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Norms of Möbius maps

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Abstract

We derive inequalities between the matrix, chordal, hyperbolic, three-point, and unitary norms of a Möbius map. These extend inequalities proved earlier by Gehring and Martin.

1. Introduction

A Möbius transformation $z \mapsto (az + b)/(cz + d)$, where $ad - bc \neq 0$, is a homeomorphism of the extended complex plane $\mathbb{C}_\infty$ onto itself with the chordal metric $q$ given by

$$q(z, w) = \frac{2|z - w|}{\sqrt{1 + |z|^2} \sqrt{1 + |w|^2}}, \quad q(z, \infty) = \frac{2}{\sqrt{1 + |z|^2}},$$

and also a conformal isometry of the upper half-space $\mathbb{H}^3$ of $\mathbb{R}^3$ endowed with the hyperbolic metric $ds = |dz|/x_3$. In [3, 4] Gehring and Martin derived inequalities between the matrix norm, the chordal norm and the hyperbolic norm of a Möbius map (all of which are defined below). Here we introduce two more norms and study the relationships between these five norms. It is known that if a sequence of Möbius transformations converges at three distinct points to three distinct values, then it converges uniformly on $\mathbb{C}_\infty$ to a Möbius transformation.

The work in this paper originated in an attempt to find a proof of this result which exhibits an explicit rate of convergence, and our inequalities provide such an estimate.

The group $\mathcal{M}$ of Möbius maps is equipped with the supremum metric $d$, where

$$d(f, g) = \sup \{q(f(z), g(z)) : z \in \mathbb{C}_\infty\}, \quad f, g \in \mathcal{M},$$

so that $d(f_n, f) \to 0$ if and only if $f_n \to f$ uniformly on $\mathbb{C}_\infty$. Following Gehring and Martin ([3, 4]), we define the chordal norm of a Möbius map $f$ to be $d(f, I)$, where $I$ denotes the identity map: thus

$$d(f, I) = \sup \{q(f(z), z) : z \in \mathbb{C}_\infty\}.$$

Given a Möbius map $f$, we can write

$$f(z) = \frac{az + b}{cz + d}, \quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad ad - bc = 1,$$

where $A$ is determined to within a factor $\pm 1$. The matrix norm of $f$ is $\|f\|$, or $\|A\|$, where

$$\|f\| = \|A\| = \sqrt{|a|^2 + |b|^2 + |c|^2 + |d|^2},$$

and this is independent of the factor $\pm 1$. Gehring and Martin introduce the norm

$$m(f) = \|A - A^{-1}\| = \sqrt{2|a - d|^2 + 4|b|^2 + 4|c|^2},$$

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but more often than not use the expression $m(f)/\|A\|$. We shall combine these ideas and define
$$M(f) = \frac{\|A - \det(A)A^{-1}\|}{\|A\|},$$
where $A$ is now any matrix which represents $f$ as in (1.1), except that we no longer insist that $ad - bc = 1$. Note that $M(f)$ is independent of the choice of $A$ from $\text{GL}(2, \mathbb{C})$ which represents $f$, and if $\det(A) = 1$, then $M(f) = m(f)/\|A\|$. In this notation, Gehring and Martin prove (see [4, (3.35)])
$$M(f) \leq d(f, I) \leq \sqrt{2} M(f).$$
(1.2)
Both constants in these inequalities are best possible.

Guided by the fact that convergence at three points implies uniform convergence we now introduce the three-point norm
$$\varepsilon(f) = \max \{ q(f(1), 1), q(f(\omega), \omega), q(f(\omega^2), \omega^2) \},$$
where $\omega = e^{2\pi i/3}$. In principle, we could take any three points here, and originally the authors proved an estimate similar to (1.3) below using the points 0, 1 and $\infty$ instead of 1, $\omega$ and $\omega^2$, and with a constant on the right between 2 and 3 (and not best possible). The authors thank the referee for providing the better inequality $M(f) \leq \sqrt{2} \varepsilon(f)$ in (1.3) and its proof.

**Theorem 1.1.** For any Möbius transformation $f$,
$$\frac{\varepsilon(f)}{\sqrt{2}} \leq M(f) \leq \sqrt{2} \varepsilon(f),$$
(1.3)
and
$$\varepsilon(f) \leq d(f, I) \leq 2 \varepsilon(f).$$
(1.4)
Further, each constant in each of the four inequalities is best possible.

In the case of parabolic Möbius maps, we can say more than Theorem 1.1. Let $\varphi$ be the stereographic projection of $\mathbb{C}_\infty$ onto the unit sphere $S$ in $\mathbb{R}^3$. Then $p$ and $\hat{p}$ in $\mathbb{C}_\infty$ are antipodal points if and only if $\varphi(p)$ and $\varphi(\hat{p})$ are the end-points of a diameter of $S$ (that is, $q(p, \hat{p}) = 2$).
In particular, the points 0 and $\infty$ are antipodal, and we define
$$\varepsilon_0(f) = \max \{ q(f(0), 0), q(f(\infty), \infty) \}.$$
The choice of the pair of antipodal points used to define $\varepsilon_0$ is insignificant (see Section 2).

**Theorem 1.2.** For any parabolic Möbius transformation $f$,
$$\frac{\varepsilon_0(f)}{\sqrt{2}} \leq M(f) \leq \sqrt{2} \varepsilon_0(f),$$
(1.5)
and
$$\varepsilon_0(f) \leq d(f, I) \leq 2 \varepsilon_0(f).$$
(1.6)
Further, each constant in each of the four inequalities is best possible.

Next, the hyperbolic norm of $f$ is $\rho(j, f(j))$, where $\rho$ is the hyperbolic metric on $\mathbb{H}^3$, and $j = (0, 0, 1)$, and Gehring and Martin obtained an inequality that is equivalent to
$$2 \tanh \frac{1}{2} \rho(j, f(j)) \leq d(f, I).$$
(1.7)
(see [4, (1.19) and Theorem 3.19]).
A Möbius map $u$ is a unitary map if its conjugate $\varphi u \varphi^{-1}$ is a rotation of the sphere $S$ or, equivalently, if $u(j) = j$. Now suppose that $u$ is unitary, and apply (1.7) to $fu^{-1}$ instead of $f$; this gives

$$2 \tanh \frac{1}{2} \rho(j, f(j)) \leq d(f, u).$$

We let $\mathcal{U}$ be the subgroup of unitary maps, and we call $d(f, U)$ the unitary norm of $f$, where

$$d(f, U) = \inf \{ d(f, u) : u \in U \};$$

this measures how far $f$ is from the subgroup $\mathcal{U}$. The Gehring Martin inequality (1.7) implies that

$$2 \tanh \frac{1}{2} \rho(j, f(j)) \leq d(f, U), \quad (1.8)$$

and our last result is that this inequality is, in fact, an equality.

**Theorem 1.3.** In the notation above,

$$d(f, U) = 2 \tanh \frac{1}{2} \rho(j, f(j)) = 2 \sqrt{\|f\|^{2} - 2}.$$

We discuss unitary maps in Section 2. The proofs of Theorems 1.1, 1.2 and 1.3 are given in Sections 3, 4 and 5, respectively. Finally, in Section 6, we return to the original motivation for this paper and obtain a quantitative version of the theorem that says convergence at three points implies uniform convergence.

## 2. Unitary maps

We devote this section to a brief discussion of unitary maps, and the reader is referred to [1] for details. A Möbius map $u$ is a unitary map if and only if $\varphi u \varphi^{-1}$ is a rotation of the sphere $S$, and a unitary Möbius map is clearly a chordal isometry. Also, a Möbius map is unitary if and only if, in its action on $\mathbb{H}^{3}$, it fixes $j = (0, 0, 1)$. Since

$$\|f\|^{2} = 2 \cosh \rho(j, f(j)) \quad (2.1)$$

for any Möbius map $f$, we also see that $f$ is unitary if and only if $\|f\|^{2} = 2$.

The metric $d$ is right invariant: for all Möbius $f$, $g$ and $h$,

$$d(fh, gh) = d(f, g).$$

The Möbius map $h$ induces the left invariance property $d(hf, hg) = d(f, g)$ for all $f$ and $g$ if and only if $h$ is unitary.

Unitary maps have other useful invariance properties; for example, if $X$ is any $2 \times 2$ complex matrix (singular or non-singular), and if $U$ is a unitary matrix (corresponding to a unitary Möbius map), then $\|UX\| = \|X\| = \|XU\|$. Now consider any Möbius map $f$, and any unitary Möbius map $u$. Then, $\|ufu^{-1}\| = \|f\|$. Also, since $u$ is a chordal isometry,

$$d(ufu^{-1}, I) = d(ufu^{-1}, uu^{-1}) = d(f, I).$$

These facts imply that any relationship between $d(f, I)$ and $\|f\|$ is invariant under conjugation by a unitary map, and this leads to a considerable simplification of our arguments. Since a unitary map $u$ fixes $j$ and is a hyperbolic isometry, we see that, for any Möbius map $f$,

$$\rho(ufu^{-1}(j), j) = \rho(f(j), j),$$

so similar comments apply to this norm too.
In conclusion, these remarks show that we could replace the three points \(1, \omega\) and \(\omega^2\) in the definition of \(\varepsilon\) by any three points equally spaced around a great circle and Theorem 1.1 would remain true. Similarly, Theorem 1.2 would remain true if 0 and \(\infty\) are replaced by any pair of antipodal points.

3. The proof of Theorem 1.1

To establish the four inequalities in (1.3) and (1.4), it suffices to prove the right hand inequality of (1.3) and the left hand inequality of (1.4), because the remaining inequalities follow from these two inequalities together with \(d(f, I) \leq \sqrt{2M(f)}\) from (1.2). The left hand inequality of (1.4), \(\varepsilon(f) \leq d(f, I)\), follows immediately from the definition of \(\varepsilon\), so we have only to prove the right hand inequality \(M(f) \leq \sqrt{2}\varepsilon(f)\) of (1.3). In fact, we shall prove the following slightly stronger result.

**Proposition 3.1.** There are positive numbers \(\mu_0, \mu_1\) and \(\mu_2\), with \(\mu_0 + \mu_1 + \mu_2 = 1\), such that

\[
M(f)^2 \leq 2 \left[ \mu_0 q(f(1), 1)^2 + \mu_1 q(f(\omega), \omega)^2 + \mu_2 q(f(\omega^2), \omega^2)^2 \right] \leq 2\varepsilon(f)^2. \tag{3.1}
\]

**Proof.** If \(|z| = 1\) then

\[
|az + b|^2 + |cz + d|^2 = \|A\|^2 + 2\text{Re}((a\bar{b} + c\bar{d})z),
\]

and since \(1 + \omega + \omega^2 = 0\), this gives

\[
\sum_{z^3 = 1} (|az + b|^2 + |cz + d|^2) = 3\|A\|^2. \tag{3.2}
\]

In a similar way we get

\[
\sum_{z^3 = 1} |(az + b) - z(cz + d)|^2 = 3 \left( |a - d|^2 + |b|^2 + |c|^2 \right).
\]

Now define

\[
\mu_j = \frac{|a\omega^j + b|^2 + |c\omega^j + d|^2}{3\|A\|^2}, \quad j = 0, 1, 2. \tag{3.3}
\]

We see from (3.2) that \(\mu_0 + \mu_1 + \mu_2 = 1\). Observe that

\[
2|a\omega^j + b - \omega^j(c\omega^j + d)|^2 = 3\mu_j \|A\|^2 q(f(\omega^j), \omega^j)^2,
\]

which means that

\[
\sum_{j=0}^2 \mu_j q(f(\omega^j), \omega^j)^2 = \frac{2}{\|A\|^2} \left( |a - d|^2 + |b|^2 + |c|^2 \right),
\]

and this gives (3.1) since

\[
M(f)^2 = \frac{2}{\|A\|^2} \left( |a - d|^2 + 2|b|^2 + 2|c|^2 \right).
\]

\(\square\)

To show that the constants in the four inequalities from (1.3) and (1.4) are best possible it suffices to prove that the constants in the left hand inequality of (1.3) and the right hand inequality of (1.4) are best possible, because then, using the inequality \(d(f, I) \leq \sqrt{2M(f)}\) from (1.2), we see that all four constants are best possible. For example, if we show that the constant
in \( d(f, I) \leq 2\epsilon(f) \) is best possible, then the constants in the two inequalities \( d(f, I) \leq \sqrt{2}M(f) \) and \( M(f) \leq \sqrt{2}\epsilon(f) \) must also be best possible.

To see that the constant \( \sqrt{2} \) in the left hand inequality of (1.3), \( \epsilon(f) \leq \sqrt{2}M(f) \), is best possible, consider the following one parameter group of Möbius transformations:

\[
f_n(z) = \frac{n z - (n-1)}{-(n+1)z + n}, \quad n = 1, 2, \ldots.
\]

Then \( f_n(1) = -1 \), so that \( \epsilon(f_n) = 2 \), and one can check that \( M(f_n) \rightarrow \sqrt{2} \) as \( n \rightarrow \infty \).

To see that the constant 2 in the right hand inequality of (1.4), \( d(f, I) \leq 2\epsilon(f) \), is best possible, consider the following one parameter group of Möbius transformations with fixed points \(-1\) and 1:

\[
f_t(z) = \frac{(1 + t)z + (1 - t)}{(1 - t)z + (1 + t)}, \quad t \in \mathbb{R}.
\]

Notice that \( f_t(z) \rightarrow -1 \) as \( t \rightarrow \infty \) for all points \( z \) other than 1. Therefore

\[
\epsilon(f_t) \rightarrow 1, \quad d(f_t, I) \rightarrow 2,
\]

as \( t \rightarrow \infty \), and this shows that the constant 2 is best possible.

### 4. The proof of Theorem 1.2

To establish the four inequalities in (1.5) and (1.6), it suffices to prove the right hand inequality of (1.5) and the left hand inequality of (1.6), because the remaining inequalities follow from these two inequalities together with \( d(f, I) \leq \sqrt{2}M(f) \) from (1.2). The left hand inequality of (1.6), \( \epsilon_0(f) \leq d(f, I) \), follows immediately from the definition of \( \epsilon_0 \), so we have only to prove the right hand inequality of (1.5), \( M(f) \leq \sqrt{2}\epsilon_0(f) \).

Since \( f \) is parabolic we may assume, with the notation of (1.1), that \( ad - bc = 1 \) and \( a + d = 2 \). This means that

\[
4bc = 4(ad - 1) = -4(a - 1)^2 = -(a - d)^2.
\]

Hence

\[
M(f)^2 = \frac{2|a - d|^2 + 4|b|^2 + 4|c|^2}{|a|^2 + |b|^2 + |c|^2 + |d|^2} \leq 2 \left( \frac{4|b|^2 + 4|c|^2}{|b|^2 + |c|^2 + |d|^2} \right) \leq 2 \max \left\{ \frac{4|b|^2}{|b|^2 + |d|^2}, \frac{4|c|^2}{|a|^2 + |c|^2} \right\} = 2\epsilon_0(f)^2,
\]

as required.

To show that the constants in the four inequalities in (1.5) and (1.6) are best possible it suffices to prove that the constants in the left hand inequality of (1.5) and the right hand inequality of (1.6) are best possible, because then, using the inequality \( d(f, I) \leq \sqrt{2}M(f) \) from (1.2), we see that all four constants are best possible.

To see that the constant \( \sqrt{2} \) in the left hand inequality of (1.3), \( \epsilon_0(f) \leq \sqrt{2}M(f) \), is best possible, consider the maps \( f_t(z) = z + t \) for \( t > 0 \). One can check that \( \epsilon_0(f_t)/M(f_t) \rightarrow \sqrt{2} \) as \( t \rightarrow 0 \). To see that the constant 2 in the right hand inequality of (1.4), \( d(f, I) \leq 2\epsilon_0(f) \), is best possible, consider the following one parameter group of parabolic Möbius transformations with fixed point \( t \):

\[
f_t(z) = \frac{(1 + it)z + t}{tz + (1 - it)}, \quad t \in \mathbb{R}.
\]
As \( t \to 0 \) we find that 
\[
q(f_t(0), 0) \sim 2t, \quad q(f_t(\infty), \infty) \sim 2t \quad \text{and} \quad q(f_t(-i), -i) \sim 4t.
\]
This means that 
\[
\limsup_{t \to 0} d(f_t, I)/\epsilon_0(f_t) \geq 2.
\]
Therefore the constant 2 in \( d(f, I) \leq 2\epsilon_0(f) \) is best possible.

5. The proof of Theorem 1.3

We begin with a decomposition result for a general Möbius \( f \); this is a straightforward consequence of the standard results on isometric spheres.

**Theorem 5.1.** Each Möbius map \( f \) can be represented in the form 
\[
f = ug,
\]
where \( u \) is a unitary map, and \( g \) is a hyperbolic map with antipodal fixed points (or \( I \)).

**Proof.** We may assume that \( f \) is not unitary (else we take \( u = f \) and \( g = I \)). Then the action of the conjugate map \( f^* = \varphi f \varphi^{-1} \) on the unit ball is given by \( f^* = \alpha \beta \), where \( \beta \) is the inversion in the isometric sphere \( S \) of \( f \), and \( \alpha \) is some orthogonal map. Let \( \ell \) be the Euclidean line that passes through 0 and the centre of \( S \), and let \( \gamma \) be the reflection in the plane through 0 that is orthogonal to \( \ell \). Then \( f = (\alpha \gamma)(\gamma \beta) \), where \( \alpha \gamma \) is unitary and \( \gamma \beta \) is hyperbolic with antipodal fixed points. \( \square \)

We can now complete the proof of Theorem 1.3, and we first prove this in the case of a hyperbolic map with antipodal fixed points.

**Lemma 5.2.** If \( g \) is hyperbolic with antipodal fixed points then 
\[
d(g, U) = 2 \tanh \frac{1}{2} \rho(j, g(j)).
\]

**Proof.** We know from (1.8) that 
\[
2 \tanh \frac{1}{2} \rho(j, g(j)) \leq d(g, U).
\]
Also, Gehring and Martin prove in [4, Theorem 3.19] that equality holds in (1.7) when \( f \) is hyperbolic with antipodal fixed points. Thus 
\[
2 \tanh \frac{1}{2} \rho(j, g(j)) = d(g, I) \geq d(g, U).
\]

Now let \( f \) be a general Möbius map, and write \( f = ug \), where \( u \) is unitary and \( g \) is hyperbolic with antipodal fixed points (or \( I \)). Then 
\[
d(f, U) = d(g, U) = 2 \tanh \frac{1}{2} \rho(j, g(j)) = 2 \tanh \frac{1}{2} \rho(j, f(j))
\]
as required. Finally, from (2.1) we obtain 
\[
d(f, U) = 2 \sqrt{\frac{\|f\|^2 - 2}{\|f\|^2 + 2}}.
\]

6. The convergence theorem

We finish by returning to the original motivation for this paper, namely that if a sequence of Möbius maps converges at three distinct points to three distinct values, then it converges uniformly on \( \mathbb{C}_\infty \) to a Möbius map. Theorem 1.1 implies that if a sequence of Möbius maps converges to \( I \) on \( \{1, \omega, \omega^2\} \), then it converges to \( I \) uniformly on \( \mathbb{C}_\infty \). The extension to the general case is easy because, for any Möbius map \( h \), 
\[
\|h\| \leq q(z, w) \leq q(h(z), h(w)) \leq \|h\|^2 q(z, w)
\]
Suppose that $z_1$, $z_2$ and $z_3$ are distinct, and that a sequence $g_n$ of Möbius maps satisfies $g_n(z_j) \rightarrow w_j$, $j = 1, 2, 3$, where the $w_j$ are distinct. We can choose Möbius maps $r$ and $s$ that map $1$, $\omega$ and $\omega^2$ to $z_1$, $z_2$ and $z_3$, and $w_1$, $w_2$ and $w_3$, respectively, and then

$$
d(g_n, sr^{-1}) \leq \|s\|^2 d(s^{-1}g_n, s^{-1}sr^{-1})$$

$$= \|s\|^2 d(s^{-1}g_nr, I)$$

$$\leq 2 \|s\|^2 \varepsilon(s^{-1}g_nr)$$

$$\leq 2 \|s\|^4 \max \{ q(g_n(z_j), w_j) : j = 1, 2, 3 \}.$$ 

We deduce that $g_n \rightarrow sr^{-1}$ with the given rate of convergence.

References