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Sets of Points Determining Only Acute Angles and Some Related Colouring Problems

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Abstract

We present both probabilistic and constructive lower bounds on the maximum size of a set of points $S \subseteq \mathbb{R}^d$ such that every angle determined by three points in $S$ is acute, considering especially the case $S \subseteq \{0,1\}^d$. These results improve upon a probabilistic lower bound of Erdős and Füredi. We also present lower bounds for some generalisations of the acute angles problem, considering especially some problems concerning colourings of sets of integers.

1 Introduction

Let us say that a set of points $S \subseteq \mathbb{R}^d$ is an acute $d$-set if every angle determined by a triple of $S$ is acute ($< \frac{\pi}{2}$). Let us also say that $S$ is a cubic acute $d$-set if $S$ is an acute $d$-set and is also a subset of the unit $d$-cube (i.e. $S \subseteq \{0,1\}^d$).

Let us further say that a triple $u, v, w \in \mathbb{R}^d$ is an acute triple, a right triple, or an obtuse triple, if the angle determined by the triple with apex $v$ is less than $\frac{\pi}{2}$, equal to $\frac{\pi}{2}$, or greater than $\frac{\pi}{2}$, respectively. Note that we consider the triples $u, v, w$ and $w, v, u$ to be the same.

We will denote by $\alpha(d)$ the size of a largest possible acute $d$-set. Similarly, we will denote by $\kappa(d)$ the size of a largest possible cubic acute $d$-set. Clearly $\kappa(d) \leq \alpha(d)$, $\kappa(d) \leq \kappa(d+1)$ and $\alpha(d) \leq \alpha(d+1)$ for all $d$. 
In [EF], Paul Erdős and Zoltán Füredi gave a probabilistic proof that $\kappa(d) \geq \left\lfloor \frac{1}{2} \left( \frac{2}{\sqrt{3}} \right)^d \right\rfloor$ (see also [AZ2]). This disproved an earlier conjecture of Ludwig Danzer and Branko Grünbaum [DG] that $\alpha(d) = 2d - 1$.

In the following two sections we give improved probabilistic lower bounds for $\kappa(d)$ and $\alpha(d)$. In section 4 we present a construction that gives further improved lower bounds for $\kappa(d)$ for small $d$. In section 5, we tabulate the best lower bounds known for $\kappa(d)$ and $\alpha(d)$ for small $d$. Finally, in sections 6–9, we give probabilistic and constructive lower bounds for some generalisations of $\kappa(d)$, considering especially some problems concerning colourings of sets of integers.

## 2 A probabilistic lower bound for $\kappa(d)$

**Theorem 2.1**

$$\kappa(d) \geq 2 \left\lfloor \frac{\sqrt{6}}{9} \left( \frac{2}{\sqrt{3}} \right)^d \right\rfloor \approx 0.544 \times 1.155^d.$$ 

For large $d$, this improves upon the result of Erdős and Füredi by a factor of $\frac{4\sqrt{2}}{9} \approx 1.089$. This is achieved by a slight improvement in the choice of parameters. This proof can also be found in [AZ3].

**Proof:** Let $m = \left\lfloor \frac{\sqrt{6}}{9} \left( \frac{2}{\sqrt{3}} \right)^d \right\rfloor$ and randomly pick a set $S$ of $3m$ point vectors from the vertices of the $d$-dimensional unit cube $\{0, 1\}^d$, choosing the coordinates independently with probability $\Pr[v_i = 0] = \Pr[v_i = 1] = \frac{1}{2}$, $1 \leq i \leq d$, for every $v = (v_1, v_2, \ldots, v_d) \in S$.

Now every angle determined by a triple of points from $S$ is non-obtuse ($\leq \frac{\pi}{2}$), and a triple of vectors $u, v, w$ from $S$ is a right triple iff the scalar product $\langle u - v, w - v \rangle$ vanishes, i.e. iff either $u_i - v_i = 0$ or $w_i - v_i = 0$ for each $i$, $1 \leq i \leq d$.

Thus $u, v, w$ is a right triple iff $u_i, v_i, w_i$ is neither 0, 1, 0 nor 1, 0, 1 for any $i$, $1 \leq i \leq d$. Since $u_i, v_i, w_i$ can take eight different values, this occurs independently with probability $\frac{3}{4}$ for each $i$, so the probability that a triple of $S$ is a right triple is $\left( \frac{3}{4} \right)^d$.

Hence, the expected number of right triples in a set of $3m$ vectors is $3 \binom{3m}{3} \left( \frac{3}{4} \right)^d$. Thus there is some set $S$ of $3m$ vectors with no more than $3 \binom{3m}{3} \left( \frac{3}{4} \right)^d$ right triples, where

$$3 \binom{3m}{3} \left( \frac{3}{4} \right)^d \leq 3 \binom{3m}{3} \left( \frac{3}{4} \right)^d = m \left( \frac{9m}{\sqrt{6}} \right)^2 \left( \frac{3}{4} \right)^d \leq m$$

due to the choice of $m$. 

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If we remove one point of each right triple from $S$, the remaining set is a cubic acute $d$-set of cardinality at least $3m - m = 2m$. □

3 A probabilistic lower bound for $\alpha(d)$

We can improve the lower bound in theorem 2.1 for non-cubic acute $d$-sets by a factor of $\sqrt{2}$ by slightly perturbing the points chosen away from the vertices of the unit cube. The intuition behind this is that a small random symmetrical perturbation of the points in a right triple is more likely than not to produce an acute triple, as the following diagram suggests.

Theorem 3.1
\[
\alpha(d) \geq 2 \left\lfloor \frac{1}{3} \left( \frac{2}{\sqrt{3}} \right)^d \right\rfloor \approx 0.770 \times 1.155^d.
\]

Before we can prove this theorem, we need some results concerning continuous random variables.

Definition 3.2 If $F(x) = \Pr[X \leq x]$ is the cumulative distribution function of a continuous random variable $X$, let $\overline{F}(x) = \Pr[X \geq x] = 1 - F(x)$.

Definition 3.3 Let us say that a continuous random variable $X$ has positive bias if, for all $t$, $\Pr[X \geq t] \geq \Pr[X \leq -t]$, i.e. $\overline{F}(t) \geq F(-t)$.

Property 3.3.1 If a continuous random variable $X$ has positive bias, it follows that $\Pr[X > 0] \geq \frac{1}{2}$.

Property 3.3.2 To show that a continuous random variable $X$ has positive bias, it suffices to demonstrate that the condition $\overline{F}(t) \geq F(-t)$ holds for all positive $t$. 
Lemma 3.4 If $X$ and $Y$ are independent continuous random variables with positive bias, then $X + Y$ also has positive bias.

Proof: Let $f$, $g$ and $h$ be the probability density functions, and $F$, $G$ and $H$ the cumulative distribution functions, for $X$, $Y$ and $X + Y$ respectively. Then,

$$
H(t) - H(-t) = \int \int_{x+y \geq t} f(x)g(y) \, dy \, dx - \int \int_{x+y \leq -t} f(x)g(y) \, dy \, dx
$$

$$
= \int \int_{x+y \geq t} f(x)g(y) \, dy \, dx - \int \int_{y-x \geq t} f(x)g(y) \, dy \, dx
$$

$$
+ \int \int_{y-x \geq t} f(x)g(y) \, dy \, dx - \int \int_{x+y \leq -t} f(x)g(y) \, dy \, dx
$$

$$
= \int_{-\infty}^{\infty} g(y) \left[ F(t-y) - F(y-t) \right] \, dy
$$

$$
+ \int_{-\infty}^{\infty} f(x) \left[ G(x+t) - G(-x-t) \right] \, dx
$$

which is non-negative because $f(t)$, $g(t)$, $F(t) - F(-t)$ and $G(t) - G(-t)$ are all non-negative for all $t$. \qed

Definition 3.5 Let us say that a continuous random variable $X$ is $\epsilon$-uniformly distributed for some $\epsilon > 0$ if $X$ is uniformly distributed between $-\epsilon$ and $\epsilon$.

Let us denote by $j$, the probability density function of an $\epsilon$-uniformly distributed random variable:

$$
j(x) = \begin{cases} 
\frac{1}{2\epsilon} & -\epsilon \leq x \leq \epsilon \\
0 & \text{otherwise}
\end{cases}
$$

and by $J$, its cumulative distribution function:

$$
J(x) = \begin{cases} 
0 & x < -\epsilon \\
\frac{1}{2} + \frac{x}{2\epsilon} & -\epsilon \leq x \leq \epsilon \\
1 & x > \epsilon
\end{cases}
$$

Property 3.5.1 If $X$ is an $\epsilon$-uniformly distributed random variable, then so is $-X$. 

Lemma 3.6 If $X$, $Y$ and $Z$ are independent $\epsilon$-uniformly distributed random variables for some $\epsilon < \frac{1}{2}$, then $U = (Y - X)(1 + Z - X)$ has positive bias.

Proof: Let $G$ be the cumulative distribution function of $U$. By 3.3.2, it suffices to show that $G(u) - G(-u) \geq 0$ for all positive $u$.

Let $u$ be positive. Because $1 + Z - X$ is always positive, $U \geq u$ iff $Y > X$ and $Z \geq -1 + X + \frac{u}{Y - X}$. Similarly, $U \leq -u$ iff $X > Y$ and $Z \geq -1 + X + \frac{u}{X - Y}$. So,

$$G(u) - G(-u) = \int \int_{y > x} j(x)j(y)J(1 + x + \frac{u}{y - x}) \, dy \, dx - \int \int_{x > y} j(x)j(y)J(1 + x + \frac{u}{x - y}) \, dy \, dx$$

$$= \int \int_{y > x} j(x)j(y) \left[ J(1 - x + \frac{u}{y - x}) - J(1 - y + \frac{u}{y - x}) \right] \, dy \, dx$$

(because $J(x) = J(-x)$, and by variable renaming)

which is non-negative because $j$ is non-negative and $J$ is non-decreasing (so the expression in square brackets is non-negative over the domain of integration).

Corollary 3.6.1 If $X$, $Y$ and $Z$ are independent $\epsilon$-uniformly distributed random variables for some $\epsilon < \frac{1}{2}$, then $(Y - X)(Z - X - 1)$ has positive bias.

Proof: $(Y - X)(Z - X - 1) = ((-Y) - (-X))(1 + (-Z) - (-X))$. The result follows from 3.5.1 and lemma 3.6.

Lemma 3.7 If $X$, $Y$ and $Z$ are independent $\epsilon$-uniformly distributed random variables, then $V = (Y - X)(Z - X)$ has positive bias.

Proof: Let $H$ be the cumulative distribution function of $V$. By 3.3.2, it suffices to show that $H(v) - H(-v) \geq 0$ for all positive $v$. 

Let \( v \) be positive, \( V \geq v \) iff \( Y > X \) and \( Z \geq X + \frac{v}{\sqrt[3]{X}} \) or \( Y < X \) and \( Z \leq X + \frac{v}{\sqrt[3]{X}} \). Similarly, \( V \leq -v \) iff \( Y > X \) and \( Z \leq X - \frac{v}{\sqrt[3]{X}} \) or \( Y < X \) and \( Z \geq X - \frac{v}{\sqrt[3]{X}} \). So,

\[
\overline{H}(v) - H(-v) = \int_{Y>Y} j(x) j(y) \overline{J}(x + \frac{v}{y-x}) \, dy \, dx
+ \int_{Y<X} j(x) j(y) \overline{J}(x + \frac{v}{y-x}) \, dy \, dx
- \int_{Y>Y} j(x) j(y) \overline{J}(x - \frac{v}{y-x}) \, dy \, dx
- \int_{Y<X} j(x) j(y) \overline{J}(x - \frac{v}{y-x}) \, dy \, dx
\]

\[
= \int_{Y>Y} j(x) j(y) \left[ J(-x - \frac{v}{y-x}) - J(-y - \frac{v}{y-x}) \right] \, dy \, dx
+ \int_{Y<X} j(x) j(y) \left[ J(x + \frac{v}{y-x}) - J(y + \frac{v}{y-x}) \right] \, dy \, dx
\]

(because \( \overline{J}(x) = J(-x) \), and by variable renaming)

which is non-negative because \( j \) is non-negative and \( J \) is non-decreasing (so the expressions in square brackets are non-negative over the domains of integration).

We are now in a position to prove the theorem.

**Proof of theorem 3.1**

Let \( m = \left\lfloor \frac{1}{3} \left( \frac{2}{\sqrt{3}} \right)^{d+1} \right\rfloor \), and randomly pick a set \( S \) of \( 3m \) point vectors, \( v_1, v_2, \ldots, v_{3m} \), from the vertices of the \( d \)-dimensional unit cube \( \{0,1\}^d \), choosing the coordinates independently with probability \( \Pr[v_{k_i} = 0] = \Pr[v_{k_i} = 1] = \frac{1}{2} \) for every \( v_k = (v_{k_1}, v_{k_2}, \ldots, v_{k_d}) \), \( 1 \leq k \leq 3m \), \( 1 \leq i \leq d \).

Now for some \( \epsilon, 0 < \epsilon < \frac{1}{2(d+1)} \), randomly pick \( 3m \) vectors, \( \delta_1, \delta_2, \ldots, \delta_{3m} \), from the \( d \)-dimensional cube \( [-\epsilon, \epsilon]^d \) of side \( 2\epsilon \) centred on the origin, choosing the coordinates \( \delta_{k_i} \), \( 1 \leq k \leq 3m \), \( 1 \leq i \leq d \), independently so that they are \( \epsilon \)-uniformly distributed, and let \( S' = \{v'_1, v'_2, \ldots, v'_{3m}\} \) where \( v'_k = v_k + \delta_k \) for each \( k, 1 \leq k \leq 3m \).

**Case 1: Acute triples in \( S \)**

Because \( \epsilon < \frac{1}{2(d+1)} \), if \( v_j, v_k, v_l \) is an acute triple in \( S \), the scalar product \( \langle v'_j - v'_k, v'_l - v'_k \rangle > \frac{1}{(d+1)^2} \), so \( v'_j, v'_k, v'_l \) is also an acute triple in \( S' \).

**Case 2: Right triples in \( S \)**

If, \( v_j, v_k, v_l \) is a right triple in \( S \) then the scalar product \( \langle v_j - v_k, v_l - v_k \rangle \) vanishes, i.e. either \( v_j - v_k = 0 \) or \( v_l - v_k = 0 \) for each \( i, 1 \leq i \leq d \). There are six possibilities for each triple of coordinates:
\[
\begin{array}{|c|c|}
\hline
\mathbf{v}_{j_i}, \mathbf{v}_{k_i}, \mathbf{v}_{l_i} & (\mathbf{v}_{j_i}' - \mathbf{v}_{k_i}')(\mathbf{v}_{l_i}' - \mathbf{v}_{k_i}') \\
\hline
0, 0, 0 & (\delta_{j_i} - \delta_{k_i})(\delta_{l_i} - \delta_{k_i}) \\
1, 1, 1 & (\delta_{j_i} - \delta_{k_i})(\delta_{l_i} - \delta_{k_i}) \\
0, 0, 1 & (\delta_{j_i} - \delta_{k_i})(1 + \delta_{l_i} - \delta_{k_i}) \\
1, 0, 0 & (\delta_{l_i} - \delta_{k_i})(1 + \delta_{j_i} - \delta_{k_i}) \\
0, 1, 1 & (\delta_{l_i} - \delta_{k_i})(\delta_{j_i} - \delta_{k_i} - 1) \\
1, 1, 0 & (\delta_{j_i} - \delta_{k_i})(\delta_{l_i} - \delta_{k_i} - 1) \\
\hline
\end{array}
\]

Now, the values of the \(\delta_{k_i}\) are independent and \(\epsilon\)-uniformly distributed, so by lemmas 3.7 and 3.6 and corollary 3.6.1, the distribution of the \((\mathbf{v}_{j_i}' - \mathbf{v}_{k_i}')(\mathbf{v}_{l_i}' - \mathbf{v}_{k_i}')\) has positive bias, and by repeated application of lemma 3.4, the distribution of the scalar product \(\langle \mathbf{v}_j' - \mathbf{v}_k', \mathbf{v}_l' - \mathbf{v}_k' \rangle = \sum_{i=1}^d (\mathbf{v}_{j_i}' - \mathbf{v}_{k_i}')(\mathbf{v}_{l_i}' - \mathbf{v}_{k_i}')\) also has positive bias.

Thus, if \(\mathbf{v}_j, \mathbf{v}_k, \mathbf{v}_l\) is a right triple in \(\mathcal{S}\), then, by 3.3.1,

\[
\Pr [\langle \mathbf{v}_j' - \mathbf{v}_k', \mathbf{v}_l' - \mathbf{v}_k' \rangle > 0] \geq \frac{1}{2},
\]

so the probability that the triple \(\mathbf{v}_j', \mathbf{v}_k', \mathbf{v}_l'\) is an acute triple in \(\mathcal{S}'\) is at least \(\frac{1}{2}\).

As in the proof of theorem 2.1, the expected number of right triples in \(\mathcal{S}\) is \(3^m \binom{3}{3}^3 \frac{3^d}{4}\), so the expected number of non-acute triples in \(\mathcal{S}'\) is no more than half this value. Thus there is some set \(\mathcal{S}'\) of \(3m\) vectors with no more than \(\frac{3^m}{2} \binom{3m}{3} \frac{3^d}{4}\) non-acute triples, where

\[
\frac{3}{2} \binom{3m}{3} \left(\frac{3}{4}\right)^d < \frac{3}{2} \binom{3m}{3} \left(\frac{3}{4}\right)^d = m(3m)^2 \left(\frac{3}{4}\right)^{d+1} \leq m
\]

by the choice of \(m\).

If we remove one point of each non-acute triple from \(\mathcal{S}'\), the remaining set is an acute \(d\)-set of cardinality at least \(3m - m = 2m\). \(\square\)

4 Constructive lower bounds for \(\kappa(d)\)

In the following proofs, for clarity of exposition, we will represent point vectors in \(\{0, 1\}^d\) as binary words of length \(d\), e.g. \(\mathcal{S}_3 = \{000, 011, 101, 110\}\) represents a cubic acute 3-set.
Concatenation of words (vectors) $v$ and $v'$ will be written $vv'$.

We begin with a simple construction that enables us to extend a cubic acute $d$-set of cardinality $n$ to a cubic acute $(d + 2)$-set of cardinality $n + 1$.

**Theorem 4.1**

$$\kappa(d + 2) \geq \kappa(d) + 1$$

**Proof:** Let $S = \{v_0, v_1, \ldots, v_{n-1}\}$ be a cubic acute $d$-set of cardinality $n = \kappa(d)$. Now let $S' = \{v'_0, v'_1, \ldots, v'_{n-1}\} \subseteq \{0,1\}^{d+2}$ where $v'_i = v_{i00}$ for $0 \leq i \leq n - 2$, $v'_{n-1} = v_{n-110}$ and $v'_n = v_{n-101}$.

If $v'_i$, $v'_j$, $v'_k$ is a triple of distinct points in $S'$ with no more than one of $i$, $j$ and $k$ greater than $n - 2$, then $v'_i$, $v'_j$, $v'_k$ is an acute triple, because $S$ is an acute $d$-set. Also, any triple $v'_k$, $v'_{n-1}$, $v'_j$ or $v'_k$, $v'_{n-1}$, $v'_i$ is an acute triple, because its $(d+1)$th or $(d+2)$th coordinates (respectively) are $0, 1, 0$. Finally, for any triple $v'_{n-1}$, $v'_k$, $v'_n$, if $v_k$ and $v_{n-1}$ differ in the $r$th coordinate, then the $r$th coordinates of $v'_{n-1}$, $v'_k$, $v'_n$ are $0, 1, 0$ or $1, 0, 1$. Thus, $S'$ is a cubic acute $(d + 2)$-set of cardinality $n + 1$. \(\square\)

Our second construction combines cubic acute $d$-sets of cardinality $n$ to make a cubic acute $3d$-set of cardinality $n^2$.

**Theorem 4.2**

$$\kappa(3d) \geq \kappa(d)^2$$

**Proof:** Let $S = \{v_0, v_1, \ldots, v_{n-1}\}$ be a cubic acute $d$-set of cardinality $n = \kappa(d)$, and let

$$T = \{w_{ij} = v_i v_{j \bmod n} : 0 \leq i, j \leq n - 1\},$$

each $w_{ij}$ being made by concatenating three of the $v_i$.

Let $w_{ps}$, $w_{qt}$, $w_{ru}$ be any triple of distinct points in $T$. They constitute an acute triple iff the scalar product $\langle w_{ps} - w_{qt}, w_{ru} - w_{qt} \rangle$ does not vanish (is positive). Now,

$$\langle w_{ps} - w_{qt}, w_{ru} - w_{qt} \rangle = \langle v_p v_s v_{s-p} - v_q v_t v_{t-q}, v_i v_{u-r} v_{u-r} - v_q v_t v_{t-q} \rangle$$

$$= \langle v_p - v_q, v_r - v_q \rangle$$

$$+ \langle v_s - v_t, v_u - v_t \rangle$$

$$+ \langle v_{s-p} - v_{t-q}, v_{u-r} - v_{t-q} \rangle$$

with all the index arithmetic modulo $n$.

If both $p \neq q$ and $q \neq r$, then the first component of this sum is positive, because $S$ is an acute $d$-set. Similarly, if both $s \neq t$ and $t \neq u$, then the second component is positive. Finally, if $p = q$ and $t = u$, then $q \neq r$ and $s \neq t$ or else the points would not be distinct, so the third component, $\langle v_{s-p} - v_{t-q}, v_{u-r} - v_{t-q} \rangle$ is positive. Similarly if $q = r$ and $s = t$.

Thus, all triples in $T$ are acute triples, so $T$ is a cubic acute $3d$-set of cardinality $n^2$. \(\square\)
Corollary 4.2.1  \( \kappa(3^d) \geq 2^{2^d} \).

Proof: By repeated application of theorem 4.2 starting with \( S_3 \), a cubic acute 3-set of cardinality 4. \( \square \)

Corollary 4.2.2  If \( d \geq 3 \),

\[
\kappa(d) \geq 10^{(d+1)^\mu} \approx 1.778^{(d+1)^{0.631}} \quad \text{where} \quad \mu = \frac{\log 2}{\log 3}.
\]

For small \( d \), this is a tighter bound than theorem 2.1.

Proof: By induction on \( d \). For \( 3 \leq d \leq 8 \), we have the following cubic acute \( d \)-sets \( (S_3, \ldots, S_8) \) that satisfy this lower bound for \( \kappa(d) \) (with equality for \( d = 8 \)):

\[
\begin{array}{c|c|c}
S_3 : \kappa(3) \geq 4 & S_4 : \kappa(4) \geq 5 & S_5 : \kappa(5) \geq 6 \\
000 & 0000 & 00000 \\
011 & 0011 & 00011 \\
101 & 0101 & 00101 \\
110 & 1001 & 01001 \\
& 1110 & 10001 \\
\end{array}
\]

\[
\begin{array}{c|c|c}
S_6 : \kappa(6) \geq 8 & S_7 : \kappa(7) \geq 9 & S_8 : \kappa(8) \geq 10 \\
000000 & 0000000 & 00000000 \\
000111 & 0000011 & 00000011 \\
011001 & 0001101 & 0000101 \\
011110 & 0110001 & 0011001 \\
101010 & 0111110 & 0110001 \\
101101 & 1010101 & 0111110 \\
110011 & 1011010 & 1010101 \\
110100 & 1100110 & 1011010 \\
& 1101001 & 1101001 \\
\end{array}
\]

If \( \kappa(d) \geq 10^{(d+1)^\mu} \), then \( \kappa(3d) \geq \kappa(d)^2 \geq 10^{2(d+1)^\mu} \) by theorem 4.2

\[
= 10^{(3d+3)^\mu} \quad \text{because} \quad 3^\mu = 2.
\]

So, since \( \kappa(3d+2) \geq \kappa(3d+1) \geq \kappa(3d) \), if the lower bound is satisfied for \( d \), it is also satisfied for \( 3d, 3d+1 \) and \( 3d+2 \). \( \square \)
Theorem 4.3 If, for each \( r \), \( 1 \leq r \leq m \), we have a cubic acute \( d_r \)-set of cardinality \( n_r \), where \( n_1 \) is the least of the \( n_r \), and if, for some dimension \( d_Z \), we have a cubic acute \( d_Z \)-set of cardinality \( n_Z \), where

\[
 n_Z \geq \prod_{r=2}^{m} n_r,
\]

then a cubic acute \( D \)-set of cardinality \( N \) can be constructed, where

\[
 D = \sum_{r=1}^{m} d_r + d_Z \quad \text{and} \quad N = \prod_{r=1}^{m} n_r.
\]

This result generalises theorem 4.2, but before we can prove it, we first need some preliminary results.

Definition 4.4 If \( n_1 \leq n_2 \leq \ldots \leq n_m \) and \( 0 \leq k_r < n_r \), for each \( r \), \( 1 \leq r \leq m \), then let us denote by \( \langle\langle k_1 k_2 \ldots k_m \rangle\rangle_{n_1 n_2 \ldots n_m} \), the number

\[
 \langle\langle k_1 k_2 \ldots k_m \rangle\rangle_{n_1 n_2 \ldots n_m} = \sum_{r=2}^{m} \left( (k_r - 1 - k_r \mod n_r) \prod_{s=r+1}^{m} n_s \right).
\]

Where the \( n_r \) can be inferred from the context, \( \langle\langle k_1 k_2 \ldots k_m \rangle\rangle \) may be used instead of \( \langle\langle k_1 k_2 \ldots k_m \rangle\rangle_{n_1 n_2 \ldots n_m} \).

The expression \( \langle\langle k_1 k_2 \ldots k_m \rangle\rangle_{n_1 n_2 \ldots n_m} \) can be understood as representing a number in a number system where the radix for each digit is a different \( n_r \) — like the old British monetary system of pounds, shillings and pennies — and the digits are the difference of two adjacent \( k_r \) (mod \( n_r \)). For example,

\[
 \langle\langle 2053 \rangle\rangle_{4668} = [2 - 0][0 - 5][5 - 3] = 2 \times 6 \times 8 + 1 \times 8 + 2 = 106,
\]

where \([a_2]_{n_2} \ldots [a_m]_{n_m}\) is place notation with the \( n_r \) the radix for each place.

By construction, we have the following results:

Property 4.4.1

\[
 \langle\langle k_1 k_2 \ldots k_m \rangle\rangle_{n_1 n_2 \ldots n_m} < \prod_{r=2}^{m} n_r.
\]

Property 4.4.2 If \( 2 \leq t \leq m \) and \( j_{t-1} - j_t \neq k_{t-1} - k_t \) (mod \( n_t \)), then

\[
 \langle\langle j_1 j_2 \ldots j_m \rangle\rangle_{n_1 n_2 \ldots n_m} \neq \langle\langle k_1 k_2 \ldots k_m \rangle\rangle_{n_1 n_2 \ldots n_m}.
\]
Lemma 4.5 If \( n_1 \leq n_2 \leq \ldots \leq n_m \) and \( 0 \leq j_r, k_r < n_r \), for each \( r, 1 \leq r \leq m \), and the sequences of \( j_r \) and \( k_r \) are neither identical nor everywhere different (i.e. there exist both \( t \) and \( u \) such that \( j_t = k_t \) and \( j_u \neq k_u \)), then
\[
\langle j_1 j_2 \ldots j_m \rangle_{n_1 n_2 \ldots n_m} \neq \langle k_1 k_2 \ldots k_m \rangle_{n_1 n_2 \ldots n_m}.
\]

Proof: Let \( u \) be the greatest integer, \( 1 \leq u < m \), such that \( j_u - j_{u+1} \neq k_u - k_{u+1} \) (mod \( n_{u+1} \)). (If \( j_m = k_m \), then \( u \) is the greatest integer such that \( j_u \neq k_u \). If \( j_m \neq k_m \), then \( u \) is at least as great as the greatest integer \( t \) such that \( j_t = k_t \).) The result now follows from 4.4.2.

We are now in a position to prove the theorem.

Proof of Theorem 4.3

Let \( n_1 \leq n_2 \leq \ldots \leq n_m \), and, for each \( r, 1 \leq r \leq m \), let \( S_r = \{v^r_0, v^r_1, \ldots, v^r_{n_r-1}\} \) be a cubic acute \( d_r \)-set of cardinality \( n_r \). Let \( Z = \{z_0, z_1, \ldots, z_{n_Z-1}\} \) be a cubic acute \( d_Z \)-set of cardinality \( n_Z \), where
\[
n_Z \geq \prod_{r=2}^m n_r,
\]
and let
\[
D = \sum_{r=1}^m d_r + d_Z \quad \text{and} \quad N = \prod_{r=1}^m n_r.
\]

Now let
\[
T = \{w_{k_1 k_2 \ldots k_m} = v^1_{k_1} v^2_{k_2} \ldots v^m_{k_m} z_{k_Z} : 0 \leq k_r < n_r, 1 \leq r \leq m\},
\]
where \( k_Z = \langle k_1 k_2 \ldots k_m \rangle_{n_1 n_2 \ldots n_m} \), be a point set of dimension \( D \) and cardinality \( N \), each element of \( T \) being made by concatenating one vector from each of the \( S_r \) together with a vector from \( Z \). (In section 5, we will denote this construction by \( d_1 \otimes \cdots \otimes d_m \otimes d_Z \).)

By 4.4.1, we know that \( k_Z < \prod_{r=2}^m n_r \leq n_Z \), so \( k_Z \) is a valid index into \( Z \).

Let \( w_{i_1 i_2 \ldots i_m}, w_{j_1 j_2 \ldots j_m}, w_{k_1 k_2 \ldots k_m} \) be any triple of distinct points in \( T \). They constitute an acute triple iff the scalar product
\[
q = \langle w_{i_1 i_2 \ldots i_m} - w_{j_1 j_2 \ldots j_m}, w_{k_1 k_2 \ldots k_m} - w_{j_1 j_2 \ldots j_m} \rangle
\]
does not vanish (is positive). Now,
\[
q = \sum_{r=1}^m \langle v^r_{i_r} - v^r_{j_r}, v^r_{k_r} - v^r_{j_r} \rangle + \langle z_{i_Z} - z_{j_Z}, z_{k_Z} - z_{j_Z} \rangle.
\]

If, for some \( r \), both \( i_r \neq j_r \) and \( j_r \neq k_r \), then the first component of this sum is positive, because \( S_r \) is an acute set.

If, however, there is no \( r \) such that both \( i_r \neq j_r \) and \( j_r \neq k_r \), then there must be some \( t \) for which \( i_t \neq j_t \) (or else \( w_{i_1 i_2 \ldots i_m} \) and \( w_{j_1 j_2 \ldots j_m} \) would not be distinct) and \( j_t = k_t \), and
also some \( u \) for which \( j_u \neq k_u \) (or else \( w_{j_1, j_2, \ldots, j_m} \) and \( w_{k_1, k_2, \ldots, k_m} \) would not be distinct) and \( i_u = j_u \). So, by lemma 4.5, \( i_Z \neq j_Z \) and \( j_Z \neq k_Z \), so the second component of the sum for the scalar product is positive, because \( Z \) is an acute set.

Thus, all triples in \( T \) are acute triples, so \( T \) is a cubic acute \( D \)-set of cardinality \( N \).

**Corollary 4.5.1**

If \( d_1 \leq d_2 \leq \ldots \leq d_m \), then

\[
\kappa \left( \sum_{r=1}^{m} r d_r \right) \geq \prod_{r=1}^{m} \kappa(d_r).
\]

**Proof:** By induction on \( m \). The bound is trivially true for \( m = 1 \).

Assume the bound holds for \( m - 1 \), and for each \( r, 1 \leq r \leq m \), let \( S_r \) be a cubic acute \( d_r \)-set of cardinality \( n_r = \kappa(d_r) \), with \( d_1 \leq d_2 \leq \ldots \leq d_m \) and thus \( n_1 \leq n_2 \leq \ldots \leq n_m \). By the induction hypothesis, there exists a cubic acute \( d_Z \)-set \( Z \) of cardinality \( n_Z \), where

\[
d_Z = \sum_{r=2}^{m} (r - 1)d_r \quad \text{and} \quad n_Z \geq \prod_{r=2}^{m} \kappa(d_r) = \prod_{r=2}^{m} n_r.
\]

Thus, by theorem 4.3, there exists a cubic acute \( D \)-set of cardinality \( N \), where

\[
D = \sum_{r=1}^{m} d_r + d_Z = \sum_{r=1}^{m} d_r + \sum_{r=2}^{m} (r - 1)d_r = \sum_{r=1}^{m} rd_r,
\]

and

\[
N = \prod_{r=1}^{m} n_r = \prod_{r=1}^{m} \kappa(d_r).
\]

**5 Lower bounds for \( \kappa(d) \) and \( \alpha(d) \) for small \( d \)**

The following table lists the best lower bounds known for \( \kappa(d) \), \( 0 \leq d \leq 69 \). For \( 3 \leq d \leq 9 \), an exhaustive computer search shows that \( S_3, \ldots, S_8 \) (corollary 4.2.2), are optimal and also that \( \kappa(9) = 16 \). For other small values of \( d \), the construction used in theorem 4.3 provides the largest known cubic acute \( d \)-set. In the table, these constructions are denoted by \( d_1 \circ d_2 \circ d_Z \) or \( d_1 \circ d_2 \circ d_3 \circ d_Z \). For \( 39 \leq d \leq 48 \), the results of a computer program, based on the ‘probabilistic construction’ of theorem 2.1, provide the largest known cubic acute \( d \)-sets. Finally, for \( d \geq 67 \), theorem 2.1 provides the best (probabilistic) lower bound. \( \kappa(d) \) is sequence A089676 in Sloane [S].

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12
**Best Lower Bounds Known for** $\kappa(d)$

<table>
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<th>$\kappa(d)$</th>
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<td>26</td>
<td>$\geq 160$</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>27</td>
<td>$\geq 256$</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>28</td>
<td>$\geq 256$</td>
</tr>
<tr>
<td>3</td>
<td>computer, $S_3$</td>
<td>29</td>
<td>$\geq 257$ theorem 4.1</td>
</tr>
<tr>
<td>4</td>
<td>computer, $S_4$</td>
<td>30</td>
<td>$\geq 257$</td>
</tr>
<tr>
<td>5</td>
<td>computer, $S_5$</td>
<td>31</td>
<td>$\geq 320$ 9*11*11</td>
</tr>
<tr>
<td>6</td>
<td>computer, $S_6$</td>
<td>32</td>
<td>$\geq 320$</td>
</tr>
<tr>
<td>7</td>
<td>computer, $S_7$</td>
<td>33</td>
<td>$\geq 400$ 11*11*11</td>
</tr>
<tr>
<td>8</td>
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<td>$\geq 400$</td>
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<td>35</td>
<td>$\geq 500$ 11*12*12</td>
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<td>36</td>
<td>$\geq 625$ 12*12*12</td>
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<tr>
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<td>$\geq 20$ 3*4*4</td>
<td>37</td>
<td>$\geq 625$</td>
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<td>$\geq 25$ 4*4*4</td>
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</tr>
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<td>$\geq 36$ 5*5*5</td>
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<td>$\geq 871$ computer</td>
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<td>16</td>
<td>$\geq 40$ 4*6*6</td>
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<td>$\geq 976$ computer</td>
</tr>
<tr>
<td>17</td>
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</tr>
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<td>18</td>
<td>$\geq 64$ 6*6*6 or 3*3*3*9</td>
<td>44</td>
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<td>19</td>
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<td>46</td>
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<td>$\geq 81$ 7*7*7</td>
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<td>$\geq 81$</td>
<td>48</td>
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<td>$\geq 2036$</td>
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<td>25</td>
<td>$\geq 144$ 7*9*9</td>
<td>51</td>
<td>$\geq 2304$ 17*17*17</td>
</tr>
<tr>
<td>52</td>
<td>$\geq 2560$ 16*18*18</td>
<td>61</td>
<td>$\geq 5184$</td>
</tr>
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<td>$\geq 3072$ 17*18*18</td>
<td>62</td>
<td>$\geq 5832$ 20*21*21</td>
</tr>
<tr>
<td>54</td>
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<td>63</td>
<td>$\geq 6561$ 21*21*21</td>
</tr>
<tr>
<td>55</td>
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<td>$\geq 5184$ 20*20*20</td>
<td>69</td>
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The following tables summarise the best lower bounds known for $\alpha(d)$. For $3 \leq d \leq 6$, the best lower bound is Danzer and Grünbaum’s $2d - 1$ [DG]. For $7 \leq d \leq 26$, the results of a computer program, based on the ‘probabilistic construction’ but using sets of points close to the surface of the $d$-sphere, provide the largest known acute $d$-sets. An acute 7-set of cardinality 14 and an acute 8-set of cardinality 16 are displayed. For $27 \leq d \leq 62$, the largest known acute $d$-set is cubic. Finally, for $d \geq 63$, theorem 3.1 provides the best (probabilistic) lower bound.

**Best Lower Bounds Known for $\alpha(d)$**

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<td>17</td>
<td>$\geq 63$ computer</td>
</tr>
<tr>
<td>2</td>
<td>$= 3$</td>
<td>18</td>
<td>$\geq 71$ computer</td>
</tr>
<tr>
<td>3</td>
<td>$= 5$ [DG]</td>
<td>19</td>
<td>$\geq 76$ computer</td>
</tr>
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<td>4–6</td>
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<td>20</td>
<td>$\geq 90$ computer</td>
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<tr>
<td>7</td>
<td>$\geq 14$ computer</td>
<td>21</td>
<td>$\geq 103$ computer</td>
</tr>
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<td>$\geq 16$ computer</td>
<td>22</td>
<td>$\geq 118$ computer</td>
</tr>
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<td>23</td>
<td>$\geq 121$ computer</td>
</tr>
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<td>24</td>
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</tr>
<tr>
<td>11</td>
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<td>25</td>
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</tr>
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<td>12</td>
<td>$\geq 30$ computer</td>
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<td>(12, 36, 28, 30, 3, 45, 48, 45)</td>
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6 Generalising $\kappa(d)$

We can understand $\kappa(d)$ to be the size of the largest possible set $S$ of binary words such that, for any ordered triple of words $(u, v, w)$ in $S$, there exists an index $i$ for which $(u_i, v_i, w_i) = (0, 1, 0)$ or $(u_i, v_i, w_i) = (1, 0, 1)$. We can generalise this in the following way:

**Definition 6.1** If $T_1, \ldots, T_m$ are ordered $k$-tuples from $\{0, \ldots, r-1\}^k$ (which we will refer to as the matching $k$-tuples), then let us define $\kappa[r, k, T_1, \ldots, T_m](d)$ to be the size of the largest possible set $S$ of $r$-ary words of length $d$ such that, for any ordered $k$-tuple of words $(w_1, \ldots, w_k)$ in $S$, there exist $i$ and $j$, $1 \leq i \leq d$, $1 \leq j \leq m$, for which $(w_{1i}, \ldots, w_{ki}) = T_j$.

Thus we have $\kappa(d) = \kappa[2, 3, (0, 1, 0), (1, 0, 1)](d)$. If the set of matching $k$-tuples is closed under permutation, we will abbreviate by writing a list of matching multisets of cardinality $k$, rather than ordered tuples. For example, instead of $\kappa[2, 3, (0, 0, 1), (0, 1, 0), (1, 0, 0)](d)$, we write $\kappa[2, 3, (0, 0, 1)](d)$.

We can find probabilistic and, in some cases, constructive lower bounds for general $\kappa[r, k, T_1, \ldots, T_m](d)$ using the approaches we used for cubic acute $d$-sets. To illustrate this, in the remainder of this paper, we will consider the set of problems in which it is simply required that at some index the $k$-tuple of words be all different (pairwise distinct). First, we express this in a slightly different form.

Let us say that an $r$-ary $d$-colouring is some colouring of the integers $1, \ldots, d$ using $r$ colours. Let us also also say that a set $R$ of $r$-ary $d$-colourings is a $k$-rainbow set, for some $k \leq r$ if for any set $\{c_1, \ldots, c_k\}$ of $k$ colourings in $R$, there exists some integer $t$, $1 \leq t \leq d$, for which the colours $c_1(t), \ldots, c_k(t)$ are all different, i.e. $c_i(t) \neq c_j(t)$ for any $i$ and $j$, $1 \leq i, j \leq k$, $i \neq j$. For conciseness, we will denote “a $k$-rainbow set of $r$-ary $d$-colourings” by “a $\text{RSC}[k, r, d]$”.

Let us further say that a set $\{c_1, \ldots, c_k\}$ of $k$ $d$-colourings is a good $k$-set if there exists some integer $t$, $1 \leq t \leq d$, for which the colours $c_1(t), \ldots, c_k(t)$ are all different, and a bad $k$-set if there exists no such $t$.

We will denote by $\rho_{r,k}(d)$ the size of the largest possible $\text{RSC}[k, r, d]$, abbreviating $\rho_{k,k}(d)$ by $\rho_k(d)$. Now, $\rho_k(d) = \kappa[k, k, \{0, 1, \ldots, k-1\}](d)$ and

$$\rho_{r,k}(d) = \kappa[r, k, \{0, \ldots, k-1\}, \ldots, \{r-k, \ldots, r-1\}](d),$$

where the matching multisets are those of cardinality $k$ with $k$ distinct members.

Clearly, $\rho_{r,k}(d) \leq \rho_{r,k}(d + 1)$. We will denote $\rho_{r+1,k}(d)$ and $\rho_{r,k}(d) \geq \rho_{r,k+1}(d)$. Also, $\rho_{r,1}(d)$ is undefined because any set of colourings is a 1-rainbow, $\rho_{r,k}(1) = r$ if $k > 1$, and $\rho_{r,2}(d) = r^d$ because any two distinct $r$-ary $d$-colourings (or $r$-ary words of length $d$) differ somewhere.
In the next two sections we will give a number of probabilistic and constructive lower bounds for $\rho_{r,k}(d)$, for various $r$ and $k$.

7 A probabilistic lower bound for $\rho_{r,k}(d)$

Theorem 7.1

$$\rho_{r,k}(d) \geq (k-1)m \text{ where } m = \left[ \frac{k^{-1} \left( \frac{(r-k)!r^k}{(r-k)!r^k-r!} \right)^d}{k^k} \right].$$

Proof: This proof is similar that of theorem 2.1.

Randomly pick a set $\mathcal{R}$ of $km$ $r$-ary $d$-colourings, choosing the colours from $\{\chi_0, \ldots, \chi_{r-1}\}$ independently with probability $\Pr[c(i) = \chi_j] = 1/r, 1 \leq i \leq d, 0 \leq j < r$ for every $c \in \mathcal{R}$.

Now the probability that a set of $k$ colourings from $\mathcal{R}$ is a bad $k$-set is

$$(1-p)^d \text{ where } p = \frac{r!(r-k)!}{r^k}.$$

Hence, the expected number of bad $k$-sets in a set of $km$ $d$-colourings is $\left(\frac{km}{k}\right)(1-p)^d$. Thus there is some set $\mathcal{R}$ of $km$ $d$-colourings with no more than $\left(\frac{km}{k}\right)(1-p)^d$ bad $k$-sets, where

$$\left(\frac{km}{k}\right)(1-p)^d < \frac{(km)^k}{k!}(1-p)^d = m \frac{k^k}{k!} m^{k-1}(1-p)^d \leq m$$

by the choice of $m$.

If we remove one colouring of each bad $k$-set from $\mathcal{R}$, the remaining set is a $\mathcal{RSC}[k, r, d]$ of cardinality at least $km - m = (k-1)m$.

The following results follow directly:

$$\rho_3(d) \geq 2 \left[ \frac{\sqrt{2}}{3} \left( \frac{3}{\sqrt[3]{7}} \right)^d \right] \approx 0.943 \times 1.134^d.$$

$$\rho_{4,3}(d) \geq 2 \left[ \frac{\sqrt{2}}{3} \left( \frac{4}{\sqrt[3]{10}} \right)^d \right] \approx 0.943 \times 1.265^d.$$

$$\rho_4(d) \geq 3 \left[ \frac{\sqrt{3}}{32} \left( \frac{3/29}{32} \right)^d \right] \approx 1.363 \times 1.033^d.$$
Constructive lower bounds for \( \rho_{r,k}(d) \)

In the following proofs, for clarity of exposition, we will represent \( r \)-ary \( d \)-colourings as \( r \)-ary words of length \( d \), e.g. \( R_{3,3,3} = \{000, 011, 102, 121, 212, 220\} \) represents a 3-rainbow set of ternary 3-colourings (using the colours \( \chi_0, \chi_1 \) and \( \chi_2 \)). Concatenation of words (colourings) \( c \) and \( c' \) will be written \( c.c' \).

We begin with a construction that enables us to extend a \( RSC[k, r, d] \) of cardinality \( n \) to one of cardinality \( n + 1 \) or greater.

**Theorem 8.1** If for some \( r \geq k \geq 3 \), and some \( d \), we have a \( RSC[k, r, d] \) of cardinality \( n \), and for some \( r' \), \( k - 2 \leq r' \leq r - 2 \), and \( d' \), we have a \( RSC[k - 2, r', d'] \) of cardinality at least \( n - 1 \), then we can construct a \( RSC[k, r, d + d'] \) of cardinality \( N = n - 1 + r - r' \).

**Proof:** Let \( R = \{c_0, c_1, \ldots, c_{n-1}\} \) be a \( RSC[k, r, d] \) of cardinality \( n \) (using colours \( \chi_0, \ldots, \chi_{r-1} \)) and \( R' = \{c'_0, c'_1, \ldots, c'_{n-1}\} \) be a \( RSC[k - 2, r', d'] \) of cardinality \( n' \geq n - 1 \) (using colours \( \chi_0, \ldots, \chi_{r'-1} \)).

Now let \( Q = \{q_0, q_1, \ldots, q_{N-1}\} \) be a set of \( r \)-ary \((d + d')\)-colourings where \( q_i = c_i.c'_i \) for \( 0 \leq i \leq n - 2 \), and \( q_{n-1+j} = c_{n-1}.(r' + j)d' \) for \( 0 \leq j < r - r' \), each element of \( Q \) being made by concatenating two component colourings, the first from \( R \) and the second being either from \( R' \) or a monochrome colouring.

If \( \{q_{i_1}, \ldots, q_{i_k}\} \) is a set of colourings in \( Q \) with no more than one of the \( i_m \) greater than \( n - 2 \), then it is a good \( k \)-set because of the first components, since \( R \) is a \( k \)-rainbow set.

On the other hand, if \( \{q_{i_1}, \ldots, q_{i_k}\} \) is a set of colourings in \( Q \) with no more than \( k - 2 \) of the \( i_m \) less than \( n - 1 \), then it too is a good \( k \)-set because of the second components, since \( R' \) is a \((k - 2)\)-rainbow set using colours \( \chi_0, \ldots, \chi_{r'-1} \) and the second components of the colourings with indices greater than \( n - 2 \) are each monochrome of a different colour, drawn from \( \chi_{r'}, \ldots, \chi_{r-1} \).

Thus \( Q \) is a \( RSC[k, r, d + d'] \) of cardinality \( N \).

**Corollary 8.1.1** \( \rho_{r,3}(d + 1) \geq \rho_{r,3}(d) + r - 2 \).

**Proof:** This follows from the theorem due to the fact that there is a 1-rainbow set of 1-ary 1-colourings of any cardinality.

**Corollary 8.1.2** \( \rho_{r,4}(d + \lfloor \log_2(\rho_{r,4}(d) - 1) \rfloor) \geq \rho_{r,4}(d) + r - 3 \).

**Proof:** Since \( \rho_{r,2}(d) = r^d \), we have \( \rho_{2,2}(d') \geq \rho_{r,4}(d) - 1 \) iff \( d' \geq \log_2(\rho_{r,4}(d) - 1) \).
Theorem 8.2 If, for each \( s, 1 \leq s \leq m \), we have a \( \mathcal{RSC}[3, r, d_s] \) of cardinality \( n_s \), where \( n_1 \) is the least of the \( n_s \), and if, for some \( d_Z \), we have a \( \mathcal{RSC}[3, r, d_Z] \) of cardinality \( n_Z \), where
\[
  n_Z \geq \prod_{s=2}^{m} \left( 1 + 2 \left\lfloor \frac{n_s}{2} \right\rfloor \right),
\]
then a \( \mathcal{RSC}[3, r, D] \) of cardinality \( N \) can be constructed, where
\[
  D = \sum_{s=1}^{m} d_s + 2d_Z \quad \text{and} \quad N = \prod_{s=1}^{m} n_s.
\]

This result for 3-rainbow sets corresponds to theorem 4.3 for cubic acute \( d \)-sets. Before we can prove it, we need some further preliminary results.

Definition 8.3 If \( n_1 \leq n_2 \leq \ldots \leq n_m \) and \( 0 \leq k_r < n_r \), for each \( r, 1 \leq r \leq m \), then let us denote by \( \langle k_1 k_2 \ldots k_m \rangle_{n_1 n_2 \ldots n_m}^{+} \), the number
\[
  \langle k_1 k_2 \ldots k_m \rangle_{n_1 n_2 \ldots n_m}^{+} = \sum_{r=2}^{m} \left( k_{r-1} + k_r \mod n_r \right) \prod_{s=r+1}^{m} n_s.
\]
The definition of \( \langle k_1 k_2 \ldots k_m \rangle_{n_1 n_2 \ldots n_m}^{+} \) is the same as that for \( \langle k_1 k_2 \ldots k_m \rangle_{n_1 n_2 \ldots n_m} \) (see 4.4), but with addition replacing subtraction. By construction, we have
\[
  \langle k_1 k_2 \ldots k_m \rangle_{n_1 n_2 \ldots n_m}^{+} < \prod_{r=2}^{m} n_r,
\]
and, if \( 2 \leq t \leq m \) and \( j_{t-1} + j_t \neq k_{t-1} + k_t \mod n_t \), then
\[
  \langle j_1 j_2 \ldots j_m \rangle_{n_1 n_2 \ldots n_m}^{+} \neq \langle k_1 k_2 \ldots k_m \rangle_{n_1 n_2 \ldots n_m}^{+}.
\]

Lemma 8.4 If \( n_1 \leq n_2 \leq \ldots \leq n_m \), with all the \( n_r \) odd except perhaps \( n_1 \), and \( 0 \leq j_r, k_r, l_r < n_r \), for each \( r, 1 \leq r \leq m \), and the sequences of \( j_r, k_r \) and \( l_r \) are neither pairwise identical nor anywhere pairwise distinct, i.e. there is some \( u, v \) and \( w \) such that \( j_u \neq k_u, k_v \neq l_v \) and \( l_w \neq j_w \) but no \( t \) such that \( j_t \neq k_t, k_t \neq l_t \) and \( l_t \neq j_t \), then either
\[
  \langle j_1 \ldots j_m \rangle_{n_1 \ldots n_m}, \langle k_1 \ldots k_m \rangle_{n_1 \ldots n_m}, \langle l_1 \ldots l_m \rangle_{n_1 \ldots n_m} \quad \text{are pairwise distinct}
\]
or
\[
  \langle j_1 \ldots j_m \rangle_{n_1 \ldots n_m}^{+}, \langle k_1 \ldots k_m \rangle_{n_1 \ldots n_m}^{+}, \langle l_1 \ldots l_m \rangle_{n_1 \ldots n_m}^{+} \quad \text{are pairwise distinct}.
\]
Proof: Without loss of generality, we can assume that we have \( j_t = k_1 \), that \( t > 1 \) is the least integer for which \( j_t \neq k_t \), and that \( k_t = l_t \). We will consider two cases:

**Case 1:** \( k_{t-1} \neq l_{t-1} \)

Since \( j_{t-1} = k_{t-1} \neq l_{t-1} \) and \( j_t \neq k_t = l_t \), we have \( j_{t-1} - j_t \neq k_{t-1} - k_t \) and \( k_{t-1} - k_t \neq l_{t-1} - l_t \), and so \( \langle j_1 \ldots j_m \rangle \neq \langle k_1 \ldots k_m \rangle \) and \( \langle k_1 \ldots k_m \rangle \neq \langle l_1 \ldots l_m \rangle \). Similarly, \( j_t + j_t \neq k_t + k_t \) and \( k_t + l_t \neq l_t + l_t \), and so \( \langle j_1 \ldots j_m \rangle^+ \neq \langle k_1 \ldots k_m \rangle^+ \) and \( \langle k_1 \ldots k_m \rangle^+ \neq \langle l_1 \ldots l_m \rangle^+ \).

If \( j_{t-1} - j_t \neq l_{t-1} - l_t \), then \( \langle j_1 \ldots j_m \rangle \neq \langle l_1 \ldots l_m \rangle \). If \( j_{t-1} - j_t = l_{t-1} - l_t \) then \((j_{t-1} + j_t) - (l_{t-1} + l_t) = (j_{t-1} - j_t + 2j_t - (l_{t-1} - l_t + 2l_t) = 2(j_t - l_t) \neq 0 \) (mod \( n_t \)) because \( j_t \neq l_t \) and \( n_t \) is odd, so \( j_{t-1} + j_t \neq l_{t-1} + l_t \) and \( \langle j_1 \ldots j_m \rangle^+ \neq \langle l_1 \ldots l_m \rangle^+ \).

**Case 2:** \( k_{t-1} = l_{t-1} \)

Since \( j_{t-1} = k_{t-1} = l_{t-1} \) and \( j_t \neq k_t = l_t \), we have \( j_{t-1} - j_t \neq k_{t-1} - k_t \) and \( j_t - j_t \neq l_{t-1} - l_t \), and so \( \langle j_1 \ldots j_m \rangle \neq \langle k_1 \ldots k_m \rangle \) and \( \langle j_1 \ldots j_m \rangle \neq \langle l_1 \ldots l_m \rangle \).

If \( k_1 = l_1 \), let \( u \) be the least integer such that \( k_u \neq l_u \). Since \( k_{u-1} = l_{u-1} \), we have \( k_u - k_u \neq l_u - l_u \). If \( k_1 \neq l_1 \), let \( u \) be the least integer such that \( k_u = l_u \). Since \( k_{u-1} \neq l_{u-1} \), we still have \( k_{u-1} - k_u \neq l_{u-1} - l_u \). Thus, \( \langle k_1 \ldots k_m \rangle \neq \langle l_1 \ldots l_m \rangle \). \( \square \)

**Proof of Theorem 8.2**

Let \( n_1 \leq n_2 \leq \ldots \leq n_m \), and, for each \( s \), \( 1 \leq s \leq m \), let \( \mathcal{R}_s = \{ c_0^s, c_1^s, \ldots, c_{n_s-1}^s \} \) be a \( \mathcal{RSC}[3, r, d_s] \) of cardinality \( n_s \), and let \( n'_s = 1 + 2 \lfloor n_s/2 \rfloor \) be the least odd integer not less than \( n_s \). Let \( \mathcal{Z} = \{ z_0, z_1, \ldots, z_{n_Z-1} \} \) be a \( \mathcal{RSC}[3, r, d_Z] \) of cardinality \( n_Z \), where

\[
n_Z \geq \prod_{s=2}^{m} n'_s,\]

and let

\[
D = \sum_{s=1}^{m} d_s + 2d_Z \quad \text{and} \quad N = \prod_{s=1}^{m} n_s.
\]

Now let

\[
\mathcal{Q} = \{ c_{k_1}^1, c_{k_2}^2, \ldots, c_{k_m}^m, z_{k_Z}^z, z_{k_2}^+ \} : 0 \leq k_s < n_s, 1 \leq s \leq m, \}
\]

where \( k_Z = \langle k_1 k_2 \ldots k_m \rangle_{n'_1 n'_2 \ldots n'_m} \) and \( k_Z^+ = \langle k_1 k_2 \ldots k_m \rangle_{n'_1 n'_2 \ldots n'_m}^+ \) be a set of \( D \)-colourings of cardinality \( N \), each element of \( \mathcal{Q} \) being made by concatenating one colouring from each of the \( \mathcal{R}_s \) together with two colourings from \( \mathcal{Z} \). (Below, we will denote this construction by \( d_1 \otimes \cdots \otimes d_m \otimes d_Z \otimes d_Z \).)

Let \( c_{i_1}^1, c_{i_2}^2, \ldots, c_{i_m}^m, z_{i_2}^z, z_{i_2}^+ \), \( c_{j_1}^1, c_{j_2}^2, \ldots, c_{j_m}^m, z_{j_2}^z, z_{j_2}^+ \), and \( c_{k_1}^1, c_{k_2}^2, \ldots, c_{k_m}^m, z_{k_2}^z, z_{k_2}^+ \) be any three distinct colourings in \( \mathcal{Q} \). If, for some \( s, t_s \neq j_s, j_s \neq k_s \) and \( k_s \neq i_s \), then these three colourings comprise a good 3-set because \( \mathcal{R}_s \) is a 3-rainbow set.
If, however, there is no $s$ such that $i_s$, $j_s$ and $k_s$ are all different, then the condition of lemma 8.4 holds, and so either $i_Z$, $j_Z$ and $k_Z$ are all different, or $i^+_Z$, $j^+_Z$ and $k^+_Z$ are all different, and the three colourings comprise a good 3-set because $Z$ is a 3-rainbow set.

Thus, any three colourings in $Q$ comprise a good 3-set, so $Q$ is a $\mathcal{RSC}[3, r, D]$ of cardinality $N$.

Corollary 8.4.1 If $\rho_{r,3}(d)$ is odd, then $\rho_{r,3}(4d) \geq \rho_{r,3}(d)^2$.

Proof: By theorem 8.2 using the construction $d \otimes d \otimes d \otimes d$.

Corollary 8.4.2 $\rho_{r,3}(4d + 2) \geq \rho_{r,3}(d)^2$.

Proof: By 8.1.1, if $n = \rho_{r,3}(d)$, we can construct a $\mathcal{RSC}[3, r, d + 1]$ of cardinality $n + 1 \geq 1 + 2 \lfloor n/2 \rfloor$. By theorem 8.2, we can then construct a $\mathcal{RSC}[3, r, 4d + 2]$ of cardinality $n^2$ using the construction $d \otimes d \otimes (d + 1) \otimes (d + 1)$.

Corollary 8.4.3 $\rho_3(4^d) \geq 3^{2^d}$.

Proof: By repeated application of 8.4.1 starting with $\rho_{3,3}(1) = 3$.

Our final construction enables us to combine $k$-rainbow sets of $r$-ary $d$-colourings for arbitrary $k$.

Theorem 8.5 If we have a $\mathcal{RSC}[k, r, d_1]$ of cardinality $n_1$, a $\mathcal{RSC}[k, r, d_2]$ of cardinality $n_2 \geq n_1$, and a $\mathcal{RSC}[k, r, d_Z]$ of cardinality $n_Z \geq n_2$, with $n_Z$ coprime to each integer in the range $[2, \ldots, h]$ where $h = \binom{k}{2} - 1$, then a $\mathcal{RSC}[k, r, D]$ of cardinality $N$ can be constructed, where $D = d_1 + d_2 + hd_Z$ and $N = n_1n_2$.

As before, we first need a preliminary result:

Lemma 8.6 Given distinct pairs of integers $(a, b)$ and $(c, d)$ with $0 \leq a, b, c, d < n$ for some $n$, and given a positive integer $h$ such that $n$ is coprime to each integer in the range $[2, \ldots, h]$, then if we let $b_{-1} = a$ and $d_{-1} = c$, and $b_r = b + ra \pmod{n}$ and $d_r = d + rc \pmod{n}$ for $0 \leq r \leq h$, then if $b_i = d_i$ for some $i$, $-1 \leq i \leq h$, we have $b_j \neq d_j$ for all $j \neq i$. 

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Proof: We consider two cases:

Case 1: \( i = -1 \)

Since \( a = c, (b + ja) - (d + jc) = b - d \neq 0 \pmod{n} \) since \((a, b)\) and \((c, d)\) are distinct, and \(b\) and \(d\) both less than \(n\).

Case 2: \( i \neq -1 \)

By the reversing the argument in case 1, \( a \neq c\), i.e. \( b_{-1} \neq d_{-1}\). For \( j \geq 0\), since \( b + ia = d + ic\), we have \((b + ja) - (d + jc) = (j - i)a - (j - i)c = (j - i)(a - c) \neq 0 \pmod{n}\) since \(a \neq c\) and \(|j - i| \leq h\) so \(j - i\) is coprime to \(n\).

Proof of Theorem 8.5

Let \(\mathcal{R}_1 = \{e_0^1, \ldots, e_{n-1}^1\}, \mathcal{R}_2 = \{e_0^2, \ldots, e_{n-1}^2\}\) and \(\mathcal{Z} = \{z_0, \ldots, z_{nZ-1}\}\) be \(k\)-rainbow sets of \(r\)-ary \(d_1\)-, \(d_2\)- and \(d_Z\)-colourings of cardinality \(n_1\), \(n_2\) and \(n_Z\), respectively.

Now let

\[
\mathcal{Q} = \{c_i^1, c^2_j: z_{j+i}, z_{j+2i} \cdots z_{j+hi} : 0 \leq i < n_1, 0 \leq j < n_2\},
\]

where \(h = \binom{k}{2} - 1\) and the subscript arithmetic is modulo \(n_Z\), be a set of \(D\)-colourings of cardinality \(N\), each element of \(\mathcal{Q}\) being made by concatenating \(h+2\) component colourings: one from \(\mathcal{R}_1\), one from \(\mathcal{R}_2\), and \(h\) from \(\mathcal{Z}\).

Let

\[
\mathcal{S} = \{c_1^1, c^2_j: z_{j+i}, z_{j+i+1} \cdots z_{j+i+h_i}, c^1_{j+2}, c^2_j: z_{j+2i} \cdots z_{j+2i+hi_2}, \cdots, c^1_{j+k}, c^2_j: z_{j+k+i} \cdots z_{j+k+hi_k}\}
\]

be any set of \(k\) distinct colourings in \(\mathcal{Q}\), and let \(b_{s,t} = j_s + ti_s \pmod{n_Z}\), for each \(s\) and \(t, 1 \leq s \leq k, 0 \leq t \leq h\), so the \(s^t\)th colouring in \(\mathcal{S}\) is \(c^1_{bs_{s-1}, c^2_{bs_{s-1}}: z_{bs_{s-1}} \cdots z_{bs_{s-1}}hi_k}\).

Now, for any \(s, s'\) and \(t, 1 \leq s, s' \leq k, -1 \leq t \leq h\), if \(b_{s,t} = b_{s',t}\), then by lemma 8.6 we know that for all \(u \neq t\), \(b_{s,u} \neq b_{s',u}\). So for each pair \(\{s, s'\}\), \(b_{s,t} = b_{s',t}\) for no more than one value of \(t\). Now there are \(h+2\) possible values of \(t\), but only \(\binom{k}{2}\) = \(h+1\) different pairs \(\{s, s'\}\), so there is some \(t\) for which \(b_{s,t} \neq b_{s',t}\) for all pairs \(\{s, s'\}\) and the \((t+2)^{th}\) component colourings of the elements in \(\mathcal{S}\) are all different. Since \(\mathcal{R}_1, \mathcal{R}_2\) and \(\mathcal{Z}\) are all \(k\)-rainbow sets, we know that \(\mathcal{S}\) is a good \(k\)-set.

Thus, any \(k\) colourings from \(\mathcal{Q}\) comprise a good \(k\)-set, so \(\mathcal{Q}\) is a \(\mathcal{RSC}[k, r, D]\) of cardinality \(N\).

\[\text{Corollary 8.6.1} \quad \rho_4(6.7^d) \geq 7^{2d}.\]

Proof: The following 4-rainbow set of 4-ary 6-colourings of cardinality 8 — a version of \(\mathcal{R}_{4,4,6}\) (see below) displayed with different symbols for each colour — shows that \(\rho_4(6) \geq 7\).
The result follows by repeated application of theorem 8.5, noting that 7 is coprime to 2, 3, 4 and $5 = \binom{4}{2} - 1$.

\[ \]  

9 Lower bounds for $\rho_{r,k}(d)$ for small $r$, $k$ and $d$

We conclude with tables of the best lower bounds known for $\rho_3(d)$, $\rho_{4,3}(d)$ and $\rho_4(d)$ for small $d$. For very small $d$, exhaustive computer searches have determined the values of $\rho_{r,k}(d)$. For other small values of $d$, the constructions used in theorems 8.2 and 8.5 provide the largest known rainbow sets. In the tables, these constructions are denoted $d_1 \oplus d_2 \oplus d_3 \oplus d_4$, etc., with superscript minus signs ($d^-$) to denote the removal of a single colouring from a largest rainbow set of $d$-colourings (to satisfy the requirement that the cardinality be odd). For $\rho_3(d)$, the probabilistic lower bound of theorem 7.1 is better than the constructions for $d \geq 71$; for $\rho_{4,3}(d)$, this is the case for $d \geq 26$. 

\[ \]
Some \(k\)-rainbow sets of \(r\)-ary \(d\)-colourings, for small \(k, r\) and \(d\)

<table>
<thead>
<tr>
<th>(\mathcal{R}_{3,3,3})</th>
<th>(\mathcal{R}_{3,3,6})</th>
<th>(\mathcal{R}_{4,3,3})</th>
<th>(\mathcal{R}_{4,3,4})</th>
<th>(\mathcal{R}_{4,4,6})</th>
</tr>
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<tbody>
<tr>
<td>(\rho_3(3) \geq 6)</td>
<td>(\rho_3(6) \geq 13)</td>
<td>(\rho_{4,3}(3) \geq 9)</td>
<td>(\rho_{4,3}(4) \geq 16)</td>
<td>(\rho_{4,6}(6) \geq 8)</td>
</tr>
<tr>
<td>000 011 102 121 212 220</td>
<td>000000 000111 000222 011012 022120 101120</td>
<td>000 011 022 0102 1013 2121</td>
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<td></td>
<td></td>
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</table>

Best Lower Bounds Known for \(\rho_3(d)\) and \(\rho_{4,3}(d)\)

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<tr>
<th>(d)</th>
<th>(\rho_3(d))</th>
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<tbody>
<tr>
<td>1</td>
<td>(= 3)</td>
</tr>
<tr>
<td>2</td>
<td>(= 4)  computer, 8.1.1</td>
</tr>
<tr>
<td>3</td>
<td>(= 6)  computer, (\mathcal{R}_{3,3,3})</td>
</tr>
<tr>
<td>4</td>
<td>(= 9)  computer, (1\text{#1}\text{#1})</td>
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<tr>
<td>5</td>
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<td>6</td>
<td>(\geq 13) computer, (\mathcal{R}_{3,3,6})</td>
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<tr>
<td>7</td>
<td>(\geq 14) 8.1.1</td>
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<td>8</td>
<td>(\geq 15) 8.1.1</td>
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<tr>
<td>9</td>
<td>(\geq 16) 8.1.1</td>
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<tr>
<td>10</td>
<td>(\geq 17) 8.1.1</td>
</tr>
<tr>
<td>11</td>
<td>(\geq 27) (1\text{#1}\text{#1}\text{#1}\text{#1})</td>
</tr>
<tr>
<td>12</td>
<td>(\geq 28) 8.1.1</td>
</tr>
<tr>
<td>13</td>
<td>(\geq 29) 8.1.1</td>
</tr>
<tr>
<td>14</td>
<td>(\geq 36) (2\text{#4}\text{#4})</td>
</tr>
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<td>15</td>
<td>(\geq 54) (3\text{#4}\text{#4})</td>
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<tr>
<td>16</td>
<td>(\geq 81) (4\text{#4}\text{#4})</td>
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<td>.....</td>
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<tr>
<td>70</td>
<td>(\geq 76723) (16\text{#18}\text{#18}\text{#18})</td>
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<tr>
<td>71</td>
<td>(\geq 7064) theorem 7.1</td>
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<table>
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<td>(= 9) computer, (\mathcal{R}_{4,3,3})</td>
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<td>(\geq 22) 8.1.1</td>
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<td>8</td>
<td>(\geq 25) (2\text{#2#2}\text{#2})</td>
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<tr>
<td>9</td>
<td>(\geq 27) 8.1.1</td>
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<tr>
<td>10</td>
<td>(\geq 36) (1\text{#3#3}\text{#3}\text{#3}) or (2\text{#2#3}\text{#3})</td>
</tr>
<tr>
<td>11</td>
<td>(\geq 54) (2\text{#3#3}\text{#3})</td>
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<tr>
<td>12</td>
<td>(\geq 81) (3\text{#3#3}\text{#3})</td>
</tr>
<tr>
<td>13</td>
<td>(\geq 83) 8.1.1</td>
</tr>
<tr>
<td>14</td>
<td>(\geq 90) (2\text{#4#4}\text{#4})</td>
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<tr>
<td>15</td>
<td>(\geq 135) (3\text{#4#4}\text{#4})</td>
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<tr>
<td>16</td>
<td>(\geq 225) (4\text{#4#4}\text{#4})</td>
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<tr>
<td>25</td>
<td>(\geq 363) 8.1.1</td>
</tr>
<tr>
<td>26</td>
<td>(\geq 424) theorem 7.1</td>
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Best Lower Bounds Known for $\rho_4(d)$

<table>
<thead>
<tr>
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<th>$\rho_4(d)$</th>
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<tbody>
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<tr>
<td>2</td>
<td>4</td>
</tr>
<tr>
<td>3</td>
<td>computer, 8.1.2</td>
</tr>
<tr>
<td>4</td>
<td>computer</td>
</tr>
<tr>
<td>5</td>
<td>computer, 8.1.2</td>
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<tr>
<td>6</td>
<td>computer, $\mathcal{R}_{4,4,6}$</td>
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<td>42</td>
<td>$\geq 49$</td>
</tr>
</tbody>
</table>

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**References**


