Rigidity of configurations of balls and points in the *N*-sphere

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RIGIDITY OF CONFIGURATIONS OF BALLS AND POINTS IN THE N-SPHERE

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ABSTRACT. We answer two questions of Beardon and Minda which arose from their study of the conformal symmetries of circular regions in the complex plane. We show that a configuration of closed balls in the N-sphere is determined up to Möbius transformations by the signed inversive distances between pairs of its elements, except when the boundaries of the balls have a point in common, and that a configuration of points in the N-sphere is determined up to Möbius transformations by the absolute cross-ratios of 4-tuples of its elements. The proofs use the hyperboloid model of hyperbolic $(N+1)$-space.

1. INTRODUCTION

Let $\mathbb{R}^N_\infty$ denote the one point extension of $\mathbb{R}^N$. A *sphere* in $\mathbb{R}^N_\infty$ refers to either an $(N-1)$-dimensional Euclidean sphere in $\mathbb{R}^N$, or an $(N-1)$-dimensional Euclidean plane in $\mathbb{R}^N$ with the point ∞ attached. An *inversion* in a sphere refers to either a Euclidean inversion, if the sphere is a Euclidean sphere, or a Euclidean reflection, if the sphere is a Euclidean plane. The group of *Möbius transformations* on $\mathbb{R}^N_\infty$ is the group generated by inversions in spheres. By an *open ball* B in $\mathbb{R}^N_\infty$ we mean one of the connected components of the complement of a sphere. We write $\widehat{B}$ for the other component. Given two distinct open balls B_1 and B_2 in $\mathbb{R}^N_\infty$, we denote the signed inversive distance between them by $[B_1, B_2]$. For any four distinct points p_1, p_2, p_3, and p_4 in $\mathbb{R}^N_\infty$ we let $|p_1, p_2, p_3, p_4|$ denote their absolute cross-ratio. The signed inversive distance and absolute cross-ratio are two geometric quantities invariant under the action of the Möbius group on $\mathbb{R}^N_\infty$. We will describe them in detail in section 2. The two main results of this paper state that these invariants suffice to rigidify a configuration of open balls, or a configuration of points, up to Möbius transformations.

Theorem 1. *Let $\{B_\alpha : \alpha \in \mathcal{A}\}$ and $\{B'_\alpha : \alpha \in \mathcal{A}\}$ be two collections of open balls in $\mathbb{R}^N_\infty$, indexed by the same set. Suppose that $\bigcap_{\alpha\in\mathcal{A}} \partial B_\alpha = \emptyset$. Then there is a Möbius transformation f such that one of the following holds: either $f(B_\alpha) = B'_\alpha$ for each α in $\mathcal{A}$, or else $f(\widehat{B_\alpha}) = B'_\alpha$ for each α in $\mathcal{A}$, if and only if $[B_\alpha, B_\beta] = [B'_\alpha, B'_\beta]$ for all pairs α and β in $\mathcal{A}$.*

Theorem 2. *Let $\{p_\alpha : \alpha \in \mathcal{A}\}$ and $\{p'_\alpha : \alpha \in \mathcal{A}\}$ be two collections of distinct points in $\mathbb{R}^N_\infty$, indexed by the same set. There is a Möbius transformation f with $f(p_\alpha) = p'_\alpha$*

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for each α in $\mathcal{A}$ if and only if $|p_\alpha, p_\beta, p_\gamma, p_\delta| = |p'_\alpha, p'_\beta, p'_\gamma, p'_\delta|$ for all ordered 4-tuples $(\alpha, \beta, \gamma, \delta)$ of distinct indices in $\mathcal{A}$.

These theorems resolve two problems posed by Beardon and Minda [2] concerning extensions and higher-dimensional generalizations of their results on the conformal symmetries of circular regions in the extended complex plane.

2. Background

A *circular region* in the extended complex plane $\mathbb{C}_\infty$ is a region bounded by a collection of pairwise disjoint circles. A *finitely connected region* in $\mathbb{C}_\infty$ is a region with a finite number of boundary components. A classical theorem of Koebe (which can be found in [3, chapter 15] or [4, chapter X]) says that a finitely connected region is conformally equivalent to a finitely connected circular region that has a finite number of punctures. A *Möbius transformation* is a conformal or anti-conformal homeomorphism of $\mathbb{C}_\infty$. Such maps can be expressed algebraically as

$$z \mapsto \frac{az+b}{cz+d}, \quad \text{or} \quad z \mapsto \frac{a\bar{z}+b}{c\bar{z}+d}, \tag{2.1}$$

where $ad - bc \neq 0$. Given two Euclidean circles C_1 and C_2, with centres c_1 and c_2, and radii r_1 and r_2, the *inversive distance* between these two circles is the positive quantity

$$(C_1, C_2) = \left| \frac{r_1^2 + r_2^2 - |c_1 - c_2|^2}{2r_1 r_2} \right|. \tag{2.2}$$

More generally, if C_1 and C_2 are two circles in $\mathbb{C}_\infty$ (that is, they are each either Euclidean circles, or Euclidean lines with the point ∞ attached), then we define the inversive distance (C_1, C_2) to be $(f(C_1), f(C_2))$, where f is any Möbius transformation that maps both C_1 and C_2 to Euclidean circles. This definition is independent of f, and the resulting quantity is invariant under Möbius transformations, in the sense that $(g(C_1), g(C_2)) = (C_1, C_2)$ for each Möbius map g. See [1, section 3.2] for information on the inversive distance.

The following result is part of [2, Thm 4.1]; the original theorem of Beardon and Minda also includes a uniqueness statement, which we will address in section 7.

Theorem A. *Suppose that Ω and Ω' are circular regions bounded by circles $C_1, \dots, C_m$ and $C'_1, \dots, C'_m$, respectively, where $m \geqslant 2$. Then there is a Möbius transformation f with $f(\Omega) = \Omega'$ and $f(C_j) = C'_j$, $1 \leqslant j \leqslant m$, if and only if $(C_j, C_k) = (C'_j, C'_k)$ for all j and k with $1 \leqslant j < k \leqslant m$.*

Beardon and Minda also gave an analogous result about punctured regions. For points a, b, c, and d in $\mathbb{C}_\infty$, let $|a, b, c, d|$ denote the *absolute cross-ratio* of a, b, c, and d; that is,

$$|a, b, c, d| = \frac{|a-b||c-d|}{|a-c||b-d|}, \tag{2.3}$$

with the usual conventions regarding the point ∞. The absolute cross-ratio is invariant under Möbius transformations. The following result is [2, Theorem 14.1].

Theorem B. *Given two collections of points $p_1, \dots, p_m$ and $p'_1, \dots, p'_m$ in $\mathbb{C}_\infty$, $m \geqslant 4$, there is a Möbius transformation f with $f(p_i) = p'_i$ for $i = 1, 2, \dots, m$ if and only if $|p_i, p_j, p_k, p_l| = |p'_i, p'_j, p'_k, p'_l|$ for all distinct i, j, k, l in $\{1, 2, \dots, m\}$.*

A weaker theorem than Theorem B in which the absolute cross-ratio is replaced by the usual complex cross-ratio is well known and straightforward to prove.

At the end of [2], Beardon and Minda asked the following questions (the second question has been paraphrased).

Question 1. Is the conclusion of Theorem A valid when $C_1, \dots, C_m$ are any set of m distinct circles in $\mathbb{C}_\infty$? Here the C_i are allowed to be intersecting, or tangent, to each other.

Question 2. Do Theorems A and B generalize to higher dimensions?

The answer to Question 1 is negative, and we provide examples to justify this in section 3. Subject to certain restrictions, however, both Theorems A and B generalize to allow arbitrarily many circles and points, and the circles may intersect. These generalizations are our main theorems (Theorems 1 and 2), and they apply in all dimensions.

To generalize Theorem A we work with the *signed* inversive distance between *discs* rather than circles (or, in higher dimensions, with balls rather than spheres). Given two Euclidean balls B_1 and B_2, with centres c_1 and c_2, and radii r_1 and r_2, the *signed inversive distance* between these two balls is the quantity

$$[B_1, B_2] = \frac{r_1^2 + r_2^2 - |c_1 - c_2|^2}{2r_1r_2}. \tag{2.4}$$

Again, $[B_1, B_2]$ can be defined for arbitrary balls B_1 and B_2 by transferring away from the point ∞ using a Möbius transformation, and again, the quantity $[B_1, B_2]$ is preserved under Möbius transformations. Notice that $[\widehat{B_1}, B_2] = -[B_1, B_2]$ and $(\partial B_1, \partial B_2) = |[B_1, B_2]|$.

To recover Theorem A from the $N = 2$ case of Theorem 1, we begin with the hypotheses of Theorem A, and define B_i to be the component of $\mathbb{C}_\infty \setminus C_i$ that contains Ω. Likewise we define B'_i to be the component of $\mathbb{C}_\infty \setminus C'_i$ that contains Ω'. This means that $[B_i, B_j] = (C_i, C_j)$ and $[B'_i, B'_j] = (C'_i, C'_j)$ for all indices i and j in $\{1, \dots, m\}$. From Theorem 1 we deduce the existence of a Möbius map f such that one of the following holds: either $f(B_i) = B'_i$ for each i, or else $f(\widehat{B_i}) = B'_i$ for each i. In the latter case, because $\widehat{B_2} \subset B_1$, we find that

$$B'_2 = f(\widehat{B_2}) \subset f(B_1) = \widehat{B'_1},$$

which is false. Therefore $f(B_i) = B'_i$ for each i, which means that $f(C_i) = C'_i$ for each i, and because $\Omega = \bigcap_{i=1}^m B_i$ and $\Omega' = \bigcap_{i=1}^m B'_i$, we also see that $f(\Omega) = \Omega'$.

Theorem 1 may fail when $\bigcap_{\alpha \in \mathcal{A}} \partial B_\alpha \neq \emptyset$; an example of its failure is given in section 3.

3. Examples

We provide a sequence of examples which answer Question 1 and explain the necessity of the conditions in Theorem 1.

Example 3.1. Here is the simplest example to show that Theorem A is invalid when the circles C_i are allowed to intersect. Let C_1 and C_2 be two Euclidean lines through the origin that cross at an angle $\pi/3$. Let $C_1' = C_1$ and $C_2' = C_2$. Define Ω to be one of the resulting sectors with angle $\pi/3$, and define Ω' to be one of the sectors with angle $2\pi/3$. See Figure 3.1. Only Möbius transformations of the form $z \mapsto \lambda z$ or $z \mapsto \lambda/z$, for non-zero real numbers λ, fix both C_1 and C_2 as sets. None of these transformations map Ω to Ω'.

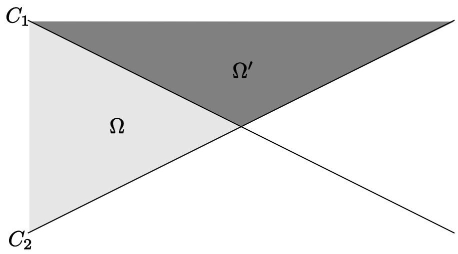

Figure 3.1

There is not a Möbius transformation mapping the set Ω in Example 3.1 to Ω'; however, there is a Möbius transformation that maps C_1 to C_1' and C_2 to C_2', namely the identity. This explains why we consider balls rather than spheres in Theorem 1.

Example 3.2. This example shows that it is necessary to use signed inversive distances. Let C_1 and C_1' both denote the line $x = -1$. Let C_2 denote the line $x = -1/2$ and let C_2' denote the line $x = 1/2$. Let C_3 and C_3' both denote the unit circle. See Figure 3.2. Then $(C_i, C_j) = (C_i', C_j')$ for all $1 \leqslant i, j \leqslant 3$, but there is not a Möbius transformation that maps C_i to C_i' for $i = 1, 2, 3$.

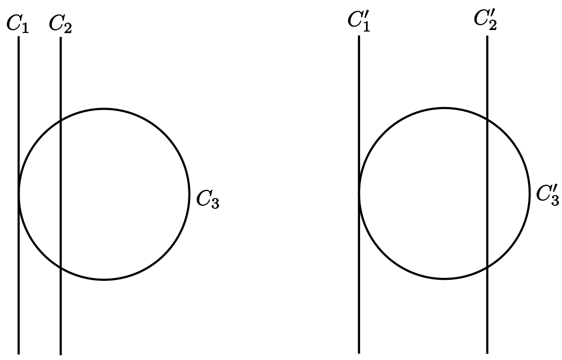

Figure 3.2

The inversive distance condition is not sensitive enough to distinguish the two geometric configurations shown in Figure 3.2. It is impossible to define discs B_i and B_i' with boundary circles C_i and C_i' in Example 3.2 such that $[B_i, B_j] = [B_i', B_j']$ for all i and j in $\{1, 2, 3\}$.

The final example in this section shows that Theorem 1 would fail if we allowed $C_1 \cap \cdots \cap C_m \neq \emptyset$.

Example 3.3. Let C_1, C_2, C_3, and C_4 be the extended sides of a square in the complex plane. Let C'_1, C'_2, C'_3, and C'_4 be the extended sides of a rectangle (that is not a square) in the complex plane. See Figure 3.3. For each i, let B_i be the half-plane with boundary C_i that contains the shaded square, and let B'_i be the half-plane with boundary C'_i that contains the shaded rectangle. Then $[B_i, B_j] = [B'_i, B'_j]$ for all pairs i, j. If there is a Möbius transformation f that maps B_i to B'_i for each i, then f must map the square to the rectangle. This cannot be.

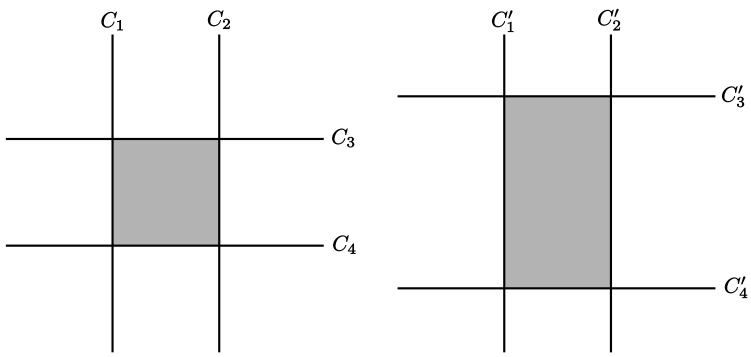

Figure 3.3

4. Hyperbolic geometry

Beardon and Minda noted that their results can be interpreted in terms of hyperbolic geometry, and this is our starting point. Refer to [1, 5] for complete introductions to hyperbolic geometry.

The action of Möbius transformations on $\mathbb{C}_\infty$ extends to an action on the upper half-space model of three-dimensional hyperbolic space, $\mathbb{H}^3$, and this action on $\mathbb{H}^3$ is isometric with respect to the hyperbolic metric, ρ. Each circle in $\mathbb{C}_\infty$ is the ideal boundary of a unique hyperbolic plane in $\mathbb{H}^3$. If Π_1 and Π_2 are two hyperbolic planes with ideal boundary circles C_1 and C_2, then

$$(C_1, C_2) = \begin{cases} \cosh \rho(\Pi_1, \Pi_2) & \text{if } \Pi_1 \text{ and } \Pi_2 \text{ are disjoint,} \\ \cos \theta & \text{if } \Pi_1 \text{ and } \Pi_2 \text{ intersect in an angle } \theta. \end{cases}$$

In higher dimensions, the situation is similar. The set

$$\left\{(x_1, \ldots, x_{N+1}) \in \mathbb{R}^{N+1} \,:\, x_{N+1} > 0\right\},$$

which we denote by $\mathbb{H}^{N+1}$, is a model of $(N+1)$-dimensional hyperbolic space when equipped with the Riemannian density $ds = |dx|/x_{N+1}$. The ideal boundary of $\mathbb{H}^{N+1}$ consists of the set $x_{N+1} = 0$, which we identify with $\mathbb{R}^N$, and the point ∞. The action of the group of Möbius transformations on $\mathbb{R}^N_\infty$ extends to an isometric action on $\mathbb{H}^{N+1}$. The boundary in $\mathbb{H}^{N+1}$ of a hyperbolic half-space Σ is an N-dimensional hyperbolic plane Π. The ideal boundary of Σ is a spherical ball B in $\mathbb{R}^N_\infty$. The boundary in $\mathbb{R}^N_\infty$

of B is a sphere C, and C is the ideal boundary of Π. We often move between spheres and balls in $\mathbb{R}^N_\infty$, and half-spaces and planes in $\mathbb{H}^{N+1}$.

There are many models of $(N+1)$-dimensional hyperbolic space, and although $\mathbb{H}^{N+1}$ is the most appropriate model for explaining how Theorem 1 can be considered as a problem in hyperbolic geometry, the *hyperboloid model* of hyperbolic space is the most appropriate model for proving the theorem. We describe the hyperboloid model in the next section.

5. The hyperboloid model of hyperbolic space

The substance of this section is taken from [5, chapter 3].

We equip $\mathbb{R}^{N+1}$ with the *Lorentz inner product* $\langle\,\cdot\,\rangle$, defined by

$$\langle (x_1,\dots,x_{N+1}),(y_1,\dots,y_{N+1})\rangle = x_1y_1+\cdots+x_Ny_N - x_{N+1}y_{N+1}.$$

This is not an inner product in the usual sense, as it is not positive definite. We write $\|x\|^2=\langle x,x\rangle$. In contrast, we denote the Euclidean scalar product of points x and y in $\mathbb{R}^{N+1}$ by $x\cdot y$, and the Euclidean norm of x by $|x|$. A vector x in $\mathbb{R}^{N+1}$ is *space-like* if $\|x\|^2>0$, *time-like* if $\|x\|^2<0$, and *light-like* if $\|x\|^2=0$. The terminology originates from the theory of relativity. A subspace V of $\mathbb{R}^{N+1}$ is *space-like* if every non-zero element of V is space-like, *time-like* if there is a time-like vector in V, and *light-like* otherwise.

A linear map of $\mathbb{R}^{N+1}$ that preserves the Lorentz inner inner product is described as a *Lorentz transformation.* A vector x in $\mathbb{R}^{N+1}$ is *positive* if $x_{N+1}>0$. A Lorentz transformation is *positive* if it maps positive time-like vectors to positive time-like vectors. The positive Lorentz transformations together form a group, denoted $\mathrm{PO}(N,1)$.

The underlying space of the hyperboloid model of N-dimensional hyperbolic space is the hyperboloid sheet

$$\mathcal{H}^N = \{\, x\in\mathbb{R}^{N+1} : \|x\|^2=-1,\ x_{N+1}>0\,\},$$

embedded in $\mathbb{R}^{N+1}$. This is a model of N-dimensional hyperbolic space with the metric ρ defined by

$$\cosh\rho(x,y) = -\langle x,y\rangle.$$

The group $\mathrm{PO}(N,1)$ consists of those Lorentz transformations that fix $\mathcal{H}^N$ (the Lorentz transformations that are not positive swap $\mathcal{H}^N$ with its twin hyperboloid sheet). The group $\mathrm{PO}(N,1)$ is the full group of hyperbolic isometries of $\mathcal{H}^N$.

The hyperbolic lines in $\mathcal{H}^N$ are intersections of $\mathcal{H}^N$ with two-dimensional time-like subspaces of $\mathbb{R}^{N+1}$. The hyperbolic planes of codimension one in $\mathcal{H}^N$ are intersections of $\mathcal{H}^N$ with N-dimensional time-like subspaces of $\mathbb{R}^{N+1}$. In future we describe such planes merely as ‘planes’, because all the planes we consider have codimension one.

Given a subspace V of $\mathbb{R}^{N+1}$, the *Lorentz complement* of V is the space

$$V^{\mathrm{L}} = \{\, y\in\mathbb{R}^{N+1} : \langle x,y\rangle = 0 \text{ for all } x \text{ in } V\,\}.$$

To each time-like Euclidean plane P there corresponds a unique line ℓ of space-like vectors in $\mathbb{R}^{N+1}$ that are Lorentz orthogonal to P, so that $P=\ell^{\mathrm{L}}$. Conversely, to a

Euclidean line ℓ of space-like vectors there corresponds a unique time-like Euclidean plane P that is Lorentz orthogonal to ℓ.

Let P_1 and P_2 be two N-dimensional time-like planes in $\mathbb{R}^{N+1}$ with non-zero space-like normals v_1 and v_2, respectively, where $\|v_1\|^2 = \|v_2\|^2 = 1$. Let $\Pi_1 = \mathcal{H}^N \cap P_1$ and $\Pi_2 = \mathcal{H}^N \cap P_2$. The ideal boundaries of Π_1 and Π_2 are spheres C_1 and C_2. The inversive distance of C_1 and C_2 defined in (2.2) satisfies

$$(C_1, C_2) = |\langle v_1, v_2 \rangle| \tag{5.1}$$

(see [5, section 3.2]). This is the simplest formula for the inversive distance so far, hinting that the hyperboloid model may be the most natural setting for considering Theorem 1.

The plane P_1 consists of all points x in $\mathbb{R}^{N+1}$ for which $\langle x, v_1 \rangle = 0$. Let Q_1 consist of all points x in $\mathbb{R}^{N+1}$ for which $\langle x, v_1 \rangle > 0$. Define the half-space Σ_1 to be equal to $\mathcal{H}^N \cap Q_1$. We define Σ_2 in a similar fashion using P_2. The ideal boundaries of Σ_1 and Σ_2 are open spherical balls B_1 and B_2. The signed inversive distance of B_1 and B_2 defined in (2.4) satisfies the formula

$$[B_1, B_2] = \langle v_1, v_2 \rangle. \tag{5.2}$$

(again, see [5, section 3.2]).

6. Canonical forms for subspaces of Lorentz space

Let $e_1, \ldots, e_{N+1}$ be the standard basis vectors. For each $p = 1, \ldots, N$, define subspaces

$$\begin{aligned} T_p &= \{\, (x_1, \ldots, x_{p-1}, 0, \ldots, 0, x_{N+1}) \in \mathbb{R}^{N+1} \;:\; x_i \in \mathbb{R} \,\}, \\ S_p &= \{\, (x_1, \ldots, x_{p-1}, x_p, 0, \ldots, 0, 0) \in \mathbb{R}^{N+1} \;:\; x_i \in \mathbb{R} \,\}, \\ L_p &= \{\, (x_1, \ldots, x_{p-1}, \lambda, 0, \ldots, 0, \lambda) \in \mathbb{R}^{N+1} \;:\; \lambda, x_i \in \mathbb{R} \,\}, \end{aligned}$$

each of dimension p. We identify the subspace S_p with $\mathbb{R}^p$, for each p. Notice that T_p is time-like, since it contains the time-like vector e_{N+1}; S_p is space-like, since each non-zero vector in S_p is space-like; and L_p is light-like, because it contains no time-like vectors, but it does contain the light-like vector $e_p + e_{N+1}$.

Lemma 6.1. *Each p-dimensional subspace of $\mathbb{R}^{N+1}$ (for $1 \leqslant p \leqslant N$) is isomorphic by a Lorentz transformation to either T_p, S_p, or L_p.*

Proof. Given a p-dimensional proper subspace V, let α be a Lorentz transformation that fixes e_{N+1} and acts as a standard orthogonal map on $\mathbb{R}^N$ in such a way that $\mathbb{R}^N \cap V$ is mapped to $\mathbb{R}^k$, where k is the dimension of $\mathbb{R}^N \cap V$. Either $k = p$, in which case V is contained in $\mathbb{R}^N$ and the proof is finished, or $k = p - 1$. In the second case, choose an element u in $\alpha(V) \setminus \mathbb{R}^N$. Let

$$v = (0, \ldots, 0, u_p, \ldots, u_{N+1});$$

this vector is also in $\alpha(V) \setminus \mathbb{R}^N$, since $\mathbb{R}^{p-1}$ is contained in $\alpha(V)$. Choose a Lorentz transformation β that fixes $e_1, \ldots, e_{p-1}$ and e_{N+1}, and acts as a standard orthogonal map on the span of $e_p, \ldots, e_N$ in such a way that v maps to

$$w = (0, \ldots, 0, w_p, 0 \ldots, 0, w_{N+1}),$$

where $w_{N+1} = u_{N+1}$. Note that $\beta\alpha(V)$ is the span of $\mathbb{R}^{p-1}$ and w. Let $A[a,b]$ denote the $(N+1)$-by-$(N+1)$ Lorentz matrix whose entries $A_{i,j}$ coincide with the entries of the identity matrix, except $A_{p,p} = A_{N+1,N+1} = a$ and $A_{p,N+1} = A_{N+1,p} = b$, where $a^2 - b^2 = 1$. We define a third Lorentz transformation γ as follows. If $w_p = w_{N+1}$ then let γ be the identity map, and if $w_p = -w_{N+1}$ then let γ be the map $(x_1,\dots,x_N,x_{N+1}) \mapsto (x_1,\dots,x_N,-x_{N+1})$. Otherwise, let $\ell(w) = \sqrt{|w_p^2 - w_{N+1}^2|}$ and define

$$\gamma = \begin{cases} A\left[\frac{w_p}{\ell(w)}, -\frac{w_{N+1}}{\ell(w)}\right] & \text{if } \|w\|^2 > 0, \\ A\left[\frac{w_{N+1}}{\ell(w)}, -\frac{w_p}{\ell(w)}\right] & \text{if } \|w\|^2 < 0. \end{cases}$$

The Lorentz transformation $\gamma\beta\alpha$ maps V to either L_p, S_p, or T_p, depending on whether $\|w\|^2 = 0$, $\|w\|^2 > 0$, or $\|w\|^2 < 0$. □

7. Proof of Theorem 1

The proofs of both Theorem 1 and Theorem 2 are based on the following proposition.

Proposition 7.1. *Let $\{v_\alpha : \alpha \in \mathcal{A}\}$ and $\{v'_\alpha : \alpha \in \mathcal{A}\}$ be two collections of vectors in $\mathbb{R}^{N+1}$ such that $\langle v_\alpha, v_\beta\rangle = \langle v'_\alpha, v'_\beta\rangle$ for all pairs α and β in $\mathcal{A}$. Suppose that the subspace spanned by the v_α is either time-like or space-like. Then there is a Lorentz transformation ϕ with $\phi(v_\alpha) = v'_\alpha$ for each α in $\mathcal{A}$.*

Proposition 7.1 fails when the subspace V spanned by the v_α is light-like because the next elementary lemma, used in the proof of Proposition 7.1, also fails when V is light-like.

Lemma 7.2. *Let V be either a time-like or a space-like subspace of $\mathbb{R}^{N+1}$. If there is an element v of V such that $\langle v, w\rangle = 0$ for all vectors w in V then $v = 0$.*

Indeed, if $\langle v, w\rangle = 0$ for all w in V then, in particular, $\langle v, v\rangle = 0$ so v is either 0 or light-like.

Proof of Proposition 7.1. Let V denote the subspace spanned by the vectors v_α, and let V' denote the subspace spanned by the vectors v'_α. By applying preliminary Lorentz transformations, we may assume that each of V and V' are either equal to $\mathbb{R}^{N+1}$ or else assume one of the canonical forms listed at the beginning of section 6. The subspace V' is time-like if V is time-like, and V' is space-like if V is space-like. The following proof is valid whether V is time-like or space-like.

Let p and q be the dimensions of V and V'. By swapping V and V' if necessary we may assume that $p \geqslant q$. Relabel the v_α so that $v_1,\dots,v_p$ span V, and the remaining vectors v_α are linearly dependent on $v_1,\dots,v_p$. Relabel the v'_α in a corresponding fashion. It will now be shown that $v'_1,\dots,v'_p$ are linearly independent. Suppose that $\lambda_1 v'_1 + \dots + \lambda_p v'_p = 0$, for real numbers $\lambda_1,\dots,\lambda_p$. Define $v = \lambda_1 v_1 + \dots + \lambda_p v_p$. Then $\langle v, v_i\rangle = \langle 0, v'_i\rangle = 0$, for $i = 1,\dots,p$. Thus $v = 0$, by Lemma 7.2. By linear independence of $v_1,\dots,v_p$, we deduce that the numbers λ_i are all 0. We can extend both $v_1,\dots,v_p$ and $v'_1,\dots,v'_p$ to bases of $\mathbb{R}^{N+1}$ using either the standard basis vectors $e_{p+1},\dots,e_{N+1}$ (if $V = S_p$) or $e_p,\dots,e_N$ (if $V = T_p$).

Let ϕ be the unique bijective linear map that fixes each of these $N+1-p$ standard basis vectors, and satisfies $\phi(v_j) = v'_j$, for $j = 1, \ldots, p$. Because the vectors v_i and v'_j are Lorentz orthogonal to the vectors e_k the map ϕ is a Lorentz transformation.

Finally, observe that for indices α other than $1, \ldots, p$, we have

$$\langle \phi(v_\alpha) - v'_\alpha, v'_j \rangle = \langle v_\alpha, v_j \rangle - \langle v'_\alpha, v'_j \rangle = 0, \text{ for } j = 1, \ldots, p.$$

Therefore $\phi(v_\alpha) = v'_\alpha$, by Lemma 7.2. □

Before we prove Theorem 1, we state a lemma that explains the significance of the condition $\bigcap_{\alpha \in \mathcal{A}} \partial B_\alpha = \emptyset$ of Theorem 1 in terms of hyperbolic geometry. (Note that, because the ideal boundary of $\mathcal{H}^N$ is $(N-1)$-dimensional, we assume that B_α and B'_α are balls in $\mathbb{R}^{N-1}_\infty$, rather than $\mathbb{R}^N_\infty$.) The spheres ∂B_α are the ideal boundaries of hyperbolic planes Π_α, and each hyperbolic plane Π_α is the intersection of $\mathcal{H}^N$ with a time-like Euclidean plane P_α in $\mathbb{R}^{N+1}$. Let Σ_α denote the hyperbolic half-space with ideal boundary B_α, and let v_α denote the unique space-like Lorentz unit normal of P_α such that $\Sigma_\alpha = \{x : \langle x, v_\alpha \rangle > 0\}$.

Lemma 7.3. *The spheres ∂B_α do not contain a common point of intersection if and only if the subspace of $\mathbb{R}^{N+1}$ spanned by the v_α is either time-like, or N-dimensional and space-like.*

Proof. Let V be the subspace spanned by the vectors v_α. By applying a Lorentz transformation we can assume that V is either $\mathbb{R}^{N+1}$, T_p, S_p, or L_p. The spheres ∂B_α contain a common point of intersection if and only if there is a light-like vector in $\bigcap_{\alpha \in \mathcal{A}} P_\alpha$. That is, if and only if there is a light-like vector in V^{L}. If $V = L_p$ then the vector $e_p - e_{N+1}$ is light-like and contained in V^{L}, and if $V = S_p$ and $p < N$ then the vector $e_N + e_{N+1}$ is light-like and contained in V^{L}. If $V = \mathbb{R}^{N+1}$, $V = T_p$, or $V = S_N$, then V^{L} does not contain any light-like vectors. □

Let $\widehat{\Sigma_\alpha}$ denote the hyperbolic half-space with ideal boundary $\widehat{B_\alpha}$.

Proof of Theorem 1. Suppose that $[B_\alpha, B_\beta] = [B'_\alpha, B'_\beta]$ for each α and β in $\mathcal{A}$. We must construct a Möbius transformation f for which either $f(B_\alpha) = B'_\alpha$ for all α, or else $f(\widehat{B_\alpha}) = B'_\alpha$ for all α. (Note that the converse implication follows immediately by preservation of the signed inversive distance under Möbius transformations.)

Since $\bigcap_{\alpha \in \mathcal{A}} \partial B_\alpha = \emptyset$, we see from Lemma 7.3 that the subspace V spanned by the v_α is not light-like. Proposition 7.1 shows that there is a Lorentz transformation ϕ such that $\phi(\Sigma_\alpha) = \Sigma'_\alpha$ for each α in $\mathcal{A}$. Either ϕ or $-\phi$ is a positive Lorentz transformation. In the first case $\phi(\Sigma_\alpha) = \Sigma'_\alpha$ for each α in $\mathcal{A}$, and in the second case $-\phi(\widehat{\Sigma_\alpha}) = \Sigma'_\alpha$ for each α in $\mathcal{A}$. The action of this positive Lorentz transformation on the ideal boundary of hyperbolic space is a Möbius transformation with the required properties. □

The map f of Theorem 1 is uniquely determined if and only if the map ϕ of Proposition 7.1 is uniquely determined. This occurs if and only if the subspace V spanned by the v_α is the whole of $\mathbb{R}^{N+1}$.

From Lemma 7.3 we know that $V \neq \mathbb{R}^{N+1}$ if and only if either V is (i) N-dimensional and space-like, or (ii) time-like of dimension less than $N+1$. Case (i) occurs if and only if V^{L} is 1-dimensional and time-like. Since $\bigcap_{\alpha\in\mathcal{A}} P_\alpha = V^{\mathrm{L}}$ and $\bigcap_{\alpha\in\mathcal{A}} \Pi_\alpha = \mathcal{H}^N \cap \left(\bigcap_{\alpha\in\mathcal{A}} P_\alpha\right)$, case (i) occurs if and only if the intersection of the planes Π_α is a single point in hyperbolic space. Case (ii) occurs if and only if there is a space-like vector that is Lorentz orthogonal to V. That is, if and only if there is a time-like Euclidean plane that is Lorentz orthogonal to all the P_α. Or, equivalently, there is a sphere in $\mathbb{R}^{N-1}_\infty$ that is orthogonal to all the ∂B_α.

Since Beardon and Minda considered only non-intersecting circles, case (i) did not arise in their study. Case (ii) did arise: they defined a collection of circles in $\mathbb{C}_\infty$ to be *strongly symmetric* if there is another circle orthogonal to each circle in the collection. Beardon and Minda prove, as we have just verified, that the map f is unique if and only if the collection of circles is not strongly symmetric.

8. Proof of Theorem 2

Proposition 7.1 can also be used to prove Theorem 2. To apply this proposition, first the $(N-1)$-dimensional unit sphere must be identified with the ideal boundary of $\mathcal{H}^N$. The N-dimensional unit ball $\mathbb{B}^N$ is also a model of hyperbolic space, and there is an isometry Φ from $\mathbb{B}^N$ to $\mathcal{H}^N$ given by

$$(x_1,\dots,x_N) \mapsto \left(\frac{2x_1}{1-|x|^2},\dots,\frac{2x_N}{1-|x|^2},\frac{1+|x|^2}{1-|x|^2}\right).$$

With this correspondence, the point $(x_1,\dots,x_N)$ in $\mathbb{S}^{N-1}$ is paired with the Euclidean line that passes through 0 and $(x_1,\dots,x_N,1)$. The absolute cross-ratio of four points p_1, p_2, p_3, and p_4 in $\mathbb{S}^{N-1}$ is

$$|p_1,p_2,p_3,p_4| = \frac{|p_1-p_2||p_3-p_4|}{|p_1-p_3||p_2-p_4|}. \tag{8.1}$$

We now wish to define the cross-ratio in terms of the Lorentz model of hyperbolic space. We use the next elementary lemma.

Lemma 8.1. *If u and v are two linearly independent positive light-like vectors in $\mathbb{R}^{N+1}$ then $\langle u,v\rangle < 0$.*

Proof. Since u and v are positive and light-like we can choose elements u_0 and v_0 of $\mathbb{R}^N$ such that $u = u_0 + |u_0|e_{N+1}$ and $v = v_0 + |v_0|e_{N+1}$. Therefore $|u| = \sqrt{2}|u_0|$ and $|v| = \sqrt{2}|v_0|$. We obtain

$$\langle u,v\rangle = u_0\cdot v_0 - |u_0||v_0| = u\cdot v - 2|u_0||v_0| < |u||v| - 2|u_0||v_0| = 0$$

by the Cauchy-Schwarz inequality. $\square$

Given light-like lines ℓ_1, ℓ_2, ℓ_3, and ℓ_4 we choose, for each i, any positive light-like vector v_i in ℓ_i and define

$$|\ell_1,\ell_2,\ell_3,\ell_4| = \frac{\langle v_1,v_2\rangle\langle v_3,v_4\rangle}{\langle v_1,v_3\rangle\langle v_2,v_4\rangle}.$$

This quantity is preserved under Lorentz transformations. If the point p_i in $\mathbb{S}^{N-1}$ corresponds to the line ℓ_i under Φ then we have

$$|\ell_1,\ell_2,\ell_3,\ell_4| = |p_1,p_2,p_3,p_4|^2.$$

It suffices to verify this formula when $p_1 = e_1$, $p_2 = e_2$, $p_3 = e_3$, and $p_4 = (x_1, \ldots, x_N)$, because Möbius transformations are triply transitive. Choose $v_1 = e_1 + e_{N+1}$, $v_2 = e_2 + e_{N+1}$, $v_3 = e_3 + e_{N+1}$, and $v_4 = (x_1, \ldots, x_N, 1)$. Then

$$|p_1, p_2, p_3, p_4| = \sqrt{\frac{1 - x_3}{1 - x_2}}$$

and

$$|\ell_1, \ell_2, \ell_3, \ell_4| = \frac{1 - x_3}{1 - x_2}.$$

To establish Theorem 2 we prove the following reformulation of Theorem 2 in terms of the hyperboloid model of hyperbolic space and Lorentz transformations.

Proposition 8.2. *Given two collections of light-like lines $\{\ell_\alpha : \alpha \in \mathcal{A}\}$ and $\{\ell'_\alpha : \alpha \in \mathcal{A}\}$, there is a positive Lorentz transformation ϕ with $\phi(\ell_\alpha) = \ell'_\alpha$ for each α in $\mathcal{A}$ if and only if $|\ell_\alpha, \ell_\beta, \ell_\gamma, \ell_\delta| = |\ell'_\alpha, \ell'_\beta, \ell'_\gamma, \ell'_\delta|$ for all ordered 4-tuples $(\alpha, \beta, \gamma, \delta)$ of distinct indices in $\mathcal{A}$.*

Proof. The proposition is true when $\mathcal{A}$ has fewer than four elements, by triple transitivity of Möbius transformations. We assume, therefore, that $\mathcal{A}$ has at least four elements.

Suppose that $|\ell_\alpha, \ell_\beta, \ell_\gamma, \ell_\delta| = |\ell'_\alpha, \ell'_\beta, \ell'_\gamma, \ell'_\delta|$ for all ordered 4-tuples $(\alpha, \beta, \gamma, \delta)$. Choose any three indices 1, 2, and 3 from the collection $\mathcal{A}$. For each $i = 1, 2, 3$, let v_i be a positive vector in ℓ_i, and let v'_i be a positive vector in ℓ'_i, chosen such that $\langle v_1, v_2\rangle = \langle v'_1, v'_2\rangle$, $\langle v_1, v_3\rangle = \langle v'_1, v'_3\rangle$, and $\langle v_2, v_3\rangle = \langle v'_2, v'_3\rangle$. This can be achieved by adjusting the v_i by positive scalar multiples. For all other lines ℓ_α, choose v_α to be the unique positive member of ℓ_α for which

$$\langle v_\alpha, v_2\rangle = -\frac{\langle v_2, v_3\rangle}{\langle v_1, v_3\rangle}.$$

Define a positive element v'_α of ℓ'_α so that $\langle v'_\alpha, v'_2\rangle$ is also equal to this quantity. Then the equation

$$|\ell_\alpha, \ell_1, \ell_2, \ell_3| = |\ell'_\alpha, \ell'_1, \ell'_2, \ell'_3|$$

ensures that $\langle v_\alpha, v_1\rangle = \langle v'_\alpha, v'_1\rangle$. Finally , given any pair α, β in $\mathcal{A}$, the equation

$$|\ell_\alpha, \ell_\beta, \ell_1, \ell_2| = |\ell'_\alpha, \ell'_\beta, \ell'_1, \ell'_2|$$

reduces to $\langle v_\alpha, v_\beta\rangle = \langle v'_\alpha, v'_\beta\rangle$.

The subspace spanned by $\{v_\alpha : \alpha \in \mathcal{A}\}$ is time-like because

$$\|v_1 + v_2\|^2 = 2\langle v_1, v_2\rangle < 0;$$

therefore Proposition 7.1 applies to yield a Lorentz transformation ψ with $\psi(v_\alpha) = v'_\alpha$ for all α. Define ϕ to be whichever of the maps ψ or $-\psi$ is positive. Then $\phi(\ell_\alpha) = \ell'_\alpha$ for each α in $\mathcal{A}$, as required. The converse implication follows by preservation of the cross-ratio under Lorentz transformations. □

The map ϕ in Proposition 8.2 is *not* unique if and only if the time-like space V spanned by the v_α is not equal to $\mathbb{R}^{N+1}$. In other words, ϕ is not unique if and only if all the v_α lie in a time-like Euclidean plane. In terms of the points p_α, this occurs if and only if there is an $(N-2)$-dimensional sphere in $\mathbb{S}^{N-1}$ that contains all the points p_α.

References

[1] A. F. Beardon, *The geometry of discrete groups*, Springer, New York, 1983.
[2] A. F. Beardon and D. Minda, Conformal automorphisms of finitely connected regions, in *Transcendental dynamics and complex analysis*, 37–73, Cambridge Univ. Press, Cambridge.
[3] J. B. Conway, *Functions of one complex variable. II*, Springer, New York, 1995.
[4] L. R. Ford, *Automorphic functions*, Chelsea, 1951.
[5] J. G. Ratcliffe, *Foundations of hyperbolic manifolds*, Springer, New York, 1994.

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