The bubble algebra: structure of a two-colour Temperley–Lieb Algebra

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The Bubble Algebra: Structure of a Two-Colour Temperley-Lieb Algebra

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Abstract. We define new diagram algebras providing a sequence of multiparameter generalisations of the Temperley-Lieb algebra, suitable for the modelling of dilute lattice systems of two-dimensional Statistical Mechanics. These algebras give a rigorous foundation to the various 'multi-colour algebras' of Grimm, Pearce and others. We determine the generic representation theory of the simplest of these algebras, and locate the nongeneric cases (at roots of unity of the corresponding parameters). We show by this example how the method used (Martin's general procedure for diagram algebras) may be applied to a wide variety of such algebras occurring in Statistical Mechanics.

We demonstrate how these algebras may be used to solve the Yang-Baxter equations.

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1. Introduction

Some time ago, motivated by the study of dilute lattice models [41, 50], Grimm and Pearce [18] introduced generalisations of certain diagram algebras (algebras with a diagrammatic formulation [33]), such as Temperley-Lieb [44] and Murakami-Birman-Wenzl [39, 4] algebras. These algebras are important in the theory of solvable lattice models of two-dimensional Statistical Mechanics [2] and are related to link and knot invariants [46]. The generalisation was conceived on the diagram level by introducing diagrams with lines in a number of colours. Each algebra was then described by generators and relations dictated, or at least suggested, by the requirement of solving the Yang-Baxter equations. However the diagrammatic (which is to say, topological) underpinning was not precisely formalised.

The classes of solvable lattice models called dilute lattice models [41, 50, 51] whose discovery motivated this generalisation are closely linked to models of dilute loops on a lattice [1, 49]. These models attract particular interest because they contain a solvable 'companion' of the two-dimensional Ising model in a magnetic field [50, 16, 17] — one of the famous unsolved problems in Statistical Mechanics. The idea here is to consider two colours, and to regard the second colour merely as a dilution of the first.

In the two colour case the requirement of solving the Yang-Baxter equations is fully satisfied by the design of the relations. The Yang-Baxter equations are sufficient to guarantee solvability in the sense of commuting transfer matrices [2, 46]. Thus representations of such algebras give rise to solvable dilute and two-colour lattice models. (More precisely, one has relations for a *tower* of algebras, and the representation must be defined for the whole tower.) Various explicit representations and associated models are considered in [18, 10, 11, 20, 21, 12].

The representations found included previously known lattice models [10, 11], but also gave rise to new series of solvable lattice models [20, 21, 12]. However, little else was discovered about these algebras and their structures. We have generators and relations, and enough representations to show that these relations do not imply a trivial algebra, but no knowledge of dimensions or even finiteness, and no analysis of irreducible representations. To this extent the representations which *were* found were a matter of luck, and there was no way to tell if the relations could engender other important but undiscovered models. This may be contrasted with our quite complete knowledge of the representation theory of the Temperley-Lieb algebra itself, which is strikingly rich and beautiful, and important in several areas of mathematics and physics [8, 27, 28, 33, 25, 29].

In this paper we define a new algebra — the *bubble* algebra. We define this algebra entirely diagrammatically, such that it is amenable to the general method of [33, §9.5],[36]. We then show that this gives a properly constituted diagrammatic realisation of the Grimm-Pearce multi-colour Temperley-Lieb algebra (i.e. it solves the Yang-Baxter equations). We hence use the general method to determine the generic representation theory of these algebras completely. We set up the machinery to investigate their exceptional representation theory (analogous to that of ordinary Hecke algebras at q a root of unity). We show how irreducible representations may be associated to physical observables in the corresponding lattice models. We conclude with a discussion of the implications of our results for Bethe ansatz on models derived using this algebra. We mainly discuss the case of two colours, as the further generalisation to more colours is straightforward. (The case of one colour is the original Temperley-Lieb algebra.)

Generalisations of the Temperley-Lieb algebra are two-a-penny [35, 36, 31, 42, 9], however there are now a number of reasons for looking at the algebras introduced in [18] again. Firstly, the diagram form of the Temperley-Lieb algebra is a deep and powerful property (cf. [27, 28, 33]), and our new realisation provides a natural generalisation on the diagram level. Secondly, they provide solutions of the Yang-Baxter equation as we have said. They are similar in some ways to the blob algebra, which has recently been shown [6] to be useful in solving the reflection equation [43]. We also expect them to be of use in constructing integrable boundary conditions for certain solvable lattice models, including 'conformal twisted' boundary conditions [19, 3, 40, 14, 15], and thus to be of relevance to boundary conformal field theory. Thirdly, we show that they are part of a class of algebras amenable to the methods of [36], so that we may now analyse them quite efficiently (and hence provide a uniform theory of such algebras). This analysis suggests (see later) that they may be relevant for Statistical Mechanics on ladders (cf. recent works [48, 47, 45]), and indeed, just recently, this was shown to be the case [22]. They also look like they should be relevant for circuit design and even transport network design (although we know of no example of their use in these areas!) as we will see. There are also similarities with Murakami-Birman-Wenzl algebras [39, 4] and Fuss-Catalan algebras [7], both of which have been used to construct integrable systems, to the extent that the same methods are applicable there. Finally, they have a number of features of technical interest in representation theory (we largely postpone comment on these to a separate paper, but see section 6 for a brief discussion).

We start with some definitions.

2. Diagram algebras

Our new algebra is a diagram algebra — an algebra with a diagrammatic formulation akin to the Temperley-Lieb algebra [44, 33]. It will be convenient to recall this familiar example in a suitable formalism, and then generalise to our case.

(2.1) Fix a rectangular subset of $\mathbb{R}^2$ such that there is an edge with a North pointing normal (e.g. $[0,1] \times [0,1]$). Label each edge by the direction (NSEW) of its normal. Consider the set of partitions of this rectangle by finitely many continuous non-crossing lines (walls) with no wall touching the E or W edge. We define an equivalence relation on this set by equivalencing two such partitions if they differ only by a continuous edge preserving deformation of the rectangle. We call (representatives of) equivalence classes *diagrams*.

We say we can *compose* two such partitions, a over b, if there lie in their equivalence classes two diagrams such that when a is juxtaposed with b from above, the southern endpoints of lines in a coincide with the northern endpoints of lines in b (NB, this requires only that the *number* of lines matches up). Each point of coincidence may then be regarded as an interior point of a continuous line passing though the juxtaposition $a|b$. The composite ab is the new partition of the combined region which results from this.

(2.2) Consider the subset of diagrams where there are precisely n endpoints on each northern and each southern edge. For q an invertible indeterminate, consider the $\mathbb{Z}[q, q^{-1}]$-linear extension of this set. Let $T_{n,\mathbb{Z}}$ denote the quotient of *this* set by the relation which equivalences any diagram with a closed (interior) loop to δ times the same diagram without, where $\delta = q + q^{-1}$. Note that $T_{n,\mathbb{Z}}$ has basis the set of diagrams where there are precisely n endpoints on each northern and each southern edge, *and no interior loops.* Note that the composition of diagrams passes to a well defined composition on this set, making it a $\mathbb{Z}[q, q^{-1}]$-algebra.

Fix K a field which is a $\mathbb{Z}[q, q^{-1}]$-algebra (for example, the complex numbers, with q acting as some specified nonzero complex number). The *Temperley-Lieb algebra* $T_n(q)$

is the K-algebra $K \otimes_{\mathbb{Z}[q,q^{-1}]} T_{n,\mathbb{Z}}$.

(2.3) A line in a diagram with one endpoint in the Northern (N) edge and one in the S edge is called a *propagating* line. The identity element of $T_n(q)$ is the unique diagram all of whose lines are propagating.

There are a number of ways of embedding T_{n-1} as a subalgebra in T_n. We will call that embedding which maps $a \in T_{n-1}$ to the same diagram, but with one extra propagating line on the right, the *natural* embedding.

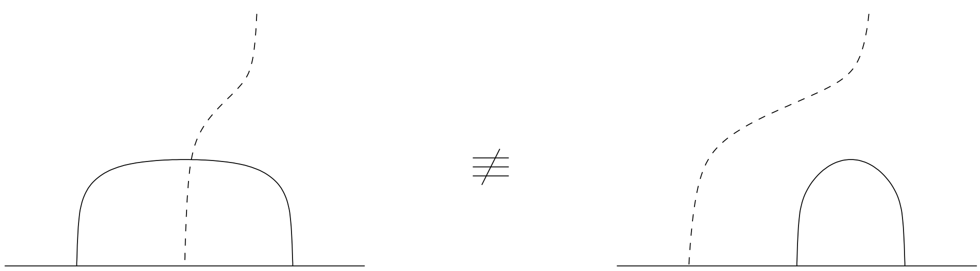
For brevity we will assume familiarity with the usual presentation of T_n by generators $\{U_1, U_2, .., U_{n-1}\}$ and relations, and the correspondence with the diagram version (see for example [32]).

The topological/diagram realisation of the Temperley-Lieb algebra is enormously useful [33, 9] and deep [27, 28, 29]. We require a similarly clearcut and intuitive construction, let us call it a *model*, for the algebra introduced in [18]. Here we will concentrate mainly on the model for two colours. The generalisation to arbitrarily many colours will be obvious. A generalisation to the Murakami-Birman-Wenzl version is also possible.

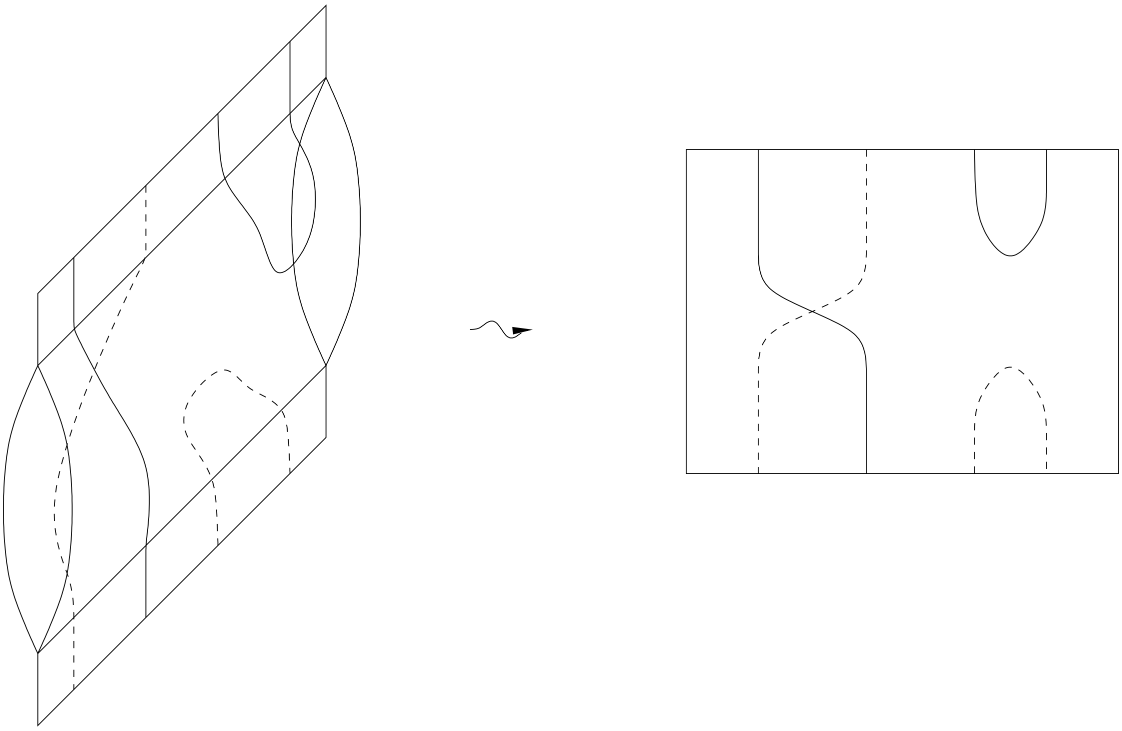
Before we introduce the model note that the Temperley-Lieb diagrams described above may be regarded as partitionings of the set of endpoints into pairs. The non-crossing rule means that they are a proper subset of the set B_n^J of all such pair partitionings in general. The full set B_n^J is a basis for the Brauer algebra J_n [5] (whose composition rule need not concern us here).

(2.4) Now consider the set each element of which consists of two independent (but simultaneous) partitionings of a rectangle as above (one, say, with red lines, one with blue). Here independence means that walls of different colours may cross, but we will *exclude* elements in which such crossings occur on the frame of the rectangle. We define an equivalence essentially as before, so for example (locally)

but (because of the exclusion) not on the frame:



This is as if we have two parallel but independent deformable rectangles (one for each colour), but they share the frame. Another way to think of this is as lines embedded not just in a rectangle, but in bubble wrap (bubble wrap is made from two sheets of polythene welded together along certain lines to trap bubbles). Red lines are allowed on the welds and the back sheet, blue lines are allowed on the welds and the front sheet:



In this realisation lines on the same sheet (or on the weld) are not allowed to touch, but otherwise may be deformed isotopically as before. Accordingly, we call the deformation equivalence 'bubble isotopy'.

Again we define composition whenever the number of endpoints (irrespective of colour) matches up. We do this as follows. We call the match up *precise* if the colours match up precisely (i.e. we can identify the touching edges and have a properly formed two-colour partition). The composite is 0 unless the colours match up precisely. If they *do* match up the composite *is* that two-colour partition.

Consider the subset of double partitionings in which the total number of endpoints (red *and* blue) on the northern edge is n, and similarly on the southern edge. The *bubble algebra* $T^2_{n,\mathbb{Z}}$ (so named to emphasise the topological diagram underpinning) is the $\mathbb{Z}[q_r, q_r^{-1}, q_b, q_b^{-1}]$-linear extension of this set and composition, with internal closed loop replacements (as in $T_n(q)$). Thus $T^2_{n,\mathbb{Z}}$ has a basis, B_n say, of two-colour partitions (up to bubble isotopy) with no internal loops. The loop replacement scalar δ here depends on the colour: $\delta_r = q_r + q_r^{-1}$ and $\delta_b = q_b + q_b^{-1}$. Fix a field K which is a $\mathbb{Z}[q_r, q_r^{-1}, q_b, q_b^{-1}]$-algebra as before (e.g. the complex numbers with q_r, q_b specified complex numbers). Denote the K-algebra $K \otimes_{\mathbb{Z}[q_r,q_r^{-1},q_b,q_b^{-1}]} T^2_{n,\mathbb{Z}}$ by $T^2_n = T^2_n(q_r, q_b)$.

The obvious generalisations T^N_n $(N = 1, 2, ..)$ include $T^1_n = T_n$.

(2.5) Let $\#_r(d)$ denote the number of **r**ed propagating lines in diagram d (and similarly for **b**lue). Extend this to apply to any non-zero scalar multiple of d. It will be evident that composing with any second diagram d' such that $dd' \neq 0$ we have

$$\#_r(dd') \leq \#_r(d) \tag{1}$$

and similarly for blue. Write $B_n(i,j)$ for the subset of B_n with $\#_r(d) = i$, $\#_b(d) = j$, and define

$$B_n(i) = \cup_j B_n(i-j,j)$$

$$B_n[i] = \cup_{j \leq i} B_n(j)$$

so $B_n(i)$ and $B_n[i]$ consist of those diagrams in B_n with exactly i and at most i propagating lines, respectively.

We say that two lines are *strictly* non-crossing when they are non-crossing even when projected into a single plane (so as to recover Grimm and Pearce's original diagrams). Write $B'_n(i,j)$ for the subset of $B_n(i,j)$ with lines all strictly non-crossing, and define $B'_n(i)$ similarly.

For example $B'_n(n)$ is the set of diagrams with all lines propagating and strictly non-crossing. It will be evident that $|B'_n(n)| = 2^n$, and that

$$1 = \sum_{d \in B'_n(n)} d$$

is an orthogonal idempotent decomposition of the identity element of T^2_n.

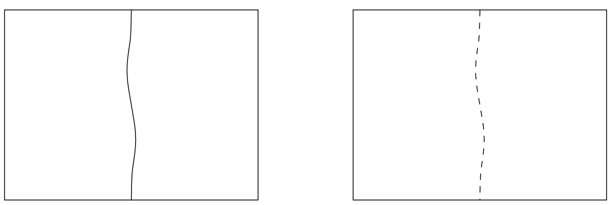
If $d \in B_{n-1}$, let $\mathcal{I}_r(d), \mathcal{I}_b(d) \in B_n$ denote the same diagram except with one extra non-crossing propagating red (resp. blue) line to the right of all other lines. Thus $\mathcal{I}_r$ and $\mathcal{I}_b$ are injective maps on bases, which extend to injective maps from T^2_{n-1} to T^2_n. Note that these maps do not preserve the identity element. There is, however, an inclusion

$$\mathcal{I} : T_{n-1} \hookrightarrow T_n$$

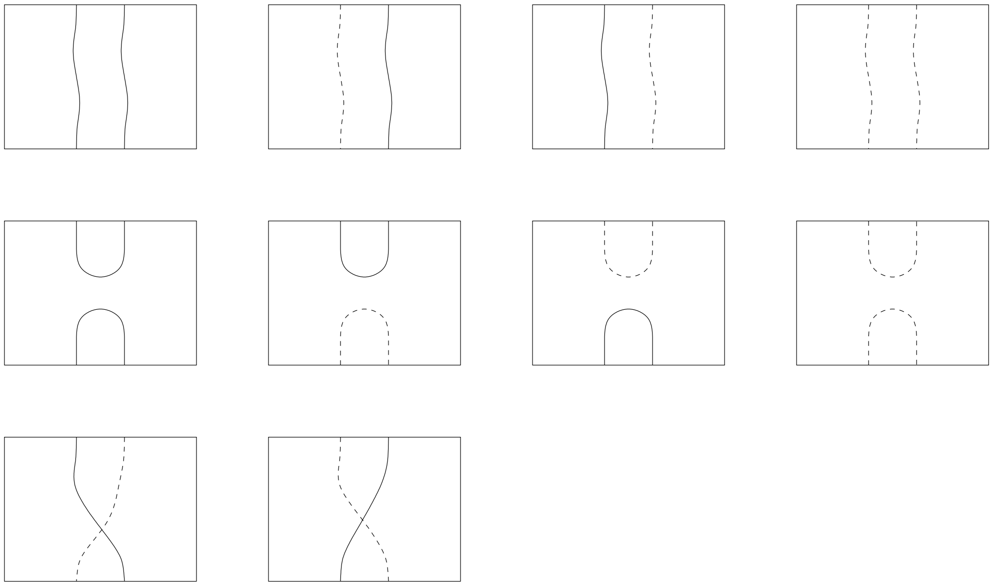
given by $d \mapsto \mathcal{I}_r(d) + \mathcal{I}_b(d)$ which we will call the 'natural' inclusion by analogy with the T_n case.

(2.6) The basis B_n may be constructed systematically for each n from that for $n-1$ using some simple combinatorial devices which we will describe shortly.

Examples: The basis B_1 of T^2_1 consists of the following diagrams



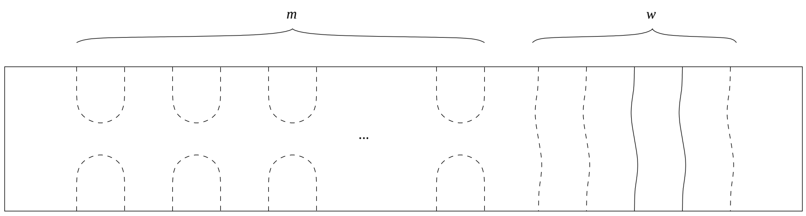
The basis B_2 of T_2^2 consists of the following diagrams



In particular $B_2(0,0)$ consists of the diagrams in the middle row; $B_2(2,0)$ consists only of the leftmost diagram in the top row; and $B_2(0,2)$ the rightmost. The remaining diagrams are in $B_2(1,1)$, thus $B_2 = B_2(0,0) \cup B_2(2,0) \cup B_2(0,2) \cup B_2(1,1)$.

Let us write U_1^r for the rightmost diagram in the middle row of B_2 diagrams above, and also for the image of this diagram under $\mathcal{I}$ (or arbitrary compositions of $\mathcal{I}$). Let w be a sequence in $\{r, b\}$ of length $n - 2m$, then write e_w^r for the following element of B_n:

$e_w^r =$



(in this example $w = rrbbr$).

3. Solutions to the Yang-Baxter Equations

In the sections *after* this one we will return to discussing the general algebra basis and irreducible representation theory. First let us briefly look at how the algebra can be used to build solutions to the Yang-Baxter equations. We will need to start with some notation and definitions.

3.1. One colour notations

Consider the ordinary spin chain representation [44, 2] of the one colour Temperley-Lieb algebra. Choose a basis $v_1 = |\,++\rangle$, $v_2 = |\,+-\rangle$, $v_3 = |\,-+\rangle$, $v_4 = |\,--\rangle$ and define

$$e = \left(\begin{smallmatrix}\cup\\ \cap\end{smallmatrix}\right) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & q & 1 & 0 \\ 0 & 1 & q^{-1} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{2}$$

We can think of this as a product of a bra and a ket

$$e = \left(\begin{smallmatrix}\cup\\ \cap\end{smallmatrix}\right) = \left(\cup\right)\left(\cap\right) = \begin{pmatrix} 0 \\ q^{\frac{1}{2}} \\ q^{-\frac{1}{2}} \\ 0 \end{pmatrix} \begin{pmatrix} 0 & q^{\frac{1}{2}} & q^{-\frac{1}{2}} & 0 \end{pmatrix} \tag{3}$$

where

$$\left(\cup\right)^t = \left(\cap\right) = \begin{pmatrix} 0 & q^{\frac{1}{2}} & q^{-\frac{1}{2}} & 0 \end{pmatrix} \tag{4}$$

3.2. Two colour representation

We now describe a (highly reducible) representation of the two colour algebra. This is a representation on the tensor product space $\mathbb{C}^{4^n} \cong \mathbb{C}^4 \otimes \mathbb{C}^4 \otimes \ldots \otimes \mathbb{C}^4$. We specify the representation by giving the explicit representation matrices of the ten elements of the basis B_2 of T_2^2 on $\mathbb{C}^{16} \cong \mathbb{C}^4 \otimes \mathbb{C}^4$. The representation matrices are thus 16×16 matrices, with entries that now depend on two parameters q_r and q_b. The matrices are given in the basis $v_1 = |r_+r_+\rangle$, $v_2 = |r_+r_-\rangle$, $v_3 = |r_+b_+\rangle$, $v_4 = |r_+b_-\rangle$, $v_5 = |r_-r_+\rangle$, $v_6 = |r_-r_-\rangle$, $v_7 = |r_-b_+\rangle$, $v_8 = |r_-b_-\rangle$, $v_9 = |b_+r_+\rangle$, $v_{10} = |b_+r_-\rangle$, $v_{11} = |b_+b_+\rangle$, $v_{12} = |b_+b_-\rangle$, $v_{13} = |b_-r_+\rangle$, $v_{14} = |b_-r_-\rangle$, $v_{15} = |b_-b_+\rangle$, $v_{16} = |b_-b_-\rangle$ of $\mathbb{C}^{16}$, where r and b refer to the two colours, and we have an additional variable living on the lines which you may think of as an arrow pointing up $(+)$ or down $(-)$, as in the usual spin chain representation of the Temperley-Lieb algebra, compare equation (2).

The representation matrices for the elements with two propagating straight red or blue lines are diagonal, with elements

$$\left(\,|\;\;|\,\right) = \mathrm{diag}(1,1,0,0,1,1,0,0,0,0,0,0,0,0,0,0)\,,$$
$$\left(\,|\;\;\vdots\,\right) = \mathrm{diag}(0,0,1,1,0,0,1,1,0,0,0,0,0,0,0,0)\,,$$
$$\left(\,\vdots\;\;|\,\right) = \mathrm{diag}(0,0,0,0,0,0,0,0,1,1,0,0,1,1,0,0)\,,$$
$$\left(\,\vdots\;\;\vdots\,\right) = \mathrm{diag}(0,0,0,0,0,0,0,0,0,0,1,1,0,0,1,1)\,,$$

those for two crossing lines of different colour are

$$
\left(\times\right) = \left(\times\right)^t == \begin{pmatrix}
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
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0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0
\end{pmatrix}.
$$

Finally, there are four colourings of the usual Temperley-Lieb generators. These can again be written as products of kets and bras, just as in equation (3) for the one colour case,

$$
\begin{aligned}
\left(\begin{smallmatrix}\cup\\ \cap\end{smallmatrix}\right) &= \left(\cup\right)\left(\cap\right),\\
\left(\begin{smallmatrix}\cup\\ \cap\end{smallmatrix}\right) &= \left(\cup\right)\left(\cap\right),\\
\left(\begin{smallmatrix}\cup\\ \cap\end{smallmatrix}\right) &= \left(\cup\right)\left(\cap\right),\\
\left(\begin{smallmatrix}\cup\\ \cap\end{smallmatrix}\right) &= \left(\cup\right)\left(\cap\right),
\end{aligned}
$$

where now

$$
\begin{aligned}
\left(\cup\right)^t &= \left(\cap\right) = \begin{pmatrix}0 & q_r^{\frac{1}{2}} & 0 & 0 & q_r^{-\frac{1}{2}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\end{pmatrix},\\
\left(\cup\right)^t &= \left(\cap\right) = \begin{pmatrix}0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & q_b^{\frac{1}{2}} & 0 & 0 & q_b^{-\frac{1}{2}} & 0\end{pmatrix}.
\end{aligned}
$$

3.3. Yang-Baxter construction

Let us elaborate on this by means of an explicit example. From certain representations of the two-colour bubble algebra, we can derive integrable vertex models. In the simplest scenario, these vertex models correspond to $\check{R}$-matrix solutions of the Yang-Baxter equation

$$
\left(\check{R}(u)\otimes \mathrm{id}\right)\left(\mathrm{id}\otimes \check{R}(u+v)\right)\left(\check{R}(v)\otimes \mathrm{id}\right) = \left(\mathrm{id}\otimes \check{R}(v)\right)\left(\check{R}(u+v)\otimes \mathrm{id}\right)\left(\mathrm{id}\otimes \check{R}(u)\right) \quad (5)
$$

which is an equation on a triple tensor product space $V\otimes V\otimes V$, with $\check{R}$ acting on $V\otimes V$. A particular vertex model on the square lattice is specified by the matrix elements of $\check{R}(u)$, which correspond to the Boltzmann weights of the respective local configurations (the variable u is the spectral parameter). The transfer matrices $\mathcal{T}_n(u)$ are built from elementary matrices $\check{R}(u)$, so as to act on the n-fold tensor product $V \otimes V \otimes \ldots \otimes V$. They commute for different values of u. The free energy of the vertex model is then obtained from the largest eigenvalue of the transfer matrix, and we are particularly interested in its behaviour as n tends to infinity.

The $\check{R}$-matrix

$$\check{R}(u) = \frac{\sin(\lambda - u)}{\sin(\lambda)} I + \frac{\sin(u)}{\sin(\lambda)} e = \frac{\sin(\lambda - u)}{\sin(\lambda)} \left(\,|\;\;|\,\right) + \frac{\sin(u)}{\sin(\lambda)} \left(\begin{smallmatrix}\cup\\ \cap\end{smallmatrix}\right), \qquad (6)$$

expressed in terms of generators of the (one-colour) Temperley-Lieb algebra with $q + q^{-1} = 2\cos\lambda$, is a well-known example of a solution of the Yang-Baxter equation [46]. In fact it follows from the relations in the Temperley-Lieb algebra that this combination satisfies the Yang-Baxter equation. Hence this 'Baxterisation' shows that any representation of the Temperley-Lieb algebra with a large n limit yields a solvable lattice model of Statistical Mechanics. For the representation at hand, this model is the well-known six-vertex model, and the $\check{R}$-matrix is related to the affine Lie algebra $A_1^{(1)}$ [24].

Let us now move on to the two-colour case. The $\check{R}$-matrix [18, 21]

$$\begin{aligned}\check{R}(u) = {} & \frac{\sin(\lambda - u)\sin(3\lambda - u)}{\sin(\lambda)\sin(3\lambda)} \left[\left(\,|\;\;|\,\right) + \left(\,\vdots\;\;\vdots\,\right)\right] + \frac{\sin(3\lambda - u)}{\sin(3\lambda)} \left[\left(\,|\;\;\vdots\,\right) + \left(\,\vdots\;\;|\,\right)\right] \\ & - \frac{\sin(u)\sin(2\lambda - u)}{\sin(\lambda)\sin(3\lambda)} \left[\left(\begin{smallmatrix}\cup\\ \cap\end{smallmatrix}\right) + \left(\begin{smallmatrix}\cup\\ \cap\end{smallmatrix}\right)_{\text{dashed}}\right] + \frac{\sin(u)}{\sin(3\lambda)} \left[\left(\begin{smallmatrix}\cup_{\text{dashed}}\\ \cap\end{smallmatrix}\right) + \left(\begin{smallmatrix}\cup\\ \cap_{\text{dashed}}\end{smallmatrix}\right)\right] \\ & + \frac{\sin(u)\sin(3\lambda - u)}{\sin(\lambda)\sin(3\lambda)} \left[\left(\,\times\,\right) + \left(\,\times\,\right)\right] \qquad (7)\end{aligned}$$

with $q + q^{-1} = -2\cos(\lambda)$, satisfies the Yang-Baxter equation (5) as a consequence of the relations of the bubble algebra. Thus any representation of the two-colour bubble algebra on a tensor product space $V \otimes V \otimes \ldots \otimes V$ with $q_r = q_b =: q$ gives rise to an integrable vertex model with an $\check{R}$ matrix given by equation (7) with Boltzmann weights that are trigonometric functions of the spectral parameter u.

For the representation at hand, the $\check{R}$ matrix turns out to be related to the affine Lie algebra $\mathrm{C}_2^{(1)}$ [24]. It differs from the vertex model of [24] by a spectral-parameter-dependent gauge transformation [18].

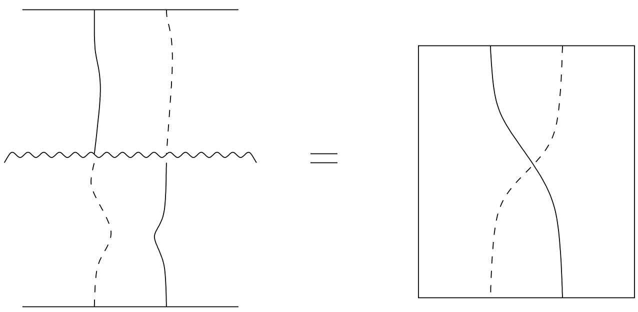
4. Combinatorics and representation theory

4.1. The algebra basis B_n

Note that B_n is somewhat like the Brauer diagram basis B_n^J of the Brauer algebra J_n [5], which in turn contains the diagram basis of T_n. From each 'seed' element of the Brauer basis we can get some number (0 or more) of diagrams of B_n by colouring the lines in the following way. First put a total order on the n lines (any one will do — for definiteness we will number the line coming out of the top left endpoint 1, then number other lines 2,3,.. as they first appear reading clockwise round the frame). Each line, considered in this order, may be coloured red or blue, unless it crosses one or more already coloured lines, in which case it must be coloured in a colour *distinct* from those of all the crossing lines. Of course, if there are three or more lines in the diagram this may not be possible (i.e. when a line crosses both of a pair of crossed lines), in which case there are *no* coloured Brauer diagrams of this type in B_n.

Apart from the seeds with no possible colourings, each seed produces 2^c elements of B_n, where c is the number of lines in the seed which are either without any crossings, or are the first line in their crossed cluster (i.e. the line whose colour is chosen freely).

(4.1) As with T_n [33] there is a bra-ket construction. Imagine cutting the bubble in half (i.e. cutting through the front and back sheets, leaving a top and bottom piece each with Y cross-section). It will be evident that it is possible to do this such that only propagating lines are cut, and these once each. Note that given two pieces in this way, because of the non-crossing within a layer rule, there is a unique way of recombining them, i.e. recovering the original diagram. Indeed any bra- (top piece) and -ket (bottom piece) such that the number of cut lines matches up (on each sheet of the bubble separately) may be combined in a unique way. For example



That is, writing $B_n^{|\rangle}(i,j)$ for the set of bra pieces obtained by cutting elements of $B_n(i,j)$ (and similarly $B_n^{\langle|}(i,j)$ for the set of ket pieces), then any element a of $B_n^{|\rangle}(i,j)$ may be combined with any element b of $B_n^{\langle|}(i,j)$ in a unique way to make a diagram ab in $B_n(i,j)$:

$$B_n(i,j) \cong B_n^{|\rangle}(i,j) \times B_n^{\langle|}(i,j). \tag{8}$$

(4.2) Following [33, §13.2], we define certain injective homomorphisms of bra sets for any appropriate n,i,j:

$$\mathcal{A}_r : B_{n-1}^{|\rangle}(i,j) \hookrightarrow B_n^{|\rangle}(i+1,j)$$

takes a diagram d to a diagram differing from d only in having an additional red propagating line at the right hand end;

$$\mathcal{A}_b : B_{n-1}^{|\rangle}(i,j) \hookrightarrow B_n^{|\rangle}(i,j+1)$$

similarly, but adding a blue line; for $i > 0$

$$\mathcal{B}_r : B_{n-1}^{|\rangle}(i,j) \hookrightarrow B_n^{|\rangle}(i-1,j)$$

takes a diagram d to a diagram differing from d only in having the Southern endpoint of the last (rightmost) red propagating line in d turn back to form a new rightmost Northern vertex; and $\mathcal{B}_b$ similarly for the last blue line.

It will be evident that

$$B_n^{|\rangle}(i,j) = \mathcal{A}_r B_n^{|\rangle}(i-1,j) \cup \mathcal{A}_b B_n^{|\rangle}(i,j-1) \cup \mathcal{B}_r B_n^{|\rangle}(i+1,j) \cup \mathcal{B}_b B_n^{|\rangle}(i,j+1) \tag{9}$$

(any undefined set here to be interpreted as the empty set).

(4.3) It will be convenient to be able to depict the sum of two diagrams in B_n differing only in the colour of one line by drawing any one of these diagrams with the relevant line replaced by a thick line (called a *white* line). We will generalise this so that a diagram with two white lines is a sum of four diagrams from B_n, and so on. Let us write U_i for that diagram which has the same shape as the diagram $U_i \in T_n$, but has all white lines. Note that this is an (unnormalised) idempotent: $U_i U_i = (\delta_r + \delta_b) U_i$.

Write e_l for an all white diagram of the same shape as e^r_w, where l is the length of the sequence w. Thus $e_{n-2m} = U_1 U_3 .. U_{2m-1}$.

Let $\tilde{n}$ denote the element of $\{0,1\}$ congruent to n modulo 2.

4.2. Standard modules

Next we construct a basic set of representations of T_n^2.

(4.4) It will be evident from equation (1) that there is a filtration of T_n^2 by ideals with sections spanned by diagrams having fixed numbers of propagating lines — and hence having the subsets $B_n(i) \subset B_n$ as bases.

Note in particular from equation (1) that $T_n^2 U_1 T_n^2$ has basis $B_n[n-2] = \cup_{i \leq n-2} B_n(i)$; $T_n^2 U_1 U_3 T_n^2$ has basis $B_n[n-4]$; and so on. (If n is big enough then $T_n^2 U_1 T_n^2 = T_n^2 U_1 U_2 T_n^2$, $T_n^2 U_1 U_3 T_n^2 = T_n^2 U_1 U_3 U_2 U_4 T_n^2$, and so on, so these ideals may be considered idempotently generated over any field K.) The filtration by propagating lines may thus be written

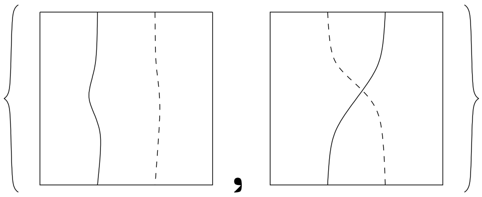
$$T_n^2 \supset T_n^2 U_1 T_n^2 \supset T_n^2 U_1 U_3 T_n^2 \supset .. \supset T_n^2 U_1 U_3 .. U_{2m-1} T_n^2 \supset ..$$

Let us write $T_n^2[i]$ for the ideal spanned by diagrams with $\leq i$ propagating lines.

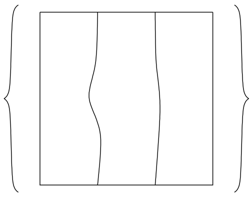
The total number $i+j$ of propagating lines is one of $n, n-2, n-4, .., 1/0$. This number may be partitioned in any way between red and blue lines, with the corresponding ideal breaking up as a direct sum accordingly. The filtration thus refines to one with sections spanned by the sets $B_n(i,j)$ of diagrams with fixed *propagating index* $(\#_r(d), \#_b(d))$.

Each such section breaks up as a sum of isomorphic left modules each with basis of the form $\{|a\rangle\langle b| \mid |a\rangle \in B_n^{|\rangle}(i,j)\}$ where $|a\rangle$ varies over all possibilities and b is fixed. (There is obviously a parallel construction for right modules.) We denote (any representative of) the equivalence class of these summands $\Delta_n(i,j)$. These left modules are called *standard* modules.

For example, $\Delta_2(1,1)$ has basis



while $\Delta_2(2,0)$ has basis



Note that, because of the sectioning, the action of $B_2(0,0)$ elements on this object *regarded as a basis element of* $\Delta_2(2,0)$ is to give *zero* (that is, any object with fewer propagating lines would lie in the next layer of the filtration, and is thus congruent to zero in this section).

(4.5) Since the bottom (ket) halves of diagrams regarded as basis elements of standard modules play no role, we have another basis for each standard module $\Delta_n(i,j)$, consisting of the set $B_n^{|\rangle}(i,j)$ of bra diagrams, with the action defined in an obvious way.

Note the following.

(4.6) The construction of standard modules is independent of the choice of field.

Each standard module $\Delta_n(i,j)$ comes with an inner product via its basis of bra diagrams (and dual basis of ket diagrams):

$$\begin{array}{cccccc} d & d' & = & k_{dd'} & d'' \\ |d\rangle\langle d| & |d'\rangle\langle d'| & = & \langle d||d'\rangle & |d\rangle\langle d'| \end{array} \qquad (10)$$

In particular, if $i+j=n$, it is easily verified that the corresponding Gram matrix, $G_n(i,j)$, is the unit matrix. Thus $\Delta_n(i,n-i)$ is irreducible for any q_r, q_b.

Note on the other hand that $G_2(0,0) = \mathrm{diag}(\delta_r, \delta_b)$, so that $\Delta_2(0,0)$ is reducible if either δ_r or δ_b vanishes.

More generally, it will be evident that $|G_n(i,j)|$ is a non-zero polynomial in δ_r, δ_b. Thus

Theorem 1 *The standard modules $\Delta_n(i,j)$ are generically simple.*

(Recall that *generically* means: in a Zariski open subset of the (δ_r, δ_b) parameter space.)

On the other hand, inspection of the diagram for e^r_w shows the following:

Proposition 1 *Let δ_r be invertible in K and let $e^r_w \in T^2_n$ have sequence $w = rr..rbb..b = r^i b^j$. Then*

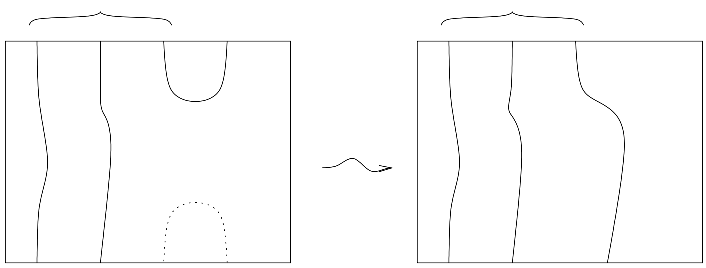
$$\Delta_n(i,j) \cong T^2_n e^r_w \qquad \textit{mod. } T^2_n[i+j-2]$$

Proposition 2 *Over any field K with δ_r invertible $\Delta_n(i,j)$ has simple head.*

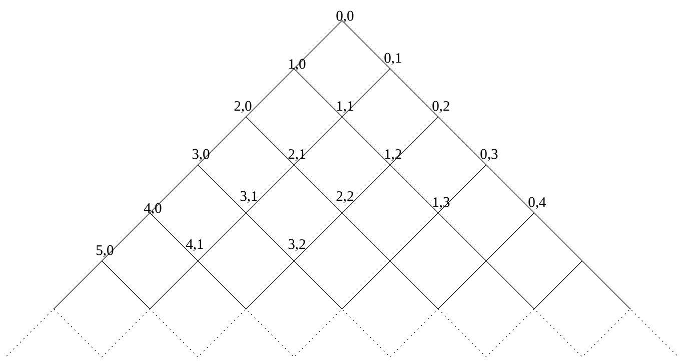
(Recall that the *head* of a module is the quotient by the intersection of all maximal proper submodules.)
Proof: The generating element provided by the previous proposition is an unnormalised (but normalisable) idempotent. It is not primitive in T_n^2, but it is primitive in a suitable quotient. This means that the induced module is indecomposable projective in some quotient, so it has a simple head. Done.

Next we consider the *completeness* of this set of representations.

(4.7) Irrespective of the choice of field, the restriction of standard module $\Delta_n(i,j)$ to T_{n-1}^2 works as follows. If the line coming out of the last (rightmost) northern endpoint is propagating and red (resp. blue), then d behaves, on restriction, like an element of the corresponding basis of $\Delta_{n-1}(i-1,j)$ (resp. $\Delta_{n-1}(i,j-1)$). That is, these modules are submodules of $\mathrm{Res}^n_{n-1}\Delta_n(i,j)$. If the line coming out of the last (rightmost) northern endpoint is red (resp. blue) and not propagating, then, *quotienting by the submodules just noted*, d behaves, on restriction, like an element of the corresponding basis of $\Delta_{n-1}(i+1,j)$ (resp. $\Delta_{n-1}(i,j+1)$). For example in $\Delta_4(2,0)$, restricting to $n=3$ as indicated by the brace we have:



This information is neatly summarised (including positivity constraints) as follows. Define a bipartite infinite graph $\mathcal{G}$ with vertex set $\mathbb{N}_0 \times \mathbb{N}_0$ by



Define $\mathcal{G}_n$ to be the full subgraph with vertices (i,j) such that $i+j \leq n$.

Proposition 3 *The restriction from T_n^2 to T_{n-1}^2 of $\Delta_n(\mu)$ may be given by*

$$Res^n_{n-1}\Delta_n(\mu) \cong +_\lambda \, \Delta_{n-1}(\lambda)$$

where the sum is over the set of nearest neighbours of $\mu = (i,j)$ on the graph $\mathcal{G}_n$.

It follows that the dimension of the module $\Delta_n(i,j)$ is the number of walks from $(0,0)$ to (i,j) of length n. It follows from this and equation (8) that the rank of the algebra (the degree of B_n) is the sum of the squares of these dimensions; and that

Theorem 2 *If the algebra is semisimple in some specialisation of the parameters (as generically, for example — see Theorem* 1*), then the standard modules are a complete set of irreducible modules in that specialisation.*

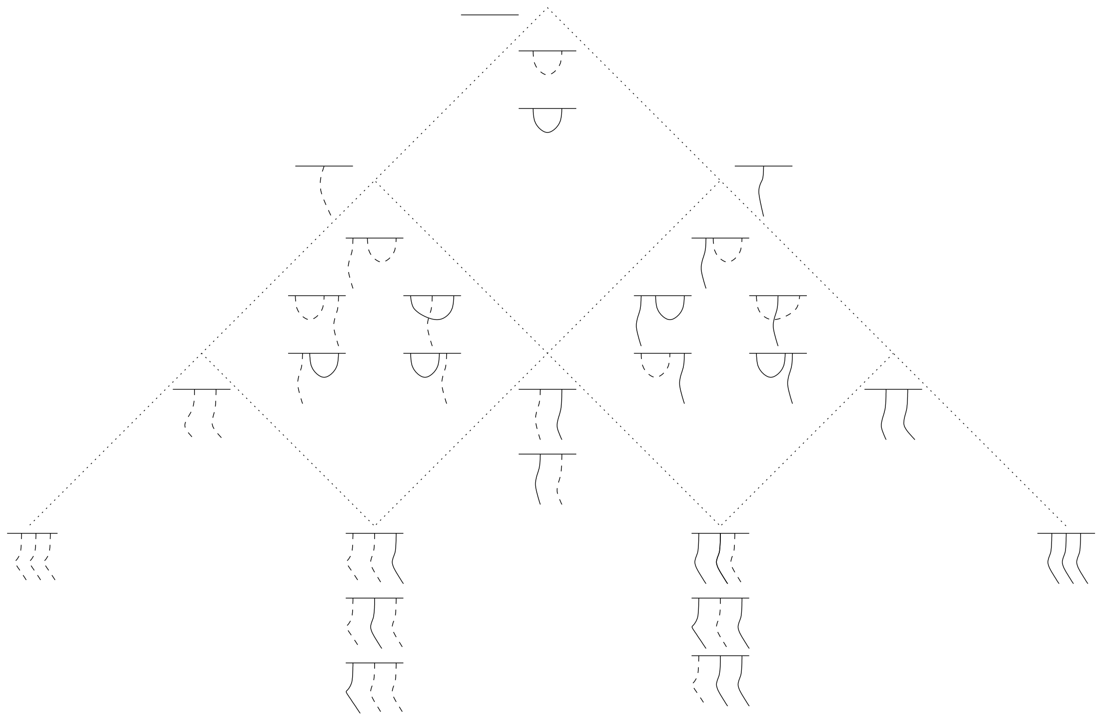
(4.8) A finite dimensional algebra (call it A) has, of course, finitely many classes of irreducible representations. Let $\mathcal{S}(A)$ be the set of these. Now suppose B is a subalgebra of A, with irreducibles $\mathcal{S}(B)$. We can restrict $R \in \mathcal{S}(A)$ to be a representation of B (every element of B has a representation matrix in R since $B \subset A$). As a representation of B this R will not in general be irreducible — in general it will be made up of a sum of one or more B irreducibles (we say, one or more factors). A *Bratteli diagram* of the pair (A,B) is a graph with a vertex for each element of $\mathcal{S}(A)$ and a vertex for each element of $\mathcal{S}(B)$ (the union of these vertices is usually but not always taken disjoint). If there is an edge between a and b say, then it means the restriction of $a \in \mathcal{S}(A)$ contains b as a factor. Thus in particular the fact that "the sum of the dimensions of all the factors of a is the dimension of a itself" becomes "the dimension at a is the sum of the dimensions of the nearest neighbours of a on the graph (possibly with multiplicities)".

A Bratteli diagram of a tower of algebras $A \supset B \supset C$... extends this in the obvious way. If the last algebra in this sequence is a one-dimensional algebra (with one 1d irrep, call it o) then the number of walks on the graph from o to point a will be the dimension of the irreducible a (so the set of these walks will be a basis for a). Again for the tower, $\mathcal{S}(A)$ and $\mathcal{S}(C)$ may (for ease of drawing, say) be allocated some vertices in common ('foreshortened' diagram). This does not spoil the counting if we are careful (each vertex corresponds to irreducibles in more than one algebra, but, on fixing a given algebra, that vertex becomes unambiguous).

(4.9) In our algebra we see that the standard modules play a special role, somewhat akin to that of simple modules, although over an arbitrary field they are not themselves necessarily simple.

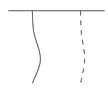
The Bratteli *basis diagram* of such an algebra is a certain embellished graph, each vertex of which corresponds to a standard module label (in this case the label consists of n and a propagating index λ); and each edge of which corresponds to a factor in the restriction of that module to $n-1$ (as above). Each vertex is embellished with a depiction of a basis for the corresponding standard module.

A *foreshortened* Bratteli basis diagram is a Bratteli basis diagram viewed from such a direction as to cause certain vertices to coincide. In our case it is those vertices for different n which have the same propagating index (thus a whole tower of embellishments will have to be drawn at the same point, but there are good reasons for this — see later). Such a diagram, where *possible* to draw, contains essentially complete information on the generic representation theory of the algebra. The (foreshortened) Bratteli basis diagram for T^2_- begins

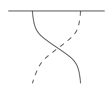


(11)

It may be helpful to emphasise that here the basis element



could also be written



— in the bra form these are equivalent, since the red and blue lines are not on the same bubble layer.

If an algebra is semisimple (as ours is generically) then its total dimension is the sum of the squares of the dimensions of the irreducibles. Here we see, for any K, that the total dimension is the sum of the squares of the dimensions of the standards. Thus the entire combinatoric is encoded in the pictures. Every possible diagram is built bra-ket from the kets in the foreshortened Bratteli diagram (and their descendents).

(4.10) In the ordinary Temperley-Lieb case it is the non-semisimple exceptions (q root of unity) which are of most interest. We conclude by setting up machinery to investigate this case (again paralleling Martin's usual approach [34, 37] to Temperley-Lieb and its generalisations).

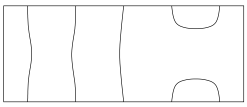
5. Categories, roots of unity, conformal series etc.

We may use a little category theory to very efficiently rederive the generic representation theory of T^2_n given above, in such a way that it can readily be extended to the exceptional cases.

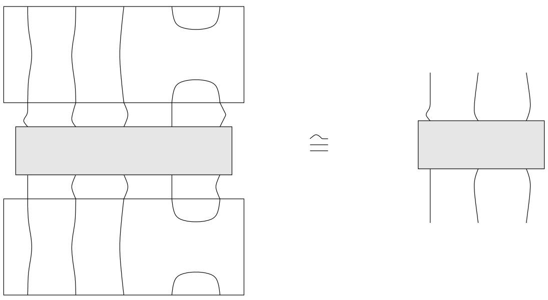
Recall U_i from section 4 and let $U = U_{n-1}$. Note that

$$UT^2_nU \cong T^2_{n-2}. \tag{12}$$

A diagrammatic version of this follows from the representation of U by



(all lines 'white') so that



Note that U commutes with $T^2_{n-2} \subset T^2_n$ so that T^2_nU is a left T^2_n right T^2_{n-2}-bimodule. Let $F : T^2_n\text{-mod} \to T^2_{n-2}\text{-mod}$ be the functor

$$F : M \to UM$$

and $G : T^2_{n-2}\text{-mod} \to T^2_n\text{-mod}$

$$G : N \to T^2_nU \otimes_{T^2_{n-2}} N\,.$$

It follows from (12) that $FG = 1_{T^2_{n-2}\text{-mod}}$. Thus (so long as U may be normalised as an idempotent) we have a full embedding of the category T^2_{n-2}-mod in T^2_n-mod. The simple modules L in T^2_n-mod *not* hit in this embedding are those for which $UL = 0$. That is, they are also the simple modules of the quotient algebra $T^2_n/T^2_nUT^2_n$.

To reiterate, equation (12) gives what is called a full embedding of T^2_{n-2} in T^2_n. This means that there is a natural injection of $\mathcal{S}(T^2_{n-2})$ into $\mathcal{S}(T^2_n)$ — another reason for the foreshortening of the Bratteli diagram. It also means that most of the representation theory of T^2_n follows from that of T^2_{n-2} via a little elementary category theory — and hence inductively from the trivial cases T^2_0 and T^2_1.

Note that $T^2_nUT^2_n$ includes every diagram except those with exactly n propagating lines. Thus $T^2_n/T^2_nUT^2_n$ is spanned by the set $B_n(n)$ of diagrams with exactly n propagating lines. Let Γ_n denote an index set for the simple modules of T^2_n, and Λ_n an index set for the quotient algebra $T_n^2/T_n^2UT_n^2$. So long as U may be renormalised as an idempotent it follows that

$$\Gamma_n = \Gamma_{n-2} \cup \Lambda_n \tag{13}$$

where the union is disjoint. The full embedding allows us to inject $\Gamma_{n-2} \hookrightarrow \Gamma_n$ in precisely the way implied by the foreshortening of our foreshortened Bratteli diagram.

By equation (13), we know Γ_n if we know Λ_m for all m. We now determine this set.

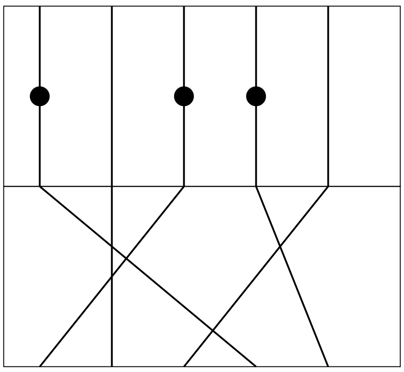
5.1. The subalgebra generated by $B_n(n)$

Note that $KB_n(n)$ is a subalgebra of T_n^2. Let T_n' denote this subalgebra, then

$$T_n' \hookrightarrow T_n^2 \twoheadrightarrow T_n^2/T_n^2UT_n^2$$

is a sequence of algebra morphisms. The composite morphism takes an element of $B_n(n)$ to the same object regarded as a basis element of $T_n^2/T_n^2UT_n^2$, hence it is an isomorphism. Provided our ground field K has characteristic different from two (we are mainly interested in $\mathbb{C}$, with characteristic zero, of course) this algebra may be identified with a certain quotient of the wreath product group algebra $KC_2 \wr S_n$, where C_2 denotes the cyclic group of order two and S_n the symmetric group (permutation group of n elements) of order $n!$, as follows.

Elements of $C_2 \wr S_n$ may be represented in the form of permutation diagrams where each line carries zero or one beads:



(i.e. there are $2^n n!$ such diagrams). The rule of composition is then as for ordinary permutations except that the number of beads on a single line is reduced modulo 2.

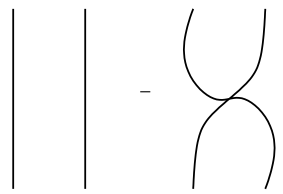
Such a diagram d in which some line is replaced by a (beadless) red (resp. blue) line is to be understood as the linear combination

$$d' = \frac{d_0 \pm d_1}{2} \tag{14}$$

where d_0, d_1 denote the diagram with that line having zero/one beads respectively. Replacing all lines in this way, in all possible ways, we have another basis of $KC_2 \wr S_n$ consisting of all possible two-colourings of permutations. The composition rule here (in consequence of (14)) is to compose permutations by juxtaposition as usual, except that if two different coloured lines are juxtaposed the composite is zero.

Consider the subset of this basis consisting of elements in which two lines may cross only if their colour is distinct. This is a basis for a subalgebra (to see this again consider the different coloured lines as living in two different layers — within a layer there are no crossings, and this is not affected by composition). Indeed it will be evident that this subalgebra may be *identified* with T'_n.

Now define two linear combinations in $KC_2 \wr S_2$, each of shape



but one with all lines coloured red, one with all lines coloured blue. Note that these are (unnormalised) idempotents. Define algebra H^2_n to be the quotient of $KC_2 \wr S_n$ (any $n > 1$) by both these objects.

It is well known that the irreducible representations of $KC_2 \wr S_n$ over $K = \mathbb{C}$ are indexed by pairs of integer partitions of combined degree n, and that the idempotents above are the primitive and central idempotents of $KC_2 \wr S_2$ corresponding to the one-dimensional irreducible representations indexed by $((1^2),)$ and $(, (1^2))$. It follows (after a little work, see [38]) that the index set for irreducibles of the quotient H^2_n is the subset of the set of pairs of integer partitions in which no partition has a second row. It follows similarly that the Bratteli diagram for the tower of these algebras as n varies is the Pascal triangle, i.e. the simples for given n lie in the n^{th} layer of the Pascal triangle.

Note that $B_n(n)$ may be regarded as a basis for H^2_n, since in H^2_n any two lines of the same colour which are crossed may be replaced by the same two lines uncrossed (i.e. a local implementation of the quotient by the diagram above in that colour). Now consider the sequence of algebra morphisms

$$T'_n \hookrightarrow KC_2 \wr S_n \twoheadrightarrow H^2_n$$

The image of a basis element in $B_n(n)$ under the composite map is the same element regarded as a basis element of H^2_n. Thus the composite is an isomorphism.

We have established a sequence of isomorphisms which allows us to identify the index set for simple modules of H^2_n with Λ_n. This reproduces the layer of the Pascal triangle in our original foreshortened Bratteli diagram and, taken layer by layer, using equation (13) reproduces the whole foreshortened Bratteli diagram.

5.2. On the exceptional structure of T^2_n

The rb-sequence of a diagram is the sequence of colours of strings, read off clockwise from the top left hand corner of the frame. The standard basis $B_n(\mu)$ may be partitioned into subsets of elements with the same rb-sequence, called rb-parts. Since colours are orthogonal, the inner product is zero on any pair from $B_n(\mu)$ unless they lie in the same

rb-part. Thus the determinant of the Gram matrix is a product of corresponding block determinants.

It will be evident that the block *determinant* depends only on the number of r's and b's, not their order in sequence. It is thus straightforward to determine the roots of the Gram determinants (which, we recall, are polynomials in δ_r, δ_b), as follows.

For $B_n(i,j)$ with $i+j=n$ we have $|G_n(i,j)|=1$. (Recall that a module is simple unless its Gram matrix is singular, thus all these modules are simple.) For $B_n(i,j)$ with $i+j=n-2$ all the lines but one are propagating, so *all* lines of one colour (red or blue) are propagating. For example, the $rrrrbb$ part of $B_6(2,2)$ has Gram block

δ_r	1	0
1	δ_r	1
0	1	δ_r

We see that the colour with all lines propagating plays no role, and that the Gram block coincides, in this example, with the $(3,1)$ Gram matrix of ordinary $T_4(\delta_r)$. Thus the set of roots of *any* such Gram determinant must be taken from the roots of the Gram determinants of $T_n(\delta_r)$ and $T_n(\delta_b)$. These are well known [33] to lie in the set of roots of unity (when expressed in terms of q_r, q_b) for each colour. That is,

Proposition 4 *If neither q_r nor q_b is a root of 1 then every module $\Delta_n(i,j)$ with $i+j=n-2$ is simple.*

Proposition 5 *If either q_r or q_b is a root of 1 then there is an n such that T_n^2 is not semisimple.*

Now suppose that for some choice of K some standard module, $\Delta_n(\mu)$ say, is not simple (as already noted, this has to happen for the algebra to fail to be semisimple). Then in particular this module has some simple module, $L(\lambda)$ say, in its socle. Take n to be at its lowest value such that this occurs.

It is easy to see that both the localisation functor F and the globalisation functor G take a standard module to a standard module with the same label (or 0 if no such module exists, in case of F). It is also easy to see that every simple module occurs as the head of some standard module. Thus the nonsimplicity of our standard $\Delta_n(\mu)$ must show up as a morphism of standard modules between that having the simple $L(\lambda)$ as its head (we might as well call it $\Delta_n(\lambda)$), and $\Delta_n(\mu)$. If neither μ nor λ has $i+j=n$ then we can localise until one of them does, whereupon the corresponding standard is simple by our earlier analysis, thus this simple standard must be $\Delta_n(\lambda)$, and $\Delta_n(\mu)$ must have $i+j<n$, that is to say, $\mu_1+\mu_2<n$. But now suppose this $\mu_1+\mu_2<n-2$ (and there is not a suitable choice of μ with $\mu_1+\mu_2=n-2$). Consider the following *Frobenius reciprocity*:

$$\mathrm{Hom}(\mathrm{Ind}^n_{n-1}\Delta_{n-1}(\lambda_1,\lambda_2-1),\Delta_n(\mu)) \cong \mathrm{Hom}(\Delta_{n-1}(\lambda_1,\lambda_2-1),\mathrm{Res}^n_{n-1}\Delta_n(\mu))$$

It is straightforward to show that $\Delta_n(\lambda)$ appears in the head of the induced module $\mathrm{Ind}^n_{n-1}\Delta_{n-1}(\lambda_1,\lambda_2-1)$. Thus the left hand hom space is not empty. But then neither is the right, and we have a nontrivial homomorphism of distinct standard modules at level $n-1$ also. This is a *contradiction* of our construction that n is the lowest value for which such a homomorphism occurs. Thus we *may not* suppose that $\mu_1+\mu_2<n-2$, i.e., we must take $\mu_1+\mu_2=n-2$. In other words, the first occurrence of such a morphism must be into a standard module with this type of label. But by proposition 4 such a morphism is only possible if at least one of q_r, q_b is a root of unity. We have established

Proposition 6 *If neither q_r nor q_b is a root of 1, then every module $\Delta_n(i,j)$ is simple, and T^2_n is semisimple.*

The determination of the complete structure of T^2_n when the parameters are roots of unity remains for now an open problem. It should be amenable to the methods we have developed, but it seems possible that considerably more donkey work remains. Given the connection between the ordinary case and conformal representation theory (cf. [30, 23, 37]) the answer should raise some interesting issues.

6. Conclusions and discussion

We have constructed the generic irreducible modules of T^2_n. Every other module (such as occurs in transfer matrices) can be built as a sum of these.§ Thus the spectrum of any transfer matrix will be (up to multiplicities) the union of the spectra computed using these smallest possible modules. This analysis thus provides the most efficient tools for explicit computation (modulo any overarching constraints imposed in practice by, for example, implementation of the Bethe ansatz), cf. [2, §12.4]. Note that it also tells us a convenient labelling scheme for types of correlation functions. The pair label (i,j) here replaces the charge sector label relevant for Bethe ansatz in ordinary spin-chains [2, §8.4]. That is, we have two naive pseudoparticle types.

In case the second colour is merely to be regarded as a dilution (i.e. it just acts as a placeholder), direct contributions to correlations involving this colour would be trivial or ignored. In more general settings we have the possibility of correlations in which distinct operators cross over in the plane (a feature which cannot occur in models built

§ Or in non-semisimple cases as a not necessarily direct sum of their simple heads.

from the ordinary Temperley-Lieb algebra, such as ordinary Potts models and Ising models). This 'thickening' of the underlying plane lattice provides scope for considering a number of further generalisations to more exotic 2d models and possibly also to 3d. We will return to these possibilities in a subsequent paper.

Certainly T_n^2 has a number of relatively obvious generalisations (T_n^N, $N = 3, 4, ..$, and certain generalisations to more exotic underlying spaces) for which the corresponding generalised analysis goes through directly. Work is in progress to find generalisations of Grimm and Pearce's original idea accordingly.

It is interesting that the exceptional structure of T_n^2 is tied to the same special parameter choices as are already widely familiar for 2d systems — q a root of unity — even though our models have multiple parameters. This contrasts sharply with direct attempts to generalise to 3d, such as the partition algebra, for which a completely different set of exceptional cases occur [34].

Finally we note two points of interest in representation theory. The new algebras have features reminiscent of a recent conjecture (see [38], cf. our section 5.1) for a basis of generalised blob algebras. These algebras have been used recently to probe the physically relevant part of the representation theory of affine Hecke algebras [6], so the connection here is intriguing. Secondly, we observe that there is no known diagram calculus for the Hecke algebra quotient associated to $U_q sl_3$ (in the sense that the Temperley-Lieb algebra is a Hecke algebra quotient associated to $U_q sl_2$). Indeed there is no calculus for sl_N for any N but 2. Such a calculus has long been sought, and would be enormously useful in a number of areas of representation theory. As a generalisation of the sl_2 case, our calculus provides some intriguing clues for sl_3 (although it is *not* itself an sl_3 calculus).

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