Sets of points determining only acute angles and some related colouring problems

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Sets of Points Determining Only Acute Angles and Some Related Colouring Problems

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Abstract

We present both probabilistic and constructive lower bounds on the maximum size of a set of points $S \subseteq \mathbb{R}^d$ such that every angle determined by three points in $S$ is acute, considering especially the case $S \subseteq \{0, 1\}^d$. These results improve upon a probabilistic lower bound of Erdős and Füredi. We also present lower bounds for some generalisations of the acute angles problem, considering especially some problems concerning colourings of sets of integers.

1 Introduction

Let us say that a set of points $S \subseteq \mathbb{R}^d$ is an acute \textit{d-set} if every angle determined by a triple of $S$ is acute ($< \frac{\pi}{2}$). Let us also say that $S$ is a cubic acute \textit{d-set} if $S$ is an acute \textit{d-set} and is also a subset of the unit \textit{d-cube} (i.e. $S \subseteq \{0, 1\}^d$).

Let us further say that a triple $u, v, w \in \mathbb{R}^d$ is an acute triple, a right triple, or an obtuse triple, if the angle determined by the triple with apex $v$ is less than $\frac{\pi}{2}$, equal to $\frac{\pi}{2}$, or greater than $\frac{\pi}{2}$, respectively. Note that we consider the triples $u, v, w$ and $w, v, u$ to be the same.

We will denote by $\alpha(d)$ the size of a largest possible acute \textit{d-set}. Similarly, we will denote by $\kappa(d)$ the size of a largest possible cubic acute \textit{d-set}. Clearly $\kappa(d) \leq \alpha(d)$, $\kappa(d) \leq \kappa(d+1)$ and $\alpha(d) \leq \alpha(d+1)$ for all $d$. 

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In [EF], Paul Erdős and Zoltán Füredi gave a probabilistic proof that $\kappa(d) \geq \left\lfloor \frac{1}{2} \left( \frac{2}{\sqrt{3}} \right)^d \right\rfloor$ (see also [AZ2]). This disproved an earlier conjecture of Ludwig Danzer and Branko Grünbaum [DG] that $\alpha(d) = 2d - 1$.

In the following two sections we give improved probabilistic lower bounds for $\kappa(d)$ and $\alpha(d)$. In section 4 we present a construction that gives further improved lower bounds for $\kappa(d)$ for small $d$. In section 5, we tabulate the best lower bounds known for $\kappa(d)$ and $\alpha(d)$ for small $d$. Finally, in sections 6–9, we give probabilistic and constructive lower bounds for some generalisations of $\kappa(d)$, considering especially some problems concerning colourings of sets of integers.

2 A probabilistic lower bound for $\kappa(d)$

Theorem 2.1

$$\kappa(d) \geq 2 \left\lfloor \frac{\sqrt{6}}{3} \left( \frac{2}{\sqrt{3}} \right)^d \right\rfloor \approx 0.544 \times 1.155^d.$$ 

For large $d$, this improves upon the result of Erdős and Füredi by a factor of $\frac{4\sqrt{6}}{9} \approx 1.089$. This is achieved by a slight improvement in the choice of parameters. This proof can also be found in [AZ3].

Proof: Let $m = \left\lfloor \frac{\sqrt{6}}{3} \left( \frac{2}{\sqrt{3}} \right)^d \right\rfloor$ and randomly pick a set $S$ of $3m$ point vectors from the vertices of the $d$-dimensional unit cube $\{0, 1\}^d$, choosing the coordinates independently with probability $\Pr[v_i = 0] = \Pr[v_i = 1] = \frac{1}{2}$, $1 \leq i \leq d$, for every $v = (v_1, v_2, \ldots, v_d) \in S$.

Now every angle determined by a triple of points from $S$ is non-obtuse ($\leq \frac{\pi}{2}$), and a triple of vectors $u, v, w$ from $S$ is a right triple iff the scalar product $\langle u - v, w - v \rangle$ vanishes, i.e. iff either $u_i - v_i = 0$ or $w_i - v_i = 0$ for each $i$, $1 \leq i \leq d$.

Thus $u, v, w$ is a right triple iff $u_i, v_i, w_i$ is neither 0, 1, 0 nor 1, 0, 1 for any $i$, $1 \leq i \leq d$. Since $u_i, v_i, w_i$ can take eight different values, this occurs independently with probability $\frac{3}{4}$ for each $i$, so the probability that a triple of $S$ is a right triple is $\left( \frac{3}{4} \right)^d$.

Hence, the expected number of right triples in a set of $3m$ vectors is $3 \binom{3m}{3} \left( \frac{3}{4} \right)^d$. Thus there is some set $S$ of $3m$ vectors with no more than $3 \binom{3m}{3} \left( \frac{3}{4} \right)^d$ right triples, where

$$3 \binom{3m}{3} \left( \frac{3}{4} \right)^d < 3 \binom{3m}{3} \left( \frac{3}{4} \right)^d = m \left( \frac{9m}{\sqrt{6}} \right)^2 \left( \frac{3}{4} \right)^d \leq m$$

by the choice of $m$. 

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If we remove one point of each right triple from \( S \), the remaining set is a cubic acute \( d \)-set of cardinality at least \( 3m - m = 2m \).

\( \square \)

### 3 A probabilistic lower bound for \( \alpha(d) \)

We can improve the lower bound in theorem 2.1 for non-cubic acute \( d \)-sets by a factor of \( \sqrt{2} \) by slightly perturbing the points chosen away from the vertices of the unit cube. The intuition behind this is that a small random symmetrical perturbation of the points in a right triple is more likely than not to produce an acute triple, as the following diagram suggests.

\[ \text{Theorem 3.1} \]

\[ \alpha(d) \geq 2 \left| \frac{1}{3} \left( \frac{2}{\sqrt{3}} \right)^{d+1} \right| \approx 0.770 \times 1.155^d. \]

Before we can prove this theorem, we need some results concerning continuous random variables.

**Definition 3.2** If \( F(x) = \Pr[X \leq x] \) is the cumulative distribution function of a continuous random variable \( X \), let \( F(x) \) denote \( \Pr[X \geq x] = 1 - F(x) \).

**Definition 3.3** Let us say that a continuous random variable \( X \) has **positive bias** if, for all \( t \), \( \Pr[X \geq t] \geq \Pr[X \leq -t] \), i.e. \( F(t) \geq F(-t) \).

**Property 3.3.1** If a continuous random variable \( X \) has positive bias, it follows that \( \Pr[X > 0] \geq \frac{1}{2} \).

**Property 3.3.2** To show that a continuous random variable \( X \) has positive bias, it suffices to demonstrate that the condition \( F(t) \geq F(-t) \) holds for all **positive** \( t \).
Lemma 3.4 If $X$ and $Y$ are independent continuous random variables with positive bias, then $X + Y$ also has positive bias.

Proof: Let $f$, $g$ and $h$ be the probability density functions, and $F$, $G$ and $H$ the cumulative distribution functions, for $X$, $Y$ and $X + Y$ respectively. Then,

$$
\overline{H}(t) - H(-t) = \int\int_{x+y \geq t} f(x)g(y) \, dx \, dy - \int\int_{x+y \leq -t} f(x)g(y) \, dx \, dy
$$

$$
= \int\int_{x+y \geq t} f(x)g(y) \, dx \, dy - \int\int_{y-x \geq t} f(x)g(y) \, dx \, dy
$$

$$
+ \int\int_{y-x \geq t} f(x)g(y) \, dx \, dy - \int\int_{x+y \leq -t} f(x)g(y) \, dx \, dy
$$

$$
= \int_{-\infty}^{\infty} g(y) \left[ F(t-y) - F(y-t) \right] \, dy
$$

$$
+ \int_{-\infty}^{\infty} f(x) \left[ \overline{G}(x+t) - G(-x-t) \right] \, dx
$$

which is non-negative because $f(t)$, $g(t)$, $F(t) - F(-t)$ and $\overline{G}(t) - G(-t)$ are all non-negative for all $t$.  

Definition 3.5 Let us say that a continuous random variable $X$ is $\epsilon$-uniformly distributed for some $\epsilon > 0$ if $X$ is uniformly distributed between $-\epsilon$ and $\epsilon$.

Let us denote by $j$, the probability density function of an $\epsilon$-uniformly distributed random variable:

$$
j(x) = \begin{cases} 
\frac{1}{2\epsilon} & -\epsilon \leq x \leq \epsilon \\
0 & \text{otherwise}
\end{cases}
$$

and by $J$, its cumulative distribution function:

$$
J(x) = \begin{cases} 
0 & x < -\epsilon \\
\frac{1}{2} + \frac{x}{2\epsilon} & -\epsilon \leq x \leq \epsilon \\
1 & x > \epsilon
\end{cases}
$$

Property 3.5.1 If $X$ is an $\epsilon$-uniformly distributed random variable, then so is $-X$. 

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Lemma 3.6  If $X$, $Y$ and $Z$ are independent $\epsilon$-uniformly distributed random variables for some $\epsilon < \frac{1}{2}$, then $U = (Y - X)(1 + Z - X)$ has positive bias.

Proof: Let $G$ be the cumulative distribution function of $U$. By 3.3.2, it suffices to show that $G(u) - G(-u) \geq 0$ for all positive $u$.

Let $u$ be positive. Because $1 + Z - X$ is always positive, $U \geq u$ iff $Y > X$ and $Z \geq -1 + X + \frac{u}{y-x}$. Similarly, $U \leq -u$ iff $X > Y$ and $Z \geq -1 + X + \frac{u}{x-y}$. So,

$$G(u) - G(-u) = \int \int_{y>x} j(x)j(y)J(-1 + x + \frac{u}{y-x})dy dx - \int \int_{x>y} j(x)j(y)J(-1 + x + \frac{u}{x-y})dy dx$$

$$= \int \int_{y>x} j(x)j(y) \left[ J(1 - x - \frac{u}{y-x}) - J(1 - y - \frac{u}{y-x}) \right] dy dx$$

(because $J(x) = J(-x)$, and by variable renaming)

which is non-negative because $j$ is non-negative and $J$ is non-decreasing (so the expression in square brackets is non-negative over the domain of integration).

Corollary 3.6.1  If $X$, $Y$ and $Z$ are independent $\epsilon$-uniformly distributed random variables for some $\epsilon < \frac{1}{2}$, then $(Y - X)(Z - X - 1)$ has positive bias.

Proof: $(Y - X)(Z - X - 1) = ((-Y) - (-X))(1 + (-Z) - (-X))$. The result follows from 3.5.1 and lemma 3.6. \qed

Lemma 3.7  If $X$, $Y$ and $Z$ are independent $\epsilon$-uniformly distributed random variables, then $V = (Y - X)(Z - X)$ has positive bias.

Proof: Let $H$ be the cumulative distribution function of $V$. By 3.3.2, it suffices to show that $H(v) - H(-v) \geq 0$ for all positive $v$. \qed
Let $v$ be positive. $V \geq v$ iff $Y > X$ and $Z \geq X + \frac{v}{\sqrt{2}}$ or $Y < X$ and $Z \leq X + \frac{v}{\sqrt{2}}$. Similarly, $V \leq -v$ iff $Y > X$ and $Z \leq X - \frac{v}{\sqrt{2}}$ or $Y < X$ and $Z \geq X - \frac{v}{\sqrt{2}}$. So,

$$
\overline{H}(v) - H(-v) = \iint_{y>x} j(x)j(y)\overline{J}(x + \frac{v}{y - x}) \, dy \, dx
+ \iint_{y<x} j(x)j(y)\overline{J}(x + \frac{v}{y - x}) \, dy \, dx
- \iint_{y>x} j(x)j(y)\overline{J}(x - \frac{v}{y - x}) \, dy \, dx
- \iint_{y<x} j(x)j(y)\overline{J}(x - \frac{v}{y - x}) \, dy \, dx
$$

which is non-negative because $j$ is non-negative and $J$ is non-decreasing (so the expressions in square brackets are non-negative over the domains of integration).

We are now in a position to prove the theorem.

**Proof of theorem 3.1**

Let $m = \left\lfloor \frac{1}{3} \left( \frac{2}{\sqrt{3}} \right)^{d+1} \right\rfloor$, and randomly pick a set $S$ of $3m$ point vectors, $v_1, v_2, \ldots, v_{3m}$, from the vertices of the $d$-dimensional unit cube $\{0,1\}^d$, choosing the coordinates independently with probability $\Pr[v_{ki} = 0] = \Pr[v_{ki} = 1] = \frac{1}{2}$ for every $v_k = (v_{k1}, v_{k2}, \ldots, v_{kd})$, $1 \leq k \leq 3m$, $1 \leq i \leq d$.

Now for some $\epsilon$, $0 < \epsilon < \frac{1}{2(d+1)}$, randomly pick $3m$ vectors, $\delta_1, \delta_2, \ldots, \delta_{3m}$, from the $d$-dimensional cube $[-\epsilon, \epsilon]^d$ of side $2\epsilon$ centred on the origin, choosing the coordinates $\delta_{ki}$, $1 \leq k \leq 3m$, $1 \leq i \leq d$, independently so that they are $\epsilon$-uniformly distributed, and let $S' = \{v'_1, v'_2, \ldots, v'_{3m}\}$ where $v'_k = v_k + \delta_k$ for each $k$, $1 \leq k \leq 3m$.

**Case 1: Acute triples in $S$**

Because $\epsilon < \frac{1}{2(d+1)}$, if $v_j, v_k, v_l$ is an acute triple in $S$, the scalar product $\langle v'_j - v'_k, v'_l - v'_k \rangle > \frac{1}{(d+1)^2}$, so $v'_j, v'_k, v'_l$ is also an acute triple in $S'$.

**Case 2: Right triples in $S$**

If, $v_j, v_k, v_l$ is a right triple in $S$ then the scalar product $\langle v_j - v_k, v_l - v_k \rangle$ vanishes, i.e. either $v_{j1} - v_{k1} = 0$ or $v_{l1} - v_{k1} = 0$ for each $i$, $1 \leq i \leq d$. There are six possibilities for each triple of coordinates:
\begin{align*}
\begin{array}{|c|c|}
\hline
\mathbf{v}_{ji}, \mathbf{v}_{ki}, \mathbf{v}_{li} & (\mathbf{v}'_{ji} - \mathbf{v}'_{ki})(\mathbf{v}'_{li} - \mathbf{v}'_{ki}) \\
0, 0, 0 & (\delta_{ji} - \delta_{ki})(\delta_{li} - \delta_{ki}) \\
1, 1, 1 & (\delta_{ji} - \delta_{ki})(\delta_{li} - \delta_{ki}) \\
0, 0, 1 & (\delta_{ji} - \delta_{ki})(1 + \delta_{li} - \delta_{ki}) \\
1, 0, 0 & (\delta_{ji} - \delta_{ki})(1 + \delta_{li} - \delta_{ki}) \\
0, 1, 1 & (\delta_{ji} - \delta_{ki})(\delta_{ji} - \delta_{ki} - 1) \\
1, 1, 0 & (\delta_{ji} - \delta_{ki})(\delta_{li} - \delta_{ki} - 1) \\
\hline
\end{array}
\end{align*}

Now, the values of the $\delta_{ki}$ are independent and $\epsilon$-uniformly distributed, so by lemmas 3.7 and 3.6 and corollary 3.6.1, the distribution of the $(\mathbf{v}'_{ji} - \mathbf{v}'_{ki})(\mathbf{v}'_{li} - \mathbf{v}'_{ki})$ has positive bias, and by repeated application of lemma 3.4, the distribution of the scalar product

\[ \sum_{i=1}^{d} (\mathbf{v}'_{ji} - \mathbf{v}'_{ki})(\mathbf{v}'_{li} - \mathbf{v}'_{ki}) \]

also has positive bias.

Thus, if $\mathbf{v}_{j}, \mathbf{v}_{k}, \mathbf{v}_{l}$ is a right triple in $\mathcal{S}$, then, by 3.3.1,

\[ \Pr [ (\mathbf{v}'_{j} - \mathbf{v}'_{k}, \mathbf{v}'_{l} - \mathbf{v}'_{k}) > 0 ] \geq \frac{1}{2}, \]

so the probability that the triple $\mathbf{v}'_{j}, \mathbf{v}'_{k}, \mathbf{v}'_{l}$ is an acute triple in $\mathcal{S}'$ is at least $\frac{1}{2}$.

As in the proof of theorem 2.1, the expected number of right triples in $\mathcal{S}$ is $3\binom{3m}{3} \left( \frac{3}{4} \right)^d$, so the expected number of non-acute triples in $\mathcal{S}'$ is no more than half this value. Thus there is some set $\mathcal{S}'$ of $3m$ vectors with no more than $\frac{3}{2} \binom{3m}{3} \left( \frac{3}{4} \right)^d$ non-acute triples, where

\[ \frac{3}{2} \binom{3m}{3} \left( \frac{3}{4} \right)^d < \frac{3}{2} \binom{3m}{3} \left( \frac{3}{4} \right)^d = m(3m)^2 \left( \frac{3}{4} \right)^{d+1} \leq m \]

by the choice of $m$.

If we remove one point of each non-acute triple from $\mathcal{S}'$, the remaining set is an acute $d$-set of cardinality at least $3m - m = 2m$. \qed

## 4 Constructive lower bounds for $\kappa(d)$

In the following proofs, for clarity of exposition, we will represent point vectors in $\{0, 1\}^d$ as binary words of length $d$, e.g. $\mathcal{S}_{3} = \{000, 011, 101, 110\}$ represents a cubic acute 3-set.
Concatenation of words (vectors) $v$ and $v'$ will be written $vv'$.

We begin with a simple construction that enables us to extend a cubic acute $d$-set of cardinality $n$ to a cubic acute $(d + 2)$-set of cardinality $n + 1$.

**Theorem 4.1**

\[ \kappa(d + 2) \geq \kappa(d) + 1 \]

**Proof:** Let $S = \{v_0, v_1, \ldots, v_{n-1}\}$ be a cubic acute $d$-set of cardinality $n = \kappa(d)$. Now let $S' = \{v'_0, v'_1, \ldots, v'_{n-1}\} \subseteq \{0, 1\}^d$ where $v'_i = v_{00}$ for $0 \leq i \leq n - 2$, $v'_{n-1} = v_{n-100}$ and $v'_n = v_{n-101}$.

If $v'_i, v'_j, v'_k$ is a triple of distinct points in $S'$ with no more than one of $i, j$ and $k$ greater than $n - 2$, then $v'_i, v'_j, v'_k$ is an acute triple, because $S$ is an acute $d$-set. Also, any triple $v'_k, v'_{n-1}, v'_r$ or $v'_k, v'_{n-1}$ is an acute triple, because its $(d+1)$th or $(d+2)$th coordinates (respectively) are 0, 1, 0. Finally, for any triple $v'_{n-1}, v'_k, v'_n$, if $v'_k$ and $v'_{n-1}$ differ in the $r$th coordinate, then the $r$th coordinates of $v'_{n-1}, v'_k, v'_n$ are 0, 1, 0 or 1, 0, 1. Thus, $S'$ is a cubic acute $(d + 2)$-set of cardinality $n + 1$. \( \square \)

Our second construction combines cubic acute $d$-sets of cardinality $n$ to make a cubic acute $3d$-set of cardinality $n^2$.

**Theorem 4.2**

\[ \kappa(3d) \geq \kappa(d)^2 \]

**Proof:** Let $S = \{v_0, v_1, \ldots, v_{n-1}\}$ be a cubic acute $d$-set of cardinality $n = \kappa(d)$, and let $T = \{w_{ij} = v_i v_j v_{j-i \mod n} : 0 \leq i, j \leq n - 1\}$, each $w_{ij}$ being made by concatenating three of the $v_i$.

Let $w_{ps}, w_{qt}, w_{ru}$ be any triple of distinct points in $T$. They constitute an acute triple iff the scalar product $\langle w_{ps} - w_{qt}, w_{ru} - w_{qt} \rangle$ does not vanish (is positive). Now,

\[
\langle w_{ps} - w_{qt}, w_{ru} - w_{qt} \rangle = \langle v_p v_s v_{s-p} - v_q v_t v_{t-q}, v_i v_u v_{u-r} - v_q v_t v_{t-q} \rangle \\
= \langle v_p - v_q, v_r - v_q \rangle \\
+ \langle v_s - v_t, v_u - v_t \rangle \\
+ \langle v_{s-p} - v_{t-q}, v_{u-r} - v_{t-q} \rangle
\]

with all the index arithmetic modulo $n$.

If both $p \neq q$ and $q \neq r$, then the first component of this sum is positive, because $S$ is an acute $d$-set. Similarly, if both $s \neq t$ and $t \neq u$, then the second component is positive. Finally, if $p = q$ and $t = u$, then $q \neq r$ and $s \neq t$ or else the points would not be distinct, so the third component, $\langle v_{s-p} - v_{t-q}, v_{u-r} - v_{t-q} \rangle$ is positive. Similarly if $q = r$ and $s = t$.

Thus, all triples in $T$ are acute triples, so $T$ is a cubic acute $3d$-set of cardinality $n^2$. \( \square \)
Corollary 4.2.1 \( \kappa(3^d) \geq 2^{2d} \).

**Proof:** By repeated application of theorem 4.2 starting with \( S_3 \), a cubic acute 3-set of cardinality 4. \( \Box \)

Corollary 4.2.2 If \( d \geq 3 \),

\[
\kappa(d) \geq 10^{\frac{(d+1)^\mu}{4}} \approx 1.778^{(d+1)^{0.631}} \quad \text{where} \quad \mu = \frac{\log 2}{\log 3}.
\]

For small \( d \), this is a tighter bound than theorem 2.1.

**Proof:** By induction on \( d \). For \( 3 \leq d \leq 8 \), we have the following cubic acute \( d \)-sets \( (S_3, \ldots, S_8) \) that satisfy this lower bound for \( \kappa(d) \) (with equality for \( d = 8 \)):

\begin{center}
\begin{tabular}{|c|c|}
\hline
\( S_3 \) & \( \kappa(3) \geq 4 \) \\
000 & 00000 \\
011 & 00011 \\
101 & 01010 \\
110 & 11010 \\
\hline
\end{tabular}
\hspace{1cm}
\begin{tabular}{|c|c|}
\hline
\( S_4 \) & \( \kappa(4) \geq 5 \) \\
0000 & 000000 \\
0111 & 000011 \\
1010 & 010101 \\
1101 & 011110 \\
\hline
\end{tabular}
\hspace{1cm}
\begin{tabular}{|c|c|}
\hline
\( S_5 \) & \( \kappa(5) \geq 6 \) \\
00000 & 0000000 \\
00011 & 0000011 \\
01100 & 0001101 \\
01110 & 0110001 \\
10101 & 0111110 \\
10110 & 1010101 \\
11001 & 1011010 \\
11010 & 1100110 \\
11110 & 1101001 \\
\hline
\end{tabular}
\end{center}

\begin{center}
\begin{tabular}{|c|c|}
\hline
\( S_6 \) & \( \kappa(6) \geq 8 \) \\
000000 & 00000000 \\
000111 & 00000111 \\
011001 & 00011011 \\
011110 & 01100011 \\
101010 & 01111100 \\
101101 & 10101011 \\
110011 & 10110101 \\
110100 & 11001110 \\
\hline
\end{tabular}
\hspace{1cm}
\begin{tabular}{|c|c|}
\hline
\( S_7 \) & \( \kappa(7) \geq 9 \) \\
0000000 & 000000000 \\
0000011 & 000001111 \\
0001101 & 000110101 \\
0011001 & 001100011 \\
0110101 & 011000011 \\
1010101 & 011111110 \\
1011010 & 101010011 \\
1100110 & 101101101 \\
1101001 & 110011110 \\
\hline
\end{tabular}
\hspace{1cm}
\begin{tabular}{|c|c|}
\hline
\( S_8 \) & \( \kappa(8) \geq 10 \) \\
00000000 & 0000000000 \\
00000111 & 0000011111 \\
00011001 & 0001101011 \\
00110010 & 0011000111 \\
01101010 & 0110000111 \\
10101010 & 0111111110 \\
10110101 & 1010100111 \\
11001101 & 1011011011 \\
11010011 & 1100111110 \\
11110001 & 1101000111 \\
\hline
\end{tabular}
\end{center}

If \( \kappa(d) \geq 10^{\frac{(d+1)^\mu}{4}} \), then \( \kappa(3d) \geq \kappa(d)^2 \) by theorem 4.2

\[
\geq 10^{\frac{2(d+1)^\mu}{4}} \quad \text{by the induction hypothesis}
\]

\[
= 10^{\frac{(3d+3)^\mu}{4}} \quad \text{because} \quad 3^\mu = 2.
\]

So, since \( \kappa(3d + 2) \geq \kappa(3d + 1) \geq \kappa(3d) \), if the lower bound is satisfied for \( d \), it is also satisfied for \( 3d, 3d + 1 \) and \( 3d + 2 \). \( \Box \)
**Theorem 4.3** If, for each \( r \), \( 1 \leq r \leq m \), we have a cubic acute \( d_r \)-set of cardinality \( n_r \), where \( n_1 \) is the least of the \( n_r \), and if, for some dimension \( d_Z \), we have a cubic acute \( d_Z \)-set of cardinality \( n_Z \), where

\[
 n_Z \geq \prod_{r=2}^{m} n_r,
\]

then a cubic acute \( D \)-set of cardinality \( N \) can be constructed, where

\[
 D = \sum_{r=1}^{m} d_r + d_Z \quad \text{and} \quad N = \prod_{r=1}^{m} n_r.
\]

This result generalises theorem 4.2, but before we can prove it, we first need some preliminary results.

**Definition 4.4** If \( n_1 \leq n_2 \leq \ldots \leq n_m \) and \( 0 \leq k_r < n_r \), for each \( r, 1 \leq r \leq m \), then let us denote by \( \langle k_1 k_2 \ldots k_m \rangle_{n_1 n_2 \ldots n_m} \), the number

\[
 \langle k_1 k_2 \ldots k_m \rangle_{n_1 n_2 \ldots n_m} = \sum_{r=2}^{m} \left( (k_{r-1} - k_r \mod n_r) \prod_{s=r+1}^{m} n_s \right).
\]

Where the \( n_r \) can be inferred from the context, \( \langle k_1 k_2 \ldots k_m \rangle \) may be used instead of \( \langle k_1 k_2 \ldots k_m \rangle_{n_1 n_2 \ldots n_m} \).

The expression \( \langle k_1 k_2 \ldots k_m \rangle_{n_1 n_2 \ldots n_m} \) can be understood as representing a number in a number system where the radix for each digit is a different \( n_r \) — like the old British monetary system of pounds, shillings and pence — and the digits are the difference of two adjacent \( k_r \) (mod \( n_r \)). For example,

\[
 \langle 2053 \rangle_{4668} = [2 - 0]_6[0 - 5]_6[5 - 3]_8 = 2 \times 6 \times 8 + 1 \times 8 + 2 = 106,
\]

where \([a_2]_{n_2} \ldots [a_m]_{n_m}\) is place notation with the \( n_r \) the radix for each place.

By construction, we have the following results:

**Property 4.4.1**

\[
 \langle k_1 k_2 \ldots k_m \rangle_{n_1 n_2 \ldots n_m} < \prod_{r=2}^{m} n_r.
\]

**Property 4.4.2** If \( 2 \leq t \leq m \) and \( j_{t-1} - j_t \neq k_{t-1} - k_t \) (mod \( n_t \)), then

\[
 \langle j_1 j_2 \ldots j_m \rangle_{n_1 n_2 \ldots n_m} \neq \langle k_1 k_2 \ldots k_m \rangle_{n_1 n_2 \ldots n_m}.
\]
Lemma 4.5 If \( n_1 \leq n_2 \leq \ldots \leq n_m \) and \( 0 \leq j_r, k_r < n_r \), for each \( r, 1 \leq r \leq m \), and the sequences of \( j_r \) and \( k_r \) are neither identical nor everywhere different (i.e. there exist both \( i \) and \( u \) such that \( j_i = k_i \) and \( j_u \neq k_u \)), then

\[
\langle j_1 j_2 \ldots j_m \rangle_{n_1 n_2 \ldots n_m} \neq \langle k_1 k_2 \ldots k_m \rangle_{n_1 n_2 \ldots n_m}.
\]

Proof: Let \( u \) be the greatest integer, \( 1 \leq u < m \), such that \( j_u - j_{u+1} \neq k_u - k_{u+1} \) (mod \( n_{u+1} \)). If \( j_m = k_m \), then \( u \) is the greatest integer such that \( j_u \neq k_u \). If \( j_m \neq k_m \), then \( u \) is at least as great as the greatest integer \( t \) such that \( j_i = k_t \). The result now follows from 4.4.2.

We are now in a position to prove the theorem.

Proof of Theorem 4.3

Let \( n_1 \leq n_2 \leq \ldots \leq n_m \), and, for each \( r, 1 \leq r \leq m \), let \( S_r = \{ v_{r_1}^r, v_{r_2}^r, \ldots, v_{r_{n_r-1}}^r \} \) be a cubic acute \( d_r \)-set of cardinality \( n_r \). Let \( Z = \{ z_0, z_1, \ldots, z_{n_Z-1} \} \) be a cubic acute \( d_Z \)-set of cardinality \( n_Z \), where

\[
n_Z \geq \prod_{r=2}^{m} n_r,
\]

and let

\[
D = \sum_{r=1}^{m} d_r + d_Z \quad \text{and} \quad N = \prod_{r=1}^{m} n_r.
\]

Now let

\[
T = \{ w_{k_1 k_2 \ldots k_m} = v_{k_1}^1 v_{k_2}^2 \ldots v_{k_m}^m z_{k_Z} : 0 \leq k_r < n_r, 1 \leq r \leq m \},
\]

where \( k_Z = \langle k_1 k_2 \ldots k_m \rangle_{n_1 n_2 \ldots n_m} \), be a point set of dimension \( D \) and cardinality \( N \), each element of \( T \) being made by concatenating one vector from each of the \( S_r \) together with a vector from \( Z \). (In section 5, we will denote this construction by \( d_1 \circ \ldots \circ d_m \circ d_Z \).

By 4.4.1, we know that \( k_Z < \prod_{r=2}^{m} n_r \leq n_Z \), so \( k_Z \) is a valid index into \( Z \).

Let \( w_{i_1 i_2 \ldots i_m}, w_{j_1 j_2 \ldots j_m}, w_{k_1 k_2 \ldots k_m} \) be any triple of distinct points in \( T \). They constitute an acute triple iff the scalar product \( q = \langle w_{i_1 i_2 \ldots i_m} - w_{j_1 j_2 \ldots j_m}, w_{k_1 k_2 \ldots k_m} - w_{j_1 j_2 \ldots j_m} \rangle \) does not vanish (is positive). Now,

\[
q = \sum_{r=1}^{m} \langle v_{i_r}^r - v_{j_r}^r, v_{k_r}^r - v_{j_r}^r \rangle + \langle z_{i_Z} - z_{j_Z}, z_{k_Z} - z_{j_Z} \rangle.
\]

If, for some \( r \), both \( i_r \neq j_r \) and \( k_r \neq j_r \), then the first component of this sum is positive, because \( S_r \) is an acute set.

If, however, there is no \( r \) such that both \( i_r \neq j_r \) and \( j_r \neq k_r \), then there must be some \( t \) for which \( i_t \neq j_t \) (or else \( w_{i_1 i_2 \ldots i_m} \) and \( w_{j_1 j_2 \ldots j_m} \) would not be distinct) and \( j_t = k_t \), and
also some \( u \) for which \( j_u \neq k_u \) (or else \( w_{j_1j_2\ldots j_m} \) and \( w_{k_1k_2\ldots k_m} \) would not be distinct) and \( i_u = j_u \). So, by lemma 4.5, \( i \neq j \) and \( j \neq k \), so the second component of the sum for the scalar product is positive, because \( Z \) is an acute set.

Thus, all triples in \( T \) are acute triples, so \( T \) is a cubic acute \( D \)-set of cardinality \( N \).

\[ \square \]

Corollary 4.5.1

If \( d_1 \leq d_2 \leq \ldots \leq d_m \), then
\[
\kappa\left(\sum_{r=1}^{m} rd_r\right) \geq \prod_{r=1}^{m} \kappa(d_r).
\]

\textbf{Proof:} By induction on \( m \). The bound is trivially true for \( m = 1 \).

Assume the bound holds for \( m - 1 \), and for each \( r, 1 \leq r \leq m \), let \( S_r \) be a cubic acute \( d_r \)-set of cardinality \( n_r = \kappa(d_r) \), with \( d_1 \leq d_2 \leq \ldots \leq d_m \) and thus \( n_1 \leq n_2 \leq \ldots \leq n_m \).

By the induction hypothesis, there exists a cubic acute \( d_Z \)-set \( Z \) of cardinality \( n_Z \), where
\[
d_Z = \sum_{r=2}^{m} (r-1)d_r \quad \text{and} \quad n_Z = \prod_{r=2}^{m} \kappa(d_r) = \prod_{r=2}^{m} n_r.
\]

Thus, by theorem 4.3, there exists a cubic acute \( D \)-set of cardinality \( N \), where
\[
D = \sum_{r=1}^{m} d_r + d_Z = \sum_{r=1}^{m} d_r + \sum_{r=2}^{m} (r-1)d_r = \sum_{r=1}^{m} rd_r,
\]

and
\[
N = \prod_{r=1}^{m} n_r = \prod_{r=1}^{m} \kappa(d_r).
\]

\[ \square \]

5 Lower bounds for \( \kappa(d) \) and \( \alpha(d) \) for small \( d \)

The following table lists the best lower bounds known for \( \kappa(d) \), \( 0 \leq d \leq 69 \). For \( 3 \leq d \leq 9 \), an exhaustive computer search shows that \( S_3, \ldots, S_8 \) (corollary 4.2.2), are optimal and also that \( \kappa(9) = 16 \). For other small values of \( d \), the construction used in theorem 4.3 provides the largest known cubic acute \( d \)-set. In the table, these constructions are denoted by \( d_1 \oplus d_2 \oplus d_Z \) or \( d_1 \oplus d_2 \oplus d_3 \oplus d_Z \). For \( 39 \leq d \leq 48 \), the results of a computer program, based on the ‘probabilistic construction’ of theorem 2.1, provide the largest known cubic acute \( d \)-sets. Finally, for \( d \geq 67 \), theorem 2.1 provides the best (probabilistic) lower bound. \( \kappa(d) \) is sequence A089676 in Sloane [S].
Best Lower Bounds Known for $\kappa(d)$

<table>
<thead>
<tr>
<th>$d$</th>
<th>$\kappa(d)$</th>
<th>$d$</th>
<th>$\kappa(d)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$= 1$</td>
<td>26</td>
<td>$\geq 160$ $8\cdot9\cdot9$</td>
</tr>
<tr>
<td>1</td>
<td>$= 2$</td>
<td>27</td>
<td>$\geq 256$ $9\cdot9\cdot9$</td>
</tr>
<tr>
<td>2</td>
<td>$= 2$</td>
<td>28</td>
<td>$\geq 256$</td>
</tr>
<tr>
<td>3</td>
<td>$= 4$ computer, $S_3$</td>
<td>29</td>
<td>$\geq 257$ theorem 4.1</td>
</tr>
<tr>
<td>4</td>
<td>$= 5$ computer, $S_4$</td>
<td>30</td>
<td>$\geq 257$</td>
</tr>
<tr>
<td>5</td>
<td>$= 6$ computer, $S_5$</td>
<td>31</td>
<td>$\geq 320$ $9\cdot11\cdot11$</td>
</tr>
<tr>
<td>6</td>
<td>$= 8$ computer, $S_6$</td>
<td>32</td>
<td>$\geq 320$</td>
</tr>
<tr>
<td>7</td>
<td>$= 9$ computer, $S_7$</td>
<td>33</td>
<td>$\geq 400$ $11\cdot11\cdot11$</td>
</tr>
<tr>
<td>8</td>
<td>$= 10$ computer, $S_8$</td>
<td>34</td>
<td>$\geq 400$</td>
</tr>
<tr>
<td>9</td>
<td>$= 16$ computer, $3\cdot3\cdot3$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>$\geq 16$</td>
<td>35</td>
<td>$\geq 500$ $11\cdot12\cdot12$</td>
</tr>
<tr>
<td>11</td>
<td>$\geq 20$ $3\cdot4\cdot4$</td>
<td>36</td>
<td>$\geq 625$ $12\cdot12\cdot12$</td>
</tr>
<tr>
<td>12</td>
<td>$\geq 25$ $4\cdot4\cdot4$</td>
<td>37</td>
<td>$\geq 625$</td>
</tr>
<tr>
<td>13</td>
<td>$\geq 25$</td>
<td>38</td>
<td>$\geq 626$ theorem 4.1</td>
</tr>
<tr>
<td>14</td>
<td>$\geq 30$ $4\cdot5\cdot5$</td>
<td>39</td>
<td>$\geq 678$ computer</td>
</tr>
<tr>
<td>15</td>
<td>$\geq 36$ $5\cdot5\cdot5$</td>
<td>40</td>
<td>$\geq 762$ computer</td>
</tr>
<tr>
<td>16</td>
<td>$\geq 40$ $4\cdot6\cdot6$</td>
<td>41</td>
<td>$\geq 871$ computer</td>
</tr>
<tr>
<td>17</td>
<td>$\geq 48$ $5\cdot6\cdot6$</td>
<td>42</td>
<td>$\geq 976$ computer</td>
</tr>
<tr>
<td>18</td>
<td>$\geq 64$ $6\cdot6\cdot6$ or $3\cdot3\cdot3\cdot9$</td>
<td>43</td>
<td>$\geq 1086$ computer</td>
</tr>
<tr>
<td>19</td>
<td>$\geq 64$</td>
<td>44</td>
<td>$\geq 1246$ computer</td>
</tr>
<tr>
<td>20</td>
<td>$\geq 72$ $6\cdot7\cdot7$</td>
<td>45</td>
<td>$\geq 1420$ computer</td>
</tr>
<tr>
<td>21</td>
<td>$\geq 81$ $7\cdot7\cdot7$</td>
<td>46</td>
<td>$\geq 1630$ computer</td>
</tr>
<tr>
<td>22</td>
<td>$\geq 81$</td>
<td>47</td>
<td>$\geq 1808$ computer</td>
</tr>
<tr>
<td>23</td>
<td>$\geq 100$ $3\cdot4\cdot4\cdot12$</td>
<td>48</td>
<td>$\geq 2036$ computer</td>
</tr>
<tr>
<td>24</td>
<td>$\geq 125$ $4\cdot4\cdot4\cdot12$</td>
<td>49</td>
<td>$\geq 2036$</td>
</tr>
<tr>
<td>25</td>
<td>$\geq 144$ $7\cdot9\cdot9$</td>
<td>50</td>
<td>$\geq 2037$ theorem 4.1</td>
</tr>
<tr>
<td>26</td>
<td>$\geq 2560$ $16\cdot18\cdot18$</td>
<td>51</td>
<td>$\geq 2304$ $17\cdot17\cdot17$</td>
</tr>
<tr>
<td>52</td>
<td>$\geq 2560$ $16\cdot18\cdot18$</td>
<td>52</td>
<td>$\geq 2560$ $16\cdot18\cdot18$</td>
</tr>
<tr>
<td>53</td>
<td>$\geq 3072$ $17\cdot18\cdot18$</td>
<td>54</td>
<td>$\geq 4096$ $18\cdot18\cdot18$ or $9\cdot9\cdot9\cdot27$</td>
</tr>
<tr>
<td>55</td>
<td>$\geq 4096$</td>
<td>56</td>
<td>$\geq 4097$ theorem 4.1</td>
</tr>
<tr>
<td>56</td>
<td>$\geq 4097$ theorem 4.1</td>
<td>57</td>
<td>$\geq 4097$</td>
</tr>
<tr>
<td>58</td>
<td>$\geq 4608$ $18\cdot20\cdot20$</td>
<td>58</td>
<td>$\geq 4608$</td>
</tr>
<tr>
<td>59</td>
<td>$\geq 4608$</td>
<td>59</td>
<td>$\geq 4608$</td>
</tr>
<tr>
<td>60</td>
<td>$\geq 5184$ $20\cdot20\cdot20$</td>
<td>60</td>
<td>$\geq 5184$ $20\cdot20\cdot20$</td>
</tr>
<tr>
<td>61</td>
<td>$\geq 5184$</td>
<td>62</td>
<td>$\geq 5832$ $20\cdot21\cdot21$</td>
</tr>
<tr>
<td>63</td>
<td>$\geq 6561$ $21\cdot21\cdot21$</td>
<td>64</td>
<td>$\geq 6561$</td>
</tr>
<tr>
<td>65</td>
<td>$\geq 6562$ theorem 4.1</td>
<td>66</td>
<td>$\geq 8000$ $11\cdot11\cdot11\cdot33$</td>
</tr>
<tr>
<td>67</td>
<td>$\geq 8342$ theorem 2.1</td>
<td>68</td>
<td>$\geq 9632$ theorem 2.1</td>
</tr>
<tr>
<td>69</td>
<td>$\geq 11122$ theorem 2.1</td>
<td>70</td>
<td>$\geq 11122$ theorem 2.1</td>
</tr>
</tbody>
</table>
The following tables summarise the best lower bounds known for $\alpha(d)$. For $3 \leq d \leq 6$, the best lower bound is Danzer and Grünbaum’s $2d - 1$ [DG]. For $7 \leq d \leq 26$, the results of a computer program, based on the ‘probabilistic construction’ but using sets of points close to the surface of the $d$-sphere, provide the largest known acute $d$-sets. An acute $7$-set of cardinality $14$ and an acute $8$-set of cardinality $16$ are displayed. For $27 \leq d \leq 62$, the largest known acute $d$-set is cubic. Finally, for $d \geq 63$, theorem 3.1 provides the best (probabilistic) lower bound.

**Best Lower Bounds Known for $\alpha(d)$**

<table>
<thead>
<tr>
<th>$d$</th>
<th>$\alpha(d)$</th>
<th>$d$</th>
<th>$\alpha(d)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$= 1$</td>
<td>16</td>
<td>$\geq 54$ computer</td>
</tr>
<tr>
<td>1</td>
<td>$= 2$</td>
<td>17</td>
<td>$\geq 63$ computer</td>
</tr>
<tr>
<td>2</td>
<td>$= 3$</td>
<td>18</td>
<td>$\geq 71$ computer</td>
</tr>
<tr>
<td>3</td>
<td>$= 5$ [DG]</td>
<td>19</td>
<td>$\geq 76$ computer</td>
</tr>
<tr>
<td>4–6</td>
<td>$\geq 2d - 1$ [DG]</td>
<td>20</td>
<td>$\geq 90$ computer</td>
</tr>
<tr>
<td>7</td>
<td>$\geq 14$ computer</td>
<td>21</td>
<td>$\geq 103$ computer</td>
</tr>
<tr>
<td>8</td>
<td>$\geq 16$ computer</td>
<td>22</td>
<td>$\geq 118$ computer</td>
</tr>
<tr>
<td>9</td>
<td>$\geq 19$ computer</td>
<td>23</td>
<td>$\geq 121$ computer</td>
</tr>
<tr>
<td>10</td>
<td>$\geq 23$ computer</td>
<td>24</td>
<td>$\geq 144$ computer</td>
</tr>
<tr>
<td>11</td>
<td>$\geq 26$ computer</td>
<td>25</td>
<td>$\geq 155$ computer</td>
</tr>
<tr>
<td>12</td>
<td>$\geq 30$ computer</td>
<td>26</td>
<td>$\geq 184$ computer</td>
</tr>
<tr>
<td>13</td>
<td>$\geq 36$ computer</td>
<td>27–62</td>
<td>$\geq \kappa(d)$</td>
</tr>
<tr>
<td>14</td>
<td>$\geq 42$ computer</td>
<td>63</td>
<td>$\geq 6636$ theorem 3.1</td>
</tr>
<tr>
<td>15</td>
<td>$\geq 47$ computer</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**$\alpha(7) \geq 14$**

- $(62, 1, 9, 10, 17, 38, 46)$
- $(38, 54, 0, 19, 38, 14, 25)$
- $(60, 33, 42, 9, 48, 3, 12)$
- $(62, 35, 41, 44, 16, 39, 44)$
- $(62, 34, 7, 45, 48, 37, 12)$
- $(28, 33, 42, 8, 49, 39, 45)$
- $(40, 16, 22, 12, 0, 0, 25)$
- $(45, 17, 26, 67, 25, 20, 29)$
- $(38, 6, 35, 0, 32, 18, 0)$
- $(62, 0, 42, 45, 49, 3, 48)$
- $(30, 0, 9, 44, 49, 37, 48)$
- $(0, 20, 31, 27, 34, 21, 28)$
- $(48, 19, 24, 22, 33, 20, 73)$
- $(43, 17, 25, 27, 32, 64, 19)$

**$\alpha(8) \geq 16$**

- $(34, 49, 14, 51, 0, 36, 46, 0)$
- $(31, 17, 14, 51, 1, 5, 44, 31)$
- $(33, 50, 48, 20, 34, 35, 15, 0)$
- $(0, 16, 16, 52, 32, 36, 45, 0)$
- $(37, 31, 46, 52, 13, 0, 0, 22)$
- $(2, 50, 13, 52, 3, 3, 46, 0)$
- $(1, 50, 48, 51, 1, 5, 46, 31)$
- $(24, 0, 43, 2, 17, 20, 32, 16)$
- $(11, 49, 0, 11, 19, 8, 32, 19)$
- $(0, 48, 48, 52, 1, 34, 12, 2)$
- $(0, 48, 47, 51, 34, 37, 47, 32)$
- $(34, 49, 14, 51, 34, 36, 13, 34)$
- $(0, 46, 31, 0, 0, 23, 29, 29)$
- $(16, 40, 29, 23, 54, 3, 17, 16)$
- $(2, 15, 14, 50, 2, 36, 15, 33)$
- $(12, 36, 28, 30, 3, 45, 48, 45)$

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14
6 Generalising \( \kappa(d) \)

We can understand \( \kappa(d) \) to be the size of the largest possible set \( S \) of binary words such that, for any ordered triple of words \((u, v, w)\) in \( S \), there exists an index \( i \) for which \((u_i, v_i, w_i) = (0, 1, 0)\) or \((u_i, v_i, w_i) = (1, 0, 1)\). We can generalise this in the following way:

**Definition 6.1** If \( T_1, \ldots, T_m \) are ordered \( k \)-tuples from \( \{0, \ldots, r-1\}^k \) (which we will refer to as the matching \( k \)-tuples), then let us define \( \kappa[r, k, T_1, \ldots, T_m](d) \) to be the size of the largest possible set \( S \) of \( r \)-ary words of length \( d \) such that, for any ordered \( k \)-tuple of words \((w_1, \ldots, w_k)\) in \( S \), there exist \( i \) and \( j \), \( 1 \leq i \leq d, 1 \leq j \leq m \), for which \((w_{i}, \ldots, w_{k_i}) = T_j\).

Thus we have \( \kappa(d) = \kappa[2, 3, (0, 1, 0), (1, 0, 1)](d) \). If the set of matching \( k \)-tuples is closed under permutation, we will abbreviate by writing a list of matching multisets of cardinality \( k \), rather than ordered tuples. For example, instead of \( \kappa[2, 3, (0, 0, 1), (0, 1, 0), (1, 0, 0)](d) \), we write \( \kappa[2, 3, \{0, 0, 1\}](d) \).

We can find probabilistic and, in some cases, constructive lower bounds for general \( \kappa[r, k, T_1, \ldots, T_m](d) \) using the approaches we used for cubic acute \( d \)-sets. To illustrate this, in the remainder of this paper, we will consider the set of problems in which it is simply required that at some index the \( k \)-tuple of words be all different (pairwise distinct). First, we express this in a slightly different form.

Let us say that an \( r \)-ary \( d \)-colouring is some colouring of the integers \( 1, \ldots, d \) using \( r \) colours. Let us also also say that a set \( R \) of \( r \)-ary \( d \)-colourings is a \( k \)-rainbow set, for some \( k \leq r \) if for any set \( \{c_1, \ldots, c_k\} \) of \( k \) colourings in \( R \), there exists some integer \( t \), \( 1 \leq t \leq d \), for which the colours \( c_1(t), \ldots, c_k(t) \) are all different, i.e. \( c_i(t) \neq c_j(t) \) for any \( i \) and \( j \), \( 1 \leq i, j \leq k \), \( i \neq j \). For conciseness, we will denote “a \( k \)-rainbow set of \( r \)-ary \( d \)-colourings” by “a \( RSC[k, r, d] \)”.

Let us further say that a set \( \{c_1, \ldots, c_k\} \) of \( d \)-colourings is a good \( k \)-set if there exists some integer \( t \), \( 1 \leq t \leq d \), for which the colours \( c_1(t), \ldots, c_k(t) \) are all different, and a bad \( k \)-set if there exists no such \( t \).

We will denote by \( \rho_{r,k}(d) \) the size of the largest possible \( RSC[k, r, d] \), abbreviating \( \rho_{k,k}(d) \) by \( \rho_k(d) \). Now, \( \rho_k(d) = \kappa[k, k, \{0, 1, \ldots, k-1\}](d) \) and

\[
\rho_{r,k}(d) = \kappa[r, k, \{0, \ldots, k-1\}, \ldots, \{r-k, \ldots, r-1\}](d),
\]

where the matching multisets are those of cardinality \( k \) with \( k \) distinct members.

Clearly, \( \rho_{r,k}(d) \leq \rho_{r,k}(d + 1) \), \( \rho_{r,k}(d) \leq \rho_{r+1,k}(d) \) and \( \rho_{r,k}(d) \geq \rho_{r,k+1}(d) \). Also, \( \rho_{r,1}(d) \) is undefined because any set of colourings is a 1-rainbow, \( \rho_{r,1}(1) = r \) if \( k > 1 \), and \( \rho_{r,2}(d) = r^d \) because any two distinct \( r \)-ary \( d \)-colourings (or \( r \)-ary words of length \( d \)) differ somewhere.
In the next two sections we will give a number of probabilistic and constructive lower bounds for \( \rho_{r,k}(d) \), for various \( r \) and \( k \).

## 7 A probabilistic lower bound for \( \rho_{r,k}(d) \)

**Theorem 7.1**

\[
\rho_{r,k}(d) \geq (k - 1)m \quad \text{where} \quad m = \left[ \frac{k-1}{k} \frac{k!}{n-1} \left( \frac{(r - k)!r^k}{(r - k)!r^k - r^k} \right)^d \right].
\]

**Proof:** This proof is similar to that of Theorem 2.1.

Randomly pick a set \( \mathcal{R} \) of \( km \) \( r \)-ary \( d \)-colourings, choosing the colours from \( \{\chi_0, \ldots, \chi_{r-1}\} \) independently with probability \( \Pr[c(i) = \chi_j] = 1/r \), \( 1 \leq i \leq d \), \( 0 \leq j < r \) for every \( c \in \mathcal{R} \).

Now the probability that a set of \( k \) colourings from \( \mathcal{R} \) is a bad \( k \)-set is

\[
(1 - p)^d \quad \text{where} \quad p = \frac{r!/(r - k)!}{r^k}.
\]

Hence, the expected number of bad \( k \)-sets in a set of \( km \) \( d \)-colourings is \( (\frac{km}{k}) (1 - p)^d \). Thus there is some set \( \mathcal{R} \) of \( km \) \( d \)-colourings with no more than \( (\frac{km}{k}) (1 - p)^d \) bad \( k \)-sets, where

\[
\left( \frac{km}{k} \right) (1 - p)^d < \left( \frac{km}{k} \right)^k \frac{k!}{(1 - p)^d} = m \frac{k^k}{k!} m^{k-1} (1 - p)^d \leq m
\]

by the choice of \( m \).

If we remove one colouring of each bad \( k \)-set from \( \mathcal{R} \), the remaining set is a \( \mathcal{RSC}[k, r, d] \) of cardinality at least \( km - m = (k - 1)m \).

The following results follow directly:

\[
\rho_3(d) \geq 2 \left[ \frac{\sqrt{2}}{3} \left( \frac{3}{\sqrt{7}} \right)^d \right] \approx 0.943 \times 1.134^d.
\]

\[
\rho_{4,3}(d) \geq 2 \left[ \frac{\sqrt{2}}{3} \left( \frac{4}{\sqrt{10}} \right)^d \right] \approx 0.943 \times 1.265^d.
\]

\[
\rho_{4}(d) \geq 3 \left[ \frac{\sqrt{3}}{32} \left( \frac{32}{\sqrt{29}} \right)^d \right] \approx 1.363 \times 1.033^d.
\]
8 Constructive lower bounds for $\rho_{r,k}(d)$

In the following proofs, for clarity of exposition, we will represent $r$-ary $d$-colourings as $r$-ary words of length $d$, e.g. $R_{3,3,3} = \{000, 011, 102, 121, 212, 220\}$ represents a 3-rainbow set of ternary 3-colourings (using the colours $\chi_0$, $\chi_1$ and $\chi_2$). Concatenation of words (colourings) $c$ and $c'$ will be written $c.c'$.

We begin with a construction that enables us to extend a $\mathcal{RSC}[k, r, d]$ of cardinality $n$ to one of cardinality $n + 1$ or greater.

**Theorem 8.1** If for some $r \geq k \geq 3$, and some $d$, we have a $\mathcal{RSC}[k, r, d]$ of cardinality $n$, and for some $r'$, $k - 2 \leq r' \leq r - 2$, and $d'$, we have a $\mathcal{RSC}[k - 2, r', d']$ of cardinality at least $n - 1$, then we can construct a $\mathcal{RSC}[k, r, d + d']$ of cardinality $N = n - 1 + r - r'$.

**Proof:** Let $\mathcal{R} = \{c_0, c_1, \ldots, c_{n-1}\}$ be a $\mathcal{RSC}[k, r, d]$ of cardinality $n$ (using colours $\chi_0, \ldots, \chi_{r-1}$) and $\mathcal{R}' = \{c'_0, c'_1, \ldots, c'_{n'-1}\}$ be a $\mathcal{RSC}[k - 2, r', d']$ of cardinality $n' \geq n - 1$ (using colours $\chi_0, \ldots, \chi_{r'-1}$).

Now let $\mathcal{Q} = \{q_0, q_1, \ldots, q_{N-1}\}$ be a set of $r$-ary $(d + d')$-colourings where $q_i = c_i.c'_i$ for $0 \leq i \leq n - 2$, and $q_{n-1+j} = c_{n-1}.(r' + j)d'$ for $0 \leq j < r - r'$, each element of $\mathcal{Q}$ being made by concatenating two component colourings, the first from $\mathcal{R}$ and the second being either from $\mathcal{R}'$ or a monochrome colouring.

If $\{q_{i_1}, \ldots, q_{i_k}\}$ is a set of colourings in $\mathcal{Q}$ with no more than one of the $i_m$ greater than $n - 2$, then it is a good $k$-set because of the first components, since $\mathcal{R}$ is a $k$-rainbow set.

On the other hand, if $\{q_{i_1}, \ldots, q_{i_k}\}$ is a set of colourings in $\mathcal{Q}$ with no more than $k - 2$ of the $i_m$ less than $n - 1$, then it too is a good $k$-set because of the second components, since $\mathcal{R}'$ is a $(k - 2)$-rainbow set using colours $\chi_0, \ldots, \chi_{r'-1}$ and the second components of the colourings with indices greater than $n - 2$ are each monochrome of a different colour, drawn from $\chi_{r'}, \ldots, \chi_{r-1}$.

Thus $\mathcal{Q}$ is a $\mathcal{RSC}[k, r, d + d']$ of cardinality $N$. \hfill $\Box$

**Corollary 8.1.1** $\rho_{r,3}(d + 1) \geq \rho_{r,3}(d) + r - 2$.

**Proof:** This follows from the theorem due to the fact that there is a 1-rainbow set of 1-ary 1-colourings of any cardinality. \hfill $\Box$

**Corollary 8.1.2** $\rho_{r,4}(d + \lceil \log_2(\rho_{r,4}(d) - 1) \rceil) \geq \rho_{r,4}(d) + r - 3$.

**Proof:** Since $\rho_{r,2}(d) = r^d$, we have $\rho_{2,2}(d') \geq \rho_{r,4}(d) - 1$ iff $d' \geq \log_2(\rho_{r,4}(d) - 1)$. \hfill $\Box$
Theorem 8.2 If, for each \( s \), \( 1 \leq s \leq m \), we have a \( RSC[3, r, d_s] \) of cardinality \( n_s \), where \( n_1 \) is the least of the \( n_s \), and if, for some \( d_Z \), we have a \( RSC[3, r, d_Z] \) of cardinality \( n_Z \), where
\[
n_Z \geq \prod_{s=2}^{m} \left( 1 + 2 \left\lfloor \frac{n_s}{2} \right\rfloor \right),
\]
then a \( RSC[3, r, D] \) of cardinality \( N \) can be constructed, where
\[
D = \sum_{s=1}^{m} d_s + 2d_Z \quad \text{and} \quad N = \prod_{s=1}^{m} n_s.
\]

This result for 3-rainbow sets corresponds to theorem 4.3 for cubic acute \( d \)-sets. Before we can prove it, we need some further preliminary results.

Definition 8.3 If \( n_1 \leq n_2 \leq \ldots \leq n_m \) and \( 0 \leq k_r < n_r \), for each \( r \), \( 1 \leq r \leq m \), then let us denote by \( \langle\langle k_1k_2\ldots k_m \rangle\rangle_{n_1n_2\ldots n_m}^+ \), the number
\[
\langle\langle k_1k_2\ldots k_m \rangle\rangle_{n_1n_2\ldots n_m}^+ = \sum_{r=2}^{m} \left( (k_{r-1} + k_r \mod n_r) \prod_{s=r+1}^{m} n_s \right).
\]

The definition of \( \langle\langle k_1k_2\ldots k_m \rangle\rangle_{n_1n_2\ldots n_m}^+ \) is the same as that for \( \langle\langle k_1k_2\ldots k_m \rangle\rangle_{n_1n_2\ldots n_m} \) (see 4.4), but with addition replacing subtraction. By construction, we have
\[
\langle\langle k_1k_2\ldots k_m \rangle\rangle_{n_1n_2\ldots n_m}^+ < \prod_{r=2}^{m} n_r,
\]
and, if \( 2 \leq t \leq m \) and \( j_{t-1} + j_t \neq k_{t-1} + k_t \mod n_t \), then
\[
\langle\langle j_1j_2\ldots j_m \rangle\rangle_{n_1n_2\ldots n_m}^+ \neq \langle\langle k_1k_2\ldots k_m \rangle\rangle_{n_1n_2\ldots n_m}^+.
\]

Lemma 8.4 If \( n_1 \leq n_2 \leq \ldots \leq n_m \), with all the \( n_r \) odd except perhaps \( n_1 \), and \( 0 \leq j_r, k_r, l_r < n_r \), for each \( r \), \( 1 \leq r \leq m \), and the sequences of \( j_r \), \( k_r \) and \( l_r \) are neither pairwise identical nor anywhere pairwise distinct, i.e. there is some \( u, v \) and \( w \) such that \( j_u \neq k_u, k_v \neq l_v \) and \( l_w \neq j_w \) but no \( t \) such that \( j_t \neq k_t, k_t \neq l_t \) and \( l_t \neq j_t \), then either
\[
\langle\langle j_1\ldots j_m \rangle\rangle_{n_1\ldots n_m}, \langle\langle k_1\ldots k_m \rangle\rangle_{n_1\ldots n_m}, \langle\langle l_1\ldots l_m \rangle\rangle_{n_1\ldots n_m} \quad \text{are pairwise distinct}
\]
or
\[
\langle\langle j_1\ldots j_m \rangle\rangle_{n_1\ldots n_m}^+, \langle\langle k_1\ldots k_m \rangle\rangle_{n_1\ldots n_m}^+, \langle\langle l_1\ldots l_m \rangle\rangle_{n_1\ldots n_m}^+ \quad \text{are pairwise distinct}.
\]
Proof: Without loss of generality, we can assume that we have $j_1 = k_1$, that $t > 1$ is the least integer for which $j_t \neq k_t$, and that $k_t = l_t$. We will consider two cases:

Case 1: $k_{t-1} \neq l_{t-1}$

Since $j_{t-1} = k_{t-1} \neq l_{t-1}$ and $j_t \neq k_t = l_t$, we have $j_{t-1} - j_t \neq k_{t-1} - k_t$ and $k_{t-1} - k_t \neq l_{t-1} - l_t$, and so $\langle j_1 \ldots j_m \rangle \neq \langle k_1 \ldots k_m \rangle$ and $\langle k_1 \ldots k_m \rangle \neq \langle l_1 \ldots l_m \rangle$. Similarly, $j_{t-1} + j_t \neq k_{t-1} + k_t$ and $k_{t-1} + k_t \neq l_{t-1} + l_t$, and so $\langle j_1 \ldots j_m \rangle^+ \neq \langle k_1 \ldots k_m \rangle^+$ and $\langle k_1 \ldots k_m \rangle^+ \neq \langle l_1 \ldots l_m \rangle^+$.

If $j_{t-1} - j_t \neq l_{t-1} - l_t$, then $\langle j_1 \ldots j_m \rangle \neq \langle l_1 \ldots l_m \rangle$. If $j_{t-1} - j_t = l_{t-1} - l_t$ then $(j_{t-1} + j_t) - (l_{t-1} + l_t) = (j_{t-1} - j_t + 2j_t) - (l_{t-1} - l_t + 2l_t) = 2(j_t - l_t) \neq 0 \pmod{n_t}$ because $j_t \neq l_t$ and $n_t$ is odd, so $j_{t-1} + j_t \neq l_{t-1} + l_t$ and $\langle j_1 \ldots j_m \rangle^+ \neq \langle l_1 \ldots l_m \rangle^+$.

Case 2: $k_{t-1} = l_{t-1}$

Since $j_{t-1} = k_{t-1} = l_{t-1}$ and $j_t \neq k_t = l_t$, we have $j_{t-1} - j_t \neq k_{t-1} - k_t$ and $j_{t-1} - j_t \neq l_{t-1} - l_t$, and so $\langle j_1 \ldots j_m \rangle \neq \langle k_1 \ldots k_m \rangle$ and $\langle j_1 \ldots j_m \rangle \neq \langle l_1 \ldots l_m \rangle$.

If $k_1 = l_1$, let $u$ be the least integer such that $k_u \neq l_u$. Since $k_{u-1} = l_{u-1}$, we have $k_{u-1} - k_u \neq l_{u-1} - l_u$. If $k_1 \neq l_1$, let $u$ be the least integer such that $k_u = l_u$. Since $k_{u-1} \neq l_{u-1}$, we still have $k_{u-1} - k_u \neq l_{u-1} - l_u$. Thus, $\langle k_1 \ldots k_m \rangle \neq \langle l_1 \ldots l_m \rangle$.

Proof of Theorem 8.2

Let $n_1 \leq n_2 \leq \ldots \leq n_m$, and, for each $s$, $1 \leq s \leq m$, let $R_s = \{c_0^s, c_1^s, \ldots, c_{n_s-1}^s\}$ be a $\mathcal{RSC}[3, r, d_s]$ of cardinality $n_s$, and let $n'_s = 1 + 2 \lceil n_s/2 \rceil$ be the least odd integer not less than $n_s$. Let $Z = \{z_0, z_1, \ldots, z_{n_Z-1}\}$ be a $\mathcal{RSC}[3, r, d_Z]$ of cardinality $n_Z$, where

$$n_Z \geq \prod_{s=2}^m n'_s,$$

and let

$$D = \sum_{s=1}^m d_s + 2d_Z \quad \text{and} \quad N = \prod_{s=1}^m n_s.$$

Now let

$$Q = \{c_{k_1}^1, c_{k_2}^2, \ldots, c_{k_m}^m, z_{k_Z}, z_{k_Z}^+ : 0 \leq k_s < n_s, 1 \leq s \leq m\},$$

where $k_Z = \langle k_1k_2 \ldots k_m \rangle_{n'_1n'_2 \ldots n'_m}$ and $k_Z^+ = \langle k_1k_2 \ldots k_m \rangle_{n'_1n'_2 \ldots n'_m}^+$ be a set of $D$-colourings of cardinality $N$, each element of $Q$ being made by concatenating one colouring from each of the $R_s$ together with two colourings from $Z$. (Below, we will denote this construction by $d_1 \oplus \cdots \oplus d_m \oplus d_Z \oplus d_Z$.)

Let $c_{i_1}^1c_{i_2}^2 \ldots c_{i_m}^m, z_{i_Z}, z_{i_Z}^+$, $c_{j_1}^1c_{j_2}^2 \ldots c_{j_m}^m, z_{j_Z}, z_{j_Z}^+$ and $c_{k_1}^1c_{k_2}^2 \ldots c_{k_m}^m, z_{k_Z}, z_{k_Z}^+$ be any three distinct colourings in $Q$. If, for some $s$, $i_s \neq j_s$ and $j_s \neq k_s$ and $k_s \neq i_s$, then these three colourings comprise a good 3-set because $R_s$ is a 3-rainbow set.
If, however, there is no $s$ such that $i_s$, $j_s$ and $k_s$ are all different, then the condition of lemma 8.4 holds, and so either $i_Z$, $j_Z$ and $k_Z$ are all different, or $i_Z^+$, $j_Z^+$ and $k_Z^+$ are all different, and the three colourings comprise a good 3-set because $Z$ is a 3-rainbow set.

Thus, any three colourings in $Q$ comprise a good 3-set, so $Q$ is a $\mathcal{RSC}[3, r, D]$ of cardinality $N$.

**Corollary 8.4.1** If $\rho_{r,3}(d)$ is odd, then $\rho_{r,3}(4d) \geq \rho_{r,3}(d)^2$.

**Proof:** By theorem 8.2 using the construction $d \oplus d \oplus d \oplus d$.

**Corollary 8.4.2** $\rho_{r,3}(4d + 2) \geq \rho_{r,3}(d)^2$.

**Proof:** By 8.1.1, if $n = \rho_{r,3}(d)$, we can construct a $\mathcal{RSC}[3, r, d + 1]$ of cardinality $n + 1 \geq 1 + 2 \lfloor n/2 \rfloor$. By theorem 8.2, we can then construct a $\mathcal{RSC}[3, r, 4d + 2]$ of cardinality $n^2$ using the construction $d \oplus d \oplus (d + 1) \oplus (d + 1)$.

**Corollary 8.4.3** $\rho_3(4^d) \geq 3^{2^d}$.

**Proof:** By repeated application of 8.4.1 starting with $\rho_{3,3}(1) = 3$.

Our final construction enables us to combine $k$-rainbow sets of $r$-ary $d$-colourings for arbitrary $k$.

**Theorem 8.5** If we have a $\mathcal{RSC}[k, r, d_1]$ of cardinality $n_1$, a $\mathcal{RSC}[k, r, d_2]$ of cardinality $n_2 \geq n_1$, and a $\mathcal{RSC}[k, r, d_2]$ of cardinality $n_Z \geq n_2$, with $n_Z$ coprime to each integer in the range $[2, \ldots, h]$ where $h = \binom{h}{2} - 1$, then a $\mathcal{RSC}[k, r, D]$ of cardinality $N$ can be constructed, where $D = d_1 + d_2 + hd_Z$ and $N = n_1n_2$.

As before, we first need a preliminary result:

**Lemma 8.6** Given distinct pairs of integers $(a, b)$ and $(c, d)$ with $0 \leq a, b, c, d < n$ for some $n$, and given a positive integer $h$ such that $n$ is coprime to each integer in the range $[2, \ldots, h]$, then if we let $b_{-1} = a$ and $d_{-1} = c$, and $b_r = b + ra \pmod n$ and $d_r = d + rc \pmod n$ for $0 \leq r \leq h$, then if $b_i = d_i$ for some $i$, $-1 \leq i \leq h$, we have $b_j \neq d_j$ for all $j \neq i$. 

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Proof: We consider two cases:

Case 1: $i = -1$

Since $a = c$, $(b + ja) - (d + jc) = b - d \neq 0 \mod n$ since $(a, b)$ and $(c, d)$ are distinct, and $b$ and $d$ both less than $n$.

Case 2: $i \neq -1$

By the reversing the argument in case 1, $a \neq c$, i.e. $b_{-1} \neq d_{-1}$. For $j \geq 0$, since $b + ia = d + ic$, we have $(b + ja) - (d + jc) = (j - i)a - (j - i)c = (j - i)(a - c) \neq 0 \mod n$ since $a \neq c$ and $|j - i| \leq h$ so $j - i$ is coprime to $n$.

Proof of Theorem 8.5

Let $R_1 = \{c_0^1, \ldots, c_{n_1-1}^1\}$, $R_2 = \{c_0^2, \ldots, c_{n_2-1}^2\}$ and $S = \{z_0, \ldots, z_{nZ-1}\}$ be $k$-rainbow sets of $r$-ary $d_1$-, $d_2$- and $d_Z$-colourings of cardinality $n_1$, $n_2$ and $n_Z$, respectively.

Now let

$$Q = \{c_i^1.c_j^2.z_{j+i}.z_{j+2i} \ldots z_{j+hi} : 0 \leq i < n_1, 0 \leq j < n_2\},$$

where $h = \binom{k}{2} - 1$ and the subscript arithmetic is modulo $n_Z$, be a set of $D$-colourings of cardinality $N$, each element of $Q$ being made by concatenating $h+2$ component colourings: one from $R_1$, one from $R_2$, and $h$ from $S$.

Let

$$S = \{c_i^1.c_j^2.z_{j+i+1} \ldots z_{j+i1}, c_i^1.c_j^2.z_{j+i+2} \ldots z_{j+i2}, \ldots, c_i^1.c_j^2.z_{j+i+k} \ldots z_{j+i+k}\}$$

be any set of $k$ distinct colourings in $Q$, and let $b_{s-1} = i_s$ and $b_{s,t} = j_s + ti_s \mod n_Z$, for each $s$ and $t$, $1 \leq s \leq k$, $0 \leq i \leq h$, so the $s^{th}$ colouring in $S$ is $c_{b_{s-1}}^1.c_{b_{s,0}}^2.z_{b_{s,1}} \ldots z_{b_{s,h}}$.

Now, for any $s$, $s'$ and $t$, $1 \leq s, s' \leq k$, $-1 \leq t \leq h$, if $b_{s,t} = b_{s',t}$, then by lemma 8.6 we know that for all $u \neq t$, $b_{s,u} \neq b_{s',u}$. So for each pair $\{s, s'\}$, $b_{s,t} = b_{s',t}$ for no more than one value of $t$. Now there are $h + 2$ possible values of $t$, but only $\binom{k}{2} = h + 1$ different pairs $\{s, s'\}$, so there is some $t$ for which $b_{s,t} \neq b_{s',t}$ for all pairs $\{s, s'\}$ and the $(t + 2)^{th}$ component colourings of the elements in $S$ are all different. Since $R_1$, $R_2$ and $S$ are all $k$-rainbow sets, we know that $S$ is a good $k$-set.

Thus, any $k$ colourings from $Q$ comprise a good $k$-set, so $Q$ is a $\text{RSC}[k, r, D]$ of cardinality $N$.

Corollary 8.6.1 $\rho_4(6.7^d) \geq 7^{2^d}$.

Proof: The following 4-rainbow set of 4-ary 6-colourings of cardinality 8 — a version of $R_{4,4,6}$ (see below) displayed with different symbols for each colour — shows that $\rho_4(6) \geq 7$. 

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The result follows by repeated application of theorem 8.5, noting that 7 is coprime to 2, 3, 4 and 5 = \( \binom{4}{2} - 1 \). \[\square\]

9 **Lower bounds for** \( \rho_{r,k}(d) \) **for small** \( r, k \) **and** \( d \)**

We conclude with tables of the best lower bounds known for \( \rho_3(d) \), \( \rho_{4,3}(d) \) and \( \rho_4(d) \) for small \( d \). For very small \( d \), exhaustive computer searches have determined the values of \( \rho_{r,k}(d) \). For other small values of \( d \), the constructions used in theorems 8.2 and 8.5 provide the largest known rainbow sets. In the tables, these constructions are denoted \( d_1 \oplus d_2 \oplus d_Z \oplus d_Z \), etc., with superscript minus signs \( (d^-) \) to denote the removal of a single colouring from a largest rainbow set of \( d \)-colourings (to satisfy the requirement that the cardinality be odd). For \( \rho_3(d) \), the probabilistic lower bound of theorem 7.1 is better than the constructions for \( d \geq 71 \); for \( \rho_{4,3}(d) \), this is the case for \( d \geq 26 \).
Some \(k\)-rainbow sets of \(r\)-ary \(d\)-colourings, for small \(k\), \(r\) and \(d\)

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Best Lower Bounds Known for \(\rho_3(d)\) and \(\rho_{4,3}(d)\)

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<th>(\rho_{4,3}(d))</th>
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<td>(1e3e3e3) or (2e2e3e3)</td>
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Best Lower Bounds Known for $\rho_d(d)$

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Acknowledgements

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References


