POLAR SPACES AND EMBEDDINGS OF CLASSICAL GROUPS

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Abstract. Given polar spaces \((V, \beta)\) and \((V, Q)\) where \(V\) is a vector space over a field \(K\), \(\beta\) a reflexive sesquilinear form and \(Q\) a quadratic form, we have associated classical isometry groups. Given a subfield \(F\) of \(K\) and an \(F\)-linear function \(L : K \to F\) we can define new spaces \((V, L\beta)\) and \((V, LQ)\) which are polar spaces over \(F\).

The construction so described gives an embedding of the isometry groups of \((V, \beta)\) and \((V, Q)\) into the isometry groups of \((V, L\beta)\) and \((V, LQ)\). In the finite field case under certain added restrictions these subgroups are maximal and form the so called field extension subgroups of Aschbacher’s class \(C_3\) [1].

We give precise descriptions of the polar spaces so defined and their associated isometry group embeddings. In the finite field case our results give extra detail to the account of maximal field extension subgroups given by Kleidman and Liebeck [3, p112].

1. Introduction

Let \((V, \beta)\) and \((V, Q)\) be polar spaces over a field \(K\) with \(\beta : V \times V \to K\) a reflexive \(\sigma\)-sesquilinear form where \(\sigma\) is a \(K\)-automorphism and \(Q : V \to K\) a quadratic form with polar form \(f_Q : V \times V \to K\). Let \(F\) be a subfield of \(K\) and \(L : K \to F\) an \(F\)-linear function. We now compose functions to get \(L\beta : V \times V \to F\) and \(LQ : V \to F\) regarding \(V\) as a vector space over \(F\). In order for these forms to be well-defined it is necessary to impose the condition \(\sigma(F) = F\) after which it is easily verified that \(LQ\) is a quadratic form with polar form \(Lf_Q\) and \(L\beta\) is a sesquilinear form. In fact if \(F \subseteq \text{Fix}(\sigma)\) then \(\beta\) is bilinear.

We present three results on this situation: In Section 2, Theorem A gives conditions on the degeneracy of our composed forms, \(L\beta\) and \(LQ\). In Section 3, Theorem B gives conditions on the type (alternating, symmetric or hermitian) of our composed forms. In sections 4 and 5 we consider the situation where our fields are finite. Theorem C summarises these results and gives the isometry group embeddings which are induced by these composed forms.

2. Results on Degeneracy

We begin by presenting results on degeneracy. Our definition of degeneracy is consistent with that of Taylor [5] and so is slightly more general than that of Kleidman and Liebeck [3]:

**Definition 2.1.** A \(\sigma\)-sesquilinear form \(\beta\) is non-degenerate if

\[
\beta(u, v) = 0, \forall v \in V \implies u = 0.
\]

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A quadratic form $Q$ is non-degenerate if its polar form $f_Q$ has the property that
\[ f_Q(u, v) = Q(u) = 0, \forall v \in V \implies u = 0. \]
The forms are called degenerate otherwise.

Our first result concerns sesquilinear forms and uses an adaptation of a proof given by Lam [4]:

**Lemma 2.2.** $L\beta$ is non-degenerate exactly when $L \neq 0$ and $\beta$ is non-degenerate.

**Proof.** If either $L = 0$ or $\beta$ is degenerate then it is clear that $L\beta$ will be degenerate. Now suppose that $L \neq 0$, $\beta$ is non-degenerate and $L\beta$ is degenerate. Then there exists nonzero $v \in V$ such that $L\beta(v, w) = 0$ for all $w \in V$. Note that there exists $w \in V$ such that $\beta(v, w) \neq 0$. Now consider, for any $c \in K$,
\[
\beta(v, \frac{\sigma(c)}{\sigma(\beta(v, w))} w) = \sigma(c) \cdot \frac{\sigma(\beta(v, w))}{\sigma(\beta(v, w))} \beta(v, w) = c
\]
Then $L\beta(v, \frac{\sigma(c)}{\sigma(\beta(v, w))} w) = Lc = 0$ for all $c \in K$. This implies that $L = 0$ which is a contradiction. \qed

We turn our attention to quadratic forms. To begin with we can apply the previous lemma directly to get the following:

**Lemma 2.3.** If $L = 0$ or $Q$ is degenerate then $LQ$ is degenerate. If $f_Q$ is non-degenerate then $LQ$ is non-degenerate.

Thus we are left with the question of what happens when $Q$ is non-degenerate and $f_Q$ is degenerate. This can only occur in characteristic 2. We are able to present results only for the case where $V$ is finite-dimensional and $K$ is finite, in which case we have the following well-known result (see, for example [5, p. 143]):

**Theorem 2.4.** A non-degenerate quadratic form $Q$ on a vector space $V$ over $GF(2^h)$ has a degenerate associated polar form if and only if $\dim V$ is odd, in which case the radical of $f_Q$, $\text{rad}(V, f_Q)$, is of dimension 1.

**Corollary 2.5.** Let $K$ be a finite field of characteristic 2. Suppose $\dim_K V$ is odd, $f_Q$ is degenerate and $Q$ is non-degenerate. Then $LQ$ is degenerate.

**Proof.** Take $x \in \text{rad}(V, f_Q)$. Then $x \in \text{rad}(V, Lf_Q)$. Hence $\text{rad}(V, Lf_Q) \supseteq \text{rad}(V, f_Q)$. But $\dim_F(\text{rad}(V, Lf_Q)) \geq \dim_F(\text{rad}(V, f_Q)) > 1$. Hence $LQ$ is degenerate. \qed

We can summarise our main results in the following:

**Theorem A.** Let $\beta : V \times V \to K$ be a reflexive $\sigma$-sesquilinear form. Let $Q : V \to K$ be a quadratic form. Let $F$ be a subfield of $K$ and $L : K \to F$ be a $F$-linear function. Then:
- $L\beta$ is non-degenerate if and only if $\beta$ is non-degenerate and $L \neq 0$;
- If $\text{char } K \neq 2$, or $K = GF(2^h)$ for some integer $h$ and $\dim_K V$ is even, then $LQ$ is non-degenerate if and only if $Q$ is non-degenerate and $L \neq 0$;
- If $K = GF(2^h)$ for some integer $h$ and $\dim_K V$ is odd then $LQ$ is degenerate;
Unsolved. We have failed to ascertain the conditions under which $LQ$ is degenerate in the case where $\text{char } K = 2$, $|K| + \dim_K V$ is infinite, $Q$ is non-degenerate and $f_Q$ is degenerate.

3. A Classification of $\beta$ Into Form

Taking reflexive sesquilinear form $\beta : V \times V \to K$ to be alternating, symmetric or hermitian, $L : K \to F$, $F$-linear and not identically zero, we seek to classify $L\beta$ into these three categories or else as being ‘atypical’, i.e. not of this form.

The conditions under which $\beta$ is hermitian, $\text{char } K = 2$ and $L\beta$ is alternating will prove to be the most difficult and we discuss this case first. Observe that we must have $F \subseteq \text{Fix}(\sigma)$.

Let $\sigma$ be the field automorphism of order 2 associated with $\beta$. It is easily shown that $K/\text{Fix}(\sigma)$ is a Galois extension and we may therefore define a trace function:

$$\text{Tr}_{K/\text{Fix}(\sigma)} : K \to \text{Fix}(\sigma), x \mapsto x + \sigma(x).$$

Now any $\text{Fix}(\sigma)$-function $L : K \to \text{Fix}(\sigma)$ can be written in the form, for some $\alpha \in K$,

$$L : K \to \text{Fix}(\sigma), x \mapsto \text{Tr}_{K/\text{Fix}(\sigma)}(\alpha x).$$

Lemma 3.1. When $\text{char } K = 2$ and $\beta$ is hermitian, $L\beta$ is alternating if and only if $F \subseteq \text{Fix}(\sigma)$ and $L \sigma = L$.

Proof. Write $L : K \to F, x \mapsto L_1 \circ \text{Tr}_{K/\text{Fix}(\sigma)}(\alpha x)$ for some $\alpha \in K$ and some $L_1 : \text{Fix}(\sigma) \to F$, $F$-linear and not identically zero. We suppose that $\text{Tr}_{K/\text{Fix}(\sigma)}(\alpha \sigma) = \text{Tr}_{K/\text{Fix}(\sigma)}(\alpha)$ and it is enough to prove that $\text{Tr}_{K/\text{Fix}(\sigma)}(\alpha \beta)$ is alternating. Now for $x \in K$,

$$\text{Tr}_{K/\text{Fix}(\sigma)}(\alpha \sigma(x)) = \text{Tr}_{K/\text{Fix}(\sigma)}(\alpha x) \implies \alpha \sigma(x) + \sigma(\alpha x) = \alpha x + \sigma(\alpha x) \implies \alpha \sigma(x) + \sigma(\alpha x) = \alpha x + \sigma(\alpha) \sigma(x) \implies (\sigma(\alpha) + \alpha)(\sigma(x) + x) = 0.$$ 

Since $\sigma(x) + x \neq 0$ for all $x \notin \text{Fix}(\sigma)$, we must have $\sigma(\alpha) = \alpha$. Then

$$\text{Tr}_{K/\text{Fix}(\sigma)}(\alpha \beta)(x, x) = \alpha \beta(x, x) + \sigma(\alpha \beta(x, x)) = \alpha \beta(x, x) + \sigma(\alpha) \beta(x, x) = (\alpha + \sigma(\alpha)) \beta(x, x) = 0.$$ 

We are now able to state our main result:

Theorem B. Let $\beta : V \times V \to K$ be a reflexive sesquilinear form. Let $K/F$ be a field extension with $L : K \to F$ a $F$-linear function which is not identically zero. Then we classify $\beta$ into type as follows:

- If $\beta$ is alternating then $L \beta$ is alternating;
- If $\beta$ is symmetric then $L \beta$ is symmetric;
- If char $K = 2$, $K$ is finite and $\beta$ is symmetric not alternating then $L \beta$ is symmetric not alternating;
- If $\beta$ is hermitian and $F \nsubseteq \text{Fix}(\sigma)$ then
  - (1) $L \beta$ is hermitian if and only if $L \sigma = \sigma L$;
  - (2) $L \beta$ is atypical if and only if $L \sigma \neq \sigma L$;
If \( \beta \) is hermitian and \( F \subseteq \text{Fix}(\sigma) \) then

1. \( L\beta \) is symmetric if and only if \( L\sigma = L \);
2. \( L\beta \) is alternating if and only if \( \text{char } K \neq 2 \) and \( L\sigma = -L \) OR \( \text{char } K = 2 \) and \( L\sigma = L \);
3. \( L\beta \) is atypical if and only if \( L\sigma \neq \pm L \).

**Proof.** The first two statements are self-evident.

We turn to the third statement. Given \( \beta \) symmetric not alternating, \( L\beta \) will be alternating if and only if \( \{ \beta(x, x) \mid x \in V \} \subseteq \text{null}(L) \). Since \( L \neq 0 \) it is enough to show that \( f : V \to K, x \to \beta(x, x) \) is onto. Take any \( x \in V \) such that \( \beta(x, x) = a \in K^* \). Take any \( c \in K \). Then \( \beta(\sqrt[n]{x}, \sqrt[n]{x}) = c \) as required.

For the remainder we assume that \( \beta \) is hermitian. First of all suppose that \( F \not\subseteq \text{Fix}(\sigma) \) so \( L\beta \) is \( \sigma \)-sesquilinear. Then \( L\beta(v_1, v_2) = L\sigma\beta(v_2, v_1) \) for any \( v_1, v_2 \in V \) and so \( L\beta \) is hermitian if and only if \( L\sigma|_{\beta} = \sigma L|_{\beta} \). Since \( \beta \) is surjective we are done.

Next suppose that \( F \subseteq \text{Fix}(\sigma) \) in which case \( L\beta(v_1, v_2) = L\sigma\beta(v_2, v_1) \). This is symmetric if and only if \( L\sigma|_{\beta} = L|_{\beta} \) and so \( L\beta \) is symmetric exactly when \( L\sigma = L \).

Now we examine when \( L\beta \) is alternating. When \( \text{char } K \) is odd this is equivalent to \( L\beta \) being skew-symmetric which, by an analogous argument to the symmetric case, occurs exactly when \( L\sigma = -L \). When \( \text{char } K = 2 \) the previous lemma gives us the required result. The only other possibility is for \( L\beta \) to be atypical hence we have our final equivalence. \( \square \)

**Unsolved.** We have failed to ascertain the conditions under which \( L\beta \) is alternating in the case where \( \text{char } K = 2 \), \( K \) is infinite and \( \beta \) is symmetric not alternating.

### 4. The Isometry Classes of \((V, LQ)\) Over Finite Fields

Define \( Q : V \to GF(q^w) \) a non-degenerate quadratic form, \( L : GF(q^w) \to GF(q) \) a \( GF(q) \)-linear function which is not the zero function and \( Tr_{GF(q^w)/GF(q)} \) the trace function. We restrict \( V \) to be a finite \( A \)-dimensional vector space over \( GF(q^w) \). In order to classify \((V, Q)\) into isometry classes we need to examine the situation when \( Aw \) is even and distinguish between the \( O^+ \) and \( O^- \) cases.

Our first lemma will be useful in distinguishing the isometry class of \( LQ \) as well as giving an application of the classification:

**Lemma 4.1.** The isometry group for \( Q \), \( \text{Isom}(Q, V) \), is a subgroup of the isometry group for \( LQ \), \( \text{Isom}(LQ, V) \).

**Proof.** Simply observe that if \( T : V \to V \) satisfies \( Q(Tu) = Q(u) \) for all \( u \in V \) then \( LQ(Tu) = LQ(u) \) for all \( u \) in \( V \). \( \square \)

Consider first the situation when \( A \) is even:

**Lemma 4.2.** Let \((V, q)\) have isometry class \( O^+(A, q^w) \). Then \((V, LQ)\) has isometry class \( O^+(Aw, q) \). Thus \( O^+(A, q^w) \leq O^+(Aw, q) \).

**Proof.** Let \( W \) be a maximal totally singular subspace of \((V, q)\). Then \( \dim_K W = \frac{1}{2} \dim_K V \). But \( W \) is also a totally singular subspace of \((V, LQ)\) and \( \dim_F W = \frac{1}{2} \dim_F V \). Thus \((V, LQ)\) is of type \( O^+(Aw, q) \). \( \square \)
Lemma 4.3. Let $(V,q)$ have isometry class $O^-(A,q^w)$. Then $(V,LQ)$ has isometry class $O^-(Aw,q)$. Thus $O^-(A,q^w) \leq O^-(Aw,q)$.

Proof. Suppose first of all that $A = 2$. Suppose in addition that $(V,LQ)$ has isometry class $O^+(2w,q)$. Then $O^+(2,q^w) \leq O^-(Aw,q)$ and so, by the theorem of Lagrange,

$$(q^w + 1)|q^{w(w-1)} \prod_{i=1}^{w-1} (q^{2i} - 1).$$

If a primitive prime divisor of $q^{2w} - 1$ exists then this is impossible hence we must deal with the exceptions given by Zsigmondy. The first possibility is that $w = 1$, in which case $L : GF(q^w) \to GF(q^w)$ has form $x \mapsto ax$ for some $a \in GF(q^w)^*$. Clearly an element of $V$ is singular under $Q$ exactly when it is singular under $LQ$. Then $(V,q)$ and $(V,LQ)$ have the same Witt index and hence share type which is a contradiction.

The second possibility is that $(q,w) = (2,3)$ in which case we must consider whether or not $O^-(2,8) \leq O^+(6,2)$. Examining the atlas [2] we see that $O^-(2,8)$ contains elements of order 9 while $O^+(6,2)$ does not, hence this possibility can be excluded.

Now suppose that $A = 2m+2$ for some $m \geq 1$. Then $V = U \perp W$ under $Q$ where $U$ is a direct sum of $m$ hyperbolic lines and $W$ is an anisotropic subspace of dimension 2. Then $Q|_U$ is of type $O^+$ and hence $LQ|_U$ is of type $O^+$. Similarly $Q|_W$ is of type $O^-$ and, since $dim_{GF(q^w)}W = 2$, $LQ|_W$ is of type $O^-$. Then $V = U \perp W$ under $LQ$, $U$ is a direct sum of $mw$ hyperbolic lines under $LQ$ and $W$ is a direct sum of $w-1$ hyperbolic lines with a 2-dimensional anisotropic subspace under $LQ$. Hence $(V,LQ)$ is of type $O^-$. \hfill $\square$

We now consider the situation when $A$ is odd. If the characteristic equals 2 then Theorem A implies that $LQ$ is degenerate so we exclude this situation. We will be interested in the situation where $w$ is even and the characteristic is odd. We will write $L : GF(q^w) \to GF(q)$ in the form, for some $a \in GF(q^w)^*$,

$$L = Tr_{GF(q^w)/GF(q)}(a) : GF(q^w) \to GF(q), x \mapsto \sum_{i=0}^{w-1} (ax)^i.$$

We will need to work with the discriminant of our form $LQ$ for which we will need two preliminary results:

Lemma 4.4. Let $q$ be odd, $k \in GF(q^2) \setminus GF(q)$ such that $k^2 \in GF(q)$. Then

$$Tr_{GF(q^2)/GF(q)}(k) = 0.$$ 

Proof. Observe that $GF(q^2) = GF(q)(k)$ and $k$ has minimum polynomial $x^2 - k^2$. Now $Gal(GF(q^2)/GF(q))$ acts on the set of roots of this minimum polynomial. Since the trace map is the sum of the elements of $Gal(GF(q^2)/GF(q))$, $Tr_{GF(q^2)/GF(q)}(k) = k - k = 0$. \hfill $\square$

Theorem 4.5. A non-degenerate quadratic form, $Q : V \to GF(q)$ where $V$ is a $2n$-dimensional vector space over $GF(q)$ and $q$ is odd, gives rise to an $O^+(2n,q)$ space if and only if $disc(Q) \equiv (-1)^n (mod GF(q)^*)$. Here $GF(q)^*$ is the subgroup of $GF(q)^*$ consisting of all square terms.
A proof of the previous theorem can be found, for instance, in [3, p.32]. We can now proceed with our study of the type of \( LQ \).

**Lemma 4.6.** Let \((V, Q)\) be of type \( O \) and be one-dimensional over \( GF(q^2) \), \( q \) odd. Then \( Q \) has form \( Q(u) = \gamma u^2 \), for some \( \gamma \in GF(q^w) \). Then \( LQ \) has type,

\[
O^+ \iff (\alpha \gamma)^{-2} \in GF(q) \backslash GF(q)^{+2} \text{ or } (\alpha \gamma)^{q+1} \not\equiv -1 (mod\ GF(q)^{+2}),
\]

\[
O^- \iff (\alpha \gamma)^{-2} \not\in GF(q) \backslash GF(q)^{+2} \text{ and } (\alpha \gamma)^{q+1} \equiv -1 (mod\ GF(q)^{+2}).
\]

**Proof.** Observe that \( LQ : V \to GF(q), u \mapsto \alpha \gamma u^2 + (\alpha \gamma u^2)^q \) has polar form \( L_{f_{O}} : V \times V \to GF(q), (u, v) \mapsto u^T M v \) where, over a basis for \( V \) over \( GF(q), \{1, \omega\} \),

\[
M = \begin{pmatrix}
2Tr_{GF(q^2)/GF(q)}(\alpha \gamma) & 2Tr_{GF(q^2)/GF(q)}(\alpha \gamma \omega) \\
2Tr_{GF(q^2)/GF(q)}(\alpha \gamma \omega) & 2Tr_{GF(q^2)/GF(q)}(\alpha \gamma)^2
\end{pmatrix}.
\]

Now take \( f \) to be an element of \( GF(q) \) such that \( \sqrt{f} \not\in GF(q) \). Then \( (\alpha \gamma)^{-2} \in GF(q) \backslash GF(q)^{+2} \) if and only if \( (\alpha \gamma)^{-1} \sqrt{f} \not\in GF(q) \).

**Suppose that** \((\alpha \gamma)^{-1} \sqrt{f} \not\in GF(q) \). Let \( \omega = (\alpha \gamma)^{-1} \sqrt{f} \). Then \( M = \begin{pmatrix}
2Tr_{GF(q^2)/GF(q)}(\alpha \gamma) & 2Tr_{GF(q^2)/GF(q)}(\sqrt{f}) \\
2Tr_{GF(q^2)/GF(q)}(\sqrt{f}) & 2Tr_{GF(q^2)/GF(q)}(\alpha \gamma^{-1} f)
\end{pmatrix}.
\]

Then the discriminant of the form \( LQ \) is

\[
4Tr_{GF(q^2)/GF(q)}(\alpha \gamma)Tr_{GF(q^2)/GF(q)}(\alpha^{-1} f) - 4(Tr_{GF(q^2)/GF(q)}(\sqrt{f}))^2.
\]

\[
= 4f(\alpha \gamma + (\alpha \gamma)^q)((\alpha \gamma)^{-1} + (\alpha \gamma)^{-q}) \quad \text{(since } Tr_{GF(q^2)/GF(q)}(\sqrt{f}) = 0)\]

\[
= \frac{f^2(Tr_{GF(q^2)/GF(q)}(\alpha \gamma))}{(\alpha \gamma)^{q+1}}.
\]

Referring to Theorem 4.5 we see that our result holds in this case.

**Suppose that** \((\alpha \gamma)^{-1} \sqrt{f} \in GF(q) \). Let \( \omega = \sqrt{f} \). Then \( \alpha \gamma = f_2 \sqrt{f} \) for some \( f_2 \in GF(q) \) and

\[
M = \begin{pmatrix}
2Tr_{GF(q^2)/GF(q)}(f_2 \sqrt{f}) & 2Tr_{GF(q^2)/GF(q)}(f_2) \\
2Tr_{GF(q^2)/GF(q)}(f_2) & 2Tr_{GF(q^2)/GF(q)}(f_2 \sqrt{f})
\end{pmatrix}.
\]

The discriminant of the form \( LQ \) is

\[
4Tr_{GF(q^2)/GF(q)}(f_2 \sqrt{f})Tr_{GF(q^2)/GF(q)}(f_2 f) - 4(Tr_{GF(q^2)/GF(q)}(f_2 f))^2.
\]

\[
= -4(Tr_{GF(q^2)/GF(q)}(f_2))^2 \quad \text{(since } Tr_{GF(q^2)/GF(q)}(\sqrt{f}) = 0)\]

Appealing to Theorem 4.5 we conclude that \( LQ \) is of isometry class \( O^+ \) in all cases here.

\( \square \)

**Lemma 4.7.** Let \((V, Q)\) be \( A \)-dimensional of type \( O \) over field \( GF(q^w) \) of odd characteristic. Let \( S \) be a non-dimensional anisotropic subspace (or germ) where \( Q \big|_S \) has form \( Q(s) = \gamma s^2 \) for some \( \gamma \in GF(q^w) \). Let \( w = 2n \). Then \( LQ \) has type

\[
O^+ \iff (\alpha \gamma)^{-2} \in GF(q^n) \backslash GF(q^n)^{+2} \text{ or } (\alpha \gamma)^{q+1} \not\equiv -1 (mod\ GF(q^n)^{+2}),
\]

\[
O^- \iff (\alpha \gamma)^{-2} \not\in GF(q^n) \backslash GF(q^n)^{+2} \text{ and } (\alpha \gamma)^{q+1} \equiv -1 (mod\ GF(q^n)^{+2}).
\]
Proof. First take \( A \) odd and \( w = 2 \). Then \((V, Q) = (R, Q|_R) \perp (S, Q|_S)\) where \( R \) is an orthogonal direct sum of orthogonal hyperbolic lines and \( S \) is a one-dimensional anisotropic orthogonal space. Then \( LQ|_S \) is of type \( O^+ \) and \( LQ|_R \) will be either of type \( O^+ \) or \( O^- \) according to the conditions of the previous lemma. Since \((V, LQ) = (R, LQ|_R) \perp (S, LQ|_S)\) the type of \( LQ \) is determined according to the conditions given.

Now take \( A \) odd, \( w \) any even number. Then \( L = \text{Tr}_{\mathbb{F}(q^n)/\mathbb{F}(q^w)} \circ \text{Tr}_{\mathbb{F}(q^w)/\mathbb{F}(q^n)} \circ K \) where \( K : \mathbb{F}(q^w) \to \mathbb{F}(q^n), x \mapsto ax \). By the previous paragraph the conditions of the theorem are the conditions under which \( \text{Tr}_{\mathbb{F}(q^w)/\mathbb{F}(q^n)} \circ K \circ Q \) will be of type \( O^+ \) or \( O^- \). By Lemmas 4.3 and 4.2 we know that further compositions with \( \text{Tr}_{\mathbb{F}(q^w)/\mathbb{F}(q^n)} \) will not change this type. The result follows.

We will summarise the results of this section and the next in Theorem C at the end of the paper.

5. The Isometry Classes of \((V, L\beta)\) Over Finite Fields

Define \( \beta : V \times V \to \mathbb{F}(q^n) \) to be a non-degenerate reflexive sesquilinear form of one of the three types, \( V \) \( A \)-dimensional over \( \mathbb{F}(q^n) \). Define \( L : \mathbb{F}(q^w) \to \mathbb{F}(q^n) \), \( \mathbb{F}(q) \)-linear and not the zero function.

If we consider \( \beta \) symmetric over a field of odd characteristic then \( \beta \) shares isometry class with the quadratic form \( Q(v) = \frac{1}{2} \beta(v, v) \) and the results of the previous section give the type of \( L\beta \).

Similarly if \( \beta \) is alternating or if the characteristic is 2 and \( \beta \) is symmetric not alternating, then Theorem B gives the type of \( L\beta \). Note that over finite fields, symmetric not alternating forms result in polar spaces which are called pseudo-symplectic.

In this section we need to consider the case where \( \beta \) is hermitian with automorphism \( \sigma \). Once again we will take \( L = \text{Tr}_{\mathbb{F}(q^n)/\mathbb{F}(q^w)}(\alpha) \) for some \( \alpha \) in \( \mathbb{F}(q^n)^* \). Consider first the case where \( \mathbb{F}(q) \not\subseteq \text{Fix}(\sigma) \) which occurs exactly when \( w \) is odd:

**Lemma 5.1.** Let \( \beta \) be hermitian. When \( w \) is odd,
1. \( L\beta \) is hermitian \( \iff \) \( \sigma(\alpha) = \alpha \);
2. \( L\beta \) is atypical \( \iff \) \( \sigma(\alpha) \neq \alpha \).

When \( w \) is even,
1. \( L\beta \) is symmetric \( \iff \) \( \sigma(\alpha) = \alpha \);
2. \( L\beta \) is alternating \( \iff \) \( \sigma(\alpha) = -\alpha \);
3. \( L\beta \) is atypical \( \iff \) \( \sigma(\alpha) \neq \pm \alpha \).

**Proof.** Observe that \( L\beta \) is bilinear if and only if \( \mathbb{F}(q) \subseteq \text{Fix}(\sigma) \) if and only if \( w \) is odd.

Suppose first that \( w \) is odd; then it is enough to prove the first equivalence. By Theorem B we know that \( L\beta \) is hermitian if and only if \( L\sigma = \sigma L \). Now

\[
L\sigma(x) = \sigma L(x) \iff \sigma \text{Tr}_{\mathbb{F}(q^w)/\mathbb{F}(q)}((\sigma(\alpha) - \alpha)x) = 0.
\]

The surjectivity of the trace function gives us our result.

Now suppose that \( w \) is even. By Theorem B it is enough to prove that \( L\sigma = \pm L \iff \sigma(\alpha) = \pm \alpha \). Let the Galois group of the field extension \( \mathbb{F}(q^w)/\mathbb{F}(q) = \)
\{\sigma_1, \ldots, \sigma_w\}. Then
\[
L\sigma = \pm L \iff \sum_{i=1}^w \sigma_i(\alpha\sigma(x)) = \pm \sum_{i=1}^w \sigma_i(\alpha x)
\]
\[
\iff \sum_{i=1}^w \sigma_i(\alpha\sigma(x)) = \pm \sum_{i=1}^w \sigma_i(\alpha x)
\]
\[
\iff \sum_{i=1}^w \sigma_i((\sigma(\alpha) \mp \alpha)x) = 0.
\]
Once again the surjectivity of the trace function gives us our result. \qed

To complete the classification we need to ascertain the isometry group of $L\beta$ in the case where it is symmetric.

**Lemma 5.2.** Suppose that $\beta$ is hermitian, $w$ is even, $\sigma(\alpha) = \alpha$ and $V$ is $A$-dimensional over $GF(q^w)$. Then the isometry class of $L\beta$ is,

\[
O^+(Aw,q) \iff A \text{ is even, } O^-(Aw,q) \iff A \text{ is odd.}
\]

**Proof.** If $A$ is even then, with respect to the hermitian form $\beta$, $V$ contains a totally isotropic subspace of dimension $\frac{4w}{2}$. This subspace is also totally isotropic with respect to $L\beta$ and over $GF(q)$ has dimension $\frac{4w}{2}$. Hence $(V, L\beta)$ is of type $O^+$.

Now take $A$ to be odd. First suppose that $A = 1$ and $w = 2$ so that $GF(q) = Fi_2(\sigma)$. Then, given a basis for $V$ over $GF(q)$, $\{\omega\}$, we have $\beta(x, y) = x\sigma(y)$, $L(x) = \alpha x + \sigma(\alpha x)$ and the matrix of $L\beta$ is
\[
\begin{pmatrix}
2Tr_{GF(q^w)/GF(q)}(\alpha) & 2Tr_{GF(q^w)/GF(q)}(\alpha \omega) \\
2Tr_{GF(q^w)/GF(q)}(\alpha \omega) & 2Tr_{GF(q^w)/GF(q)}(\alpha \omega)\sigma(\omega)
\end{pmatrix}.
\]
Now put $\omega = \sqrt{J}$ where $GF(q^2) = GF(q)(\sqrt{J})$ and the discriminant of $L\beta$ is $-4\alpha^2$. Since this is minus a non-square, $L\beta$ is of type $O^-$. Now let $A$ be any odd integer, $w = 2$. Then $(V, \beta) = (R, \beta|_S) \perp (S, \beta|_R)$ where $R$ is an orthogonal direct sum of orthogonal hyperbolic lines and $S$ is a one-dimensional unitary space. Then $L\beta|_R$ is of type $O^+$ by the first part of this lemma, $L\beta|_S$ is of type $O^-$ by the previous argument and hence $(V, L\beta)$ is of type $O^-$. Finally suppose that $w > 2$, in which case $L = Tr_{GF(q^w)/GF(q)}(\alpha)\sigma Tr_{GF(q^w)/GF(q)}(\alpha \omega)$.

*We know that $Tr_{GF(q^w)/GF(q)}(\alpha \beta)$ is of type $O^-$; then Lemma 4.3 implies that $(V, L\beta)$ is of type $O^-$. \qed*

We are now in a position to summarise the results of the last two sections.

**Theorem C.** Let $V$ be an $A$-dimensional polar space over $GF(q^w)$. Take $L : GF(q^w) \rightarrow GF(q), x \mapsto Tr_{GF(q^w)/GF(q)}(\alpha x)$ for some $\alpha \in GF(q^w)^*$.

Suppose first of all that $V$ is defined via a quadratic form $Q : V \rightarrow GF(q^w)$. If the form has a germ $U$ then $Q|_U(x) = \gamma x^2$ for some $\gamma \in GF(q^w)^*$. Then we classify $LQ$ into type, including the classical group embedding, as follows:
Suppose next that $V$ is defined via a reflexive $\sigma$-sesquilinear form $\beta : V \times V \rightarrow GF(q^w)$. If the characteristic is odd and $\beta$ is symmetric then the type of $L\beta$ and its associated classical group embedding is given in the previous table taking $Q$ to be the quadratic form $Q(v) = \frac{1}{2}\beta(v,v)$.

In all other cases the type of $L\beta$, with associated classical group embedding, is as follows:

<table>
<thead>
<tr>
<th>Type of $\beta$</th>
<th>Type of $L\beta$</th>
<th>Conditions</th>
<th>Embedding</th>
</tr>
</thead>
<tbody>
<tr>
<td>hermitian</td>
<td>hermitian</td>
<td>$w$ odd, $\sigma(\alpha) = \alpha$</td>
<td>$U(A, q^w) \leq U(Aw, q)$</td>
</tr>
<tr>
<td>hermitian</td>
<td>atypical</td>
<td>$w$ odd, $\sigma(\alpha) \neq \alpha$</td>
<td>-</td>
</tr>
<tr>
<td>hermitian</td>
<td>alternating</td>
<td>$w$ even, $q$ even, $\sigma(\alpha) = \alpha$</td>
<td>$U(A, q^w) \leq Sp(Aw, q)$</td>
</tr>
<tr>
<td>hermitian</td>
<td>alternating</td>
<td>$w$ even, $q$ odd, $\sigma(\alpha) = -\alpha$</td>
<td>$U(A, q^w) \leq Sp(Aw, q)$</td>
</tr>
<tr>
<td>hermitian</td>
<td>atypical</td>
<td>$w$ even, $\sigma(\alpha) \neq \pm\alpha$</td>
<td>-</td>
</tr>
<tr>
<td>hermitian</td>
<td>$O^+$</td>
<td>$w$ even, $q$ odd, $A$ even, $\sigma(\alpha) = \alpha$</td>
<td>$U(A, q^w) \leq O^+(Aw, q)$</td>
</tr>
<tr>
<td>hermitian</td>
<td>$O^-$</td>
<td>$w$ even, $q$ odd, $A$ odd, $\sigma(\alpha) = \alpha$</td>
<td>$U(A, q^w) \leq O^-(Aw, q)$</td>
</tr>
<tr>
<td>alternating</td>
<td>alternating</td>
<td>always</td>
<td>Sp($A, q^w) \leq Sp(Aw, q)$</td>
</tr>
<tr>
<td>pseudo</td>
<td>pseudo</td>
<td>$q$ even</td>
<td>-</td>
</tr>
<tr>
<td>-symplectic</td>
<td>-symplectic</td>
<td></td>
<td></td>
</tr>
</tbody>
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References

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