leq u < m \), such that \( j_u - j_{u+1} \neq k_u - k_{u+1} \) (mod \( n_{u+1} \)). (If \( j_m = k_m \), then \( u \) is the greatest integer such that \( j_u \neq k_u \). If \( j_m \neq k_m \), then \( u \) is at least as great as the greatest integer \( t \) such that \( j_t = k_t \).) The result now follows from 4.4.2. \( \square \)

We are now in a position to prove the theorem.

**Proof of Theorem 4.3**

Let \( n_1 \leq n_2 \leq \ldots \leq n_m \), and, for each \( r, 1 \leq r \leq m \), let \( S_r = \{ v^{r}_{1}, v^{r}_{1}, \ldots, v^{r}_{n_r-1} \} \) be a cubic acute \( d_r \)-set of cardinality \( n_r \). Let \( Z = \{ z_0, z_1, \ldots, z_{n_Z-1} \} \) be a cubic acute \( d_Z \)-set of cardinality \( n_Z \), where
\[
n_Z \geq \prod_{r=2}^{m} n_r.
\]
and let
\[
D = \sum_{r=1}^{m} d_r + d_Z \quad \text{and} \quad N = \prod_{r=1}^{m} n_r.
\]

Now let
\[
T = \{ w_{k_1 k_2 \ldots k_m} = v^{k_1}_{1} v^{k_2}_{2} \ldots v^{k_m}_{n_m} z_{k_Z} : 0 \leq k_r < n_r, 1 \leq r \leq m \},
\]
where \( k_Z = \langle k_1 k_2 \ldots k_m \rangle_{n_1 n_2 \ldots n_m} \) be a point set of dimension \( D \) and cardinality \( N \), each element of \( T \) being made by concatenating one vector from each of the \( S_r \) together with a vector from \( Z \). (In section 5, we will denote this construction by \( d_1 \circ \cdots \circ d_m \circ d_Z \).)

By 4.4.1, we know that \( k_Z \leq \prod_{r=2}^{m} n_r \leq n_Z \), so \( k_Z \) is a valid index into \( Z \).

Let \( w_{i_1 i_2 \ldots i_m}, w_{j_1 j_2 \ldots j_m}, w_{k_1 k_2 \ldots k_m} \) be any triple of distinct points in \( T \). They constitute an acute triple iff the scalar product \( q = \langle w_{i_1 i_2 \ldots i_m} - w_{j_1 j_2 \ldots j_m}, w_{k_1 k_2 \ldots k_m} - w_{j_1 j_2 \ldots j_m} \rangle \) does not vanish (is positive). Now,
\[
q = \langle v^{i_1}_{1} v^{i_2}_{2} \ldots v^{i_m}_{n_m} z_{i_Z} - v^{j_1}_{1} v^{j_2}_{2} \ldots v^{j_m}_{n_m} z_{j_Z}, v^{k_1}_{1} v^{k_2}_{2} \ldots v^{k_m}_{n_m} z_{k_Z} - v^{j_1}_{1} v^{j_2}_{2} \ldots v^{j_m}_{n_m} z_{j_Z} \rangle
\]
\[
= \sum_{r=1}^{m} \langle v^{r}_{i_r} - v^{r}_{j_r}, v^{r}_{k_r} - v^{r}_{j_r} \rangle + \langle z_{i_Z} - z_{j_Z}, z_{k_Z} - z_{j_Z} \rangle.
\]

If, for some \( r \), both \( i_r \neq j_r \) and \( j_r \neq k_r \), then the first component of this sum is positive, because \( S_r \) is an acute set.

If, however, there is no \( r \) such that both \( i_r \neq j_r \) and \( j_r \neq k_r \), then there must be some \( t \) for which \( i_t \neq j_t \) (or else \( w_{i_1 i_2 \ldots i_m} \) and \( w_{j_1 j_2 \ldots j_m} \) would not be distinct) and \( j_t = k_t \), and
also some \( u \) for which \( j_u \neq k_u \) (or else \( w_{j_1 j_2 \ldots j_m} \) and \( w_{k_1 k_2 \ldots k_m} \) would not be distinct) and \( i_u = j_u \). So, by lemma 4.5, \( i_Z \neq j_Z \) and \( j_Z \neq k_Z \), so the second component of the sum for the scalar product is positive, because \( Z \) is an acute set.

Thus, all triples in \( T \) are acute triples, so \( T \) is a cubic acute \( D \)-set of cardinality \( N \). □

**Corollary 4.5.1**

If \( d_1 \leq d_2 \leq \ldots \leq d_m \), then

\[
\kappa \left( \sum_{r=1}^{m} rd_r \right) \geq \prod_{r=1}^{m} \kappa(d_r).
\]

**Proof:** By induction on \( m \). The bound is trivially true for \( m = 1 \).

Assume the bound holds for \( m - 1 \), and for each \( r, 1 \leq r \leq m \), let \( S_r \) be a cubic acute \( d_r \)-set of cardinality \( n_r = \kappa(d_r) \), with \( d_1 \leq d_2 \leq \ldots \leq d_m \) and thus \( n_1 \leq n_2 \leq \ldots \leq n_m \). By the induction hypothesis, there exists a cubic acute \( d_Z \)-set \( Z \) of cardinality \( n_Z \), where

\[
d_Z = \sum_{r=2}^{m} (r-1)d_r \quad \text{and} \quad n_Z \geq \prod_{r=2}^{m} \kappa(d_r) = \prod_{r=2}^{m} n_r.
\]

Thus, by theorem 4.3, there exists a cubic acute \( D \)-set of cardinality \( N \), where

\[
D = \sum_{r=1}^{m} d_r + d_Z = \sum_{r=1}^{m} d_r + \sum_{r=2}^{m} (r-1)d_r = \sum_{r=1}^{m} rd_r,
\]

and

\[
N = \prod_{r=1}^{m} n_r = \prod_{r=1}^{m} \kappa(d_r).
\]

□

5 Lower bounds for \( \kappa(d) \) and \( \alpha(d) \) for small \( d \)

The following table lists the best lower bounds known for \( \kappa(d) \), \( 0 \leq d \leq 69 \). For \( 3 \leq d \leq 9 \), an exhaustive computer search shows that \( S_2, \ldots, S_8 \) (corollary 4.2.2), are optimal and also that \( \kappa(9) = 16 \). For other small values of \( d \), the construction used in theorem 4.3 provides the largest known cubic acute \( d \)-set. In the table, these constructions are denoted by \( d_1 \odot d_2 \odot d_Z \) or \( d_1 \odot d_2 \odot d_3 \odot d_Z \). For \( 39 \leq d \leq 48 \), the results of a computer program, based on the ‘probabilistic construction’ of theorem 2.1, provide the largest known cubic acute \( d \)-sets. Finally, for \( d \geq 67 \), theorem 2.1 provides the best (probabilistic) lower bound. \( \kappa(d) \) is sequence A089676 in Sloane [S].
Best Lower Bounds Known for $\kappa(d)$

<table>
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<th>$\kappa(d)$</th>
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</tr>
<tr>
<td>1</td>
<td>2</td>
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<tr>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>3</td>
<td>4 computer, $S_4$</td>
</tr>
<tr>
<td>4</td>
<td>5 computer, $S_4$</td>
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<td>6 computer, $S_5$</td>
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<td>8 computer, $S_6$</td>
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<tr>
<td>7</td>
<td>9 computer, $S_7$</td>
</tr>
<tr>
<td>8</td>
<td>10 computer, $S_8$</td>
</tr>
<tr>
<td>9</td>
<td>16 computer, $3\cdot 3\cdot 3$</td>
</tr>
<tr>
<td>10</td>
<td>$\geq 16$</td>
</tr>
<tr>
<td>11</td>
<td>$\geq 20$ 3\cdot 4\cdot 4</td>
</tr>
<tr>
<td>12</td>
<td>$\geq 25$ 4\cdot 4\cdot 4</td>
</tr>
<tr>
<td>13</td>
<td>$\geq 25$</td>
</tr>
<tr>
<td>14</td>
<td>$\geq 30$ 4\cdot 5\cdot 5</td>
</tr>
<tr>
<td>15</td>
<td>$\geq 36$ 5\cdot 5\cdot 5</td>
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<tr>
<td>16</td>
<td>$\geq 40$ 4\cdot 6\cdot 6</td>
</tr>
<tr>
<td>17</td>
<td>$\geq 48$ 5\cdot 6\cdot 6</td>
</tr>
<tr>
<td>18</td>
<td>$\geq 64$ 6\cdot 6\cdot 6 or 3\cdot 3\cdot 3\cdot 9</td>
</tr>
<tr>
<td>19</td>
<td>$\geq 64$</td>
</tr>
<tr>
<td>20</td>
<td>$\geq 72$ 6\cdot 7\cdot 7</td>
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<tr>
<td>21</td>
<td>$\geq 81$ 7\cdot 7\cdot 7</td>
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<tr>
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<tr>
<td>25</td>
<td>$\geq 144$ 7\cdot 9\cdot 9</td>
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<tr>
<td>26</td>
<td>$\geq 160$ 8\cdot 9\cdot 9</td>
</tr>
<tr>
<td>27</td>
<td>$\geq 256$ 9\cdot 9\cdot 9</td>
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<tr>
<td>28</td>
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</tr>
<tr>
<td>29</td>
<td>$\geq 257$ theorem 4.1</td>
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<tr>
<td>30</td>
<td>$\geq 257$</td>
</tr>
<tr>
<td>31</td>
<td>$\geq 320$ 9\cdot 11\cdot 11</td>
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<tr>
<td>36</td>
<td>$\geq 625$ 12\cdot 12\cdot 12</td>
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<tr>
<td>37</td>
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<tr>
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</tr>
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<td>41</td>
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<td>42</td>
<td>$\geq 976$ computer</td>
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<td>43</td>
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</tr>
<tr>
<td>44</td>
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</tr>
<tr>
<td>45</td>
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</tr>
<tr>
<td>46</td>
<td>$\geq 1630$ computer</td>
</tr>
<tr>
<td>47</td>
<td>$\geq 1808$ computer</td>
</tr>
<tr>
<td>48</td>
<td>$\geq 2036$ computer</td>
</tr>
<tr>
<td>49</td>
<td>$\geq 2036$</td>
</tr>
<tr>
<td>50</td>
<td>$\geq 2037$ theorem 4.1</td>
</tr>
<tr>
<td>51</td>
<td>$\geq 2304$ 17\cdot 17\cdot 17</td>
</tr>
<tr>
<td>52</td>
<td>$\geq 2560$ 16\cdot 18\cdot 18</td>
</tr>
<tr>
<td>53</td>
<td>$\geq 3072$ 17\cdot 18\cdot 18</td>
</tr>
<tr>
<td>54</td>
<td>$\geq 4096$ 18\cdot 18\cdot 18 or 9\cdot 9\cdot 9\cdot 27</td>
</tr>
<tr>
<td>55</td>
<td>$\geq 4096$</td>
</tr>
<tr>
<td>56</td>
<td>$\geq 4097$ theorem 4.1</td>
</tr>
<tr>
<td>57</td>
<td>$\geq 4097$</td>
</tr>
<tr>
<td>58</td>
<td>$\geq 4608$ 18\cdot 20\cdot 20</td>
</tr>
<tr>
<td>59</td>
<td>$\geq 4608$</td>
</tr>
<tr>
<td>60</td>
<td>$\geq 5184$ 20\cdot 20\cdot 20</td>
</tr>
<tr>
<td>61</td>
<td>$\geq 5184$</td>
</tr>
<tr>
<td>62</td>
<td>$\geq 5832$ 20\cdot 21\cdot 21</td>
</tr>
<tr>
<td>63</td>
<td>$\geq 6561$ 21\cdot 21\cdot 21</td>
</tr>
<tr>
<td>64</td>
<td>$\geq 6561$</td>
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<tr>
<td>65</td>
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<tr>
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<td>$\geq 8000$ 11\cdot 11\cdot 11\cdot 33</td>
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<tr>
<td>67</td>
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<tr>
<td>68</td>
<td>$\geq 9632$ theorem 2.1</td>
</tr>
<tr>
<td>69</td>
<td>$\geq 11122$ theorem 2.1</td>
</tr>
</tbody>
</table>
The following tables summarise the best lower bounds known for $\alpha(d)$. For $3 \leq d \leq 6$, the best lower bound is Danzer and Grünbaum’s $2d - 1$ [DG]. For $7 \leq d \leq 26$, the results of a computer program, based on the ‘probabilistic construction’ but using sets of points close to the surface of the $d$-sphere, provide the largest known acute $d$-sets. An acute 7-set of cardinality 14 and an acute 8-set of cardinality 16 are displayed. For $27 \leq d \leq 62$, the largest known acute $d$-set is cubic. Finally, for $d \geq 63$, theorem 3.1 provides the best (probabilistic) lower bound.

### Best Lower Bounds Known for $\alpha(d)$

<table>
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<tr>
<th>$d$</th>
<th>$\alpha(d)$</th>
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<tbody>
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<tr>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>3</td>
<td>5 [DG]</td>
</tr>
<tr>
<td>4-6</td>
<td>$\geq 2d - 1$ [DG]</td>
</tr>
<tr>
<td>7</td>
<td>$\geq 14$ computer</td>
</tr>
<tr>
<td>8</td>
<td>$\geq 16$ computer</td>
</tr>
<tr>
<td>9</td>
<td>$\geq 19$ computer</td>
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<tr>
<td>10</td>
<td>$\geq 23$ computer</td>
</tr>
<tr>
<td>11</td>
<td>$\geq 26$ computer</td>
</tr>
<tr>
<td>12</td>
<td>$\geq 30$ computer</td>
</tr>
<tr>
<td>13</td>
<td>$\geq 36$ computer</td>
</tr>
<tr>
<td>14</td>
<td>$\geq 42$ computer</td>
</tr>
<tr>
<td>15</td>
<td>$\geq 47$ computer</td>
</tr>
</tbody>
</table>

### $\alpha(7) \geq 14$

- (62, 1, 9, 10, 17, 38, 46)
- (38, 54, 0, 19, 38, 14, 25)
- (60, 33, 42, 9, 48, 3, 12)
- (62, 35, 41, 44, 16, 39, 44)
- (62, 34, 7, 45, 48, 37, 12)
- (28, 33, 42, 8, 49, 39, 45)
- (40, 16, 22, 12, 0, 0, 25)
- (45, 17, 26, 67, 25, 20, 29)
- (38, 6, 35, 0, 32, 18, 0)
- (62, 0, 42, 45, 49, 3, 48)
- (30, 0, 9, 44, 49, 37, 48)
- (0, 20, 31, 27, 34, 21, 28)
- (48, 19, 24, 22, 33, 20, 73)
- (43, 17, 25, 27, 32, 64, 19)

### $\alpha(8) \geq 16$

- (34, 49, 14, 51, 0, 36, 46, 0)
- (31, 17, 14, 51, 1, 5, 44, 31)
- (33, 50, 48, 20, 34, 35, 15, 0)
- (0, 16, 16, 52, 32, 36, 45, 0)
- (37, 31, 46, 52, 13, 0, 0, 22)
- (2, 50, 13, 52, 3, 3, 46, 0)
- (1, 50, 48, 51, 1, 5, 46, 31)
- (24, 0, 43, 2, 17, 20, 32, 16)
- (11, 49, 0, 11, 19, 8, 32, 19)
- (0, 48, 48, 52, 1, 34, 12, 2)
- (0, 48, 47, 51, 34, 37, 47, 32)
- (34, 49, 14, 51, 34, 36, 13, 34)
- (0, 46, 31, 0, 0, 23, 29, 29)
- (16, 40, 29, 23, 54, 3, 17, 16)
- (2, 15, 14, 50, 2, 36, 15, 33)
- (12, 36, 28, 30, 3, 45, 48, 45)
6 Generalising \( \kappa(d) \)

We can understand \( \kappa(d) \) to be the size of the largest possible set \( S \) of binary words such that, for any ordered triple of words \((u, v, w)\) in \( S \), there exists an index \( i \) for which \((u_i, v_i, w_i) = (0, 1, 0)\) or \((u_i, v_i, w_i) = (1, 0, 1)\). We can generalise this in the following way:

**Definition 6.1** If \( T_1, \ldots, T_m \) are ordered \( k \)-tuples from \( \{0, \ldots, r-1\}^k \) (which we will refer to as the matching \( k \)-tuples), then let us define \( \kappa[r,k,T_1,\ldots,T_m](d) \) to be the size of the largest possible set \( S \) of \( r \)-ary words of length \( d \) such that, for any ordered \( k \)-tuple of words \((w_1, \ldots, w_k)\) in \( S \), there exist \( i \) and \( j \), \( 1 \leq i \leq d \), \( 1 \leq j \leq m \), for which \((w_{i1}, \ldots, w_{ki}) = T_j\).

Thus we have \( \kappa(d) = \kappa[2,3,(0,1,0),(1,0,1)](d) \). If the set of matching \( k \)-tuples is closed under permutation, we will abbreviate by writing a list of matching *multisets* of cardinality \( k \), rather than ordered tuples. For example, instead of \( \kappa[2,3,(0,0,1),(0,1,0),(1,0,0)](d) \), we write \( \kappa[2,3,(0,0,1)](d) \).

We can find probabilistic and, in some cases, constructive lower bounds for general \( \kappa[r,k,T_1,\ldots,T_m](d) \) using the approaches we used for cubic acute \( d \)-sets. To illustrate this, in the remainder of this paper, we will consider the set of problems in which it is simply required that at some index the \( k \)-tuple of words be all different (pairwise distinct). First, we express this in a slightly different form.

Let us say that an *\( r \)-ary \( d \)-colouring* is some colouring of the integers \( 1, \ldots, d \) using \( r \) colours. Let us also also say that a set \( R \) of \( r \)-ary \( d \)-colourings is a *\( k \)-rainbow set*, for some \( k \leq r \) if for any set \( \{c_1, \ldots, c_k\} \) of \( k \) colourings in \( R \), there exists some integer \( t \), \( 1 \leq t \leq d \), for which the colours \( c_1(t), \ldots, c_k(t) \) are all different, i.e. \( c_i(t) \neq c_j(t) \) for any \( i \) and \( j \), \( 1 \leq i, j \leq k \), \( i \neq j \). For conciseness, we will denote “a \( k \)-rainbow set of \( r \)-ary \( d \)-colourings” by “\( RSC[k,r,d] \)”.

Let us further say that a set \( \{c_1, \ldots, c_k\} \) of \( k \) \( d \)-colourings is a *good \( k \)-set* if there exists some integer \( t \), \( 1 \leq t \leq d \), for which the colours \( c_1(t), \ldots, c_k(t) \) are all different, and a *bad \( k \)-set* if there exists no such \( t \).

We will denote by \( \rho_{r,k}(d) \) the size of the largest possible \( RSC[k,r,d] \), abbreviating \( \rho_{k,k}(d) \) by \( \rho_k(d) \). Now, \( \rho_k(d) = \kappa[k,k,\{0,1,\ldots,k-1\}](d) \) and

\[
\rho_{r,k}(d) = \kappa[r,k,\{0,1,\ldots,k-1\},\ldots,\{r-k,\ldots,r-1\}](d),
\]

where the matching multisets are those of cardinality \( k \) with \( k \) distinct members.

Clearly, \( \rho_{r,k}(d) \leq \rho_{r,k}(d+1) \), \( \rho_{r,k}(d) \leq \rho_{r+1,k}(d) \) and \( \rho_{r,k}(d) \geq \rho_{r,k+1}(d) \). Also, \( \rho_{r,1}(d) \) is undefined because any set of colourings is a 1-rainbow, \( \rho_{r,1}(1) = r \) if \( k > 1 \), and \( \rho_{r,2}(d) = r^d \) because any two distinct \( r \)-ary \( d \)-colourings (or \( r \)-ary words of length \( d \)) differ somewhere.
In the next two sections we will give a number of probabilistic and constructive lower bounds for $\rho_{r,k}(d)$, for various $r$ and $k$.

### 7 A probabilistic lower bound for $\rho_{r,k}(d)$

**Theorem 7.1**

$$\rho_{r,k}(d) \geq (k-1)m \quad \text{where} \quad m = \left[ k^{-1} \sqrt[2]{\sqrt[k]{\frac{k!}{k^k}} \left( k^{-1} \sqrt[2]{\frac{(r-k)! r^k}{(r-k)! r^k - r!}} \right)^d} \right].$$

**Proof:** This proof is similar to that of theorem 2.1.

Randomly pick a set $\mathcal{R}$ of $km$ $r$-ary $d$-colourings, choosing the colours from $\{\chi_0, \ldots, \chi_{r-1}\}$ independently with probability $Pr[c(i) = \chi_j] = 1/r$, $1 \leq i \leq d$, $0 \leq j < r$ for every $c \in \mathcal{R}$.

Now the probability that a set of $k$ colourings from $\mathcal{R}$ is a bad $k$-set is

$$(1 - p)^d \quad \text{where} \quad p = \frac{r! / (r-k)!}{r^k}.$$  

Hence, the expected number of bad $k$-sets in a set of $km$ $d$-colourings is $\left(\frac{km}{k}\right)(1 - p)^d$.

Thus there is some set $\mathcal{R}$ of $km$ $d$-colourings with no more than $\left(\frac{km}{k}\right)(1 - p)^d$ bad $k$-sets, where

$$\left(\frac{km}{k}\right)(1 - p)^d < \frac{(km)^k}{k!} (1 - p)^d = m \frac{k^k}{k!} m^{k-1} (1 - p)^d \leq m$$

by the choice of $m$.

If we remove one colouring of each bad $k$-set from $\mathcal{R}$, the remaining set is a $\mathcal{RSC}[k, r, d]$ of cardinality at least $km - m = (k-1)m$.

The following results follow directly:

$$\rho_3(d) \geq 2 \left[ \frac{\sqrt{2}}{3} \left( \frac{3}{\sqrt{7}} \right)^d \right] \approx 0.943 \times 1.134^d.$$  

$$\rho_{4,3}(d) \geq 2 \left[ \frac{\sqrt{2}}{3} \left( \frac{4}{\sqrt{10}} \right)^d \right] \approx 0.943 \times 1.265^d.$$  

$$\rho_4(d) \geq 3 \left[ \frac{\sqrt{3}}{32} \sqrt[29]{32^d} \right] \approx 1.363 \times 1.033^d.$$
8 Constructive lower bounds for $\rho_{r,k}(d)$

In the following proofs, for clarity of exposition, we will represent $r$-ary $d$-colourings as $r$-ary words of length $d$, e.g. $R_{3,3,3} = \{000, 011, 102, 121, 212, 220\}$ represents a 3-rainbow set of ternary 3-colourings (using the colours $\chi_0$, $\chi_1$ and $\chi_2$). Concatenation of words (colourings) $c$ and $c'$ will be written $c.c'$.

We begin with a construction that enables us to extend a $\textsc{RSC}[k, r, d]$ of cardinality $n$ to one of cardinality $n + 1$ or greater.

**Theorem 8.1** If for some $r \geq k \geq 3$, and some $d$, we have a $\textsc{RSC}[k, r, d]$ of cardinality $n$, and for some $r'$, $k - 2 \leq r' \leq r - 2$, and $d'$, we have a $\textsc{RSC}[k - 2, r', d']$ of cardinality at least $n - 1$, then we can construct a $\textsc{RSC}[k, r, d + d']$ of cardinality $N = n - 1 + r - r'$.

**Proof:** Let $\mathcal{R} = \{c_0, c_1, \ldots, c_{n-1}\}$ be a $\textsc{RSC}[k, r, d]$ of cardinality $n$ (using colours $\chi_0, \ldots, \chi_{r-1}$) and $\mathcal{R}' = \{c'_0, c'_1, \ldots, c'_{n'-1}\}$ be a $\textsc{RSC}[k - 2, r', d']$ of cardinality $n' \geq n - 1$ (using colours $\chi_0, \ldots, \chi_{r-1}$).

Now let $\mathcal{Q} = \{q_0, q_1, \ldots, q_{N-1}\}$ be a set of $r$-ary $(d + d')$-colourings where $q_i = c_i . c'_i$ for $0 \leq i \leq n - 2$, and $q_{n-1+j} = c_{n-1} . (r' + j)^d$ for $0 \leq j < r - r'$, each element of $\mathcal{Q}$ being made by concatenating two component colourings, the first from $\mathcal{R}$ and the second being either from $\mathcal{R}'$ or a monochrome colouring.

If $\{q_{i_1}, \ldots, q_{i_k}\}$ is a set of colourings in $\mathcal{Q}$ with no more than one of the $i_m$ greater than $n - 2$, then it is a good $k$-set because of the first components, since $\mathcal{R}$ is a $k$-rainbow set.

On the other hand, if $\{q_{i_1}, \ldots, q_{i_k}\}$ is a set of colourings in $\mathcal{Q}$ with no more than $k - 2$ of the $i_m$ less than $n - 1$, then it too is a good $k$-set because of the second components, since $\mathcal{R}'$ is a $(k - 2)$-rainbow set using colours $\chi_0, \ldots, \chi_{r'-1}$ and the second components of the colourings with indices greater than $n - 2$ are each monochrome of a different colour, drawn from $\chi_{r'}, \ldots, \chi_{r-1}$.

Thus $\mathcal{Q}$ is a $\textsc{RSC}[k, r, d + d']$ of cardinality $N$.

**Corollary 8.1.1** $\rho_{r,3}(d + 1) \geq \rho_{r,3}(d) + r - 2$.

**Proof:** This follows from the theorem due to the fact that there is a 1-rainbow set of 1-ary 1-colourings of any cardinality.

**Corollary 8.1.2** $\rho_{r,4}(d + \lceil \log_2(\rho_{r,4}(d) - 1) \rceil) \geq \rho_{r,4}(d) + r - 3$.

**Proof:** Since $\rho_{r,2}(d) = r^d$, we have $\rho_{2,2}(d') \geq \rho_{r,4}(d) - 1$ iff $d' \geq \log_2(\rho_{r,4}(d) - 1)$.
Theorem 8.2 If, for each $s, 1 \leq s \leq m$, we have a $\mathcal{RSC}[3, r, d_s]$ of cardinality $n_s$, where $n_1$ is the least of the $n_s$, and if, for some $d_Z$, we have a $\mathcal{RSC}[3, r, d_Z]$ of cardinality $n_Z$, where

$$n_Z \geq \prod_{s=2}^{m}(1 + 2 \left\lfloor \frac{n_s}{2} \right\rfloor),$$

then a $\mathcal{RSC}[3, r, D]$ of cardinality $N$ can be constructed, where

$$D = \sum_{s=1}^{m} d_s + 2d_Z \quad \text{and} \quad N = \prod_{s=1}^{m} n_s.$$

This result for 3-rainbow sets corresponds to theorem 4.3 for cubic acute $d$-sets. Before we can prove it, we need some further preliminary results.

Definition 8.3 If $n_1 \leq n_2 \leq \ldots \leq n_m$ and $0 \leq k_r < n_r$, for each $r, 1 \leq r \leq m$, then let us denote by $\langle \langle k_1 k_2 \ldots k_m \rangle \rangle_{n_1 n_2 \ldots n_m}^+$, the number

$$\langle \langle k_1 k_2 \ldots k_m \rangle \rangle_{n_1 n_2 \ldots n_m}^+ = \sum_{r=2}^{m} \left( (k_{r-1} + k_r \mod n_r) \prod_{s=r+1}^{m} n_s \right).$$

The definition of $\langle \langle k_1 k_2 \ldots k_m \rangle \rangle_{n_1 n_2 \ldots n_m}^+$ is the same as that for $\langle \langle k_1 k_2 \ldots k_m \rangle \rangle_{n_1 n_2 \ldots n_m}$ (see 4.4), but with addition replacing subtraction. By construction, we have

$$\langle \langle k_1 k_2 \ldots k_m \rangle \rangle_{n_1 n_2 \ldots n_m}^+ < \prod_{r=2}^{m} n_r,$$

and, if $2 \leq t \leq m$ and $j_{t-1} + j_t \neq k_{t-1} + k_t \ (\mod n_t)$, then

$$\langle \langle j_1 j_2 \ldots j_m \rangle \rangle_{n_1 n_2 \ldots n_m}^+ \neq \langle \langle k_1 k_2 \ldots k_m \rangle \rangle_{n_1 n_2 \ldots n_m}^+.$$

Lemma 8.4 If $n_1 \leq n_2 \leq \ldots \leq n_m$, with all the $n_r$ odd except perhaps $n_1$, and $0 \leq j_r, k_r, l_r < n_r$, for each $r, 1 \leq r \leq m$, and the sequences of $j_r$, $k_r$ and $l_r$ are neither pairwise identical nor anywhere pairwise distinct, i.e. there is some $u, v$ and $w$ such that $j_u \neq k_u, k_v \neq l_v$ and $l_w \neq j_w$ but no $t$ such that $j_t \neq k_t, k_t \neq l_t$ and $l_t \neq j_t$, then either

$$\langle \langle j_1 \ldots j_m \rangle \rangle_{n_1 \ldots n_m}, \langle \langle k_1 \ldots k_m \rangle \rangle_{n_1 \ldots n_m}, \langle \langle l_1 \ldots l_m \rangle \rangle_{n_1 \ldots n_m} \text{ are pairwise distinct}$$

or

$$\langle \langle j_1 \ldots j_m \rangle \rangle_{n_1 \ldots n_m}^+, \langle \langle k_1 \ldots k_m \rangle \rangle_{n_1 \ldots n_m}^+, \langle \langle l_1 \ldots l_m \rangle \rangle_{n_1 \ldots n_m}^+ \text{ are pairwise distinct.}$$
Proof: Without loss of generality, we can assume that we have \( j_t \neq k_t \), that \( t > 1 \) is the least integer for which \( j_t \neq k_t \), and that \( k_t = l_t \). We will consider two cases:

Case 1: \( k_{t-1} \neq l_{t-1} \)

Since \( j_{t-1} = k_{t-1} \neq l_{t-1} \) and \( j_t \neq k_t = l_t \), we have \( j_{t-1} - j_t \neq k_{t-1} - k_t \) and \( k_{t-1} - k_t \neq l_{t-1} - l_t \), and so \( \langle j_1 \ldots j_m \rangle \neq \langle k_1 \ldots k_m \rangle \) and \( \langle k_1 \ldots k_m \rangle \neq \langle l_1 \ldots l_m \rangle \). Similarly, \( j_{t-1} + j_t \neq k_{t-1} + k_t \) and \( k_{t-1} + k_t \neq l_{t-1} + l_t \), and so \( \langle j_1 \ldots j_m \rangle^+ \neq \langle k_1 \ldots k_m \rangle^+ \) and \( \langle k_1 \ldots k_m \rangle^+ \neq \langle l_1 \ldots l_m \rangle^+ \).

If \( j_{t-1} - j_t \neq l_{t-1} - l_t \), then \( \langle j_1 \ldots j_m \rangle \neq \langle l_1 \ldots l_m \rangle \). If \( j_{t-1} - j_t = l_{t-1} - l_t \) then \( (j_{t-1} + j_t) - (l_{t-1} + l_t) = (j_{t-1} - j_t + 2j_t) - (l_{t-1} - l_t + 2l_t) = 2(j_t - l_t) \neq 0 \mod n_t \) because \( j_t \neq l_t \) and \( n_t \) is odd, so \( j_{t-1} + j_t \neq l_{t-1} + l_t \) and \( \langle j_1 \ldots j_m \rangle^+ \neq \langle l_1 \ldots l_m \rangle^+ \).

Case 2: \( k_{t-1} = l_{t-1} \)

Since \( j_{t-1} = k_{t-1} = l_{t-1} \) and \( j_t \neq k_t = l_t \), we have \( j_{t-1} - j_t \neq k_{t-1} - k_t \) and \( j_{t-1} - j_t \neq l_{t-1} - l_t \), and so \( \langle j_1 \ldots j_m \rangle \neq \langle k_1 \ldots k_m \rangle \) and \( \langle j_1 \ldots j_m \rangle \neq \langle l_1 \ldots l_m \rangle \).

If \( k_1 = l_1 \), let \( u \) be the least integer such that \( k_u \neq l_u \). Since \( k_{u-1} = l_{u-1} \), we have \( k_{u-1} - k_u \neq l_{u-1} - l_u \). If \( k_1 \neq l_1 \), let \( u \) be the least integer such that \( k_u = l_u \). Since \( k_{u-1} \neq l_{u-1} \), we still have \( k_{u-1} - k_u \neq l_{u-1} - l_u \). Thus, \( \langle k_1 \ldots k_m \rangle \neq \langle l_1 \ldots l_m \rangle \). \( \square \)

Proof of Theorem 8.2

Let \( n_1 \leq n_2 \leq \ldots \leq n_m \), and, for each \( s \), \( 1 \leq s \leq m \), let \( \mathcal{R}_s = \{c_0^n, c_1^n, \ldots, c_{n_s - 1}^n\} \) be a \( \mathcal{RSC}[3, r, d_s]\) of cardinality \( n_s \), and let \( n_s' = 1 + 2 \floor{n_s/2} \) be the least odd integer not less than \( n_s \). Let \( \mathcal{Z} = \{z_0, z_1, \ldots, z_{n_2-1}\} \) be a \( \mathcal{RSC}[3, r, d_z]\) of cardinality \( n_z \), where

\[
n_z \geq \prod_{s=2}^{m} n_s',
\]

and let

\[
D = \sum_{s=1}^{m} d_s + 2d_z \quad \text{and} \quad N = \prod_{s=1}^{m} n_s.
\]

Now let

\[
\mathcal{Q} = \{c_{k_1}^1, c_{k_2}^2, \ldots, c_{k_m}^m, z_{k_2}, z_{k_2}^+ : 0 \leq k_s < n_s, 1 \leq s \leq m\},
\]

where \( k_Z = \langle k_1k_2 \ldots k_m \rangle_n' \) and \( k_Z^+ = \langle k_1k_2 \ldots k_m \rangle_n' \) be a set of \( D\)-colours of cardinality \( N \), each element of \( \mathcal{Q} \) being made by concatenating one colouring from each of the \( \mathcal{R}_s \) together with two colourings from \( \mathcal{Z} \). (Below, we will denote this construction by \( d_1 \otimes \cdots \otimes d_m \otimes d_Z \).)

Let \( c_{i_1}^1, c_{i_2}^2, \ldots, c_{i_m}^m, z_{i_2}, z_{i_2}^+ \), \( c_{j_1}^1, c_{j_2}^2, \ldots, c_{j_m}^m, z_{j_2}, z_{j_2}^+ \) and \( c_{k_1}^1, c_{k_2}^2, \ldots, c_{k_m}^m, z_{k_2}, z_{k_2}^+ \) be any three distinct colourings in \( \mathcal{Q} \). If, for some \( s, i_s \neq j_s, j_s \neq k_s \) and \( k_s \neq i_s \), then these three colourings comprise a good 3-set because \( \mathcal{R}_s \) is a 3-rainbow set.
If, however, there is no $s$ such that $i_s$, $j_s$, and $k_s$ are all different, then the condition of lemma 8.4 holds, and so either $i_Z$, $j_Z$, and $k_Z$ are all different, or $i_Z^+$, $j_Z^+$, and $k_Z^+$ are all different, and the three colourings comprise a good 3-set because $Z$ is a 3-rainbow set.

Thus, any three colourings in $Q$ comprise a good 3-set, so $Q$ is a $RSC[3, r, D]$ of cardinality $N$.

**Corollary 8.4.1** If $\rho_{r,3}(d)$ is odd, then $\rho_{r,3}(4d) \geq \rho_{r,3}(d)^2$.

**Proof:** By theorem 8.2 using the construction $d \oplus d \oplus d \oplus d$. 

**Corollary 8.4.2** $\rho_{r,3}(4d + 2) \geq \rho_{r,3}(d)^2$.

**Proof:** By 8.1.1, if $n = \rho_{r,3}(d)$, we can construct a $RSC[3, r, d + 1]$ of cardinality $n + 1 \geq 1 + 2 \lceil n/2 \rceil$. By theorem 8.2, we can then construct a $RSC[3, r, 4d + 2]$ of cardinality $n^2$ using the construction $d \oplus d \oplus (d + 1) \oplus (d + 1)$. 

**Corollary 8.4.3** $\rho_3(4^d) \geq 3^{2^d}$.

**Proof:** By repeated application of 8.4.1 starting with $\rho_{3,3}(1) = 3$. 

Our final construction enables us to combine $k$-rainbow sets of $r$-ary $d$-colourings for arbitrary $k$.

**Theorem 8.5** If we have a $RSC[k, r, d_1]$ of cardinality $n_1$, a $RSC[k, r, d_2]$ of cardinality $n_2 \geq n_1$, and a $RSC[k, r, d_Z]$ of cardinality $n_Z \geq n_2$, with $n_Z$ coprime to each integer in the range $[2, \ldots, h]$ where $h = \binom{k}{2} - 1$, then a $RSC[k, r, D]$ of cardinality $N$ can be constructed, where $D = d_1 + d_2 + hd_Z$ and $N = n_1 n_2$.

As before, we first need a preliminary result:

**Lemma 8.6** Given distinct pairs of integers $(a, b)$ and $(c, d)$ with $0 \leq a, b, c, d < n$ for some $n$, and given a positive integer $h$ such that $n$ is coprime to each integer in the range $[2, \ldots, h]$, then if we let $b_{-1} = a$ and $d_{-1} = c$, and $b_r = b + ra \pmod{n}$ and $d_r = d + rc \pmod{n}$ for $0 \leq r \leq h$, then if $b_i = d_i$ for some $i$, $-1 \leq i \leq h$, we have $b_j \neq d_j$ for all $j \neq i$. 


Proof: We consider two cases:

Case 1: \( i = -1 \)

Since \( a = c, (b + ja) - (d + jc) = b - d \neq 0 \pmod{n} \) since \((a, b)\) and \((c, d)\) are distinct, and \( b \) and \( d \) both less than \( n \).

Case 2: \( i \neq -1 \)

By the reversing the argument in case 1, \( a \neq c \), i.e. \( b_{-1} \neq d_{-1} \). For \( j \geq 0 \), since \( b + ia = d + ic \), we have \( (b + ja) - (d + jc) = (j - i)a - (j - i)c = (j - i)(a - c) \neq 0 \pmod{n} \) since \( a \neq c \) and \( |j - i| \leq h \) so \( j - i \) is coprime to \( n \).

Proof of Theorem 8.5

Let \( R_1 = \{c_1^1, \ldots, c_{n_1}^1\} \), \( R_2 = \{c_0^2, \ldots, c_{n_2}^2\} \) and \( S = \{z_0, \ldots, z_{n_S-1}\} \) be \( k \)-rainbow sets of \( r \)-ary \( d_1 \)-, \( d_2 \)- and \( d_S \)-colours of cardinality \( n_1, n_2 \) and \( n_S \), respectively.

Now let
\[
Q = \{c_i^1, c_j^2, z_{j+i}, z_{j+2i}, \ldots, z_{j+hi} : 0 \leq i < n_1, 0 \leq j < n_2\},
\]
where \( h = \binom{k}{2} - 1 \) and the subscript arithmetic is modulo \( n_S \), be a set of \( D \)-colours of cardinality \( N \), each element of \( Q \) being made by concatenating \( h+2 \) component colourings: one from \( R_1 \), one from \( R_2 \), and \( h \) from \( S \).

Let
\[
S = \{c_i^1, c_j^2, z_{j+i}, i_1, \ldots, z_{j+i}, i_1, c_i^2, c_j^2, z_{j+i}, i_2, \ldots, z_{j+i}, i_2, \ldots, c_i^1, c_j^2, z_{j+i}, i_k, \ldots, z_{j+i}, i_k\}
\]
be any set of \( k \) distinct colourings in \( Q \), and let \( b_{s, -1} = i_s \) and \( b_{s, t} = j_s + ti_s \pmod{n_S} \), for each \( s \) and \( t \), \( 1 \leq s \leq k \), \( 0 \leq i \leq h \), so the \( s \)th colouring in \( S \) is \( c_{b_{s, -1}, c_{b_{s,0}}, z_{b_{s,1}}, \ldots, z_{b_{s,h}}} \).

Now, for any \( s, s' \) and \( t, 1 \leq s, s' \leq k, -1 \leq t \leq h \), if \( b_{s, t} = b_{s', t} \), then by lemma 8.6 we know that for all \( u \neq t \), \( b_{s, u} \neq b_{s', u} \). So for each pair \( \{s, s'\} \), \( b_{s, t} = b_{s', t} \) for no more than one value of \( t \). Now there are \( h + 2 \) possible values of \( t \), but only \( \binom{k}{2} = h + 1 \) different pairs \( \{s, s'\} \), so there is some \( t \) for which \( b_{s, t} \neq b_{s', t} \) for all pairs \( \{s, s'\} \) and the \((t + 2)\)th component colourings of the elements in \( S \) are all different. Since \( R_1, R_2 \) and \( S \) are all \( k \)-rainbow sets, we know that \( S \) is a good \( k \)-set.

Thus, any \( k \) colourings from \( Q \) comprise a good \( k \)-set, so \( Q \) is a \( RSC[k, r, D] \) of cardinality \( N \).

\[\text{Corollary 8.6.1 } \rho_4(6.7^d) \geq 7^2^d.\]

Proof: The following 4-rainbow set of 4-ary 6-colourings of cardinality 8 — a version of \( R_{4,4,6} \) (see below) displayed with different symbols for each colour — shows that \( \rho_4(6) \geq 7 \).
The result follows by repeated application of theorem 8.5, noting that 7 is coprime to 2, 3, 4 and 5 = \binom{4}{2} - 1.

\square

9 Lower bounds for \( r, k(d) \) for small \( r, k \) and \( d \)

We conclude with tables of the best lower bounds known for \( r_3(d) \), \( r_{4,3}(d) \) and \( r_4(d) \) for small \( d \). For very small \( d \), exhaustive computer searches have determined the values of \( r_{r,k}(d) \). For other small values of \( d \), the constructions used in theorems 8.2 and 8.5 provide the largest known rainbow sets. In the tables, these constructions are denoted \( d_1 \oplus d_2 \oplus d_3 \oplus d_4 \), etc., with superscript minus signs \( (d^-) \) to denote the removal of a single colouring from a largest rainbow set of \( d \)-colourings (to satisfy the requirement that the cardinality be odd). For \( r_3(d) \), the probabilistic lower bound of theorem 7.1 is better than the constructions for \( d \geq 71 \); for \( r_{4,3}(d) \), this is the case for \( d \geq 26 \).
Some $k$-rainbow sets of $r$-ary $d$-colourings, for small $k$, $r$ and $d$

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Best Lower Bounds Known for $\rho_3(d)$ and $\rho_{4,3}(d)$

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**Best Lower Bounds Known for \( \rho_4(d) \)**

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**References**


