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Abstract

A Lebesgue measurable set in the real line has Lebesgue density 0 or 1 at almost every point. Kolyada showed that there is a positive constant δ such that for non-trivial measurable sets there is at least one point with upper and lower densities lying in the interval $(\delta, 1-\delta)$. Both Kolyada and later Szenes gave bounds for the largest possible value of this δ. In this note we reduce the best known upper bound, disproving a conjecture of Szenes.

1 Introduction

If $E \subseteq \mathbb{R}$ is a measurable set for the usual Lebesgue measure $\mathcal{L}$, then it is well known that for (Lebesgue) almost every $x \in \mathbb{R}$, the density of E at x given by

$$d_E(x) = \lim_{r \downarrow 0} d_E(x,r), \text{ where } d_E(x,r) := \frac{\mathcal{L}(E \cap B(x,r))}{2r} \text{ for } r > 0,$$

exists and is either zero or one. If either E or its complement is a Lebesgue null set, then for all $x \in \mathbb{R}$, the density $d_E(x)$ exists and is trivially either identically zero or one. When neither E nor $\mathbb{R} \setminus E$ are Lebesgue null, then there may be points where the density does not exist and we introduce the upper and lower Lebesgue density of E at a point $x \in \mathbb{R}$ by

$$\underline{d}_E(x) := \liminf_{r \to 0} d_E(x,r) \quad \text{and} \quad \overline{d}_E(x) := \limsup_{r \to 0} d_E(x,r), \text{ respectively.}$$

For Lebesgue measurable sets that are non-trivial in the sense that both the set and its complement have positive Lebesgue measure, Kolyada [1] asked what is the supremum of those δ for which the following statement is true:

($*$) For every set E with $\mathcal{L}(E) > 0$, $\mathcal{L}(\mathbb{R} \backslash E) > 0$, there is $x \in \mathbb{R}$ for which $\underline{d}_E(x) > \delta$ and $\overline{d}_E(x) < 1 - \delta$.

Kolyada showed that such a δ exists and is at least $1/4$. It is clear that if the statement holds for some $\delta_1 > 0$, then it holds for any $\delta_2 < \delta_1$. We say that a non-trivial Lebesgue measurable set E is a *δ-exceptional set* if statement ($*$) does not hold for E and δ: that is, for each $x \in \mathbb{R}$ either $\underline{d}_E(x) \le \delta$ or $\overline{d}_E(x) \ge 1 - \delta$. Thus Kolyada's problem is equivalent to finding δ_0, the infimum of those δ for which there is a δ-exceptional set.

In [2], Szenes proves that $0.263 < \delta_0 < 0.272$ where the exact lower bound is the positive solution of the cubic

$$4x^3 + 2x^2 + 3x - 1 = 0$$

and the exact upper bound is the positive solution of

$$8x^3 + 4x^2 + 2x - 1 = 0.$$

In this paper, he also conjectures that δ_0 is given by this upper bound.

Szenes also shows that we can characterise δ_0 using a discrete analogue of the above formulation. A *configuration* C is a subset of $\mathbb{R}$ comprising of the half-line $(-\infty, 0]$ together with some finite collection of intervals contained in $[0, 1]$:

$$C = (-\infty, 0] \cup \bigcup_{k=1}^{n} I_k.$$

An $r > 0$ is a *δ-good radius* for a point $x \in \mathbb{R}$ and a set E if either $d_E(x,r) \ge 1-\delta$ or $d_E(x,r) \le \delta$. A *δ-exceptional configuration* is a configuration for which every $x \in \mathbb{R}$ has a δ-good radius. Clearly every interior and exterior point of a configuration has a δ-good radius and so to show a configuration is δ-exceptional, it is enough to find δ-good radii for the endpoints of the configuration. To find δ_0 we can rely on the following restatement of Proposition 2 from [2]:

($**$) δ_0 is the infimum of those δ for which there is a δ-exceptional configuration.

In the following section, we find a sequence of exceptional configurations for values of δ that are eventually strictly less than the positive solution to $8x^3 + 4x^2 + 2x - 1 = 0$, thus showing that Szenes's conjecture is false. There is no good reason for believing that the configurations that we construct are optimal, and the problem of determining the best value for δ remains open.

2 A $0.2710\ldots$-exceptional configuration

Theorem 2.1 *Let δ be the positive solution to*

$$2\delta^3 + 2\delta^2 + 3\delta = 1. \tag{1}$$

Then $\delta_0 \leq \delta$.

Proof. The proof is by construction. We exhibit a sequence of configurations where the n^{th} configuration is δ_n-exceptional for $(\delta_n)_{n=1}^{\infty}$ a decreasing sequence whose limit is δ.

2.1 Periodic part

Let $\lambda \in (\frac{1}{2}, \frac{2}{3})$, and set $\varepsilon := \frac{1}{2} - \frac{3\lambda}{4} > 0$. Then for $I_1 := [0, \frac{\lambda}{2}]$ and $I_2 := [\frac{\lambda}{2} + \varepsilon, \lambda + \varepsilon] = [\frac{1}{2} - \frac{\lambda}{4}, \frac{1}{2} + \frac{\lambda}{4}]$, define

$$S_\lambda := \bigcup_{i \in \{0\} \cup \mathbb{N}} (i + (I_1 \cup I_2)).$$

Claim 2.2 *For any $\delta \geq \frac{1-\lambda}{1+\frac{\lambda}{2}}$, each positive real number has a δ-good radius for S_λ.*

Proof of claim: As observed earlier, every positive number that is not an endpoint always has a good radius so suppose that x is some endpoint of an interval of S_λ, other than 0.

If $x \in \mathbb{N}$, then the ball around x with radius $\lambda + \varepsilon = \frac{1}{2} + \frac{\lambda}{4}$ contains three intervals of S_λ, each of length $\frac{\lambda}{2}$. Thus the density of S_λ in this ball, $d_{S_\lambda}(x, \lambda+\varepsilon)$, is

$$\frac{\frac{3\lambda}{2}}{1 + \frac{\lambda}{2}} = 1 - \frac{1-\lambda}{1+\frac{\lambda}{2}} \geq 1 - \delta$$

and so the radius is δ-good. Exactly the same radius and calculation apply for the symmetric case where $x = n + \lambda + \varepsilon$ with $n \in \mathbb{N}$.

If x is one of the remaining endpoints, of the form $n + \frac{\lambda}{2}$ or $n + \frac{\lambda}{2} + \varepsilon$, then the ball around x of radius $\frac{\lambda}{2}$ meets only one component of the complement of S_λ and this component has length ε. Thus

$$d_{S_\lambda}(x, \lambda/2) = 1 - \frac{\varepsilon}{\lambda} = 1 - \frac{\frac{1}{2} - \frac{3\lambda}{4}}{\lambda} = \frac{7}{4} - \frac{1}{2\lambda} \geq \frac{3}{4} \geq 1 - \delta$$

since $\delta \geq \frac{1-\lambda}{1+\frac{\lambda}{2}} \geq \frac{1}{4}$.

Remark 2.3 *Notice that, since the largest radius in the above argument can be chosen to be at most $\lambda + \varepsilon$, for each $n \in \mathbb{N}$, any point x of*

$$J_n := (0, n + \lambda + \varepsilon)$$

has a δ-good radius r for S_λ so that the interval $(x - r, x + r)$ is contained in J_n.

We now proceed to the construction of our configuration.

2.2 The construction for a given λ

Suppose that $\lambda \in (\frac{1}{2}, \frac{2}{3})$ and $m \in (0, \frac{1}{2})$ have been given. For $n \in \mathbb{N}$, we define the configuration C_n to be the half-line $(-\infty, 0]$ together with the image of $S_\lambda \cap J_n$ under the affine transformation that sends 0 to m and $n + \lambda + \varepsilon$ to 1. The invariance of densities under this transformation means that, by Remark 2.3, every point in $(m, 1)$ has an associated δ-good radius for $\delta \geq \frac{1-\lambda}{1+\frac{\lambda}{2}}$. Thus it is enough to show that the remaining three endpoints $0, m$ and 1 of C_n have δ-good radii (given by $r(0) = 1$, $r(m) = 1 - m$ and $r(1) = 1$, respectively) in order to conclude that C_n is a δ-exceptional configuration.

Since $B(0,1) \supset B(m, 1-m)$ and C_n has full measure in $B(0,1) \backslash B(m, 1-m)$, we deduce that $d_{C_n}(0,1) > d_{C_n}(m, 1-m)$. Hence it is enough to show that $d_{C_n}(m, 1-m) \geq 1 - \delta$ and $d_{C_n}(1,1) \leq \delta$.

Let λ_n denote the density of C_n in the interval $(m, 1)$, that is $\lambda_n = \frac{\mathcal{L}(C_n \cap (m,1))}{1-m}$. Then

$$\lambda_n = \frac{(n+1)\lambda}{n + \lambda + \varepsilon} \quad \left(= \frac{4(n+1)\lambda}{4n + 2 + \lambda} \right). \tag{2}$$

The density of C_n within the relevant radius of each of the two endpoints is then given by

$$d_{C_n}(m, 1-m) = 1 - \frac{m + (1-m)(1-\lambda_n)}{2(1-m)} \tag{3}$$

$$d_{C_n}(1,1) = \frac{\lambda_n(1-m)}{2}. \tag{4}$$

Thus for C_n to be a δ-exceptional configuration, we need only to choose δ so that

$$\delta \geq \max\left(\frac{1-\lambda}{1+\frac{\lambda}{2}}, \frac{m + (1-m)(1-\lambda_n)}{2(1-m)}, \frac{\lambda_n(1-m)}{2} \right).$$

(Here the first term comes from Claim 2.2 and the remaining two come from (3) and (4).) We set δ equal to the last term, and suppose m was chosen so that δ is also equal to the penultimate term. That is, we choose $\delta = \delta_n$, say, so that

$$2\delta_n := \lambda_n(1-m) = \frac{1 - \lambda_n + \lambda_n m}{1-m}$$

which gives

$$2\delta_n = \frac{1 - 2\delta_n}{2\delta_n/\lambda_n}$$

and so

$$4\delta_n^2 + 2\lambda_n\delta_n - \lambda_n = 0.$$

This last equation determines the value of δ_n for the configuration C_n. It still remains to choose $\lambda \in (\frac{1}{2}, \frac{2}{3})$ so that $\delta_n \geq \frac{1-\lambda}{1+\frac{\lambda}{2}}$ and $m = 1 - 2\delta_n/\lambda_n \in (0, \frac{1}{2})$.

2.3 Checking for the optimal λ

Let λ and δ be the positive solution of

$$4\delta^2 + 2\lambda\delta - \lambda = 0 \quad \text{and} \quad \delta = \frac{1-\lambda}{1+\frac{\lambda}{2}}. \tag{5}$$

(So $\delta = 0.2710\ldots$ and $\lambda = 0.6419\ldots \in (\frac{1}{2}, \frac{2}{3})$.) We can see from (2) that $\lambda_n \searrow \lambda$ as $n \to \infty$. Since $f(\delta) = \frac{4\delta^2}{1-2\delta}$ is an increasing function of δ on $(0, \frac{1}{2})$, we deduce that $\delta_n \searrow \delta$ as $n \to \infty$. Hence for each $n \in \mathbb{N}$,

$$\delta_n > \delta = \frac{1-\lambda}{1+\frac{\lambda}{2}}.$$

It only remains to verify that if n is sufficiently large, then $m = 1-2\frac{\delta_n}{\lambda_n} \in (0, \frac{1}{2})$. However, since

$$4\delta_n^2 + 2\lambda_n\delta_n - \lambda_n = 0,$$

we deduce that

$$m = 1 - 2\delta_n/\lambda_n = \frac{1}{2}\left(3 - \sqrt{1 + 4/\lambda_n}\right).$$

But $\frac{1}{2}\left(3 - \sqrt{1+4/\lambda}\right) = 0.310\ldots \in (0, \frac{1}{2})$, and so $m \in (0, \frac{1}{2})$ if n is sufficiently large.

Since we can choose δ_n to be arbitrarily close to δ, we conclude

$$\delta_0 \le \delta = 0.2710..,$$

as required.

References

[1] V. I. Kolyada, *On the metric Darboux property*, Analysis Math., **9**, no. 4, (1983) 291-312 (in Russian).

[2] A. Szenes, *Exceptional points for Lebesgue's density theorem in the real line*, http://arxiv.org/abs/math/0702432.