Hausdorff dimension of sets of divergence arising from continued fractions

PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 00, Number 0, Pages 000–000
S 0002-9939(XX)0000-0

HAUSDORFF DIMENSION OF SETS OF DIVERGENCE ARISING FROM CONTINUED FRACTIONS

IAN SHORT

Abstract. A complex continued fraction can be represented by a sequence of Möbius transformations in such a way that the continued fraction converges if and only if the sequence converges at the origin. The set of divergence of the sequence of Möbius transformations is equivalent to the conical limit set from Kleinian group theory, and it is closely related to the Julia set from complex dynamics. We determine the Hausdorff dimensions of sets of divergence for sequences of Möbius transformations corresponding to certain important classes of continued fractions.

1. Introduction

This paper is about infinite continued fractions

$$\mathbf{K}(a_n|\, b_n) = \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cfrac{a_3}{b_3 + \cdots}}}\,,$$

where a_i and b_j are complex numbers, and each $a_i \neq 0$. We define $s_n(z) = a_n/(b_n + z)$ and $S_n = s_1 \circ s_2 \circ \cdots \circ s_n$ (or, more briefly, $S_n = s_1 s_2 \cdots s_n$) for each n, so that convergence of $\mathbf{K}(a_n|\, b_n)$ (in the usual sense) is equivalent to convergence of the sequence $S_n(0)$ within the extended complex plane $\mathbb{C}_\infty$. If $S_n(0)$ converges to a point p then, because $S_n(\infty) = S_{n-1}(0)$, $S_n(\infty)$ also converges to p, and in fact with exception of at most one point z, the limit $S_n(z)$, if it exists, is p (see Theorem 3.3). The *set of divergence* of S_n is the set of points z in $\mathbb{C}_\infty$ for which the sequence $S_n(z)$ diverges. Sets of divergence for general sequences of Möbius transformations have been studied in [5, 6, 14]. The set of divergence is closely related to the *limit set* from Kleinian group theory and the *Julia set* from complex dynamics. Indeed, sequences such as S_n associated with continued fractions share similar properties with sequences arising in Kleinian group theory and complex dynamics (see [1, 3]). Whilst there is a large body of literature on the Hausdorff dimension of limit sets and Julia sets, there seems to be little known about the Hausdorff dimension of

---

2000 *Mathematics Subject Classification.* Primary: 37F35, 40A15; Secondary: 30B70, 30F45, 51B10.

Key words and phrases. Continued fractions, convergence, divergence, Hausdorff dimension, Möbius transformations, sets of divergence.

I thank the referee for useful remarks; in particular, for suggestions which strengthened Theorem 3.4.

sets of divergence arising from continued fractions. We take a first step in filling this gap by calculating the dimensions of sets of divergence for three well known theorems on complex and integer valued continued fractions.

The *Śleszyński-Pringsheim Theorem* states that $\mathbf{K}(a_n | b_n)$ converges provided that $|b_n| \geqslant 1 + |a_n|$ for each positive integer n. Our first result is a generalisation of this theorem. We denote the unit disc by $\mathbb{D}$.

Theorem 1.1. *If $|b_n| \geqslant 1 + |a_n|$ for $n = 1, 2, \dots$ then the set of divergence of S_n is a subset of $\mathbb{C}_\infty \setminus \mathbb{D}$ of Hausdorff dimension less than or equal to 1. Furthermore, there exists such a sequence S_n whose set of divergence is a subset of $\mathbb{C}_\infty \setminus \mathbb{D}$ of Hausdorff dimension 1.*

Since $0 \in \mathbb{D}$ it follows immediately from Theorem 1.1 that the sequence $S_n(0)$ converges, and hence $\mathbf{K}(a_n | b_n)$ converges. Therefore Theorem 1.1 generalises the Śleszyński-Pringsheim Theorem. In fact, the Śleszyński-Pringsheim Theorem is a special case of the *Hillam-Thron Theorem*, and we also consider Hausdorff dimensions of sets of divergence in this more general setting.

We next turn to the *Seidel-Stern Theorem*, which states that if $b_1, b_2, \dots$ are positive numbers, and $\sum_n b_n$ diverges, then $\mathbf{K}(1 | b_n)$ converges. There is also a converse to the Seidel-Stern Theorem, but that does not concern us here. We define $t_n(z) = 1/(b_n + z)$ and $T_n = t_1 t_2 \cdots t_n$, and retain this notation throughout the paper.

Theorem 1.2. *If $b_n \geqslant 0$ for $n = 1, 2, \dots,$ and $\sum_n b_n$ diverges, then the set of divergence of T_n is a subset of $(-\infty, 0)$. Furthermore, there exists such a sequence T_n which diverges everywhere on $(-\infty, 0)$.*

Theorem 1.2 contains the Seidel-Stern Theorem because 0 lies outside the set of divergence of T_n. The first part of Theorem 1.2 is known (see, for example, [4, Theorem 1.8 (ii)]).

Our final result is about continued fractions $\mathbf{K}(1 | b_n)$ in which the b_n are positive integers. By the Seidel-Stern Theorem, all such continued fractions converge. The usual Hausdorff dimension is not sensitive enough for sets of divergence associated with these continued fractions, so instead we use *logarithmic* Hausdorff dimension, which is defined in Section 2. Note that a set with finite logarithmic Hausdorff dimension has (usual) Hausdorff dimension 0.

Theorem 1.3. *If $b_1, b_2, \dots$ are positive integers then the set of divergence of T_n is a subset of $(-\infty, -1)$ of logarithmic Hausdorff dimension less than or equal to 1. Furthermore, there exists such a sequence T_n whose set of divergence contains a closed, uncountable set within $(-\infty, -1)$.*

We do not have an example of a sequence T_n from Theorem 1.3 whose set of divergence has logarithmic Hausdorff dimension 1.

The novelty of each of these three theorems is that we obtain precise information about the sets on which our sequences of Möbius transformations converge. We make use of hyperbolic geometry and techniques borrowed from the theory of Kleinian groups. Because of this geometric approach, our results and methods generalise to higher dimensions in a straightforward fashion.

Finally, we remark that in the metrical theory of continued fractions there is also interest in Hausdorff dimensions of sets associated with continued fractions (see, for example, [12, 17]). Typically, this theory involves study of the set of all possible values of $\mathbf{K}(1|\,b_n)$ when the coefficients b_n are restricted in some fashion (for instance, they may be bounded positive integers). In contrast, we focus on sets of divergence associated to individual continued fractions.

2. Hausdorff dimension

This section contains background information on Hausdorff dimension, which can be found in more detail in [7, 8, 16]. We use the *chordal metric* on $\mathbb{C}_\infty$ to define Hausdorff dimension, which is the metric inherited from the Euclidean metric on the unit sphere by stereographic projection. Given a subset X of $\mathbb{C}_\infty$, and $\varepsilon > 0$, an *ε-cover of X* is a collection of subsets $U_1, U_2, \dots$ of $\mathbb{C}_\infty$ for which $X \subseteq \bigcup_n U_n$, and such that, for each n, the chordal diameter $\operatorname{diam}(U_n)$ of U_n does not exceed ε. A *dimension function* is a function $f : [0, +\infty) \to [0, +\infty)$ that is increasing, continuous, and satisfies $f(0) = 0$. Given a dimension function f let

$$H^f_\varepsilon(X) = \inf\left\{\sum_{n=1}^{\infty} f(\operatorname{diam}(U_n)) \,:\, U_1, U_2, \dots \text{ is an } \varepsilon\text{-cover of } X\right\}$$

and

$$H^f(X) = \lim_{\varepsilon\to 0} H^f_\varepsilon(X).$$

The function H^f is known as *Hausdorff measure with respect to f*, and it is an outer measure on $\mathbb{C}_\infty$.

Given two dimension functions f and g such that $f(t)/g(t) \to 0$ as $t \to 0$ it is straightforward to show that if $H^g(X) < +\infty$ then $H^f(X) = 0$. The dimension functions most widely used are the collection $f_s(t) = t^s$, for $s > 0$. Since, for $s_1 < s_2$, $f_{s_2}(t)/f_{s_1}(t) \to 0$ as $t \to 0$, there is a unique value d of s in $[0, +\infty]$ such that $H^{f_s}(X) = +\infty$ for $s < d$, and $H^{f_s}(X) = 0$ for $s > d$. This value d is the *Hausdorff dimension of X*.

An alternative collection of dimension functions are

$$g_s(t) = \begin{cases} 1/\left(\log \frac{1}{t}\right)^s & \text{if } 0 < t \leqslant 1/e, \\ et & \text{otherwise,} \end{cases}$$

for $s > 0$. It is only the value of g_s near 0 that matters; the extension to $[0, +\infty)$ is chosen for convenience. Since, for $s_1 < s_2$, $g_{s_2}(t)/g_{s_1}(t) \to 0$ as $t \to 0$, there is a unique value d of s in $[0, +\infty]$ such that $H^{g_s}(X) = +\infty$ for $s < d$, and $H^{g_s}(X) = 0$ for $s > d$. This value d is the *logarithmic Hausdorff dimension of X*. Notice that, for any positive numbers s_1 and s_2, $f_{s_2}(t)/g_{s_1}(t) \to 0$ as $t \to 0$, which means that if X has finite logarithmic Hausdorff dimension, then it has Hausdorff dimension zero.

Let k be one of the maps f_s or g_s. Each Möbius transformation ϕ is a bi-Lipschitz map of $\mathbb{C}_\infty$ with respect to the chordal metric (see [3, Section 3]), and it follows that $H^k(X) = 0$ if and only if $H^k(\phi(X)) = 0$. This means that X and $\phi(X)$ have the same Hausdorff dimension, and they also have the same logarithmic Hausdorff dimension.

3. Hyperbolic geometry

This section contains background information on hyperbolic geometry, which can be found in more detail in [2, 3, 13, 15]. Let $\mathbb{R}^3_\infty$ denote the one point compactification of $\mathbb{R}^3$. We identify $\mathbb{C}$ with the plane $x_3 = 0$ in $\mathbb{R}^3$ by the correspondence $x_1 + ix_2 \mapsto (x_1, x_2, 0)$, and we define

$$\mathbb{H}^3 = \{(x_1, x_2, x_3) : x_3 > 0\},$$

so that the closure $\overline{\mathbb{H}^3}$ of $\mathbb{H}^3$ in $\mathbb{R}^3_\infty$ is equal to $\mathbb{H}^3 \cup \mathbb{C}_\infty$. We endow $\overline{\mathbb{H}^3}$ with the chordal metric χ which is inherited from the Euclidean metric on the closed unit ball in $\mathbb{R}^3$ by stereographic projection. For points $z, w \neq \infty$, χ is given by the formulae

$$\chi(z,w) = \frac{2|z-w|}{\sqrt{1+|z|^2}\sqrt{1+|w|^2}}\,, \quad \chi(z,\infty) = \frac{2|z|}{\sqrt{1+|z|^2}}\,. \tag{3.1}$$

The set $\mathbb{H}^3$ is a model of three-dimensional hyperbolic space when equipped with the Riemannian metric $|dx|/x_3$, and we denote hyperbolic distance in $\mathbb{H}^3$ by ϱ. For points $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$ in $\mathbb{H}^3$, ϱ satisfies

$$\cosh \varrho(x,y) = 1 + \frac{|x-y|^2}{2x_3y_3}. \tag{3.2}$$

The group $\mathcal{M}$ of Möbius transformations acts on both $\mathbb{C}_\infty$ and $\mathbb{H}^3$. It is the full group of conformal automorphisms of $\mathbb{C}_\infty$, and the full group of conformal hyperbolic isometries of $\mathbb{H}^3$. We switch freely between the two actions.

Let $j = (0, 0, 1)$. A sequence $F_1, F_2, \dots$ of Möbius transformations *converges generally* to a point p in $\mathbb{C}_\infty$ if the sequence $F_n(j)$ converges to p (in the chordal metric). This terminology was introduced by Jacobsen (now Lorentzen) in [9, Definition 3.1]. Our definition is taken from [1, Theorem 3.5] and [3, Section 6], and it differs from (but is equivalent to) Lorentzen's original definition. Given another point w in $\mathbb{H}^3$, elementary hyperbolic geometry can be used to show that $F_n(j)$ converges to p if and only if $F_n(w)$ converges to p. The next basic lemma on general convergence, which we do not prove, follows from [3, Theorem 6.6].

Lemma 3.1. *Suppose that a sequence of Möbius transformations F_n converges at two distinct points of $\mathbb{C}_\infty$ to a constant p in $\mathbb{C}_\infty$. Then F_n converges generally to p.*

In particular, for a convergent continued fraction with value p, the sequence S_n from Section 1 converges to p at both 0 and ∞, and hence S_n converges generally to p.

Suppose that a sequence F_n converges generally to a point p. Since F_n preserves hyperbolic distance, $\varrho(j, F_n^{-1}(j)) = \varrho(F_n(j), j) \to \infty$ as $n \to \infty$. Therefore the backwards orbit $F_n^{-1}(j)$ accumulates only on $\mathbb{C}_\infty$. We describe the set of accumulation points as the *Julia set* of F_n, and the complement of the Julia set as the *Fatou set* of F_n. The Fatou set is the largest open set in $\mathbb{C}_\infty$ on which F_n converges locally uniformly to p [3, Theorem 9.6]. The point j has no special significance in these definitions; the set of accumulation points of the backwards orbit $F_n^{-1}(w)$ is the same for all points w in $\overline{\mathbb{H}^3}$ other than p (see, for example, [3, Lemma 9.4]). We often use the next elementary lemma.

Lemma 3.2. *Suppose that F_n is a generally convergent Möbius sequence, and there is a subset X of $\mathbb{C}_\infty$ that contains at least two points, such that $F_n^{-1}(X) \subseteq X$ for each n. Then the Julia set of F_n is contained in $\overline{X}$.*

Proof. Choose a geodesic γ that joins two distinct points x and y of X, and choose an element w of γ. If z is a member of the Julia set of F_n then there is a sequence $n_1 < n_2 < \cdots$ such that $\chi(F_{n_i}^{-1}(w), z) \to 0$. This means that, given any open chordal disc D in $\mathbb{C}_\infty$ that contains z, for sufficiently large n_i the point $F_{n_i}^{-1}(w)$ lies in the hyperbolic convex hull of D (a hyperbolic half-space). For such n_i, one of the end-points of the geodesic $F_{n_i}^{-1}(\gamma)$ lies in D. Hence $\min\{\chi(F_{n_i}^{-1}(x), z), \chi(F_{n_i}^{-1}(y), z)\} \to 0$. Since $F_n^{-1}(X) \subseteq X$ for each n, we see that $F_{n_i}^{-1}(x)$ and $F_{n_i}^{-1}(y)$ belong to X. Therefore $z \in \overline{X}$. □

Of more interest to us than the Julia and Fatou sets is the *conical limit set* of F_n, which is a subset of the Julia set. A point z in $\mathbb{C}_\infty$ is a *conical limit point* of the sequence F_n if, given a hyperbolic geodesic γ that lands at z, and a point ζ in $\mathbb{H}^3$, there is a subsequence $w_1, w_2, \dots$ of $F_1^{-1}(\zeta), F_2^{-1}(\zeta), \dots$ that converges to z in the chordal metric, and satisfies $\sup_n \varrho(\gamma, w_n) < +\infty$. This definition is independent of the choice of point ζ and geodesic γ. The *conical limit set* of F_n consists of all its conical limit points. The next theorem indicates that, for a generally convergent sequence, the conical limit set is equal to the set of divergence (with one exceptional case).

Theorem 3.3 ([1, Theorem 5.2 and Proposition 5.3])**.** *Let F_n be a sequence of Möbius transformations that converges generally to a point p. Then either (i) the conical limit set of F_n equals the set of divergence of F_n, and $F_n(z)$ converges to p for points outside this set, or (ii) the conical limit set consists of a single point w, the set of divergence is empty, and $F_n(z) \to p$ if and only if $z \neq w$.*

The following theorem is about the Hausdorff measure of conical limit sets. We sketch the details of a proof, because although the theorem is likely to be familiar to Kleinian group theorists (it is a more general version of [13, Theorem 9.3.1]), we are unable to find a suitable reference. In this sketch proof (and nowhere else in the paper) we assume that the sequence F_n is acting on the unit ball model of hyperbolic space, so that $\mathbb{C}_\infty$ is replaced by the unit sphere $\mathbb{S}^2$, the point j in the statement of the theorem is replaced by 0, and the chordal metric is replaced by the restriction of the Euclidean metric to the closed ball.

Theorem 3.4. *Suppose that g is a dimension function and F_n is a generally convergent sequence of Möbius transformations such that $\sum_n g(-\exp[\varrho(j, F_n(j))]) < +\infty$. Then the Hausdorff measure with respect to g of the conical limit set of F_n is 0.*

Proof. Let X denote the conical limit set of F_n. For each point z in $\mathbb{S}^2$ let γ_z denote the geodesic half-line from 0 to z (a Euclidean radius). Next, for each positive integer m, let X_m consist of those points z in $\mathbb{S}^2$ for which there is a subsequence $w_1, w_2, \dots$ of $F_1^{-1}(0), F_2^{-1}(0), \dots$ that converges to z, and satisfies $\varrho(\gamma_z, w_n) < m$ for each n. We must prove that $H^g(X) = 0$, but because $X = \bigcup_m X_m$ and H^g is countably subadditive, we need only show that $H^g(X_m) = 0$ for each m.

Define $U_n = \{z \in \mathbb{S}^2 : \varrho(\gamma_z, F_n^{-1}(0)) < m\}$ for $n = 1, 2, \dots$. Observe that, for any k, $U_k, U_{k+1}, \dots$ is a cover of X_m. Each set U_n is a spherical disc, and using hyperbolic trigonometry (see [2, Theorem 7.11.2 (ii)]) we find that for sufficiently large n the Euclidean diameter $\mathrm{diam}(U_n)$ of U_n satisfies

$$\mathrm{diam}(U_n) = \frac{2\sinh m}{\sinh \varrho(0, F_n^{-1}(0))} < M \exp[-\varrho(0, F_n(0))],$$

where $M = 8\sinh m$. It follows from this equation that there is a positive integer N such that, for each n, U_n can be covered by N spherical discs $V_{n1}, V_{n2}, \dots, V_{nN}$, each of Euclidean diameter $\exp[-\varrho(0, F_n(0))]$. Now

$$\sum_{n=k}^{\infty}\sum_{i=1}^{N} g(\mathrm{diam}(V_{ni})) = N \sum_{n=k}^{\infty} g(\exp[-\varrho(0, F_n(0))]).$$

Since the sum on the right-hand side converges, and $\{V_{ni} : n \geqslant k, 1 \leqslant i \leqslant N\}$ is a cover of X_m for each k, we see that $H^g(X_m) = 0$. Thus $H^g(X) = 0$. □

We apply Theorem 3.4 first to the dimension functions $f_s(t) = t^s$ from Section 2, and then to the dimension functions g_s from Section 2, to obtain the following two corollaries.

Corollary 3.5 ([13, Theorem 9.3.1])**.** *Suppose that F_n is a generally convergent sequence of Möbius transformations such that $\sum_n \exp[-s\varrho(j, F_n(j))] < +\infty$. Then the conical limit set of F_n has Hausdorff dimension at most s.*

Corollary 3.6. *Suppose that F_n is a generally convergent sequence of Möbius transformations such that $\sum_n 1/\left(\varrho(j, F_n(j))\right)^s < +\infty$. Then the conical limit set of F_n has logarithmic Hausdorff dimension at most s.*

Corollary 3.5 relates to the theory of Kleinian groups. A Kleinian group G is said to be of *convergence type* if $\sum_{g\in G} \exp[-\varrho(j, g(j))]$ converges (see [13, Section 1.6]). The *critical exponent* of G is

$$\inf\left\{ s > 0 \, : \, \sum_{g\in G} \exp[-s\varrho(j, F_n(j))] < +\infty \right\},$$

and if G is geometricaly finite then this equals the Hausdorff dimension of the limit set of G (see [13, Theorem 9.3.6]). We do not use any of the terminology from this paragraph again.

4. Proof of Theorem 1.1: part I

In this section we prove the following generalisation of the Hillam-Thron Theorem.

Theorem 4.1. *Suppose that D is an open disc in $\mathbb{C}_\infty$, u is a point in D, v is a point in $\mathbb{C}_\infty \setminus \overline{D}$, and $s_1, s_2, \dots$ is a sequence of Möbius maps that satisfies $s_n(v) = u$ and $s_n(D) \subseteq D$ for each n. Then $S_n = s_1 s_2 \cdots s_n$ is generally convergent, and the conical limit set of S_n is a subset of $\mathbb{C}_\infty \setminus D$ of Hausdorff dimension less than or equal to* 1*.*

Notice that, by Theorem 3.3, the sequence S_n of Theorem 4.1 converges on D to a constant. Usual statements of the Hillam-Thron Theorem, such as [10, Theorem 4.37], have essentially the same hypotheses as Theorem 4.1, but they have only this weaker conclusion that S_n converges on D to a constant. There are variants on the Hillam-Thron Theorem (such as [11, Lemma 3.8]) in which the hypothesis $s_n(v) = u$ is weakened, and our observations on sets of divergence apply to most, if not all, of these alternative theorems.

We now explain how the first part of Theorem 1.1 follows from Theorem 4.1. Let $s_n(z) = a_n/(b_n + z)$ for $n = 1, 2, \dots$. It is easily shown that $|b_n| \geqslant 1 + |a_n|$ if and only if $s_n(\mathbb{D}) \subseteq \mathbb{D}$. Thus, given a sequence $s_1, s_2, \dots$ such that $|b_n| \geqslant 1 + |a_n|$ for each n, the hypotheses of Theorem 4.1 are satisfied with $D = \mathbb{D}$, $u = 0$, and $v = \infty$. Hence, by Theorem 4.1, the set of divergence of S_n is a subset of $\mathbb{C}_\infty \setminus \mathbb{D}$ of Hausdorff dimension less than or equal to 1. In fact, Theorem 1.1 is just the special case of Theorem 4.1 in which u and v are inverse points with respect to ∂D.

The key ingredient needed to move from existing statements of the Hillam-Thron Theorem to Theorem 4.1 is the following result. Recall that $j = (0, 0, 1)$.

Theorem 4.2. *Given a sequence F_n of Möbius transformations, suppose there are two points x and y in $\mathbb{C}_\infty$ such that $\sum_n \chi(F_n(x), F_n(y))$ converges. Then $\sum_n \exp[-\varrho(j, F_n(j))]$ also converges.*

The proof of Theorem 4.2 uses the next two lemmas on hyperbolic geometry. The first is well known, so we omit the proof.

Lemma 4.3 (Theorem 7.9.1, [2])**.** *Let γ be a geodesic in $\mathbb{H}^3$ that lands at points a and b in $\mathbb{C}_\infty$. Then*

$$\cosh \varrho(j, \gamma) = \frac{2}{\chi(a, b)}.$$

Lemma 4.4. *For a Möbius map F we have*

$$\exp[-\varrho(j, F(j))] \leqslant \tfrac{1}{2}\chi(F(0), F(\infty)).$$

Proof. Let γ denote the geodesic between 0 and ∞. Since $j \in \gamma$ we have, by Lemma 4.3,

$$\cosh \varrho(j, F(j)) \geqslant \cosh \varrho(j, F(\gamma)) = \frac{2}{\chi(F(0), F(\infty))}.$$

The result follows because $\exp[\varrho(j, F(j))] \geqslant \cosh \varrho(j, F(j))$. □

Proof of Theorem 4.2. Choose a Möbius transformation g such that $g(x) = 0$ and $g(y) = \infty$, and define $G_n = gF_ng^{-1}$. Since Möbius transformations are bi-Lipshitz maps of $\mathbb{C}_\infty$ with respect to the chordal metric, we see that $\sum_n \chi(G_n(0), G_n(\infty))$ converges. Hence, by Lemma 4.4, $\sum_n \exp[-\varrho(j, G_n(j))]$ converges. Let $u = g(j)$. Then, by preservation of hyperbolic distance under Möbius transformations, we deduce that $\sum_n \exp[-\varrho(u, F_n(u))]$ converges. Since $\varrho(j, F_n(j))$ differs from $\varrho(u, F_n(u))$ by no more than $2\varrho(j, u)$, we conclude that $\sum_n \exp[-\varrho(j, F_n(j))]$ converges. □

Theorem 4.1 could now be proven by modifying existing proofs of the Hillam-Thron Theorem, and then applying Theorem 4.2; however, for clarity we provide all the details in a new, complete proof which uses a mixture of hyperbolic and Euclidean geometry, and which generalises easily to higher dimensions. Two lemmas are required.

Lemma 4.5. *Suppose that a Euclidean circle A, with centre a and radius s, is contained within, and possibly internally tangent to, another Euclidean circle B, with centre b and radius t. If a point x that lies strictly inside A is the inverse point in A of another point y that lies strictly inside B, then $|x-y| \leqslant 4(t-s)$.*

Proof. That A lies inside B is equivalent to the equation $|a-b|+s \leqslant t$. This inequality, together with $s \leqslant |y-a|$ and $|y-b| \leqslant t$, gives

$$\begin{aligned} |x-y| &= |y-a| - \frac{s^2}{|y-a|} \\ &= \left(1+\frac{s}{|y-a|}\right)(|y-a|-s) \\ &\leqslant 2(|y-b|+|a-b|-s) \\ &\leqslant 4(t-s). \end{aligned}$$

□

In the next lemma we use the notation ϱ_D for the hyperbolic metric of a disc D which is strictly contained in $\mathbb{C}_\infty$. Notice that a Möbius transformation f is an isometry from the metric space (D, ϱ_D) to the metric space $(f(D), \varrho_{f(D)})$.

Lemma 4.6. *Let E be an open Euclidean disc with centre c and radius r. If x and y are two points in E then*

$$|x-y| < \exp[\varrho_E(x,y)](r-|x-c|).$$

Proof. By applying a Euclidean similarity we can assume that $c=0$ and $r=1$. Next, by applying a hyperbolic rotation in E about x, we can assume that y lies on the Euclidean line ℓ through x and 0, in the component of $\ell \setminus \{x\}$ that contains 0. This rotation increases the quantity $|x-y|$, but preserves $\varrho_E(x,y)$. Either y lies between 0 and x, or it lies on the other side of 0. In the first case we have

$$\varrho_E(x,y) = \log\left(\frac{1+|x|}{1-|x|}\right) - \log\left(\frac{1+|y|}{1-|y|}\right),$$

which implies that

$$|x-y| < 1-|y| = \left(\frac{1+|y|}{1+|x|}\right)\exp[\varrho_E(x,y)](1-|x|) \leqslant \exp[\varrho_E(x,y)](1-|x|).$$

In the second case we have

$$\varrho_E(x,y) = \log\left(\frac{1+|x|}{1-|x|}\right) + \log\left(\frac{1+|y|}{1-|y|}\right),$$

which implies that

$$|x-y| = |x|+|y| < \frac{(1+|x|)(1+|y|)}{1-|y|} = \exp[\varrho_E(x,y)](1-|x|).$$

□

Proof of Theorem 4.1. Recall that Hausdorff dimension is preserved under Möbius transformations. Therefore, after conjugating by a Möbius transformation, we may assume that D is the Euclidean unit disc centred on the origin. This means that, for each integer n, $S_n(D)$ is a Euclidean disc. Let $S_n(D)$ have centre c_n and radius r_n. Since $S_n(D) \subseteq S_{n-1}(D)$ we obtain

$$|c_n - c_{n-1}| + r_n \leqslant r_{n-1}, \tag{4.1}$$

from which it follows that r_n is a decreasing (and hence convergent) sequence.

Let v^* denote the inverse point to v in ∂D, and let $k = \varrho_D(v^*, u)$. Inverse points are preserved under Möbius transformations, and therefore $S_n(v)$ and $S_n(v^*)$ are inverse points in $\partial S_n(D)$. Since $S_n(v) = S_{n-1}(u)$ lies outside $\overline{S_n(D)}$, but strictly inside $S_{n-1}(D)$, we can apply Lemma 4.5 with $A = \partial S_n(D)$, $B = \partial S_{n-1}(D)$, $x = S_n(v^*)$, and $y = S_n(v)$ to obtain

$$|S_n(v) - S_n(v^*)| \leqslant 4(r_{n-1} - r_n). \tag{4.2}$$

Next, since $\varrho_{S_n(D)}(S_n(u), S_n(v^*)) = \varrho_D(u, v^*) = k$, we can apply Lemma 4.6 with $E = S_n(D)$, $x = S_n(v^*)$, and $y = S_n(u)$ to obtain

$$|S_n(v^*) - S_n(u)| < e^k(r_n - |S_n(v^*) - c_n|) < e^k|S_n(v) - S_n(v^*)|. \tag{4.3}$$

Given that $S_n(v) = S_{n-1}(u)$, we may combine (4.2) and (4.3) using the triangle inequality to obtain

$$|S_n(u) - S_{n-1}(u)| \leqslant 4(1 + e^k)(r_{n-1} - r_n).$$

Summing this equation over all positive integers n we deduce that $\sum_n |S_n(u) - S_{n-1}(u)|$ converges. Since $\chi(z, w) \leqslant 2|z - w|$ for complex numbers z and w, we see that $\sum_n \chi(S_n(u), S_{n-1}(u))$ also converges. It follows that $S_n(u)$ is a Cauchy sequence which converges to a point p. Since $S_n(v)$ also converges to p we see from Lemma 3.1 that S_n converges generally to p. What is more, since S_n^{-1} maps $\mathbb{C}_\infty \setminus D$ within itself for each n, we deduce from Lemma 3.2 that both the Julia set and conical limit set of S_n are contained in $\mathbb{C}_\infty \setminus D$. According to Theorem 4.2, convergence of $\sum_n \chi(S_n(u), S_n(v))$ implies convergence of $\sum_n \exp[-\varrho(j, S_n(j))]$. Hence, by Corollary 3.5, the Hausdorff dimension of the conical limit set of S_n does not exceed 1. □

In fact, we have proved the slightly stronger result that the conical limit set of a sequence S_n satisfying the hypotheses of Theorem 4.1 is a subset of $\mathbb{C}_\infty \setminus D$ of H^{f_1}-measure 0.

5. Proof of Theorem 4.1: part II

We must construct a sequence s_n satisfying the hypotheses of Theorem 1.1 for which S_n diverges on a set of Hausdorff dimension 1. We begin by describing a suitable set of Hausdorff dimension 1. Consider a positive constant $\kappa < 1/2$, and a null sequence of positive numbers $1 = \delta_0, \delta_1, \delta_2, \dots$ such that $\delta_n/\delta_{n-1} < \kappa$ for $n \geqslant 1$. We construct a nested sequence of sets $E_0 \supset E_1 \supset E_2 \supset \cdots$ such that each set E_n is a disjoint union of 2^n closed subintervals of $[0, 1]$, each of length δ_n. Let $E_0 = [0, 1]$. The sets $E_1, E_2, \dots$ are defined precisely by the following rule. Each interval $[a, b]$ of E_{n-1} (of length δ_{n-1}) contains exactly two intervals of E_n, namely $[a, a + \delta_n]$ and $[b - \delta_n, b]$.

This construction is a special case of a more general construction described in [7, Example 4.6], and there it is proven that the limit set $F = \bigcap_n E_n$ is a Cantor set with Hausdorff dimension

$$\liminf_{n\to\infty} \frac{n\log 2}{-\log \delta_n}.$$

Let us choose $\delta_n = 1/(n^2 2^{n-1})$, so that F has Hausdorff dimension 1. Let $K_1, K_2, \dots$ be a sequence consisting of all the intervals from all the sets E_n, $n \geqslant 1$, each interval occurring only once in the sequence. If $|K_i|$ denotes the length of K_i then

$$\sum_{n=1}^{\infty} |K_n| = \sum_{n=1}^{\infty} 2^n \delta_n = \sum_{n=1}^{\infty} \frac{2}{n^2} = \frac{\pi^2}{3}.$$

We scale the sets E_n and K_n linearly by a factor $3/\pi^2$ so that now, for example, $E_0 = [0, 3/\pi^2]$ and $\sum_n |K_n| = 1$ (but F still has Hausdorff dimension 1).

So far we have described a Cantor set of Hausdorff dimension 1 on the real line. Now we transfer that Cantor set to the unit circle $\partial\mathbb{D}$. For $n = 1, 2, \dots$, let $I_n = \{e^{it} : t \in K_n\}$, and let θ_n measure *half* the angle I_n makes with the origin; that is, $\theta_n = \frac{1}{2}|K_n|$. Therefore $\sum_n \theta_n = \frac{1}{2}$. Let X consist of those points in $\partial\mathbb{D}$ that are contained in infinitely many of the intervals I_n. In other words, X is the image of F under the map from $[0, 3/\pi^2]$ to $\mathbb{C}$ given by $t \mapsto e^{it}$, and since this map is bi-Lipschitz with respect to the Euclidean metric (and chordal metric), we see from [7, Corollary 2.4] that X has Hausdorff dimension 1.

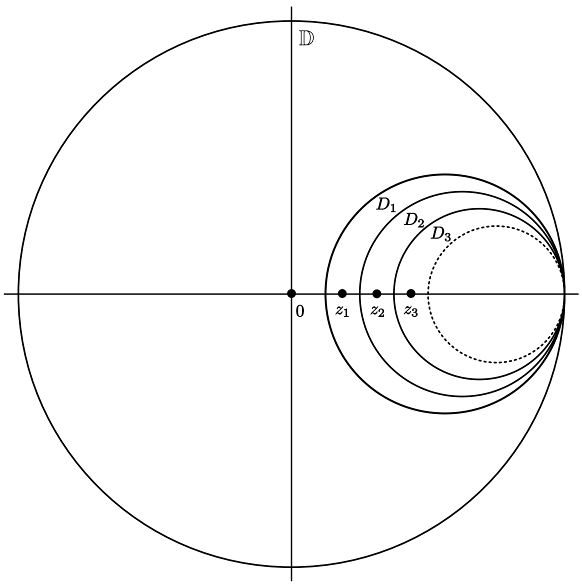

FIGURE 5.1

We move on to describe a sequence S_n that diverges on this set X of Hausdorff dimension 1. Let $z_0 = 0$ and, for $n \geqslant 1$, let z_n be the point $\theta_1 + \cdots + \theta_n$ in the complex plane. Define, for $n \geqslant 1$,

$$r_n = \frac{(1 - z_{n-1})(1 - z_n)}{2 - z_{n-1} - z_n}, \quad c_n = 1 - r_n.$$

Let D_n be the open Euclidean disc with centre c_n and radius r_n. Then z_{n-1} is exterior to D_n, z_n is interior to D_n, and z_{n-1} and z_n are inverse points with respect to D_n. The points z_n and discs D_n are shown in Figure 5.1.

Let S_n be a Möbius transformation which maps $\mathbb{D}$ to D_n, ∞ to z_{n-1}, and 0 to z_n. This is possible because 0 and ∞ are inverse points with respect to $\partial\mathbb{D}$, and z_{n-1} and z_n are inverse points with respect to ∂D_n. In fact there is still freedom in the choice of S_n as we can post compose S_n with a hyperbolic rotation of the disc D_n about the point z_n. Thus we may further assume that S_n maps the interval I_n to an interval J_n in ∂D_n that is centred on the point 1 (1 is the point on ∂D_n that is farthest from z_n).

Let J_n have end points a_n and b_n. Consider the hyperbolic triangle Δ with vertices z_n, c_n, and a_n. By conformality of S_n, the angle at z_n of Δ is θ_n. Let ϕ_n denote the angle of Δ at c_n. See Figure 5.2.

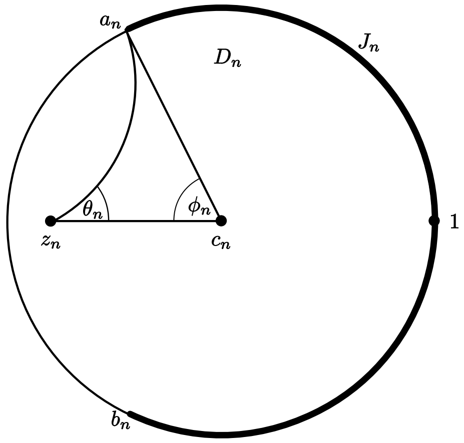

FIGURE 5.2

We can apply a cosine formula (see [2, Section 7.12]) to Δ to obtain

$$\cosh \varrho_{D_n}(c_n, z_n) = \frac{1+\cos\theta_n \cos\phi_n}{\sin\theta_n \sin\phi_n}.$$

Also, using a standard formula for hyperbolic distance in D_n, namely

$$\varrho_{D_n}(c_n, z_n) = \log\left(\frac{r_n + |z_n - c_n|}{r_n - |z_n - c_n|}\right),$$

we find that $\exp[\varrho_{D_n}(c_n, z_n)] = (2 - z_n - z_{n-1})/\theta_n$, and hence

$$\cosh \varrho_{D_n}(c_n, z_n) = \frac{(1-z_n)^2 + (1-z_{n-1})^2}{(2 - z_{n-1} - z_n)\theta_n}.$$

Therefore

$$1 + \cos\theta_n \cos\phi_n = \frac{(1-z_n)^2 + (1-z_{n-1})^2}{2 - z_{n-1} - z_n} \frac{\sin\theta_n}{\theta_n} \sin\phi_n.$$

Since $\theta_n \to 0$, $z_n \to \frac{1}{2}$, and $(\sin\theta_n)/\theta_n \to 1$ as $n \to \infty$ we see that any limit point ϕ of the sequence ϕ_n is a solution of the equation

$$2 + 2\cos\phi = \sin\phi.$$

The only solutions of this equation lie in $[\pi/2, \pi]$, so there is a neighbourhood of the limit point $\frac{1}{2}$ of z_n that intersects only finitely many of the intervals J_n. This means that if x is a point that lies in infinitely many of the sets I_n then $S_n(x)$ belongs to J_n for infinitely many n. Hence $S_n(x) \nrightarrow \frac{1}{2}$ as $n \to \infty$. That is, S_n does not converge to $\frac{1}{2}$ on the set X of Hausdorff dimension 1 defined earlier.

It remains to verify that S_n is of the form specified in Theorem 1.1. Define S_0 to be the identity map, and define $s_n = S_{n-1}^{-1}S_n$ for $n = 1, 2, \dots$. It follows from $S_n(\infty) = S_{n-1}(0)$ that $s_n(\infty) = 0$. Therefore we can write s_n in the form $s_n(z) = a_n/(b_n + z)$. Next, since $S_n(\mathbb{D}) \subseteq S_{n-1}(\mathbb{D})$ we see that $s_n(\mathbb{D}) \subseteq \mathbb{D}$, and this implies that $|b_n| \geqslant 1 + |a_n|$. Thus S_n *is* of the required form, so it converges generally to $\frac{1}{2}$, and since X has more than one point, we conclude, by Theorem 3.3, that S_n diverges on X. □

In contrast to this example, there are sequences s_n satisfying $|b_n| \geqslant 1 + |a_n|$ for each n such that S_n converges *everywhere* on $\mathbb{C}_\infty$ to a constant. For example, suppose that $a_n = -1$ and $b_n = 2$ for each n. Then $s_n(z) = -1/(2 + z)$ is a parabolic Möbius transformation with fixed point -1, so S_n converges everywhere to -1.

6. Proof of Theorem 1.2: part I

Let $t_n(z) = 1/(b_n + z)$ and $T_n = t_1 t_2 \cdots t_n$, and suppose that $b_n \geqslant 0$. We prove that if $\sum_n b_n$ diverges, then the set of divergence of T_n is a subset of $(-\infty, 0)$.

The Seidel-Stern Theorem ([4, Theorem 1.8] or [11, Theorem 3.13]) states that, with our hypotheses, T_n converges at 0 to a point p. Since $T_n(\infty) = T_{n-1}(0)$ we also have that $T_n(\infty) \to p$ as $n \to \infty$. Since T_n converges to p at two distinct points we see from Lemma 3.1 that T_n converges generally to p. Next, observe that $t_n^{-1}(-\infty, 0) \subseteq (-\infty, 0)$ for each n so that, by Lemma 3.2, the Julia set of T_n is contained in $[-\infty, 0]$. Since the conical limit set is a subset of the Julia set, we see from Theorem 3.3 that the set of divergence is contained in $(-\infty, 0)$.

7. Proof of Theorem 1.2: part II

In this section we construct a sequence T_n, with $b_n \geqslant 0$ and $\sum_n b_n = +\infty$, such that T_n diverges *everywhere* on $(-\infty, 0)$.

Let $h(z) = 1/z$. For $n = 1, 2, \dots$, define $g_n(z) = 1/(n + z)$ and $G_n = g_1 g_2 \cdots g_n$. Observe that $g_n(0, +\infty) \subseteq (0, 1)$ and $g_n^{-1}(-\infty, 0) \subseteq (-\infty, -n)$. Then $G_n(0, +\infty) \subseteq (0, 1)$ and

$$G_n^{-1}(-\infty, 0) \subseteq g_n^{-1}(-\infty, 0) = (-\infty, -n).$$

This means that G_n is a loxodromic Möbius transformation with repelling fixed point in $[-\infty, -n]$, and attracting fixed point in $[0, 1]$. Furthermore, because $G_n(0) \in (0, 1)$ for each n, and using the equations $G_n(\infty) = G_{n-1}(0)$ and $G_n(-n) = G_{n-1}(\infty)$, we see that neither ∞ nor $-n$ are fixed by G_n. Let U_n be an open interval in $(-\infty, -n)$ that contains the repelling fixed point of G_n, and which is chosen to be sufficiently small that $G_n(U_n) \subseteq (-\infty, 0)$. Note that $h(U_n) \subseteq (-1/n, 0)$.

For each positive integer n, choose a positive integer k_{n+1} such that the quantity $\varepsilon_{n+1} = (n + 1)/k_{n+1}$ is less than half the length of the interval $h(U_n)$. Let $k_1 = 1$, and $\varepsilon_1 = 1$. Define $r_n(z) = 1/(z + \varepsilon_n)$. Define a sequence $t_1, t_2, \dots$ of maps to be

$$h, h, r_1, h, r_1, \dots, h, r_1, \quad h, h, r_2, h, r_2, \dots, h, r_2, \quad h, h, r_3, h, r_3, \dots, h, r_3, \dots$$

where the map r_n occurs k_n times in the above sequence. Let $T_n = t_1 t_2 \cdots t_n$. Observe that $g_n = h(hr_n)^{k_n}$. This means that G_n is a subsequence of T_n.

Each map t_n can be expressed uniquely in the form $t_n(z) = 1/(b_n + z)$ (where $b_n = \varepsilon_m$ if $t_n = r_m$, and $b_n = 0$ if $t_n = h$). Clearly $b_n \geqslant 0$ and $\sum_n b_n$ diverges. We must show that T_n diverges on $(-\infty, 0)$. To this end, choose a positive number u (so that $-u < 0$). We will show that $T_n(-u)$ diverges. Observe that, for sufficiently large n, $g_n(-u) > 0$. Because each transformation g_i maps $(0, +\infty)$ within itself, we deduce that $G_n(-u) \in G_2(0, +\infty)$. Thus we have found a subsequence of $T_n(-u)$, each element of which lies in $(2/3, 1)$. Next we identify a subsequence of $T_n(-u)$ consisting of negative elements, which means that $T_n(-u)$ diverges.

Consider sufficiently large values of n that $1/n < u < n$. The inequality $1/n < u$ ensures that $-u$ lies to the *left* of $h(U_n)$ on the real axis. Therefore $u > \varepsilon_{n+1}$. Let q_{n+1} be the positive integer such that $0 \leqslant u - q_{n+1}\varepsilon_{n+1} < \varepsilon_{n+1}$. Observe that

$$q_{n+1} \leqslant \frac{u}{\varepsilon_{n+1}} < \frac{n+1}{\varepsilon_{n+1}} = k_{n+1}.$$

Since $h(U_n)$ has length greater than $2\varepsilon_{n+1}$ we can choose a positive integer $c_{n+1} \leqslant q_{n+1} < k_{n+1}$ such that $-u + \varepsilon_{n+1}c_{n+1}$ lies in $h(U_n)$. Therefore

$$G_n h(hr_{n+1})^{c_{n+1}}(-u) = G_n h(-u + \varepsilon_{n+1}c_{n+1}) \in G_n h(h(U_n)) = G_n(U_n).$$

Since $G_n(U_n) \subseteq (-\infty, 0)$, the sequence $G_n h(hr_{n+1})^{c_{n+1}}(-u)$ is a subsequence of $T_n(-u)$ consisting of negative elements, as required. Thus $T_n(-u)$ diverges. □

In contrast to this example, there are sequences t_n satisfying $\sum_n b_n = +\infty$ such that T_n converges *everywhere* on $\mathbb{C}_\infty$ to a constant. For instance, let t_n be the sequence g_n from the previous example. Because $(0, +\infty) \supseteq T_1(0, +\infty) \supseteq T_2(0, +\infty) \supseteq \cdots$, and $T_n(0)$ and $T_n(\infty)$ both converge to the same limit p, we see that T_n converges uniformly on $[0, +\infty]$ to p. Given $u > 0$, for sufficiently large n we have $t_n(-u) \in [0, +\infty]$. Therefore $T_n(-u)$ also converges to p. Since T_n also converges to p away from the negative real axis, we deduce that T_n converges to p everywhere on $\mathbb{C}_\infty$.

8. Proof of Theorem 1.3: part I

In this section we prove that the set of divergence of the sequence T_n from Theorem 1.3 is a subset of $(-\infty, -1)$ of logarithmic Hausdorff dimension less than or equal to 1.

The *cross-ratio* $[a, b, c, d]$ of four distinct points a, b, c, and d in $\mathbb{C}_\infty$, is given by

$$[a, b, c, d] = \frac{\chi(a, b)\chi(c, d)}{\chi(a, c)\chi(b, d)}.$$

When a, b, c, and d are all distinct from ∞ this equation reduces to

$$[a, b, c, d] = \frac{|a - b||c - d|}{|a - c||b - d|}.$$

Cross-ratios are preserved under Möbius transformations.

Lemma 8.1. *We have*

$$\frac{|T_n(0)-T_{n-1}(0)|}{|T_{n-1}(0)-T_{n-2}(0)|}=\frac{1}{|b_nT_{n-1}^{-1}(\infty)-1|}.$$

Proof. Observe that $[0,t_n(0),\infty,T_{n-1}^{-1}(\infty)]=[T_{n-1}(0),T_n(0),T_{n-2}(0),\infty]$, which follows from applying T_{n-1} to the left hand cross-ratio. The result follows immediately. □

We can now prove the first part of Theorem 1.3. Note that $t_n^{-1}(-\infty,-1)\subseteq(-\infty,-1)$, and hence $T_n^{-1}(\infty)\in(-\infty,-1)$. Using Lemma 8.1 this means that

$$\frac{|T_n(0)-T_{n-1}(0)|}{|T_{n-1}(0)-T_{n-2}(0)|}=\frac{1}{|b_nT_{n-1}^{-1}(\infty)-1|}\leqslant\frac{1}{2}$$

Hence

$$\tfrac{1}{2}\chi(T_n(0),T_{n-1}(0))\leqslant|T_n(0)-T_{n-1}(0)|\leqslant\frac{1}{2^n}. \tag{8.1}$$

If we sum this equation over n then we deduce that $T_n(0)$ is a Cauchy sequence, and hence it converges to a point p. Since $T_n(\infty)=T_{n-1}(0)$, the sequence $T_n(\infty)$ also converges to p. Therefore, by Lemma 3.1, T_n is generally convergent to p. Since $t_n^{-1}(-\infty,-1)\subseteq(-\infty,-1)$ for each n we see that, by Lemma 3.2, the Julia set of T_n is contained in $[-\infty,-1]$. Hence T_n converges locally uniformly to p on the complement of $[-\infty,-1]$. We have already seen that $T_n(\infty)\to p$, and because $t_{n-1}t_n(-1)\in[0,1)$ for each n, we see that $T_n(-1)\to p$ also. In summary, the set of divergence of T_n is contained in $(-\infty,-1)$, and it remains only to show that it has logarithmic Hausdorff dimension less than or equal to 1.

Using Lemma 4.4 and (8.1) we find that $\exp[-\varrho(j,T_n(j))]\leqslant 1/2^n$, and hence, for $s>1$,

$$\sum_{n=1}^{\infty}\frac{1}{(\varrho(j,T_n(j)))^s}\leqslant\frac{1}{(\log 2)^s}\sum_{n=1}^{\infty}\frac{1}{n^s}<+\infty.$$

It follows from Corollary 3.6 that the logarithmic Hausdorff dimension of the conical limit set of T_n is less than or equal to 1. Thus, by Theorem 3.3, the logarithmic Hausdorff dimension of the set of divergence of T_n is less than or equal to 1. □

9. Proof of Theorem 1.3: part II

In this section we construct a sequence of positive integers $b_1,b_2,\dots$ such that the associated sequence of Möbius maps T_n diverges on a closed, uncountable subset of $(-\infty,-1)$. The set we construct is a Cantor set (Cantor sets are both closed and uncountable). Let $b_1,b_2,\dots$ be a sequence in $\{1,2\}$ with the property that each finite sequence in $\{1,2\}$ appears as a subsequence of consecutive terms in $b_1,b_2,\dots$.

Let $I=[-3,-4/3]$. Define $h_1(z)=-1+1/z$ and $h_2(z)=-2+1/z$, and observe that $h_1(I)$ and $h_2(I)$ are disjoint subsets of I. Since each map t_n^{-1} is equal to either h_1 or h_2 we see that $t_n^{-1}(I)\subseteq I$. Let Ω consist of those points in I that lie in infinitely many of the sets $T_n^{-1}(I)$. If $\omega\in\Omega$ then $T_n(\omega)\in I$ for infinitely many n, which means that the sequence $T_n(\omega)$ does not converge to the (positive) value of the continued fraction, so it must diverge. Hence Ω is contained in the set of divergence of T_n.

Now, given any interval $T_n^{-1}(I)$ we can choose an integer p larger than n such that the first $n+1$ terms of the sequence $b_p, b_{p-1}, \dots, b_1$ are, in order, $b_n, b_{n-1}, \dots, b_1, 1$. In which case

$$T_p^{-1}(I) = T_n^{-1} h_1 T_{p-n-1}^{-1}(I) \subseteq T_n^{-1}(h_1(I)).$$

Likewise we can choose $q > n$ such that $T_q^{-1}(I) \subseteq T_n^{-1}(h_2(I))$. Since $h_1(I)$ and $h_2(I)$ are disjoint subintervals of I it follows that $T_p^{-1}(I)$ and $T_q^{-1}(I)$ are disjoint subintervals of $T_n^{-1}(I)$. By repeatedly choosing pairs of subintervals in this way we see that Ω, and hence the set of divergence of T_n, contains a Cantor set in $(-\infty, -1)$. □

Since Ω contains a set that is both closed and uncountable, it follows from [16, Corollary 1, Section 3] that there exists a dimension function k for which $H^k(\Omega) > 0$. We are unable to construct, for any $s \leqslant 1$, a set of divergence X associated with Theorem 1.3 such that $H^{g_s}(X) > 0$.

References

[1] B. Aebischer, The limiting behavior of sequences of Möbius transformations, Math. Z. **205** (1990), no. 1, 49–59.

[2] A. F. Beardon, *The geometry of discrete groups*, Springer, New York, 1983.

[3] A. F. Beardon, Continued fractions, discrete groups and complex dynamics, Comput. Methods Funct. Theory **1** (2001), no. 2, 535–594.

[4] A. F. Beardon and I. Short, The Seidel, Stern, Stolz and Van Vleck Theorems on continued fractions, Bull. London Math. Soc. **42** (2010), no. 3, 457–466.

[5] E. Crane and I. Short, Conical limit sets and continued fractions, Conform. Geom. Dyn. **11** (2007), 224–249 (electronic).

[6] P. Erdős and G. Piranian, Sequences of linear fractional transformations, Michigan Math. J **6** (1959), 205–209.

[7] K. Falconer, *Fractal geometry*, Second edition, Wiley, Hoboken, NJ, 2003.

[8] W. K. Hayman and P. B. Kennedy, *Subharmonic functions. Vol. I*, Academic Press, London, 1976

[9] L. Jacobsen, General convergence of continued fractions, Trans. Amer. Math. Soc. **294** (1986), no. 2, 477–485.

[10] W. B. Jones and W. J. Thron, *Continued fractions*, Addison-Wesley Publishing Co., Reading, Mass., 1980.

[11] L. Lorentzen and H. Waadeland, *Continued fractions. Vol. 1*, Second edition, Atlantis Press, Paris, 2008.

[12] R. D. Mauldin and M. Urbański, Conformal iterated function systems with applications to the geometry of continued fractions, Trans. Amer. Math. Soc. **351** (1999), no. 12, 4995–5025.

[13] P. J. Nicholls, *The ergodic theory of discrete groups*, Cambridge Univ. Press, Cambridge, 1989.

[14] G. Piranian and W. J. Thron, Convergence properties of sequences of linear fractional transformations, Michigan Math. J. **4** (1957), 129–135.

[15] J. G. Ratcliffe, *Foundations of hyperbolic manifolds*, Springer, New York, 1994.

[16] C. A. Rogers, *Hausdorff measures*, Reprint of the 1970 original, Cambridge Univ. Press, Cambridge, 1998.

[17] B.-W. Wang and J. Wu, Hausdorff dimension of certain sets arising in continued fraction expansions, Adv. Math. **218** (2008), no. 5, 1319–1339.

Department of Mathematics and Statistics, The Open University, Milton Keynes MK7 6AA, United Kingdom

E-mail address: `i.short@open.ac.uk`