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SIMILAR SUBLATTICES AND COINCIDENCE ROTATIONS OF THE ROOT LATTICE A_4 AND ITS DUAL

MANUELA HEUER

Abstract. A natural way to describe the Penrose tiling employs the projection method on the basis of the root lattice A_4 or its dual. Properties of these lattices are thus related to properties of the Penrose tiling. Moreover, the root lattice A_4 appears in various other contexts such as sphere packings, efficient coding schemes and lattice quantizers.

Here, the lattice A_4 is considered within the icosian ring, whose rich arithmetic structure leads to parametrisations of the similar sublattices and the coincidence rotations of A_4 and its dual lattice. These parametrisations, both in terms of a single icosian, imply an index formula for the corresponding sublattices. The results are encapsulated in Dirichlet series generating functions. For every index, they provide the number of distinct similar sublattices as well as the number of coincidence rotations of A_4 and its dual.

1. Introduction and generalities

A *lattice* Γ in $\mathbb{R}^d$, denoted by $\Gamma = \langle b_1, \ldots, b_d \rangle_{\mathbb{Z}}$, consists of all integer linear combinations of a basis $\{b_1, \ldots, b_d\}$ of $\mathbb{R}^d$. In crystallography, coincidence site lattices of a lattice Γ are used to describe and understand grain boundaries; compare [2] and references given there. Mathematically, a *coincidence site lattice* (CSL) of a lattice Γ is defined as $\Gamma \cap R\Gamma$, where R is an isometry and the corresponding *coincidence index*

$$\Sigma(R) = [\Gamma : (\Gamma \cap R\Gamma)]$$

is finite, i.e. R and $R\Gamma$ share a common sublattice. Denote the orthogonal group by $\mathrm{O}(d, \mathbb{R})$ and define the set of all *coincidence isometries* as

$$\mathrm{OC}(\Gamma) := \{R \in \mathrm{O}(d, \mathbb{R}) \mid \Sigma(R) < \infty\}.$$

According to [2], it is a subgroup of $\mathrm{O}(d, \mathbb{R})$. $\mathrm{OC}(\Gamma)$ and $\mathrm{SOC}(\Gamma)$, which is the subgroup of all coincidence rotations, can be characterised as subgroups of another important group of isometries, namely

$$\mathrm{OS}(\Gamma) := \{R \in \mathrm{O}(d, \mathbb{R}) \mid \alpha R\Gamma \subset \Gamma, \alpha \in \mathbb{R}, \alpha > 0\}.$$

This group consists of all isometries that emerge form similarity mappings of the lattice Γ into itself; see [3] for details. Its subgroup of rotations, denoted by $\mathrm{SOS}(\Gamma)$, contains $\mathrm{SOC}(\Gamma)$ as a normal subgroup; see [10] for a detailed analysis of their relation. A sublattice of Γ of the form $\alpha R\Gamma$ is called a *similar sublattice* (SSL) of Γ and has obviously the index $[\Gamma : \alpha R\Gamma] = \alpha^d$. If the lattice Γ possesses a rich point symmetry, there are usually interesting SSLs, beyond the trivial ones of the form $m\Gamma$ with $m \in \mathbb{N}$. For $R \in \mathrm{OS}(\Gamma)$ define the *denominator* relative to the lattice Γ as

$$\mathrm{den}_{\Gamma}(R) := \min\{\alpha \in \mathbb{R} \mid \alpha > 0,\ \alpha R\Gamma \subset \Gamma\}.$$

As R is an isometry, one always has $\mathrm{den}_{\Gamma}(R) \geq 1$, and from $\mathrm{den}_{\Gamma}(R) R\Gamma \subset \Gamma$ one concludes that $\big(\mathrm{den}_{\Gamma}(R)\big)^d$ is an integer. Consequently, $\mathrm{den}_{\Gamma}(R)$ is either a positive integer or an irrational number, but still algebraic and of degree $\leq d$. Moreover, one has

$$\{\alpha \in \mathbb{R} \mid \alpha > 0,\ \alpha R\Gamma \subset \Gamma\} \,=\, \mathrm{den}_{\Gamma}(R)\, \mathbb{N}\,.$$

This leads to the following characterisation of $\mathrm{OC}(\Gamma)$ within $\mathrm{OS}(\Gamma)$, compare [3] for details:

$$\mathrm{OC}(\Gamma) = \{R \in \mathrm{OS}(\Gamma) \mid \mathrm{den}_{\Gamma}(R) \in \mathbb{N}\}. \tag{1}$$

When a lattice $\Gamma \subset \mathbb{R}^d$ is given, one is interested in the number of distinct SSLs of Γ of index n as well as in the number of distinct CSLs of Γ of index n. If these arithmetic functions are multiplicative, they are often encapsulated into Dirichlet series generating functions, because of their Euler product decomposition. A detailed introduction to arithmetic functions, the corresponding Dirichlet series and Euler products can be found in [1]. For many lattices in $d \leq 4$ the arithmetic functions which count the number of distinct SSLs and CSLs of each index have already been derived; see for instance [2, 6, 7]. One lattice for which this problem has not been solved completely is the root lattice A_4. This lattice is of particular interest, because it forms the natural setting, in the sense of a minimal embedding, for the description of the Penrose tiling as a cut and project set, see for example [5]. Properties of the root lattice A_4 are thus directly related to the Penrose tiling. Some of the various other applications of the root lattice A_4 are described in [9].

In this paper recent results about the similar sublattices and coincidence rotations of the root lattice A_4 are reviewed and extended to its dual lattice A_4^*; details can be found in [4, 3]. In the following sections we first describe the root lattice A_4 and its dual A_4^* in a suitable setting. Then we derive their Dirichlet series generating functions for the number of distinct SSLs of index n, which turn out to be the same for A_4 and A_4^*. For these lattices it is particularly difficult to determine the Dirichlet series generation functions for the number of distinct CSLs of index n. Therefore, we consider the slightly simpler problem to derive the generating functions for the number of distinct coincidence rotation of index n, which turn out to coincide, too.

2. The root lattice A_4

The root lattice A_4 is usually defined as

$$A_4 := \{(x_1, \ldots, x_5) \in \mathbb{Z}^5 \mid x_1 + \ldots + x_5 = 0\},$$

which lies in a 4−dimensional hyperplane of $\mathbb{R}^5$. One description of A_4 as a lattice in $\mathbb{R}^4$ is given by

$$L := \big\langle (1,0,0,0), \tfrac{1}{2}(-1,1,1,1), (0,-1,0,0), \tfrac{1}{2}(0,1,\tau{-}1,-\tau) \big\rangle_{\mathbb{Z}}$$

where $\tau = (1{+}\sqrt{5})/2$ is the golden ratio. Relative to the inner product $\mathrm{tr}(x\bar{y}) = 2\langle x|y\rangle$, where $\langle x|y\rangle$ denotes the standard Euclidean inner product, this lattice is the root lattice A_4; see [8, 4, 3] for details. This particular description of the root lattice A_4 in $\mathbb{R}^4$ is very convenient for our problem, as it enables us to use the arithmetic of the quaternion algebra $\mathbb{H}(\mathbb{Q}(\sqrt{5}\,))$; see [12] for a detailed introduction to Hamilton's quaternions. For brevity we use from now on the notation

$$K := \mathbb{Q}(\sqrt{5}\,) = \{q + r\sqrt{5} \mid q, r \in \mathbb{Q}\},$$

which is a quadratic number field. The algebra $\mathbb{H}(K)$ is explicitly given as

$$\mathbb{H}(K) = K \oplus \mathrm{i}K \oplus \mathrm{j}K \oplus \mathrm{k}K,$$

where the generating elements satisfy Hamilton's relations $\mathrm{i}^2 = \mathrm{j}^2 = \mathrm{k}^2 = \mathrm{ijk} = -1$. It is equipped with a *conjugation* $\bar{\cdot}$ which is the unique mapping that fixes the elements of the centre of the algebra K and reverses the sign on its complement. If we write

$$q = (a, b, c, d) = a + \mathrm{i}b + \mathrm{j}c + \mathrm{k}d, \quad \text{this means} \quad \bar{q} = (a, -b, -c, -d).$$

The reduced norm and trace in $\mathbb{H}(K)$ are defined by

$$\mathrm{nr}(q) := q\bar{q} = |q|^2 \quad \text{and} \quad \mathrm{tr}(q) := q + \bar{q},$$

where we canonically identify an element $\alpha \in K$ with the quaternion $(\alpha, 0, 0, 0)$. For any $q \in \mathbb{H}(K)$, $|q|$ is its Euclidean length, which need not be an element of K. Nevertheless, one has $|rs| = |r||s|$ for arbitrary $r, s \in \mathbb{H}(K)$. Due to the geometric meaning, we use the notations $|q|^2$ and $\mathrm{nr}(q)$ in parallel. An element $q \in \mathbb{H}(K)$ is called *integral* when both $\mathrm{nr}(q)$ and $\mathrm{tr}(q)$ are elements of

$$\mathfrak{o} := \mathbb{Z}[\tau] := \{m + n\tau \mid m, n \in \mathbb{Z}\},$$

which is the ring of integers of the quadratic field K.

The *icosian ring* $\mathbb{I}$ consists of all linear combinations of the vectors

$$(1,0,0,0), (0,1,0,0), \tfrac{1}{2}(1,1,1,1), \tfrac{1}{2}(1-\tau, \tau, 0, 1)$$

with coefficients in $\mathfrak{o}$. The elements of $\mathbb{I}$ are called *icosians.* It is a remarkable object; see for example [15, 13, 14, 4] and references given there. In particular, $\mathbb{I}$ is a maximal order in $\mathbb{H}(K)$, all elements of $\mathbb{I}$ are integral in $\mathbb{H}(K)$ and the lattice L is contained in $\mathbb{I}$. Since $\mathbb{H}(K)$ has class number 1, compare [15, 16], all ideals of $\mathbb{I}$ are principal. The detailed arithmetic structure of $\mathbb{I}$ is the key to the characterisation of the similar sublattices and coincidence rotations [4, 3] for L. What is more, one significantly profits from another map, called the *twist map* in [4], which is an involution of the second kind for $\mathbb{H}(K)$; see [11] for details. If $q = (a, b, c, d)$, it is defined by the mapping

$$q \mapsto \widetilde{q} := (a', b', d', c'),$$

where $'$ denotes the algebraic conjugation in K, as defined by the mapping $\sqrt{5} \mapsto -\sqrt{5}$. The algebraic conjugation in K is also needed to define the absolute norm on K, via $\mathrm{N}(\alpha) = |\alpha\alpha'|$. For the various properties of the twist map, we refer the reader to [4] and references therein. The most important ones in our present context are summarised in the following Lemma, whose proof can be found in [4]. It describes the relations between L and $\mathbb{I}$.

Lemma 2.1. *Within $\mathbb{H}(K)$, one has $\widetilde{\mathbb{I}} = \mathbb{I}$ and*

$$L = \{x \in \mathbb{I} \mid \widetilde{x} = x\} = \{x + \widetilde{x} \mid x \in \mathbb{I}\} = \phi_+(\mathbb{I}),$$

where the $\mathbb{Q}$-linear mapping $\phi_+ \colon \mathbb{H}(K) \longrightarrow \mathbb{H}(K)$, is defined by $\phi_+(x) = x + \widetilde{x}$.

The dual A_4^* of the the root lattice A_4, here in form of the dual of the lattice L, is given by

$$L^* := \{x \in \mathbb{R}^4 \mid \langle x|y\rangle \in \mathbb{Z} \text{ for all } y \in L\}.$$

3. Similar Sublattices

We are interested in the SSLs of the root lattice A_4 and its dual lattice A_4^*. According to [4], there is an index preserving bijection between the SSLs of A_4^* and A_4, as well as between the SSLs of A_4 and L. Therefore, it is sufficient to concentrate on the SSLs of the lattice L. For convenience, we define $\mathbb{H}(K)^\bullet = \mathbb{H}(K) \setminus \{0\}$. Proofs for this section can be found in [4].

Lemma 3.1. *All SSLs of the lattice L are images of L under orientation preserving mappings of the form $x \mapsto pxq$, with $p, q \in \mathbb{H}(K)^\bullet$.*

This characterisation tells us that we only need to select an appropriate subset of $\mathbb{H}(K)^\bullet \times \mathbb{H}(K)^\bullet$ in order to reach all SSLs of L. A first step is provided by the following observation.

Lemma 3.2. *If $p \in \mathbb{I}$, $pL\widetilde{p}$ is an SSL of L.*

On the other hand there is the following strengthening of Lemma 3.1.

Proposition 3.3. *If $pLq \subset L$ with $p, q \in \mathbb{H}(K)^\bullet$, there is an $\alpha \in \mathbb{Q}$ such that $q = \alpha\,\widetilde{p}$.*

To improve this characterisation we need the following definitions. An element $p \in \mathbb{I}$ is called $\mathbb{I}$-*primitive* when $\alpha p \in \mathbb{I}$, with $\alpha \in K$, is only possible with $\alpha \in \mathfrak{o}$. Similarly, a sublattice $\varLambda$ of L is called L-*primitive* when $\alpha\varLambda \subset L$, with $\alpha \in \mathbb{Q}$, implies $\alpha \in \mathbb{Z}$. Whenever the context is clear, we simply use the term "primitive" in both cases.

Corollary 3.4. *All SSLs of the lattice L are images of mappings of the form $x \mapsto \alpha p x \widetilde{p}$ with $p \in \mathbb{I}$ primitive and $\alpha \in \mathbb{Q}$.*

We now need to understand how an SSL of L of the form $pL\widetilde{p}$ with an $\mathbb{I}$-primitive quaternion relates to the primitive sublattices of L.

Proposition 3.5. *If $p \in \mathbb{I}$ is $\mathbb{I}$-primitive, $pL\widetilde{p}$ is an L-primitive sublattice of L.*

Combining the results of Corollary 3.4 and Proposition 3.5, we obtain the following important observation.

Proposition 3.6. *A similar sublattice of L is primitive if and only if it is of the form $pL\widetilde{p}$ with a primitive element $p \in \mathbb{I}$. Moreover, all SSLs of L are of the form $qL\widetilde{q}$ with $q \in \mathbb{I}$.*

The next step is to find a suitable bijection that permits us to count the primitive SSLs of L of a given index. Recall from [14, 8] that the unit group of $\mathbb{I}$ has the form

$$\mathbb{I}^\times \;=\; \{x \in \mathbb{I} \mid \mathrm{N}(\mathrm{nr}(x)) = \pm 1\},$$

which leads to the following equivalence.

Lemma 3.7. *For $p \in \mathbb{I}$, one has $pL\widetilde{p} = L$ if and only if $p \in \mathbb{I}^\times$.*

Observe now that $p\mathbb{I} = q\mathbb{I}$ with $p, q \in \mathbb{I}$ holds if and only if $q^{-1}p \in \mathbb{I}^\times$. The relevance of this fact in our context comes from the observation that

$$[\mathbb{I} : p\mathbb{I}] \;=\; \mathrm{N}\bigl(|p|^4\bigr) \;=\; [L : pL\widetilde{p}\,]\,,$$

where the index of $p\mathbb{I}$ in $\mathbb{I}$ follows from the determinant formula in [4, Fact 3] followed by taking the norm in $\mathbb{Z}[\tau]$. This means that $p\mathbb{I} \longleftrightarrow pL\widetilde{p}$ describes an index preserving bijection between primitive right ideals of $\mathbb{I}$ (meaning right ideals $q\mathbb{I}$ with $q \in \mathbb{I}$ primitive) and primitive SSLs of L. Due to the definition of L-primitivity, a general SSL can be described as an integer multiple of a primitive SSL. This leads together with the observation that all possible indices of SSLs are squares, to the following central result.

Theorem 3.8. *The number of SSLs of a given index m is the same for the lattices A_4^*, A_4 and L. The possible indices are the squares of non-zero integers of the form $k^2 + k\ell - \ell^2$. If $f_{\mathrm{SSL}}(m)$ denotes the number of SSLs of index m^2, the corresponding Dirichlet series generating function reads*

$$D_{\mathrm{SSL}}(s) \;:=\; \sum_{m=1}^{\infty} \frac{f_{\mathrm{SSL}}(m)}{m^{2s}} \;=\; \zeta(4s)\,\frac{\zeta_{\mathbb{I}}(s)}{\zeta_K(4s)},$$

where $\zeta(s) = \prod\limits_{p} \frac{1}{1-p^{-s}}$ is the Riemann zeta function,

$$\zeta_K(s) = \tfrac{1}{1-5^{-s}} \prod_{p\equiv\pm2(5)} \tfrac{1}{1-p^{-2s}} \prod_{p\equiv\pm1(5)} \tfrac{1}{(1-p^{-s})^2}$$

is the Dedekind zeta function of the quadratic field K, and $\zeta_{\mathbb{I}}(s) = \zeta_K(2s)\,\zeta_K(2s-1)$ denotes the Dirichlet series for the right ideals of $\mathbb{I}$, which is the zeta function of $\mathbb{I}$.

Inserting the Euler products of $\zeta(s)$ and $\zeta_K(s)$, one finds the expansion of the Dirichlet series $D_{\mathrm{SSL}}(s)$ as an Euler product

$$D_{\mathrm{SSL}}(s) \;=\; \tfrac{1}{(1-5^{-2s})(1-5^{1-2s})} \prod_{p\equiv\pm1\ (5)} \tfrac{1+p^{-2s}}{1-p^{-2s}}\,\tfrac{1}{(1-p^{1-2s})^2} \prod_{p\equiv\pm2\ (5)} \tfrac{1+p^{-4s}}{1-p^{-4s}}\,\tfrac{1}{1-p^{2-4s}}.$$

Consequently, the arithmetic function $f_{\mathrm{SSL}}(m)$ is multiplicative and therefore completely specified by its values at prime powers p^r with $r \geq 1$. These are given by

$$f_{\mathrm{SSL}}(p^r) \;=\; \begin{cases} \frac{5^{r+1}-1}{4}, & p = 5, \\ \frac{2\,(1-p^{r+1})-(r+1)(1-p^2)p^r}{(1-p)^2}, & p \equiv \pm1\ (5), \\ \frac{2-p^r-p^{r+2}}{1-p^2}, & p \equiv \pm2\ (5),\ r \text{ even}, \\ 0, & p \equiv \pm2\ (5),\ r \text{ odd}. \end{cases}$$

The first few terms of the Dirichlet series thus read

$$D_{SSL}(s) \;=\; 1 + \tfrac{6}{4^{2s}} + \tfrac{6}{5^{2s}} + \tfrac{11}{9^{2s}} + \tfrac{24}{11^{2s}} + \tfrac{26}{16^{2s}} + \tfrac{40}{19^{2s}} + \tfrac{36}{20^{2s}} + \tfrac{31}{25^{2s}} + \tfrac{60}{29^{2s}} + \tfrac{64}{31^{2s}} + \tfrac{66}{36^{2s}} + \ldots$$

where the denominators are the squares of the integers previously identified in [9].

4. Coincidences and Rotations

According to [2, Th. 2.2], $\mathrm{OC}(A_4) = \mathrm{OC}(A_4^*)$ and the coincidence index of any conincidence isometry is the same for both lattices. Therefore, it is sufficient to investigate the CSLs of the lattice A_4. The best representation of this lattice is again the lattice L. First of all the investigation can be restricted to coincidence rotations without missing any CSLs, because $\overline{L} = L$, i.e. any orientation reversing operation can be obtained from an orientation preserving one after applying conjugation first. According to Proposition 3.6, a given SSL of L is of the form $qL\widetilde{q}$, with $q \in \mathbb{I}$. The corresponding rotation is given by the mapping $x \mapsto \frac{1}{|q\widetilde{q}|}\, qx\widetilde{q}$. Of course, many different icosians q result in the same rotation. Our aim is to restrict q to suitable subsets of $\mathbb{I}$ without missing any rotation. In general, $\mathrm{SOC}(L)$ and $\mathrm{SOS}(L)$ are related according to Eq. (1). Therefore, we have to identify the elements of $\mathrm{SOC}(L)$ within $\mathrm{SOS}(L)$. We would like to refer the reader to [3] for the proofs of this section.

Lemma 4.1. *Let $0 \neq q \in \mathbb{I}$ be an arbitrary icosian. The lattice $L \cap \frac{1}{|q\widetilde{q}|}\, qL\widetilde{q}$ is a CSL of L if and only if $|q\widetilde{q}| \in \mathbb{N}$.*

Let us call an icosian $q \in \mathbb{I}$ *admissible* when $|q\widetilde{q}| \in \mathbb{N}$. As $\mathrm{nr}(\widetilde{q}) = \mathrm{nr}(q)'$, the admissibility of q implies that $\mathrm{N}\big(\mathrm{nr}(q)\big)$ is a square in $\mathbb{N}$. With this definition the CSLs of L can be characterised as follows.

Theorem 4.2. *The CSLs of L are precisely the lattices of the form $L \cap \frac{1}{|q\widetilde{q}|}\, qL\widetilde{q}$ with $q \in \mathbb{I}$ primitive and admissible.*

This is the first step to define a bijection between certain primitive right ideals $q\mathbb{I}$ of the icosian ring and the CSLs of L. The next step in this direction is the following Lemma.

Lemma 4.3. *Let $r, s \in \mathbb{I}$ be primitive and admissible icosians, with $r\mathbb{I} = s\mathbb{I}$. Then, one has*

$$L \cap \frac{rL\tilde{r}}{|r\tilde{r}|} = L \cap \frac{sL\tilde{s}}{|s\tilde{s}|}.$$

For our further discussion we need to replace the primitive and admissible icosian p, and with it $\tilde{p}$, by certain $\mathfrak{o}$-multiples, such that their norms have the same prime divisors in $\mathfrak{o}$. In view of the form of the rotation $x \mapsto \frac{1}{|p\widetilde{p}|}\, px\widetilde{p}$, this is actually rather natural because it restores some kind of symmetry of the expressions in relation to the two quaternions involved. For a primitive and admissible icosian $p \in \mathbb{I}$ we choose explicitly

$$\alpha_q = \sqrt{\frac{\mathrm{lcm}(\mathrm{nr}(q), \mathrm{nr}(\widetilde{q}\,))}{\mathrm{nr}(q)}} \in \mathfrak{o}\,, \tag{2}$$

where we assume a suitable standardisation for the lcm of two elements of $\mathfrak{o}$; see [3] for details. Moreover, we have the relation $\alpha_{\tilde{q}} = \widetilde{\alpha_q} = \alpha_q'$. The icosian $\alpha_q q$ is called the *extension* of the primitive admissible element $q \in \mathbb{I}$, and $(\alpha_q q, \alpha_q' \widetilde{q}\,)$ the corresponding *extension pair*. Since α_q and α_q' are central, the extension does not change the rotation, i.e.

$$\frac{qx\widetilde{q}}{|q\widetilde{q}|} = \frac{q_\alpha x \widetilde{q_\alpha}}{|q_\alpha \widetilde{q_\alpha}|}$$

holds for all quaternions x. Note that the definition of the extension pair is unique up to units of $\mathfrak{o}$, and that one has the relation

$$\mathrm{nr}(q_\alpha) = \mathrm{lcm}\big(\mathrm{nr}(q), \mathrm{nr}(\widetilde{q}\,)\big) = \mathrm{nr}(\widetilde{q_\alpha}) = |q_\alpha \widetilde{q_\alpha}| \in \mathbb{N}\,,$$

which is crucial in the proof of the following Theorem. For the further characterisation of the CSLs of L, it is convenient to define the set

$$L(q) = \{qx + \widetilde{x}\widetilde{q} \mid x \in \mathbb{I}\} = \phi_+(q\mathbb{I})\,, \tag{3}$$

which is a sublattice of L, compare Lemma 2.1. Note that, due to $\widetilde{\mathbb{I}} = \mathbb{I}$, one has $L(q) = \widetilde{L(q)}$.

Theorem 4.4. *Let $q \in \mathbb{I}$ be admissible and primitive, and let $q_\alpha = \alpha_q q$ be its extension. Then, the CSL defined by q is given by*

$$L \cap \frac{1}{|q\widetilde{q}|} qL\widetilde{q} = L(q_\alpha)\,,$$

with $L(q_\alpha)$ defined as in Eq. (3).

This explicit identification of the CSL provides access to the corresponding index

Theorem 4.5. *If $q \in \mathbb{I}$ is an admissible primitive icosian, the rotation $x \mapsto \frac{1}{|q\widetilde{q}|} qx\widetilde{q}$ is a coincidence isometry of L. Moreover, the corresponding coincidence index is*

$$\Sigma(q) \;=\; \mathrm{nr}(q_\alpha) \;=\; \mathrm{lcm}(\mathrm{nr}(q), \mathrm{nr}(q)')\,,$$

which is, with our above convention from Eq. (2), always an element of $\mathbb{N}$.

At this point, it is possible to determine the number of distinct coincidence rotations of index n, where n is a prime power. The rotations come in multiples of 120, the order of the rotation symmetry group of A_4.

Theorem 4.6. *The number of coincidence rotations of a given index n is the same for the lattices A_4^*, A_4 and L. Let $120 f_{\mathrm{SOC}}(n)$ be the number of coincidence rotations of index n. Then, $f_{\mathrm{SOC}}(n)$ is a multiplicative arithmetic function, given at prime powers p^r with $r \geq 1$ by*

$$f_{\mathrm{SOC}}(p^r) \;=\; \begin{cases} 6 \cdot 5^{2r-1}, & p = 5\,, \\ \frac{p+1}{p-1}\, p^{r-1}(p^{r+1} + p^{r-1} - 2), & p \equiv \pm 1 \; (5)\,, \\ p^{2r} + p^{2r-2}, & p \equiv \pm 2 \; (5)\,. \end{cases}$$

Its Dirichlet series generating function reads

$$\begin{aligned} D_{\mathrm{SOC}}(s) \;&=\sum_{n=1}^{\infty} \frac{f_{\mathrm{SOC}}(n)}{n^s} \;=\; \frac{\zeta_K(s-1)}{1+5^{-s}}\, \frac{\zeta(s)\,\zeta(s-2)}{\zeta(2s)\,\zeta(2s-2)} \\ &= \tfrac{1+5^{1-s}}{1-5^{2-s}} \prod_{p \equiv \pm 1\,(5)} \tfrac{(1+p^{-s})\,(1+p^{1-s})}{(1-p^{1-s})\,(1-p^{2-s})} \prod_{p \equiv \pm 2\,(5)} \tfrac{1+p^{-s}}{1-p^{2-s}} \\ &= 1 + \tfrac{5}{2^s} + \tfrac{10}{3^s} + \tfrac{20}{4^s} + \tfrac{30}{5^s} + \tfrac{50}{6^s} + \tfrac{50}{7^s} + \tfrac{80}{8^s} + \tfrac{90}{9^s} + \tfrac{150}{10^s} + \tfrac{144}{11^s} + \ldots, \end{aligned}$$

where $\zeta(s)$ is the Riemann zeta function and $\zeta_K(s)$ denotes the Dedekind zeta function of the quadratic field K; see Theorem 3.8 for their explicit expressions.

5. Outlook

Due to the factorisations of icosians into irreducible elements, which are difficult to access, we have not yet found a way to approach the number of CSLs of A_4 of index n systematically. Nevertheless, it is possible to derive the number of CSLs up to index 120, see [3]. We hope to report on some progress soon.

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Department of Mathematics and Statistics, The Open University,
Walton Hall, Milton Keynes MK7 6AA, UK
E-mail address: `m.heuer@open.ac.uk`